# Some Properties of Iterated Brownian Motion and Weak Approximation

**Shumpei Sakuragi**

**Abstract**

Algorithms by stochastic methods to partial differential equations of the fourth order involving biharmonic operators are stated. The author considered a construction of the solution of a partial differential equation using a certain probability space and stochastic process. There are two algorithms for the fourth-order partial differential equations by stochastic methods. The first one is the method using signed measures. This is a method which constructs a signed measure by a solution using the Fourier transform and obtains a coordinate mapping process. The second method uses iterated Brownian motion. The latter is treated in this paper. The definition of iterated Brownian motion was modified to investigate the properties of its distribution. The author also defined an iterated random walk corresponding to discretization of that, and showed that it converges to an iterated Brownian motion in law to the iterated Brownian motion, and obtained its order.In the conventional method, the partial differential equation of the fourth order corresponding to iterated Brownian motion, the Laplacian of the boundary condition arises in the remainder term. In other words, if the boundary condition is harmonic, the representation of the partial differential equation involving the biharmonic operator is possible.By focusing on the distribution of the iterated Brownian motion, the representation of the partial differential equation including the biharmonic operator is possible when the boundary condition is biharmonic.

# 1 Introduction

## 1.1 Stochastic Approach to Partial Differential Equations

Given a solution $u(t,x) : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ of a certain partial differential equation, along with the initial condition $u(0,x) = u_0(x), (\forall x \in \mathbb{R}^N)$, we consider representing the solution in terms of the expectation value using a certain probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X : [0,\infty) \times \Omega \to \mathbb{R}^N$ defined on this space, as follows:

$$u(t,x) = E^{\Omega}(u_0(X(t+x))) \quad (\forall t \geq 0, , x \in \mathbb{R}^N).$$

Let $(\Omega, \mathcal{F}, \mathcal{F}tt \geq 0, P)$ be a filtered probability space, and let $B$ be a $d$-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{F}tt \geq 0, P)$. Consider $V_0, \cdots, V_d \in C_b^{\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and define the second-order linear differential operator $\mathcal{L} = \frac{1}{2}\sum k = 1^d V_k^2 + V_0$. We denote $X$ as a stochastic process on $(\Omega, \mathcal{F})$ that satisfies the following Stratonovich-type stochastic differential equation:

$$dX_t = \sum_{\alpha=1}^{d} V_\alpha(X_t) \circ dB_t^\alpha + V_0(X_t), dt, \quad X_0 = 0.$$

If a function $f : \mathbb{R}^N \to \mathbb{R}$ satisfies appropriate conditions, setting $v(t,x) = E(f(X(t)+x)), (\forall t \geq 0, , x \in \mathbb{R}^N)$, we find that $v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following heat equation [?Taniguchi_SDE]:

$$\frac{\partial}{\partial t} v(t,x) = \mathcal{L}v(t,x) \; v(0,x) \qquad = f(x). \tag{1.1}$$

As demonstrated above, partial differential equations determined by second-order linear operators derived from the coefficient functions of stochastic differential equations can have solutions represented by the expectation value. In this paper, we focus on boundary value problems of fourth-order partial differential equations involving the biharmonic operator $\Delta^2$.

## 1.2 Motivation of the Iterated Brownian Motion

There is a fundamental relationship between the Brownian motion and the Laplacian operator. In equation (1.1), consider the special case where $V_0 = 0$ and $V_\alpha = 1 \, (\alpha = 1, \cdots, d)$. Under suitable conditions on $f$, this equation reduces to the following basic heat equation:

$$\frac{\partial}{\partial t} v(t,x) = \frac{1}{2}\Delta v(t,x).$$

The motivation for the iterated Brownian motion (i.B.m.) is to analyze the relationship between the biharmonic operator and a stochastic process whose time parameter is itself driven by another independent Brownian motion. This concept was first introduced by Funaki [5], and the properties of the sample paths of the iterated Brownian motion were further studied by Burdzy, Krzysztof [1]. In this paper, we refer to the iterated Brownian motion as the i.B.m.

## 1.3 Funaki's Method

In Funaki [5], the following approach is employed. Let $B$ and $w$ be independent Brownian motions, and define

$$\overline{B}(t) = \begin{cases} B(t) & (t > 0) \\ \sqrt{-1}B(-t) & (t \leq 0). \end{cases}$$

For a real-valued function $g$ that can be extended to an entire function, let $\hat{g} : \mathbb{C} \to \mathbb{C}$ denote its extension. If $\hat{g}$ satisfies specific growth conditions, setting $u(t,x) = E(\hat{g}(x + \overline{B}(w(t)))) \quad (\forall t \in \mathbb{R}, \, x \in \mathbb{R})$, the function $u$ satisfies the following heat equation:

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{8}\Delta^2 u(t,x) \quad (\forall t \in \mathbb{R} \setminus \{0\},\, \forall x \in \mathbb{R})$$
$$u(0,x) = g(x) \quad (\forall x \in \mathbb{R}).$$

Here, note that $g : \mathbb{R} \to \mathbb{R}$ is a harmonic function. Moreover, the growth conditions imposed on $\hat{g}$ require that $\left|\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$, $\left|\frac{\partial}{\partial x_i}\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$, and $\left|\frac{\partial}{\partial y_j}\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$ are bounded on $\mathbb{C}$ for any $h > 0$.

Generally, the following is stated. Let $A$ be an elliptic differential operator on $\mathbb{R}^d$ of the following form:

$$A = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \tag{1.2}$$

where $a_{ij},\, b_i \in C_b(\mathbb{R}^d)\,(\forall i,j)$. For $x \in \mathbb{R}^d$, let $\sigma(x) = (\sigma_{ij}(x))$, $(a_{ij}) = \sigma(x)\sigma^\top(x)$, $\sigma_0(x) = (b_1(x), \cdots, b_d(x))^\top$. Let $X^x$ be a diffusion process with $A$ as its infinitesimal generator:

$$dX^x(t) = \sum_{k=1}^{d} \sigma_k(X^x(t))\, dB^k(t) + \sigma_0(X^x(t))\, dt \quad (t > 0).$$

Given $p \geq 0$ and $q \in \mathbb{R}$, the objective is to construct a probability process on $\mathbb{R}^d$ using the diffusion process $X^x$, which solves

$$\frac{\partial}{\partial t}u(t,x) = (pA^2 + qA)u(t,x),$$

along with the initial condition $u(0,\cdot) = u_0(\cdot)$.

**Definition 1.1** (Setting the Boundary Condition)**.** *Let $\mathcal{D}$ be a set of functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying the following conditions. There exists $n \in \mathbb{N}$ and a function $\bar{f} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ such that:*

- $\bar{f}(x,0) = f(x) \quad (\forall x \in \mathbb{R}^d)$,
- *There exists a second-order linear differential operator $\tilde{A}$ on $\mathbb{R}^n$ of the same form as* (1.2) *such that:*

$$(A_x + \tilde{A}_y)\bar{f}(x,y) = 0 \quad ((x,y) \in \mathbb{R}^d \times \mathbb{R}^n).$$

- $\left|\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$, $\left|\frac{\partial}{\partial x_i}\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$, *and* $\left|\frac{\partial}{\partial y_j}\bar{f}(x,y)\right| \exp\{-h(|x|^2+|y|^2)\}$ *are bounded on $\mathbb{R}^d \times \mathbb{R}^n$ for any $h > 0$.*

Let $\{\tilde{X}_t\}_{t\geq 0}$ be an $n$-dimensional diffusion process starting from $\tilde{X}_0 = 0 \in \mathbb{R}^n$ with $\tilde{A}$ as its infinitesimal generator.

**Definition 1.2** (Extension of the Diffusion Process)**.** *Let $\{\bar{X}_t(x)\}_{t\in\mathbb{R}, x\in\mathbb{R}^d}$ be a family of random variables in $\mathbb{R}^d \times \mathbb{R}^n$ defined as follows:*

$$\bar{X}_t(x) = \begin{cases} (X_t(x), 0) & (t \geq 0) \\ (x, \tilde{X}_{-t}) & (t \leq 0). \end{cases}$$

Let $\{w_t\}_{t\geq 0}$ be a one-dimensional Brownian motion independent of $\bar{X}$. Define $Y(t) = \sqrt{2p}w_t + qt\,(t \geq 0)$.

**Theorem 1.3** (Theorem 3 of Funaki, 1979 [5])**.** *For each $f \in \mathcal{D}$, let $u(t,x) = E(\bar{f}(\bar{X}_{Y_t}(x)))\,(t \geq 0, x \in \mathbb{R}^d)$. Then, the following holds:*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = (pA^2 + qA)u(t,x) & (t > 0, x \in \mathbb{R}^d) \\ u(0,x) = f(x) & (x \in \mathbb{R}^d). \end{cases}$$

### 1.4 Contents of This Paper

Let $Z$ denote the iterated Brownian motion, and for $f \in C_b^\infty(\mathbb{R}^d)$ and for each $t \geq 0$, $x \in \mathbb{R}^d$, define $v(t,x) = E(f(Z(t)+x))$. We have obtained the following representation of the solution for the partial differential equation:

$$\left(\frac{\partial}{\partial t} - \frac{1}{8}\Delta^2\right) v(t,x) = \frac{1}{2}\frac{\Delta f(x)}{\sqrt{2\pi t}} \qquad (t > 0,\ x \in \mathbb{R}^d)$$
$$v(0,x) = f(x) \qquad (x \in \mathbb{R}^d).$$

For each $t \geq 0$, $x \in \mathbb{R}^d$, define $v'(t,x) = v(t,x) - \frac{1}{\sqrt{c}}v(ct,x)$. We have shown that if the boundary condition $f : \mathbb{R}^d \to \mathbb{R}$ is a biharmonic function, then the following equation holds:

$$\frac{\partial}{\partial t}v'(t,x) = \frac{1}{8}\Delta^2 v'(t,x) \quad (t > 0,\ x \in \mathbb{R}^d).$$

Furthermore, we discretize the iterated Brownian motion using a random walk and demonstrate that, under certain conditions, the weak approximation error with respect to the step number $n$ is $O\left(\frac{1}{\sqrt{n}}\right)$.

## 2 Fundamentals

In this section, we summarize the fundamental results in stochastic analysis that are relevant to this paper. All the results are referenced from Karatzas, Ioannis and Shreve, Steven [6].

**Definition 2.1.** *Let $d \in \mathbb{N}$ and $\mu$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. A stochastic process $B : [0,\infty) \times \Omega \to \mathbb{R}^d$ is called a d-dimensional standard Brownian motion with initial distribution $\mu$ if it satisfies the following conditions:*

- *For $P$-almost every $\omega$, $B(\cdot,\omega) : [0,\infty) \to \mathbb{R}^d$ is continuous.*
- *$P(B_0 \in \Gamma) = \mu(\Gamma)$ for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.*
- *For $0 \leq s \leq t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$, and follows a normal distribution with mean $0$ and covariance matrix $(t-s)I_d$.*

*If there exists an $x \in \mathbb{R}^d$ such that $P(B_0 = x) = 1$, then $B$ is called a d-dimensional standard Brownian motion starting from $x$.*

There are multiple measurable spaces and probability measures that can be used to construct a Brownian motion, but we introduce the following framework in this paper. Define $C[0,\infty) = \{f : [0,\infty) \to \mathbb{R} \mid f \text{ is continuous}\}$. By introducing the following distance, $C[0,\infty)$ becomes a complete separable metric space:

$$\rho(\omega_1,\omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\left(\max_{0\leq t\leq n} |\omega_1(t) - \omega_2(t)|\right) \wedge 1\right).$$

Let $\mathcal{B}(C[0,\infty))$ denote the Borel $\sigma$-algebra generated by the distance $\rho$ on $C[0,\infty)$.

Let $\sigma > 0$ and $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent and identically distributed square-integrable random variables satisfying $E(\xi_1) = 0$ and $E(\xi_1^2) = \sigma^2 < \infty$. Define the sequence of random variables $\{S_k\}_{k=0}^\infty$ as follows:

$$S_0 = 0, \quad S_k = \sum_{m=1}^{k} \xi_m.$$

Let $\{Y(t)\}_{t\geq 0}$ be a stochastic process defined by

$$Y(t) = S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor)\xi_{\lfloor t\rfloor+1} \quad (t \geq 0),$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to $t$.

For each $n \geq 1$, define a stochastic process $\{X^{(n)}(t)\}_{t\geq 0}$ by

$$X^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} Y(nt) \quad (t \geq 0),$$

and treat $\{X^{(n)}\}_{n=1}^{\infty}$ as a sequence of random variables taking values in $C[0,\infty)$. For each $n \in \mathbb{N}$, let $\hat{P}_n$ denote the distribution of the random variable $X^{(n)}$. Under this setup, the following result holds.

**Theorem 2.2.** *The sequence of measures $\{\hat{P}_n\}_{n=1}^{\infty}$ converges weakly to a probability measure $P_*$. On the probability space $(C[0,\infty), \mathcal{B}(C[0,\infty)), P_*)$, define a stochastic process $W : [0,\infty) \times C[0,\infty) \to \mathbb{R}$ by*

$$W(t,\omega) = \omega(t) \quad (\forall t \geq 0,\, \omega \in C[0,\infty)).$$

*Then, $\{W(t)\}_{t\geq 0}$ is a one-dimensional standard Brownian motion starting from the origin on the filtered probability space $(C[0,\infty), \mathcal{B}(C[0,\infty)), \{\mathcal{F}_t^W\}_{t\geq 0}, P_*)$.*

# 3 On Iterated Brownian Motion

The study of sample paths of iterated Brownian motion (i.B.m.) has been primarily conducted by Burdzy, Krzysztof [1], [2]. In order to drive the time parameter of a Brownian motion by another independent Brownian motion, it is necessary to define Brownian motion for negative times.

**Definition 3.1.** *Let $X_1 = \{X_1(t)\}_{t\geq 0}$ and $X_2 = \{X_2(t)\}_{t\geq 0}$ be independent $d$-dimensional standard Brownian motions starting from the origin on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. For each $(t,\omega_1) \in \mathbb{R} \times \Omega_1$, define a stochastic process $X : \mathbb{R} \times \Omega_1 \to \mathbb{R}^d$ as follows:*

$$X(t,\omega_1) = \begin{cases} X_1(t,\omega_1) & \text{if } t \geq 0 \\ X_2(-t,\omega_1) & \text{if } t \leq 0. \end{cases}$$

*In this context, we refer to $X$ as a **two-sided Brownian motion**.*

In [1] and [2], iterated Brownian motion is defined as a stochastic process $B^1(B^2)$ on a probability space $(\Omega, \mathcal{F}, P)$, where $B^1$ is a two-sided Brownian motion and $B^2$ is a standard Brownian motion, both defined on $(\Omega, \mathcal{F}, P)$.

In this paper, we focus on the distribution of iterated Brownian motion and define it as follows:

**Definition 3.2.** *Let $X$ be a two-sided Brownian motion on $(\Omega_1, \mathcal{F}_1, P_1)$ as defined in Definition 3.1. Let $Y = \{Y(t)\}_{t\geq 0}$ be a standard one-dimensional Brownian motion starting from the origin on the probability space $(\Omega_2, \mathcal{F}_2, P_2)$. Define $\Omega := \Omega_1 \times \Omega_2, \mathcal{F} := \mathcal{F}_1 \times \mathcal{F}_2$, and $P := P_1 \times P_2$. For each $t \in [0,\infty)$ and $\omega = (\omega_1, \omega_2) \in \Omega$, define a stochastic process $Z = \{Z(t)\}_{t\geq 0} : [0,\infty) \times \Omega \to \mathbb{R}^d$ by*

$$Z(t,\omega) := X(Y(t,\omega_2), \omega_1).$$

*We call $Z$ a **$d$-dimensional iterated Brownian motion starting from the origin**. For $x \in \mathbb{R}^d$, the process $Z + x$ is referred to as a **$d$-dimensional iterated Brownian motion starting from $x$**.*

For each $d \in \mathbb{N}$, $(\xi_1, \xi_2) \in (0,\infty) \times \mathbb{R}^d$, we define $p_d(\xi_1, \xi_2) \in \mathbb{R}$ as follows:

$$p_d(\xi_1,\xi_2) = \frac{1}{(2\pi\xi_1)^{\frac{d}{2}}} \exp\left(-\frac{|\xi_2|^2}{2\xi_1}\right) \tag{3.1}$$

In particular, for $d = 1$, we denote it as $p(\xi_1,\xi_2) = p_1(\xi_1,\xi_2) = \frac{1}{\sqrt{2\pi\xi_1}} \exp\left(-\frac{\xi_2^2}{2\xi_1}\right)$. Throughout this paper, we will use the above notation without declaration.

**Lemma 3.3.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function. We define $F : \mathbb{R} \to \mathbb{R}$ as follows:*

$$F(u) = \begin{cases} \int_{\mathbb{R}^d} f(y) p_d(|u|, y)\, dy & (u \neq 0) \\ f(0) & (u = 0) \end{cases} \tag{3.2}$$

*Then, $F : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.*

*Proof.* From the boundedness of $f$, there exists a constant $C > 0$ such that for each $y \in \mathbb{R}^d$, the following holds:

$$|f(y)| \leq C$$

When $u \neq 0$, we have:

$$|F(u)| \leq \int_{\mathbb{R}^d} |f(y)| p_d(|u|, y)\, dy \leq \int_{\mathbb{R}} C\, p_d(|u|, y)\, dy = C$$

Thus, we conclude that $F : \mathbb{R} \to \mathbb{R}$ is bounded.

Let $u_0 \in \mathbb{R}$ be arbitrary, and consider a sequence of real numbers $\{u_n\}_{n=1}^{\infty}$ such that:

$$\lim_{n\to\infty} u_n = u_0$$

For each $n \geq 1$, it is important to note that:

$$F(u_n) = E(f(X_1(|u_n|)))$$
$$E((f(X_1(|u_n|)))^2) \leq C^2$$

Consequently, the sequence $\{f(X_1(|u_n|))\}_{n=1}^{\infty}$ is uniformly integrable. Additionally, from the continuity of the sample function $X_1$ and the continuity of $f$, we obtain:

$$\lim_{n\to\infty} E(f(X_1(|u_n|))) = E(f(X_1(|u_0|)))$$

Therefore, we can conclude that:

$$\lim_{n\to\infty} F(u_n) = \lim_{n\to\infty} E(f(X_1(|u_n|))) = E(f(X_1(|u_0|))) = F(u_0)$$

□

**Proposition 3.4.** *Let $t > 0$. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is Borel measurable and satisfies the following conditions:*

- *For $P_2$-almost surely $\omega_2 \in \Omega_2$, we have $E^{\Omega_1}(|f(X(Y(t, \omega_2)))|) < \infty$.*
- *The mapping $\Omega_2 \ni \omega_2 \mapsto E^{\Omega_1}(|f(X_1(Y(t, \omega_2)))|) \in \mathbb{R}$ is integrable.*

*Then, it holds that $E(|f(Z(t))|) < \infty$, and we have the following relation:*

$$E^{\Omega}(f(Z(t))) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} f(y) p_d(|u|, y)\, dy \right) p(t, u)\, du$$

*where $E^{\Omega}$ denotes the integral over the probability space $(\Omega, \mathcal{F}, P)$.*

*Proof.* Fix an arbitrary $t > 0$.

$$\begin{aligned}
&\int_{\Omega_2} dP_2(\omega_2) \int_{\Omega_1} |f(X(Y(t, \omega_2), \omega_1))|\, dP_1(\omega_1) \\
&= \int_{Y(t)>0} dP_2(\omega_2) \int_{\Omega_1} |f(X_1(Y(t, \omega_2), \omega_1))|\, dP_1(\omega_1) + \int_{Y(t)<0} dP_2(\omega_2) \int_{\Omega_1} |f(X_2(-Y(t, \omega_2), \omega_1))|\, dP_1(\omega_1) \\
&= \int_{\Omega_2} E^{\Omega_1}(f(X_1(|Y(t, \omega_2)|)))\, dP_2(\omega_2) \\
&= \int_{\mathbb{R}} E^{\Omega_1}(|f(X_1(|u|))|) p_d(t, u)\, du < \infty
\end{aligned}$$

Thus, by Fubini's theorem, we obtain:

$$\begin{aligned}E^{\Omega}(f(Z(t))) &= \int_{\Omega_2} dP_2(\omega_2) \int_{\Omega_1} f(X(Y(t,\omega_2),\omega_1))\, dP_1(\omega_1) \\ &= \int_{\mathbb{R}} E^{\Omega_1}(|f(X_1(|u|))|)p(t,u)\, du \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} f(y)p_d(|u|,y)\, dy \right) p(t,u)\, du\end{aligned}$$

□

**Corollary 3.5.** *Assume that $f : \mathbb{R}^d \to \mathbb{R}$ satisfies the conditions of Proposition 3.4. For any $x \in \mathbb{R}^d$, the following relation holds:*

$$E^{\Omega}(f(Z(t)+x)) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} f(y+x)p_d(|u|,y)\, dy \right) p(t,u)\, du$$

**Proposition 3.6.** *For any $c > 0$, the iterative Brownian motion $Z = \{Z(t)\}_{t\geq 0}$ and the process $\{c^{-\frac{1}{4}}Z(ct)\}_{t\geq 0}$ have the same distribution. In other words, for $f \in C_b(\mathbb{R})$, the following holds:*

$$E(f(Z(t))) = E\left( f\left( \frac{1}{c^{\frac{1}{4}}} Z(ct) \right) \right)$$

*Proof.* Let $c > 0$. Due to the invariance of the distribution under scaling transformations of Brownian motion, the following holds for each $g \in C_b(\mathbb{R})$ and $\alpha > 0$:

$$\int_{\mathbb{R}} g(y)p(t,y)\, dy = \int_{\mathbb{R}} g\left( \frac{y}{\sqrt{\alpha}} \right) p(\alpha t, y)\, dy \tag{3.3}$$

Noting Lemma 3.3, we obtain:

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)p(|u|,y)\, dy \right) p(t,u)\, du = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)p\left( \frac{|u|}{\sqrt{c}}, y \right)\, dy \right) p(ct,u)\, du$$

From (3.3), for $u \in \mathbb{R} \setminus \{0\}$, it holds that:

$$\int_{\mathbb{R}} f(y)p\left( \frac{|u|}{\sqrt{c}}, y \right)\, dy = \int_{\mathbb{R}} f\left( \frac{y}{c^{\frac{1}{4}}} \right) p(|u|,y)\, dy$$

Combining these results, we get:

$$\begin{aligned}E(f(Z(t))) &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)p(|u|,y)\, dy \right) p(t,u)\, du \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y)p\left( \frac{|u|}{\sqrt{c}}, y \right)\, dy \right) p(ct,u)\, du \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f\left( \frac{y}{c^{\frac{1}{4}}} \right) p(|u|,y)\, dy \right) p(ct,u)\, du \\ &= E\left( f\left( \frac{Z(ct)}{c^{\frac{1}{4}}} \right) \right)\end{aligned}$$

□

Next, we establish the following based on Theorem 1.1 from DeBlassie, R. Dante [3].

**Theorem 3.7.** *Let $f \in C_b^\infty(\mathbb{R}^d)$. For each $t \geq 0$ and $x \in \mathbb{R}^d$, define*

$$u(t,x) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} f(y+x) p_d(|u|, y) dy \right) p(t,u)\, du.$$

*Then, the function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following for each $t > 0$ and $x \in \mathbb{R}^d$:*

$$\left( \frac{\partial}{\partial t} - \frac{1}{8}\Delta^2 \right) u(t,x) = \frac{1}{2} \frac{\Delta f(x)}{\sqrt{2\pi t}}, \tag{3.4}$$

$$u(0,x) = f(x). \tag{3.5}$$

*Proof.* We demonstrate the case for $d = 1$. The case for $d \geq 2$ has been proven in [3], but we will approach the case $d = 1$ in a similar manner. Let $\Omega' := (0,\infty) \times \mathbb{R}$. For any $\phi \in C_0^\infty(\Omega')$, we will show:

$$\int_{\Omega'} \left( u(t,x) \left( \frac{1}{8}\Delta^2 + \frac{\partial}{\partial t} \right) \phi(t,x) + \frac{1}{2} \cdot \frac{\Delta f(x)}{\sqrt{2\pi t}} \phi(t,x) \right) dx\, dt = 0.$$

Define

$$I_1 := \int_{\Omega'} u(t,x) \Delta^2 \phi(t,x)\, dx\, dt,$$

$$I_2 := \int_{\Omega'} u(t,x) \frac{\partial}{\partial t} \phi(t,x)\, dx\, dt,$$

$$I_3 := \int_{\Omega'} \frac{\Delta f(x)}{\sqrt{2\pi t}} \phi(t,x)\, dx\, dt.$$

Then we have:

$$\int_{\Omega'} \left( u(t,x) \left( \frac{1}{8}\Delta^2 + \frac{\partial}{\partial t} \right) \phi(t,x) + \frac{1}{2} \cdot \frac{\Delta f(x)}{\sqrt{2\pi t}} \phi(t,x) \right) dx\, dt = \frac{1}{8} I_1 + I_2 + \frac{1}{2} I_3. \tag{3.6}$$

Noticing that

$$\begin{aligned} u(t,x) = E^\Omega(f(Z(t)+x)) &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y+x) p(|u|, y)\, dy \right) p(t,u)\, du \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(y) p(|u|, y-x)\, dy \right) p(t,u)\, du, \end{aligned}$$

we find that:

$$\begin{aligned} I_1 &= 2 \int_{\Omega'} \int_0^\infty \int_{\mathbb{R}} f(y) p(u, y-x) p(t,u) \Delta^2 \phi(t,x)\, dy\, du\, dx\, dt \\ &= 2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) p(u, y-x) p(t,u) \Delta^2 \phi(t,x)\, dx\, dy\, du\, dt. \end{aligned}$$

Now, for each $(y,t,u) \in \mathbb{R} \times [0,\infty) \times [0,\infty)$, we have:

$$\begin{aligned} &\int_{\mathbb{R}} f(y) p(u, y-x) p(t,u) \Delta^2 \phi(t,x)\, dx \\ &= \int_{B_\epsilon(y)} f(y) p(u, y-x) p(t,u) \Delta^2 \phi(t,x)\, dx + \int_{\mathbb{R} \setminus B_\epsilon(y)} f(y) p(u, y-x) p(t,u) \Delta^2 \phi(t,x)\, dx. \end{aligned} \tag{3.7}$$

Thus, it follows that:

$$\begin{aligned}
I_1 &= 2\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}} 1_{B_\epsilon(y)}(x)f(y)p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\,dt\,dx \\
&\quad + 2\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}} 1_{\mathbb{R}\setminus B_\epsilon(y)}(x)f(y)p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\,dt\,dx \\
&= 2\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}} 1_{B_\epsilon(x)}(y)f(y)p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\,dt\,dx \\
&\quad + 2\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}} 1_{\mathbb{R}\setminus B_\epsilon(y)}(x)f(y)p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\,dt\,dx \\
&=: 2J_1 + 2J_2.
\end{aligned}$$

There exists a constant $K > 0$ such that for any $(t,x) \in (0,\infty)\times\mathbb{R}$, the following holds:

$$\left|\int_0^\infty\int_{\mathbb{R}} 1_{B_\epsilon(x)}(y)f(y)p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\right| \le K|\Delta\phi(t,x)|.$$

Therefore, by Fatou's lemma, we obtain:

$$\varlimsup_{\epsilon\downarrow 0}|J_1| \le \int_{\mathbb{R}}\int_0^\infty \varlimsup_{\epsilon\downarrow 0}\int_0^\infty\int_{\mathbb{R}} 1_{B_\epsilon(x)}(y)|f(y)|p(u,y-x)p(t,u)\Delta^2\phi(t,x)\,dy\,du\,dt\,dx.$$

For each $(t,x)\in(0,\infty)\times\mathbb{R}$ and $(u,y)\in(0,\infty)\times\mathbb{R}$, we have:

$$\begin{aligned}
&1_{B_\epsilon(y)}|f(y)|p(u,y-x)p(t,u) \le K_2\,p(u,y-x)p(t,u), \\
&\int_{\mathbb{R}}\int_{\mathbb{R}} |p(u,y-x)p(t,u)|\,dy\,du < \infty.
\end{aligned}$$

By the dominated convergence theorem, we conclude that:

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{B_\epsilon(y)}|f(y)|p(u,y-x)p(t,u)\,dy\,du = 0.$$

Thus, we have:

$$\lim_{\epsilon\downarrow 0} J_1 = 0. \tag{3.8}$$

For each $(u,t,x)\in(0,\infty)\times(0,\infty)\times\mathbb{R}$, the following holds:

$$\begin{aligned}
&\int_{y+\epsilon}^\infty p(u,y-x)\Delta^2\phi(t,x)\,dx \\
&= \left[p(u,y-x)\left(\frac{\partial}{\partial x}\right)^3\phi(t,x)\right]_{y+\epsilon}^\infty - \int_{y+\epsilon}^\infty \frac{\partial}{\partial x}p(u,y-x)\left(\frac{\partial}{\partial x}\right)^3\phi(t,x)\,dx \\
&= -p(u,-\epsilon)\left(\frac{\partial}{\partial x}\right)^3\phi(t,x)\bigg|_{x=y+\epsilon} - \left[\frac{\partial}{\partial x}p(u,y-x)\Delta\phi(t,x)\right]_{y+\epsilon}^\infty + \int_{y+\epsilon}^\infty\left(\frac{\partial}{\partial x}\right)^2 p(u,y-x)\Delta\phi(t,x)\,dx \\
&= -p(u,-\epsilon)\left(\frac{\partial}{\partial x}\right)^3\phi(t,x)\bigg|_{x=y+\epsilon} + \frac{\partial}{\partial x}p(u,y-x)\Delta\phi(t,x)\bigg|_{x=y+\epsilon} + \int_{y+\epsilon}^\infty \Delta p(u,y-x)\Delta\phi(t,x)\,dx.
\end{aligned}$$

Similarly, we obtain:

$$\int_{-\infty}^{y-\epsilon} p(u,y-x)\Delta^2\phi(t,x)\,dx = p(u,\epsilon)\left(\frac{\partial}{\partial x}\right)^3\phi(t,x)\bigg|_{x=y-\epsilon} - \frac{\partial}{\partial x}p(u,y-x)\Delta\phi(t,x)\bigg|_{x=y-\epsilon} + \int_{-\infty}^{y-\epsilon}\Delta p(u,y-x)\Delta\phi(t,x)\,dx.$$

Therefore, for any $(u, t, y) \in (0, \infty) \times (0, \infty) \times \mathbb{R}$, the following holds:

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus B_\epsilon(y)} p(u, y-x)\Delta^2\phi(t,x)\,dx \\
&= p(u,\epsilon)\left( -\left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y+\epsilon} + \left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y-\epsilon} \right) \\
&\quad + \frac{\partial}{\partial x} p(u, y-x)\Delta\phi(t,x)\Big|_{x=y+\epsilon} - \frac{\partial}{\partial x} p(u, y-x)\Delta\phi(t,x)\Big|_{x=y-\epsilon} \\
&\quad + \int_{\mathbb{R}\setminus B_\epsilon(y)} \Delta p(u, y-x)\Delta\phi(t,x)\,dx \\
&= p(u,\epsilon)\left( -\left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y+\epsilon} + \left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y-\epsilon} \right) \\
&\quad - \frac{\epsilon}{u} p(u,\epsilon)\left( \Delta\phi(t,x)\Big|_{x=y+\epsilon} + \Delta\phi(t,x)\Big|_{x=y-\epsilon} \right) \\
&\quad + \int_{\mathbb{R}\setminus B_\epsilon(y)} \Delta p(u, y-x)\Delta\phi(t,x)\,dx.
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}} 1_{\mathbb{R}\setminus B_\epsilon(y)}(x) f(y) p(u,y-x) p(t,u)\Delta^2\phi(t,x)\,dx\,du\,dt\,dy \\
&= \int_{\mathbb{R}}\int_0^\infty\int_0^\infty f(y)p(t,u)p(u,\epsilon)\left( -\left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y+\epsilon} + \left(\frac{\partial}{\partial x}\right)^3 \phi(t,x)\Big|_{x=y-\epsilon} \right) du\,dt\,dy \\
&- \int_{\mathbb{R}}\int_0^\infty\int_0^\infty f(y)p(t,u)\frac{\epsilon}{u}p(u,\epsilon)\left( \Delta\phi(t,x)\Big|_{x=y+\epsilon} + \Delta\phi(t,x)\Big|_{x=y-\epsilon} \right) du\,dt\,dy \\
&+ \int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}\setminus B_\epsilon(y)} \Delta f(y) p(t,u) p(u,y-x)\Delta\phi(t,x)\,dx\,du\,dt\,dy \\
&=: J_3 - J_4 + J_5
\end{aligned}
$$

$$
\Delta p(u, y-x) = 2\frac{\partial}{\partial u} p(u, y-x)
$$

Additionally, we have:

$$
\begin{aligned}
\int_0^\infty p(t,u)\frac{\partial}{\partial u} p(u,y-x)\,dx &= [p(t,u)p(u,y-x)]_0^\infty - \int_0^\infty p(u,y-x)\frac{\partial}{\partial u}p(t,u)\,du \\
&= \int_0^\infty \frac{u}{t} p(u,y-x)p(t,u)\,du
\end{aligned}
$$

From this, it follows that:

$$\begin{aligned}
J_5 &= \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)p(t,u)\Delta p(u,y-x)\Delta\phi(t,x)\,dx\,du\,dt\,dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)p(t,u)\frac{\partial}{\partial u}p(u,y-x)\Delta\phi(t,x)\,dx\,du\,dt\,dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} \int_0^\infty f(y)\Delta\phi(t,x)p(t,u)\frac{\partial}{\partial u}p(u,y-x)\,du\,dx\,dt\,dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)\Delta\phi(t,x)p(t,u)\frac{u}{t}p(u,y-x)\,dx\,du\,dt\,dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)\Delta\phi(t,x)p(t,u)\frac{u}{t}p(u,y-x)\,dx\,du\,dt\,dy
\end{aligned}$$

For any $(u,t,y) \in (0,\infty)\times(0,\infty)\times\mathbb{R}$, the following holds:

$$\begin{aligned}
\int_{-\infty}^{y-\epsilon} \Delta\phi(t,x)p(u,y-x)\,dx &= \left[\frac{\partial}{\partial x}\phi(t,x)p(u,y-x)\right]_{-\infty}^{y-\epsilon} - \int_{-\infty}^{y-\epsilon} \frac{\partial}{\partial x}p(u,y-x)\,dx \\
&= \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y-\epsilon}\cdot p(u,\epsilon) - \left[\phi(t,x)\frac{\partial}{\partial x}p(u,y-x)\right]_{-\infty}^{y-\epsilon} + \int_{-\infty}^{y-\epsilon}\phi(t,x)\Delta p(u,y-x)\,dx \\
&= \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y-\epsilon}\cdot p(u,\epsilon) - \phi(t,y-\epsilon)\frac{\epsilon}{u}p(u,\epsilon) + \int_{-\infty}^{y-\epsilon}\phi(t,x)\Delta p(u,y-x)\,dx
\end{aligned}$$

Similarly, we have:

$$\int_{y+\epsilon}^{\infty} \Delta\phi(t,x)p(u,y-x)\,dx = -\frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon}\cdot p(u,-\epsilon) - \phi(t,y+\epsilon)\frac{\epsilon}{u}p(u,\epsilon) + \int_{y+\epsilon}^{\infty}\phi(t,x)\Delta p(u,y-x)\,dx$$

We can state the following:

$$\begin{aligned}
\int_{\mathbb{R}\setminus B_\epsilon(y)} \Delta\phi(t,x)p(u,y-x)\,dx &= p(u,\epsilon)\left(\frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y-\epsilon} - \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon}\right) \\
&\quad - \frac{\epsilon}{u}\left(\phi(t,y-\epsilon)+\phi(t,y+\epsilon)\right) + \int_{\mathbb{R}\setminus B_\epsilon(y)}\phi(t,x)\Delta p(u,y-x)\,dx \\
&= p(u,\epsilon)\left(\frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y-\epsilon} - \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon}\right) \\
&\quad - \frac{\epsilon}{u}\left(\phi(t,y-\epsilon)+\phi(t,y+\epsilon)\right) + 2\int_{\mathbb{R}\setminus B_\epsilon(y)}\phi(t,x)\frac{\partial}{\partial u}p(u,y-x)\,dx \\
&\quad \text{for all } (u,t,y) \in (0,\infty)\times(0,\infty)\times\mathbb{R}
\end{aligned}$$

Now, consider:

$$\begin{aligned}
J_5 &= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)\Delta\phi(t,x)p(t,u)\frac{u}{t}p(u,y-x)\,dx\,du\,dt\,dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \frac{u}{t}f(y)p(t,u)p(u,\epsilon)\left(\frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y-\epsilon} - \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon}\right)du\,dt\,dy \\
&\quad - 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \frac{\epsilon}{t}f(y)p(t,u)\left(\phi(t,y-\epsilon)+\phi(t,y+\epsilon)\right)\,du\,dt\,dy \\
&\quad + 4\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} \frac{u}{t}f(y)p(t,u)\phi(t,x)\frac{\partial}{\partial u}p(u,y-x)\,dx\,du\,dt\,dy \\
&=: 2J_6 - 2J_7 + 4J_8
\end{aligned}$$

Next, we will demonstrate that $\lim_{\epsilon\downarrow 0} J_6 = 0$. From the assumptions on $\phi$, there exist constants $M_1 > 0$ and $M_2 > \sqrt{3}$ such that:

$$J_6 = \int_{-M_1}^{M_1}\int_0^{M_2}\int_0^\infty \frac{u}{t} f(y)p(t,u)p(u,\epsilon)\left(\frac{\partial}{\partial x}\phi(t,x)\bigg|_{x=y-\epsilon} - \frac{\partial}{\partial x}\phi(t,x)\bigg|_{x=y+\epsilon}\right) du\,dt\,dy$$

Given the conditions on $f$ and $\phi$, there exists a constant $K > 0$ such that:

$$\begin{aligned}
&\int_{-M_1}^{M_1}\int_0^{M_2}\int_0^\infty \frac{u}{t}|f(y)|p(t,u)p(u,\epsilon)\left|\frac{\partial}{\partial x}\phi(t,x)\bigg|_{x=y-\epsilon} - \frac{\partial}{\partial x}\phi(t,x)\bigg|_{x=y+\epsilon}\right| du\,dt\,dy \\
&\le 2M_1K\int_0^{M_2}\int_0^\infty \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt \\
&= 2M_1K\int_0^1\int_0^\infty \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt + 2M_1K\int_1^{M_2}\int_0^\infty \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt.
\end{aligned}$$

Note that:

$$\frac{\partial}{\partial t}\frac{p(t,u)}{t} = \frac{1}{\sqrt{2\pi}}t^{-\frac{5}{2}}\exp\left(-\frac{u^2}{2t}\right)\cdot\frac{u^2-3t}{2t}.$$

From this, we obtain the following estimate:

$$\begin{aligned}
&\int_0^1\int_0^\infty \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt \\
&= \int_0^1\int_0^{\sqrt{3}} \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt + \int_0^1\int_{\sqrt{3}}^\infty \frac{u}{t}p(t,u)p(u,\epsilon)\,du\,dt \\
&\le \int_0^1\int_0^{\sqrt{3}} u\cdot\frac{3}{u^2}p\left(\frac{u^2}{3},u\right)p(u,\epsilon)\,du\,dt + \int_0^1\int_{\sqrt{3}}^\infty u\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)p(u,\epsilon)\,du\,dt \\
&\le \frac{3\sqrt{3}}{2\pi}\int_0^{\sqrt{3}}\frac{1}{u^2\sqrt{u}}\exp\left(-\frac{1}{2u}\right)du + \int_{\sqrt{3}}^\infty\frac{1}{2\pi}\sqrt{u}\exp\left(-\frac{u^2}{2}\right)du \\
&= \frac{3\sqrt{3}}{2\pi}\int_{\frac{1}{\sqrt{3}}}^\infty\sqrt{u}\exp\left(-\frac{u}{2}\right)du + \int_{\sqrt{3}}^\infty\frac{1}{2\pi}\sqrt{u}\exp\left(-\frac{u^2}{2}\right)du < \infty.
\end{aligned}$$

Thus, by the Dominated Convergence Theorem, we conclude:

$$\lim_{\epsilon\downarrow 0} J_6 = 0. \tag{3.9}$$

Similarly, we have:

$$\lim_{\epsilon\downarrow 0} J_3 = 0. \tag{3.10}$$

And,

$$\lim_{\epsilon\downarrow 0} J_7 = 0. \tag{3.11}$$

For each $(t,x,y) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}$, the following holds:

$$\begin{aligned}
&\int_0^\infty u\,p(t,u)\frac{\partial}{\partial u}p(u,y-x)\,du \\
&= [u\,p(t,u)p(u,y-x)]_0^\infty - \int_0^\infty\left(p(t,u) + u\frac{\partial}{\partial u}p(t,u)\right)p(u,y-x)\,du \\
&= \int_0^\infty\left(\frac{u^2}{t}p(t,u) - p(t,u)\right)p(u,y-x)\,du.
\end{aligned}$$

Now consider:

$$
\begin{aligned}
J_8 &= \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} \frac{u}{t} f(y) p(t,u) \phi(t,x) \frac{\partial}{\partial u} p(u, y-x)\, dx\, du\, dt\, dy \\
&= \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)\phi(t,x) \left(\frac{u^2}{t^2} - \frac{1}{t}\right) p(t,u) p(u,y-x)\, dx\, du\, dt\, dy \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}\setminus B_\epsilon(y)} f(y)\phi(t,x) \frac{\partial}{\partial t} p(t,u) p(u,y-x)\, dx\, du\, dt\, dy.
\end{aligned}
$$

The following holds:

$$
\lim_{\epsilon\downarrow 0} J_8 = 2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} f(y)\phi(t,x)\frac{\partial}{\partial t} p(t,u) p(u,y-x)\, dx\, du\, dt\, dy.
$$

On the other hand, we have:

$$
\begin{aligned}
\int_{\Omega'} u(t,x) \frac{\partial \phi}{\partial t}(t,x)\, dx\, dt &= 2\int_{\Omega'} \left(\int_0^\infty f(y) p(u,y-x) p(t,u)\, du\right) \cdot \frac{\partial \phi}{\partial t}(t,x)\, dx\, dt \\
&= 2\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} f(y) p(u,y-x) p(t,u) \frac{\partial \phi}{\partial t}(t,x)\, dy\, du\, dx\, dt \\
&= 2\int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty f(y) p(u,y-x) p(t,u) \frac{\partial \phi}{\partial t}(t,x)\, dy\, du\, dx\, dt \\
&= -2\int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty f(y) p(u,y-x) p(t,u) \frac{\partial}{\partial t} p(t,u) \cdot \phi(t,x)\, dt\, dy\, du\, dx \\
&= -2\int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} f(y) p(u,y-x) p(t,u) \frac{\partial}{\partial t} p(t,u) \cdot \phi(t,x)\, dt\, dy\, du\, dx.
\end{aligned}
$$

Thus, we conclude that:

$$
\lim_{\epsilon\downarrow 0} J_8 = -\int_{\Omega'} u(t,x) \frac{\partial}{\partial t}\phi(t,x)\, dx\, dt = -I_2. \tag{3.12}
$$

For each $t > 0$, the following holds:

$$
\begin{aligned}
\int_{\mathbb{R}} f(y) \Delta\phi(t,x)\Big|_{x=y+\epsilon} dy &= \left[ f(y) \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{d}{dy} f(y) \frac{\partial}{\partial x}\phi(t,x)\Big|_{x=y+\epsilon} \\
&= -\left[\frac{d}{dy} f(y)\phi(t,y+\epsilon)\right]_{-\infty}^{\infty} + \int_{\mathbb{R}} \Delta_y f(y) \phi(t,y-\epsilon)\, dy \\
&= \int_{\mathbb{R}} \Delta_y f(y)\phi(t,y+\epsilon)\, dy.
\end{aligned}
$$

Similarly, we have:

$$
\int_{\mathbb{R}} f(y)\Delta\phi(t,x)\Big|_{x=y-\epsilon} dy = \int_{\mathbb{R}} \Delta_y f(y)\phi(t,y-\epsilon)\, dy.
$$

Thus, for each $t > 0$, we conclude that:

$$
\int_{\mathbb{R}} f(y) \left( \Delta\phi(t,x)\Big|_{x=y+\epsilon} + \Delta\phi(t,x)\Big|_{x=y-\epsilon} \right) dy = \int_{\mathbb{R}} \Delta_y f(y) \left(\phi(t,y-\epsilon) + \phi(t,y+\epsilon)\right) dy.
$$

Now, we have:

$$\int_0^\infty \left(-\frac{\epsilon}{u}\right) p(t,u)p(u,\epsilon)\,du = \frac{1}{\epsilon}\int_\infty^0 (-2v)p(t,\frac{\epsilon^2}{2v})p(\frac{\epsilon^2}{2v},\epsilon)\cdot\left(-\frac{\epsilon}{2v^2}\right)\,dv$$
$$= \epsilon\int_0^\infty \left(-\frac{1}{v}\right) p(t,\frac{\epsilon^2}{2v})p(\frac{\epsilon^2}{2v},\epsilon)\,dv$$
$$= -\frac{1}{\sqrt{\pi}}\int_0^\infty p(t,\frac{\epsilon^2}{2v})v^{-\frac{1}{2}}e^{-v}\,dv.$$

Thus, we conclude:

$$\lim_{\epsilon\downarrow 0}\int_0^\infty \left(-\frac{\epsilon}{u}\right) p(t,u)p(u,\epsilon)\,du = -\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{2\pi t}}v^{-\frac{1}{2}}e^{-v}\,dv$$
$$= -\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2\pi t}}\Gamma\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{2\pi t}},$$

where we have set $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt \quad (a>0)$.

Now consider:

$$J_4 = \int_{\mathbb{R}}\int_0^\infty\int_0^\infty f(y)p(t,u)\frac{\epsilon}{u}p(u,\epsilon)\left(\Delta\phi(t,x)\Big|_{x=y+\epsilon} + \Delta\phi(t,x)\Big|_{x=y-\epsilon}\right)du\,dt\,dy$$
$$= \int_0^\infty\left(\int_{\mathbb{R}} f(y)\left(\Delta\phi(t,x)\Big|_{x=y+\epsilon} + \Delta\phi(t,x)\Big|_{x=y-\epsilon}\right)dy\right)\cdot\left(\int_0^\infty \frac{\epsilon}{u}p(t,u)p(u,\epsilon)\,du\right)dt$$
$$= \int_0^\infty\left(\int_{\mathbb{R}} \Delta f(y)\left(\phi(t,y+\epsilon)+\phi(t,y-\epsilon)\right)dy\right)\cdot\left(\int_0^\infty \frac{\epsilon}{u}p(t,u)p(u,\epsilon)\,du\right)dt.$$

Thus, we conclude:

$$\lim_{\epsilon\downarrow 0} J_4 = \int_0^\infty \lim_{\epsilon\downarrow 0}\left(\int_{\mathbb{R}} \Delta f(y)\left(\phi(t,y+\epsilon)+\phi(t,y-\epsilon)\right)dy\right)\cdot\left(\int_0^\infty \frac{\epsilon}{u}p(t,u)p(u,\epsilon)\,du\right)dt$$
$$= \int_0^\infty 2\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\Delta f(y)\phi(t,y)\,dy\,dt \tag{3.13}$$
$$= 2\int_0^\infty\int_{\mathbb{R}}\frac{\Delta f(x)}{\sqrt{2\pi t}}\phi(t,x)\,dx\,dt = 2I_3.$$

We have:

$$I_1 = 2J_1 + 2J_2 = 2J_1 + 2J_3 - 2J_4 + 2J_5$$
$$= 2J_1 + 2J_3 - 2J_4 + 4J_6 - 4J_7 + 8J_8$$
$$= \lim_{\epsilon\downarrow 0}(2J_1 + 2J_3 - 2J_4 + 4J_6 - 4J_7 + 8J_8)$$
$$= \lim_{\epsilon\downarrow 0}(2J_3 + 4J_6 - 4J_7) - 4I_3 - 8I_2 \quad \text{(from (3.8), (3.13))}$$
$$= -4I_3 - 8I_2 \quad \text{(from (3.10), (3.9), (3.11))}.$$

□

**Proposition 3.8.** *Let $T>0$. For $P$-almost surely $\omega\in\Omega$, the function $Z(\cdot,\omega):[0,T]\to\mathbb{R}$ is locally Hölder continuous of order $\frac{1}{4}-\epsilon$ for any $\epsilon\in(0,\frac{1}{4})$. Specifically, for any $\epsilon\in(0,\frac{1}{4})$, there exists a $\delta>0$ such that the following holds:*

$$P\left(\left\{\omega\in\Omega\,\middle|\,\sup_{\substack{0<s-t<\xi(\omega)\\ s,t\in[0,T]}}\frac{|Z(t)-Z(s)|}{|t-s|^{\frac{1}{4}-\epsilon}}\le\delta\right\}\right) = 1.$$

*Proof.* Fix arbitrary $T > 0$ and $\epsilon \in (0, \frac{1}{2})$. There exist constants $\delta_1, \delta_2 > 0$ satisfying the following:

$$\exists h^{(i)} : \Omega_i \to \mathbb{R}, \text{ s.t.}$$
$$P_1\left(\left\{\omega_1 \in \Omega_1 \,\middle|\, \sup_{\substack{0<s-t<h^{(1)}(\omega_1)\\ s,t\in\mathbb{R}}} \frac{|X(t,\omega_1) - X(s,\omega_1)|}{|t-s|^{\frac{1}{2}-\epsilon}} \leq \delta_1\right\}\right) = 1,$$
$$P_2\left(\left\{\omega_2 \in \Omega_2 \,\middle|\, \sup_{\substack{0<s-t<h^{(2)}(\omega_2)\\ s,t\in[0,T]}} \frac{|Y(t,\omega_2) - Y(s,\omega_2)|}{|t-s|^{\frac{1}{2}-\epsilon}} \leq \delta_2\right\}\right) = 1.$$

Define $\widetilde{\Omega_1} = \left\{\omega_1 \in \Omega_1 \,\middle|\, \sup_{\substack{0<s-t<h^{(1)}(\omega_1)\\ s,t\in\mathbb{R}}} \frac{|X(t,\omega_1)-X(s,\omega_1)|}{|t-s|^{\frac{1}{2}-\epsilon}} \leq \delta_1\right\}$ and $\widetilde{\Omega_2} = \left\{\omega_2 \in \Omega_2 \,\middle|\, \sup_{\substack{0<s-t<h^{(2)}(\omega_2)\\ s,t\in[0,T]}} \frac{|Y(t,\omega_2)-Y(s,\omega_2)|}{|t-s|^{\frac{1}{2}-\epsilon}} \leq \delta_2$
Now, fix an arbitrary $\omega = (\omega_1, \omega_2) \in \widetilde{\Omega_1} \times \widetilde{\Omega_2}$. Let $f(u) = X(u, \omega_1)$ and $g(t) = Y(t, \omega_2)$. For each $t \in [0, T]$, define $h(t) := f(g(t))$. In this case, if $0 < s - t < h^{(2)}(\omega_2)$, then the following holds:

$$|g(t) - g(s)| \leq \delta_2 |t-s|^{\frac{1}{2}-\epsilon}.$$

Thus, there exists $\xi(\omega) > 0$ such that if $0 < s - t < \xi(\omega)$, then

$$|g(t) - g(s)| < h^{(1)}(\omega_1).$$

Then,

$$\begin{aligned} |h(t) - h(s)| = |f(g(t)) - f(g(s))| &\leq \delta_1 |g(t) - g(s)|^{\frac{1}{2}-\epsilon} \\ &\leq \delta_1 \delta_2^{\frac{1}{2}-\epsilon} |t-s|^{(\frac{1}{2}-\epsilon)^2}. \end{aligned} \tag{3.14}$$

□

**Proposition 3.9.** *Let $Z$ be a one-dimensional iterated Brownian motion. For any $n \geq 1$, the following holds:*

$$\begin{aligned} E(Z(t)^{2n-1}) &= 0, \\ E(Z(t)^{4n}) &= (4n-1)!! \cdot (2n-1)!!\, t^n, \\ E(Z(t)^{4n-2}) &= (4n-3)!! \cdot (2n-2)!!\, t^{n-\frac{1}{2}} \sqrt{\frac{2}{\pi}}. \end{aligned}$$

*In particular, the following holds:*

$$E(Z(t)^2) = \sqrt{\frac{2t}{\pi}}, \quad E(Z(t)^4) = 3t. \tag{3.15}$$

*Proof.* Let $n \geq 1$.

$$\begin{aligned} E(Z(t)^{4n}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} y^{4n} p(|u|, y)\, dy\, p(t, u)\, du \\ &= \int_{\mathbb{R}} (4n-1)!! u^{2n} p(t, u)\, du \\ &= (4n-1)!! \cdot (2n-1)!!\, t^n. \end{aligned}$$

$$
\begin{aligned}
E(Z(t)^{4n-2}) &= \int_{\mathbb{R}}\int_{\mathbb{R}} y^{4n-2} p(|u|, y)\, dy\, p(t,u)\, du \\
&= \int_{\mathbb{R}} (4n-3)!! |u|^{2n-1} p(t,u)\, du \\
&= (4n-3)!! \cdot (2n-2)!!\, t^{n-\frac{1}{2}} \sqrt{\frac{2}{\pi}}.
\end{aligned}
$$

□

**Corollary 3.10.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial. Then, the equation (3.4) holds.*

**Proposition 3.11.** *For each $t > 0$, the characteristic function of $Z(t)$ is given by:*

$$
\phi_{Z(t)}(\xi) := E(e^{\sqrt{-1}\xi Z(t)}) = 2 \exp\left(\frac{1}{8}\xi^4 t\right) \int_{\frac{1}{2}\xi^2 t}^{\infty} p(t,u)\, du. \tag{3.16}
$$

*Proof.*

$$
\begin{aligned}
E(e^{\sqrt{-1}\xi Z(t)}) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{\sqrt{-1}\xi y} p(|u|, y)\, dy\, p(t,u)\, du \\
&= \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}|u|} p(t,u)\, du \\
&= 2\int_0^{\infty} e^{-\frac{\xi^2}{2}u} p(t,u)\, du \\
&= 2 \cdot \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{1}{8}\xi^4 t\right) \int_0^{\infty} \exp\left(-\frac{1}{2t}\left(u + \frac{1}{2}\xi^2 t\right)^2\right) du \\
&= 2 \cdot \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{1}{8}\xi^4 t\right) \int_{\frac{1}{2}\xi^2 t}^{\infty} \exp\left(-\frac{u^2}{2t}\right) du \\
&= 2 \exp\left(\frac{1}{8}\xi^4 t\right) \int_{\frac{1}{2}\xi^2 t}^{\infty} p(t,u)\, du.
\end{aligned}
$$

□

**Theorem 3.12.** *For each $f \in C_0^{\infty}(\mathbb{R})$ and $t \geq 0$, the following holds:*

$$
|E(f(Z(t))) - E(f_4(Z(t)))| \leq \frac{3K}{4!} t,
$$

*where $f_4(y) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} y^k$ for all $y \in \mathbb{R}$ and $K > 0$ is a constant.*

*Proof.* For each $y \in \mathbb{R}$, the following holds:

$$
\exists c = c_y \in \mathbb{R} \text{ such that } f(y) = f_4(y) + \frac{f^{(4)}(c)}{4!} y^4.
$$

Thus,

$$
|f(y) - f_4(y)| \leq \frac{\sup_{c\in\mathbb{R}} |f^{(4)}(c)|}{4!} |y|^4 = \frac{K}{4!}|y|^4 \quad \forall y \in \mathbb{R},\ K := \sup_{c\in\mathbb{R}} |f^{(4)}(c)| < \infty.
$$

Then,

$$\begin{aligned}|E(f(Z(t))) - E(f_4(Z(t)))| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} |f(y) - f_4(y)| p(|u|, y)\, dy\, p(t,u)\, du\right| \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{K}{4!}|y|^4 p(|u|, y)\, dy\, p(t,u)\, du \\ &= \frac{K}{4!}\int_{\mathbb{R}} 3u^2\, p(t,u)\, du \\ &\leq \frac{3K}{4!}t.\end{aligned} \tag{3.17}$$

□

In general, let $B$ be a $d$-dimensional Brownian motion and let $f : \mathbb{R}^d \to \mathbb{R}$ be a harmonic function. By Itô's formula, we have $E(f(B(t) + x)) = f(x)$, which means that the Brownian motion maps harmonic functions to harmonic functions. The corresponding properties of iterated Brownian motion are shown next. The iterated Brownian motion also maps harmonic functions to harmonic functions and biharmonic functions to biharmonic functions.

**Proposition 3.13.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a harmonic function. Then, for any $x \in \mathbb{R}^d$ and $t \geq 0$, the following holds:*

$$E(f(Z(t) + x)) = f(x).$$

*Proof.* Let $x \in \mathbb{R}^d$ and $t \geq 0$. By Itô's formula, for each $u \in \mathbb{R}$, the following holds:

$$f(X_1(|u|) + x) = f(x) + \sum_{i=1}^{d}\int_0^{|u|} \frac{\partial}{\partial y_i} f(X_1(|s|))\, dX_1(s) + \frac{1}{2}\int_0^{|u|} \Delta f(X_1(s))\, ds.$$

Thus,

$$E(f(X_1(|u|) + x)) = f(x).$$

$$\begin{aligned}E(f(Z(t) + x)) &= \int_{\mathbb{R}} E^{\Omega_1}(f(X_1(|u|) + x)) p(t,u)\, du \\ &= \int_{\mathbb{R}} f(x) p(t,u)\, du = f(x).\end{aligned}$$

□

**Proposition 3.14.** *Proof.* Let $f : \mathbb{R}^d \to \mathbb{R}$ be a biharmonic function. Then, for any $x \in \mathbb{R}^d$ and $t \geq 0$, the following holds:

$$E(f(Z(t) + x)) = f(x) + \sqrt{\frac{t}{2\pi}}\Delta f(x).$$

*Proof.* By Itô's formula, for each $u \in \mathbb{R}$, the following holds:

$$f(X_1(|u|) + x) = f(x) + \sum_{i=1}^{d}\int_0^{|u|} \frac{\partial}{\partial y_i} f(X_1(|s|))\, dX_1(s) + \frac{1}{2}\int_0^{|u|} \Delta f(X_1(s))\, ds.$$

Thus,

$$\begin{aligned}E(f(X_1(|u|) + x)) &= f(x) + \frac{1}{2}E\left(\int_0^{|u|} \Delta f(X_1(s) + x)\, ds\right) \\ &= f(x) + \frac{1}{2}\int_0^{|u|} E(\Delta f(X_1(s) + x))\, ds \\ &= f(x) + \frac{1}{2}\Delta f(x)|u|.\end{aligned}$$

□

$$E(f(Z(t)+x)) = \int_{\mathbb{R}} E^{\Omega_1}(f(X_1(|u|)+x))p(t,u)\,du = \int_{\mathbb{R}} \left\{ f(x) + \frac{1}{2}\Delta f(x)|u| \right\} p(t,u)\,du$$
$$= f(x) + \frac{1}{2}\sqrt{\frac{2t}{\pi}} = f(x) + \sqrt{\frac{t}{2\pi}}\Delta f(x)$$

□

**Theorem 3.15.** *Let $c > 0$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a biharmonic function. Define*

$$v(t,x) = E(f(Z(t)+x))$$

*for $t \geq 0$ and $x \in \mathbb{R}^d$. For any $t \geq 0$ and $x \in \mathbb{R}^d$, define*

$$u(t,x) = v(t,x) - \frac{1}{\sqrt{c}}v(ct,x).$$

*Then the following holds:*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{8}\Delta^2 u(t,x) \quad (t > 0,\ x \in \mathbb{R}^d)$$
$$u(0,x) = \left(1 - \frac{1}{\sqrt{c}}\right) f(x) \quad (x \in \mathbb{R}^d).$$

*Proof.* For any $t \geq 0$ and $x \in \mathbb{R}^d$, the following holds:

$$u(t,x) = v(t,x) - \frac{1}{\sqrt{c}}v(ct,x)$$
$$= \left(f(x) + \sqrt{\frac{t}{2\pi}}\Delta f(x)\right) - \frac{1}{\sqrt{c}}\left(f(x) + \sqrt{\frac{ct}{2\pi}}\Delta f(x)\right)$$
$$= \left(1 - \frac{1}{\sqrt{c}}\right) f(x).$$

□

## 3.1 Relation to Funaki's Method

If $f : \mathbb{R}^d \to \mathbb{R}$ is biharmonic, it is possible to construct a solution to $\left(\partial_t - \frac{\Delta^2}{8}\right) v = 0$ using the method from Theorem 3.15 with iterated Brownian motion. On the other hand, Funaki's method requires a growth condition on the function of the boundary condition. Considering the case when $d = n = 1$ in Funaki's method, we adopt a method to extend the Brownian motion $B$ to the complex plane. For $z \in \mathbb{C}$, let $\hat{f}(z) = \exp(z^3)$ and restrict it to real values as $f : \mathbb{R} \to \mathbb{R}$ (i.e., $f(x) = \exp(x^3)$). In this case, $\hat{f}$ does not satisfy the growth condition.

## 4 Discretization of Iterated Brownian Motion

In this paper, iterated Brownian motion is defined as the composition of a time-defined Brownian motion and a space-defined Brownian motion. Discretization is performed on both the time and space sides using random walks. First, we provide notation for the random walk on the spatial side.

**Notation 4.1.** *Let $(A_1, \mathcal{G}_1, Q_1)$ be a probability space, and let the sequence of independent and identically distributed random variables $(\zeta_i^j)_{i\in\mathbb{N}, j=1,\cdots,d}$ satisfy*

$$Q_1(\zeta_i^j = \pm 1) = \frac{1}{2}.$$

*For each $j = 1, \cdots, d$ and $m \in \mathbb{N}$, the random walk on $A_1$ is defined as*

$$S^{(m,j)} = \sum_{i=1}^{m} \zeta_i^j \quad (\forall j = 1, \cdots, d).$$

*For each $m \in \mathbb{N}$, the $d$-dimensional random walk scaled on the probability space $(A_1^d, \sigma(\mathcal{G}_1^d), Q_1^d)$ is defined by*

$$S_d^{(m)}(t) = \sqrt{\frac{t}{m}} \left( S^{(m,1)}, \cdots, S^{(m,d)} \right).$$

Next, we define notation for the time-defined random walk.

**Notation 4.2.** *Let $T > 0$. Let $(A_2, \mathcal{G}_2, Q_2)$ be a probability space, and let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables on this space that satisfies:*

$$Q_2(\eta_i = \pm 1) = \frac{1}{2}.$$

*For each $m \in \mathbb{N}$, the scaled random walk on $(A_2, \mathcal{G}_1, Q_2)$ is defined by*

$$R^{(m)}(T) = \sqrt{\frac{T}{m}} \sum_{i=1}^{m} \eta_i.$$

Let $A = A_1^d \times A_2, \mathcal{G} = \sigma(\mathcal{G}_1^d \times \mathcal{G}_2), Q = Q_1^d \times Q_2$.

**Definition 4.3.** *For each $T > 0$, define the random variable $\widehat{R}^{(m,d)}(T)$ on the probability space $(A, \mathcal{G}, Q)$ as follows, and denote it as the iterated random walk:*

$$\widehat{R}_d^{(m,n)}(T, \omega') = S_d^{(m)}(|R^{(n)}(T, \omega_2')|, \omega_1').$$

*Here, $\omega' = (\omega_1', \omega_2') \in A_1^d \times A_2$.*

**Corollary 4.4.** *The distribution of the iterated random walk is given by: For each $b \in \bigcup_{a\in R^{(m)}(T)(A_2)} S_d^{(m)}(|a|)(A_1^d)$,*

$$Q(\widehat{R}_d^{(m,n)}(T) = b) = \sum_{a\in R^{(m)}(T)(A)} Q_2(R^{(m)}(T) = a) Q_1^d(S_d^{(m)}(|a|) = b).$$

**Theorem 4.5.** *Let $f \in C_b(\mathbb{R})$ and $T \geq 0$. As $n, m \to \infty$,*

$$|E^{\Omega}(f(Z(t))) - E^A(f(\widehat{R}_d^{(m,n)}(T)))| \to 0.$$

*Proof.*

$$\begin{aligned} E^A\left(f(\widehat{R}_d^{(m,n)}(T))\right) &= \int_{A_2}\int_{A_1^d} f\left(S_d^{(m)}\left(|R^{(n)}(T, \omega_2')|, \omega_1'\right)\right) dQ_1^d(\omega_1')\, dQ_2(\omega_2') \\ &= \sum_{k\in R^{(n)}(T)(A_2)} \int_{\{R^{(m)}(T)=k\}} \int_{A_1^d} f\left(S_d^{(m)}\left(|k|, \omega_1'\right)\right) dQ_1^d(\omega_1')\, dQ_2(\omega_2'). \end{aligned}$$

Since $\#R^{(n)}(T)(A_2) < \infty$, the following holds:

$$\forall \epsilon > 0, \exists N_1^{(m)} \in \mathbb{N} \text{ such that } \forall k \in R^{(m)}(T)(A_2),\ \forall \omega_2 \in \{R^{(m)}(T) = k\},\ \forall n \geq N_1^{(m)}$$
$$\Rightarrow \left| \int_{A_1^d} f\left(S_d^{(m)}(|k|, \omega_1')\right) dQ_1^d(\omega_1') - \int_{\mathbb{R}} f(y) p_d(|k|, y)\, dy \right| < \frac{\epsilon}{2}.$$

Thus, for each $m \geq 1$, the following holds:

$$\begin{aligned}
&\left| E^A(f(\widehat{R}_d^{(m,n)}(T))) - \int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') \right| \\
&\leq \sum_{k \in R^{(n)}(T)(A_2)} \int_{\{R^{(n)}(T)=k\}} \left| \int_{A_1^d} f\left(S_d^{(m)}(|k|, \omega_1')\right) dQ_1^d(\omega_1') - \int_{\mathbb{R}} f(y) p_d(|R^{(n)}(T)(\omega_2')|, y)\, dy \right| dQ_2(\omega_2') \\
&\leq \sum_{k \in R^{(n)}(T)(A_2)} \int_{\{R^{(n)}(T)=k\}} \epsilon\, dQ_2(\omega_2') = \frac{\epsilon}{2}.
\end{aligned}$$

By equation (3.2), we define $F$ as follows:

$$\int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') = E^{A_2}(F(R^{(n)}(T))).$$

Thus, there exists $N_2 \in \mathbb{N}$ such that for each $m \geq N_2$, the following holds:

$$\left| \int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') - \int_{\Omega_2} \int_{\mathbb{R}} f(y) p(|Y(t, \omega_2)|, y)\, dy\, dP_2(\omega_2) \right| < \frac{\epsilon}{2}.$$

In other words,

$$\left| \int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') - E(f(Z(t))) \right| < \frac{\epsilon}{2}.$$

Therefore, for each $m \geq N_2$ and $n \geq N_1^{(m)}$, the following holds:

$$\begin{aligned}
|E(f(Z(t))) - E^A(f(\widehat{R}_d^{(m,n)}(T)))| &\leq \left| \int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') - E(f(Z(t))) \right| \\
&\quad + \left| E^A(f(\widehat{R}_d^{(m,n)}(T))) - \int_{A_2} \int_{\mathbb{R}} f(y) p(|R^{(n)}(T)(\omega_2')|, y)\, dy\, dQ_2(\omega_2') \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

**Proposition 4.6.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be bounded, twice differentiable, and assume that $f'$ and $f''$ are also bounded. Define $F : \mathbb{R} \to \mathbb{R}$ as follows:*

$$F(u) = \begin{cases} \int_{\mathbb{R}} f(y) p_d(|u|, y)\, dy & (u \neq 0) \\ f(0) & (u = 0) \end{cases}$$

*In this case, $F$ has a bounded derivative on $\mathbb{R} \setminus \{0\}$ and is Lipschitz continuous on $\mathbb{R}$.*

*Proof.* We will demonstrate this for the case $d = 1$. Note that for any $u \in \mathbb{R}$, $F(u) = F(-u)$ holds. For any $a > 0$, the following condition is satisfied:

$$\int_{\mathbb{R}} e^{-ax^2} |f(x)|\, dx < \infty.$$

Thus, by Problem 3.1 in [6], $F$ is differentiable on $\mathbb{R} \setminus \{0\}$ and the following holds:

$$F'(u) = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial u} p(|u|, y)\, dy = \frac{1}{2} \int_{\mathbb{R}} f(y) \left(\frac{\partial}{\partial y}\right)^2 p(|u|, y)\, dy \quad (\forall u \neq 0).$$

$$\begin{aligned} 2F'(u) &= \int_{\mathbb{R}} f(y) \left(\frac{\partial}{\partial y}\right)^2 p(|u|, y)\, dy \\ &= \left[f(y) \frac{\partial}{\partial y} p(u, y)\right]_{y=-\infty}^{y=\infty} - \left[f'(y) p(u, y)\right]_{y=-\infty}^{y=\infty} + \int_{\mathbb{R}} f''(y) p(u, y)\, dy. \end{aligned}$$

Since $\sup_{y \in \mathbb{R}} |f(y)|$, $\sup_{y \in \mathbb{R}} |f'(y)|$, $\sup_{y \in \mathbb{R}} |f''(y)| < \infty$, it follows that

$$\begin{aligned} \frac{\partial}{\partial y} p(u, y) &= \frac{1}{\sqrt{2\pi u}} \left(-\frac{y}{u}\right) \exp\left(-\frac{y^2}{2u}\right), \\ \left(\frac{\partial}{\partial y}\right)^2 p(u, y) &= \left(-\frac{1}{u} + \frac{y^2}{u^2}\right) \frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{y^2}{2u}\right), \end{aligned}$$

demonstrating that $F'$ is bounded on $\mathbb{R} \setminus \{0\}$. Let $L_1 = \frac{1}{2} \sup_{y \in \mathbb{R}} |f''(y)| < \infty$. For each $u \neq 0$, we will show that

$$|F(u) - F(0)| \leq L_1 |u|.$$

Assuming $u \neq 0$, by Taylor's theorem, for any $y \in \mathbb{R}$, there exists $c_y \in (0, y)$ such that

$$f(y) - f(0) = f'(0) y + \frac{f''(c_y)}{2} y^2.$$

Thus,

$$\begin{aligned} F(u) - F(0) &= \int_{\mathbb{R}} (f(y) - f(0)) p(|u|, y)\, dy \\ &= \int_{\mathbb{R}} \left(f'(0) y + \frac{f''(c_y)}{2} y^2\right) p(|u|, y)\, dy \\ &= \frac{1}{2} \int_{\mathbb{R}} f''(c_y) y^2 p(|u|, y)\, dy. \end{aligned}$$

Thus, the following holds:

$$|F(u) - F(0)| \leq \frac{\sup_{y \in \mathbb{R}} |f''(y)|}{2} \int_{\mathbb{R}} y^2 p(|u|, y)\, dy = \frac{\sup_{y \in \mathbb{R}} |f''(y)|}{2} |u|.$$

Let $L_2 = \sup_{u \in \mathbb{R} \setminus \{0\}} |F'(u)| < \infty$. For any $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, we will show that

$$|F(u_1) - F(u_2)| \leq L_2 |u_1 - u_2|.$$

Assuming $u_1, u_2 \in \mathbb{R} \setminus \{0\}$ and $|u_1| < |u_2|$, $F$ is continuous on $[|u_1|, |u_2|]$ and differentiable on $(|u_1|, |u_2|)$. By the Mean Value Theorem, there exists $d = d(u_1, u_2) \in (|u_1|, |u_2|)$ such that

$$F(|u_2|) - F(|u_1|) = F'(d)(|u_2| - |u_1|).$$

Thus,

$$|F(u_1) - F(u_2)| = |F(|u_2|) - F(|u_1|)| \leq |F'(d)|\, ||u_2| - |u_1|| \leq L_2 |u_1 - u_2|.$$

□

The iterated random walk converges to the true value of the iterated Brownian motion in the order of $-\frac{1}{2}$.

**Theorem 4.7.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be bounded and twice differentiable, and assume that both $f'$ and $f''$ are also bounded, satisfying:*

$$\lim_{|x|\to\infty} f(x) = 0.$$

*Then for any real number $K > 0$ and $t \geq 0$, there exist constants $M = M(t) > 0$ and $M' = M'(t, K) > 0$ such that for any natural numbers $n$ and $m$, the following holds:*

$$\left|E(f(Z(t))) - E^A(f(\widehat{R}_d^{(m,n)}(t)))\right| \leq \frac{M}{\sqrt{n}} + \frac{M'}{\sqrt{m}} + \frac{2}{K}.$$

*Proof.* Let $t > 0$ and $K > 0$ be arbitrary. We have:

$$\begin{aligned} E(f(Z(t))) &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(y) p_d(|u|, y)\, dy\, p(t, u)\, du \\ &= \int_{\mathbb{R}} F(u) p(t, u)\, du \\ &= E(F(X_1(t))). \end{aligned}$$

By assumption, since $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, there exists a constant $M = M(t) > 0$ such that for any $n \in \mathbb{N}$, the following holds:

$$|E(f(Z(t))) - E^A(F(R^{(n)}(t)))| = |E(F(X_1(t))) - E^A(F(R^{(n)}(t)))| \leq \frac{M}{\sqrt{n}}.$$

Additionally, note that:

$$E^A(F(R^{(n)}(t))) = \sum_{a \in R^{(n)}(t)(A)} F(a) Q(R^{(n)}(t) = a).$$

There exists a function $f_K : \mathbb{R}^d \to \mathbb{R}$ satisfying the following conditions:

- $\sup_{x\in\mathbb{R}^d} |f(x) - f_K(x)| \leq \frac{1}{K}$.
- $f_K : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous.
- The support of $f_K$ is bounded.

For each $a \in R^{(n)}(t)(A)$, let $F_K(a) = E(f_K(X_1(|a|)))$. Now, we have:

$$\left|E^A(F(R^{(n)}(t))) - \sum_{a \in R^{(n)}(t)(A) \cap \operatorname{supp} f_K} F_K(|a|) Q(R^{(n)}(t) = a)\right| \leq \frac{1}{K}.$$

Since $R^{(n)}(t)(A) \cap \operatorname{supp} f_K \subset \operatorname{supp} f_K$, there exists a constant $M' = M'(t, K) > 0$ such that for any $n, m \in \mathbb{N}$ and $a \in R^{(n)}(t)(A)$, the following holds:

$$|F_K(|a|) - E^A(f_K(S^{(m)}(|a|)))| \leq \frac{M'}{\sqrt{m}}.$$

Also, for each $a \in R^{(n)}(t)(A)$, we have:

$$|E^A(f_K(S_d^{(m)}(|a|))) - E^A(f(S_d^{(m)}(|a|)))| \leq \frac{1}{K}.$$

Thus, for each $a \in R^{(n)}(t)(A)$, the following holds:

$$|F_K(|a|) - E^A(f(R^{(n)}(t)))| \leq \frac{M'}{\sqrt{m}} + \frac{1}{K}.$$

Consequently, we have:

$$\left|\sum_{a\in R^{(n)}(t)(A)\cap \operatorname{supp} f_K} F_K(|a|)Q(R^{(n)}(t)=a) - E^A(f(R_d^{(m,n)}(t)))\right| \le \left(\frac{M'}{m}+\frac{1}{K}\right)\sum_{a\in R^{(n)}(t)(A)\cap \operatorname{supp} f_K} Q(R^{(n)}(t)=a)$$
$$\le \frac{M'}{\sqrt{m}}+\frac{1}{K}.$$

From the above, we obtain:

$$\begin{aligned}
&|E(f(Z(t))) - E^A(f(R_d^{(m,n)}(t)))| \\
&\le |E(f(Z(t))) - E^A(F(R^{(n)}(t)))| \\
&\quad + \left|E^A(F(R^{(n)}(t))) - \sum_{a\in R^{(n)}(t)(A)\cap \operatorname{supp} f_K} F_K(|a|)Q(R^{(n)}(t)=a)\right| \\
&\quad + \left|\sum_{a\in R^{(n)}(t)(A)\cap \operatorname{supp} f_K} F_K(|a|)Q(R^{(n)}(t)=a) - E^A(f(R_d^{(m,n)}(t)))\right| \\
&\le \frac{M}{\sqrt{n}}+\frac{1}{K}+\frac{M'}{\sqrt{m}}+\frac{1}{K} \\
&= \frac{M}{\sqrt{n}}+\frac{M'}{\sqrt{m}}+\frac{2}{K}.
\end{aligned}$$

□

**Example 4.8.** *Let $d = 1$ and $f(x) = x^2$. In this case, for each $u \in \mathbb{R}$,*

$$F(u) = |u|.$$

*Thus, $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Moreover, from equation* (3.15)*,*

$$E(f(Z(t))) = E((Z(t))^2) = \sqrt{\frac{2t}{\pi}}.$$

*For each $a \in R^{(n)}(t)(A)$, we have:*

$$|F(a) - E(f(R^{(n)}(|a|)))| = ||a| - E((R^{(n)}(|a|))^2)| = ||a| - |a|| = 0.$$

*Therefore,*

$$\begin{aligned}
&\left|E(f(Z(t))) - \sum_{a\in R^{(n)}(t)(A)} Q(R^{(n)}(t)=a)E^A(f(R^{(n)}(|a|)))\right| \\
&\le \frac{K_1}{\sqrt{n}}.
\end{aligned}$$

**Example 4.9.** *Consider the case where $f(x) = x^4 \quad (x \in \mathbb{R})$. From equation* (3.15)*, we have:*

$$E(f(Z(t))) = E((Z(t))^4) = 3t.$$

$$F(u) = 3u^2 \quad (\forall u \in \mathbb{R}).$$

*Let $n \ge 1$. Then,*

$$E(F(R^{(n)}(t))) = 3E((R^{(n)}(t))^2) = 3t.$$

*Let the characteristic function of* $\sum_{i=1}^n \xi_i$ *be* $\phi_n$*. By the independence of* $\{\xi_i\}_{i=1}^n$*,*

$$\phi_n(v) = \cos^n v \quad (v \in \mathbb{R}).$$

$$\begin{aligned}
\frac{d^4}{dv^4}\phi_n(v) &= \frac{d^3}{dv^3}\left(-n\cos^{n-1} v \sin v\right) \\
&= \frac{d^2}{dv^2}\left(n(n-1)\cos^{n-2} v \sin^2 v - n\cos^n v\right) \\
&= \frac{d}{dv}\left(-n(n-1)(n-2)\cos^{n-3} v \sin^3 v + (3n^2-2n)\cos^{n-1} v \sin v\right) \\
&= n(n-1)(n-2)(n-3)\cos^{n-4} v \sin^4 v - 2n(n-1)(3n-2)\cos^{n-2} v \sin^2 v + (3n^2-2n)\cos^n v.
\end{aligned}$$

*Therefore, for any* $a \in R^{(n)}(t)(\Omega)$*, the following holds:*

$$E((R^{(n)}(a))^4) = \frac{a^2}{n^2} E\left(\left(\sum_{i=1}^n \xi_i\right)^4\right) = \frac{a^2}{n^2}\frac{d^4}{dv^4}\phi_n(v)\Big|_{v=0} = 3a^2 - \frac{2a^2}{n}.$$

$$F(a) - E^A(f(R^{(n)}(|a|))) = \frac{2a^2}{n}.$$

$$\begin{aligned}
&|E(F(R^{(n)}(t))) - \sum_{a \in R^{(n)}(t)(A)} Q(R^{(n)}(t)=a)E^A(f(R^{(n)}(|a|)))| \\
&\leq \sum_{a \in R^{(n)}(t)(A)} Q(R^{(n)}(t)=a)\left|F(a) - E(f(R^{(n)}(|a|)))\right| \\
&\leq \frac{1}{n}\sum_{a \in R^{(n)}(t)(A)} a^2 Q(R^{(n)}(t)=a) \\
&\leq \frac{t}{n}.
\end{aligned}$$

**Example 4.10.** *Consider the case where* $f(x) = \exp\left(-\frac{x^2}{2}\right) \quad (x \in \mathbb{R})$*. In this case,* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the conditions of Theorem 4.7.*

$$\begin{aligned}
E(f(Z(t))) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(-\frac{y^2}{2}\right) p(|u|, y)\, dy\, p(t,u)\, du \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi|u|}}\int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(\frac{1}{|u|}+1\right)y^2\right) dy\, p(t,u)\, du \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi|u|}}\sqrt{\frac{2|u|}{|u|+1}\pi}\, p(t,u)\, du \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} \frac{1}{\sqrt{1+|u|}}\exp\left(-\frac{u^2}{2t}\right) du.
\end{aligned} \tag{4.1}$$

# 5 Future Challenges

The future challenges can be broadly classified into three points. The first point is to extend the solutions to fourth-order partial differential equations for which an expression is possible. This study demonstrated that when the boundary condition $f$ is a biharmonic function, a solution representation via iterative Brownian motion is achievable for the equation $\left(\frac{\partial}{\partial t} - \frac{1}{8}\Delta^2\right)u = 0$. The goal is to relax the conditions on $f$ to obtain a similar assertion.

The second point is the construction of Ito's formula for iterative Brownian motion. There is a possibility of inferring a structure from Proposition 3.14. In this paper, weak approximation via random walks for iterative Brownian motion was obtained. There is a background that allows the construction of Ito's formula for Brownian motion as the limit of the discrete Ito formula for random walks in the case of standard Brownian motion [4].


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