Negative Time is Real, Physicists Confirm.

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Abstract

Recently a group of physicists measured a negative "group delay" for a pulse of light transmitted in a cold cloud of Rubidium atoms [1]. In this paper we will study the propagation of a wave packet incident on a dispersive and dissipative medium, determining the "transit time" of the packet, which is the analogue of the group delay. We will show that if the phase of the transmitted wave is decreasing in correspondence with the value of the wavenumber that determines the peak value of the amplitude of the spectral density of the packet, the transit time is negative.

1 Spectral Density. Dispersion Law

Let us consider the one-dimensional propagation of a wave packet [2]:

$$\psi\left(x,t\right)=\int_{-\infty}^{+\infty}A\left(k\right)e^{i\left(kx-\omega\left(k\right)t\right)}dk \tag{1}$$

where k is the wavenumber related to the wavelength by $k=2\pi/\lambda$. Let $\psi_0\left(x\right)=\psi\left(x,0\right)$ (initial profile of the packet) with $\psi_0\in\mathcal{L}^2\left(\mathbb{R}\right)$, by the Fourier integral theorem:

$$A\left(k\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_0\left(x\right)e^{-ikx}dx \tag{2}$$

for which $A\left(k\right):\mathbb{R}\to\mathbb{C}$.

$d\mathcal{A}\overset{def}{=}\left|A\left(k\right)\right|dk$ is the (infinitesimal) amplitude of the monochromatic components of wavenumber belonging to the infinitesimal interval $[k,k+dk]$. Therefore

$$\left|A\left(k\right)\right|=\frac{d\mathcal{A}}{dk}$$

i.e. $\left|A\left(k\right)\right|$ is the *spectral density* of the wave packet. We assume the real function $|A(k)|$ extremely peaked around a given[1] $k_0\in\mathbb{R}$. This implies that the dominant contribution to $\psi\left(x,t\right)$ comes from the monochromatic components with

$$k\in\left(k_0-\delta k,k_0+\delta k\right),\quad\frac{\delta k}{|k_0|}\ll 1. \tag{3}$$

In (1) $\omega\left(k\right)$ is the pulsation of the single monochromatic component. We assume the real function $\omega\left(k\right)$ analytic in $\mathbb{R}$. This function expresses the *dispersion law* of the medium in which the packet propagates.

2 Propagation velocity. Phase velocity and group velocity

Let us consider the scheme in Fig. 1 iin which a one-dimensional wave packet strikes the region represented by the segment $[0,\delta]$ of the x-axis. Let us then set:

$$\psi_{inc}\left(x,t\right)=\int_{-\infty}^{+\infty}A\left(k\right)e^{i\left(kx-\omega\left(k\right)t\right)}dk \tag{4}$$

[1] For example, $|A(k)|$ can be a Gaussian centered at k_0.

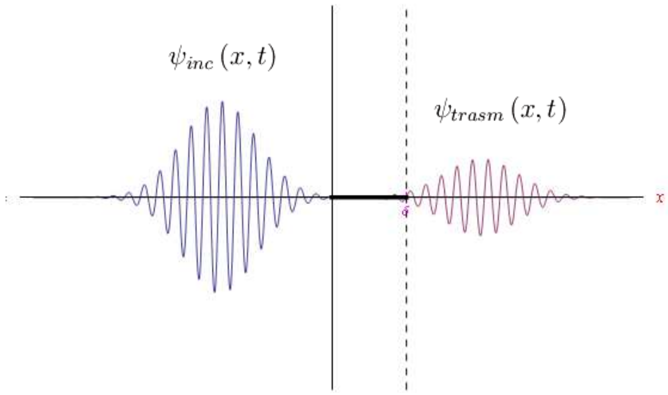

Figure 1: A wave packet strikes the region $[0, \delta]$ of the x-axis.

Definition 1 *We call the propagation speed of the monochromatic component* $A(k)\, e^{i(kx-\omega(k)t)}$ *the* **phase velocity** *of the wave packet:*

$$v_p(k) = \frac{\omega(k)}{k} \tag{5}$$

For the above, we assume $|A(k)|$ extremely steep around a given k_0. Since the dispersion law $\omega(k)$ is by hypothesis analytic, we can develop this function in Taylor series with initial point k_0. In a neighborhood of the type (3) is permissible to truncate the expansion to first order:

$$\omega(k) \simeq \omega_0 + v_g(k - k_0) \tag{6}$$

where

$$\omega_0 \stackrel{def}{=} \omega(k_0), \quad v_g \stackrel{def}{=} \left.\frac{d\omega(k)}{dk}\right|_{k=k_0} \tag{7}$$

with v_g having the dimensions of a velocity. Substituting in (4):

$$\begin{aligned}
\psi_{inc}(x,t) &= \int_{-\infty}^{+\infty} A(k) \exp\{kx - [\omega_0 + v_g(k-k_0)]\, t\}\, dk \\
&= \int_{-\infty}^{+\infty} A(k) \exp(kx - \omega_0 t - v_g t k + v_g t k_0)\, dk \\
&= \int_{-\infty}^{+\infty} A(k)\, e^{ik(x-v_g t)} e^{-i(\omega_0 - v_g k_0)t} dk \\
&= e^{-i(\omega_0 - v_g k_0)t} \int_{-\infty}^{+\infty} A(k)\, e^{ik(x-v_g t)} dk
\end{aligned}$$

$e^{-i(\omega_0 - v_g k_0)t}$ is an inessential phase factor, so

$$\psi_{inc}(x,t) = \int_{-\infty}^{+\infty} A(k)\, e^{ik(x-v_g t)} dk \tag{8}$$

This wave function tells us that the profile of the incident packet translates rigidly (i.e. without deforming) and uniformly with velocity v_g. This circumstance suggests:

Definition 2 *We call* **group velocity** *of the wave packet (8) the quantity:*

$$v_g = \left.\frac{d\omega(k)}{dk}\right|_{k=k_0} \tag{9}$$

In order for the propagation illustrated in fig. 1 to be sensible, must be $v_g > 0$ i.e. i.e. the function $\omega(k)$ must be increasing in ω_0. In the opposite case, $\omega(k)$ decreasing in k_0, the propagation is regressive since $v_g < 0$. There remains the case in which the dominant wave number k_0 is the critical point for $\omega(k)$ for which $v_g = 0$. Since there is no propagation (packet cut-off) the wave function $\psi_{inc}(x,t)$ describes a standing wave.

The transmitted wave packet is:

$$\psi_{trasm}(x,t) = \int_{-\infty}^{+\infty} B(k) \exp\left[k(x-\delta) - v_g k t\right] dk \tag{10}$$

Here we assume that the propagation medium $[0,\delta]$ sis dissipative. It follows that the output packet is attenuated. We then define a (complex) *transmission coefficient* of a single monochromatic component:

$$\tau(k) = \frac{B(k)}{A(k)} = \frac{|B(k)| e^{i\phi_2(k)}}{|A(k)| e^{i\phi_2(k)}} = \frac{|B(k)|}{|A(k)|} e^{i\phi(k)},$$

where $\phi(k) = \phi_2(k) - \phi_1(k)$. Without loss of generality, we assume $A(k)$ to be a real function.[2] for which $\phi(k)$ is the phase of the complex amplitude $B(k)$ or of the monochromatic component of wavenumber k. So

$$\psi_{trasm}(x,t) = \int_{-\infty}^{+\infty} A(k) |\tau(k)| \exp\left[k(x-\delta) - v_g k t + \phi(k)\right] dk \tag{11}$$

We develop $\phi(k)$ in Taylor series with initial point k_0. In a neighborhood of the type (3) is permissible to truncate the expansion to first order:

$$\phi(k) \simeq \phi_0 + \Lambda_0 (k - k_0) \tag{12}$$

where

$$\phi_0 \stackrel{def}{=} \phi(k_0), \quad \Lambda_0 \stackrel{def}{=} \left.\frac{d\phi(k)}{dk}\right|_{k=k_0} \tag{13}$$

(11) becomes

$$\psi_{trasm}(x,t) = e^{i\beta_0} \int_{-\infty}^{+\infty} A(k) |\tau(k)| e^{ik[(x-\delta)-v_g(t-T)]} dk \tag{14}$$

where

$$T \stackrel{def}{=} \frac{\Lambda_0}{v_g} = \frac{\left.\frac{d\phi(k)}{dk}\right|_{k=k_0}}{\left.\frac{d\omega(k)}{dk}\right|_{k=k_0}} = \left.\frac{d\phi(\omega)}{d\omega}\right|_{\omega=\omega_0}, \quad \beta_0 = \phi_0 + \Lambda_0 k_0 \tag{15}$$

So except for an inessential phase factor, the transmitted packet is:

$$\psi_{trasm}(x,t) = \int_{-\infty}^{+\infty} A(k) |\tau(k)| e^{ik[(x-\delta)-v_g(t-T)]} dk \tag{16}$$

3 Conclusions

From (16)-(15) follows il tempo impiegato

$$T = \frac{\Lambda_0}{v_g} \tag{17}$$

[2]This occurs if $\psi_0(-x) \equiv \psi_0(x)$.

On the other hand, the profile propagates uniformly with velocityv_g, so the transit time in $[0, \delta]$ is $T = \delta / v_g$. Comparing with (17)

$$\left. \frac{d\phi(k)}{dk} \right|_{k=k_0} = \delta \tag{18}$$

which constrains the value of the derivative of the phase $\phi(k)$ of the transmitted wave at the point $k0$ which is the relative maximum for the spectral density $A(k)$. Follows that as the thickness d increases, the slope of the curve $\phi = \phi(k)$. Therefore, as d increases, the transmitted wave is further out of phase (as well as attenuated). Alternatively, we can conjecture the existence of a dispersive and dissipative medium such as to violate the (18). This forces us to assume as the transit time the (17):

$$T = \frac{\left. \frac{d\phi(k)}{dk} \right|_{k=k_0}}{v_g} \neq \delta$$

In a medium characterized by a $\omega(k)$ such as to have an anomalous dispersion corresponding to a reduction in the phase of the transmitted wave as the wave number k increases (and therefore as the wavelength decreases), we necessarily have

$$\left. \frac{d\phi(k)}{dk} \right|_{k=k_0} < 0 \underset{v_g > 0}{\Longrightarrow} T = \frac{\left. \frac{d\phi(k)}{dk} \right|_{k=k_0}}{v_g} < 0 \tag{19}$$

References

[1] Experimental evidence that a photon can spend a negative amount of time in an atom cloud .

[2] Colozzo M., Motion of a Wave Packet.

[3] Jackson J.D., *Classical Electrodynamics*. Wiley.