# A new continued fraction approximation and bounds for the psi function

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**ABSTRACT:** In this paper, we provide some useful lemmas for construction continued fraction based on a given power series. Then we establish a new continued fraction approximation and bounds for the psi function. Especially, we analytically determine all parameters of the continued fraction by Bernoulli numbers.

**Keywords:** Euler connection, Continued fraction, Approximation, Psi function

## 1. Introduction

The classical Euler gamma function $\Gamma$ defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0, \tag{1.1}$$

was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non-integer values.

The logarithmic derivative $\psi(x)$ of the gamma function $\Gamma(x)$ given by

$$\psi(x) = \frac{\Gamma(x)'}{\Gamma(x)} \qquad \text{or} \qquad \ln\Gamma(x) = \int_1^x \psi(t)dt$$

is well-known as the psi(or digamma) function.
The following recurrence formula is well known for the psi function (see [1, p. 258]):

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{1.2}$$

The psi function is connected to the Euler-Mascheroni constant and harmonic numbers through the well-known relation (see [1, p. 258, Eq. (6.3.2)]):

$$\psi(n+1) = -\gamma + \mathrm{H}_n, \ \ \mathrm{n} \in \mathbb{N} \ , \tag{1.3}$$

where

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \, (\mathrm{n} \in \mathbb{N})$$

is the $n^{\text{th}}$ harmonic number and $\gamma$ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty} D_n = 0.577215664\cdots,$$

where

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$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n. \tag{1.4}$$

The constant $\gamma$ is deeply related to the gamma function $\Gamma(x)$ thanks to the Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [18, Chapter 1, Section 1.1]):

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{x}{k}\right)^{-1} e^{x/k} \right\} \tag{1.5}$$

As you can see, the gamma function, psi function and Euler-Mascheroni constant are related to each other.

In the study of special functions, the remarkable trend is to find more accurate approximations and bounds for them, so during the past several decades, many mathematicians and scientists have worked on this subject. Up to now, many researchers have made great efforts in this area of establishing more accurate approximations and bounds for the special functions and had lots of inspiring results.[2-7,9,10,12,13]

Recently, some authors have focused on continued fractions in order to obtain new asymptotic formulas.

For example, on the one hand, Mortici [16] found Stieltjes' continued fraction

$$\Gamma(x+1) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^{x} exp\left( \cfrac{a_0}{x + \cfrac{a_1}{x + \cfrac{a_2}{x + \ddots}}} \right), \tag{1.6}$$

where

$$a_0 = \frac{1}{12}, \quad a_1 = \frac{1}{30}, \quad a_2 = \frac{53}{210}, \quad \cdots .$$

Also Mortici [17] provided a new continued fraction approximation starting from the Nemes' formula as follows,

$$\Gamma(x+1) \approx \sqrt{2\pi x} e^{-x} \left( x + \cfrac{1}{12x - \cfrac{1}{10x + \cfrac{a}{x + \cfrac{b}{x + \cfrac{c}{x + \cfrac{d}{x + \ddots}}}}}} \right)^{x}, \tag{1.7}$$

where

$$a=-\frac{2369}{252},b=\frac{2117009}{1193976},c=\frac{3930321915\ 11}{1324011300\ 744},d=\frac{3326589616\ 4277124002\ 451}{1427802410\ 4089641878\ 840}\cdots.$$

On the other hand, Lu [14] provided a new continued fraction approximation based on the Burnside's formula as follows,

$$n!\approx\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\frac{a_1}{n^2+\frac{a_2 n}{n+\frac{a_3 n}{n+\ddots}}}\right)^{\frac{n-\frac{1}{2}}{k}}, \tag{1.8}$$

where

$$a_1=-\frac{k}{24},\quad a_2=\frac{k}{48}-\frac{23}{120},\quad a_3=\frac{14}{5k-46},\quad \cdots.$$

Also Lu [19] found two asymptotic formulas

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}+\frac{1}{x^2}\frac{a_1}{x+\frac{a_2}{x+\frac{a_3}{x+\ddots}}}}\right)^{x^2+\frac{53}{210}}, \tag{1.9}$$

where

$$a_1=\frac{2117}{35280},a_2=\frac{188098}{116435},a_3=\frac{1681526854\ 38}{1993008347\ 33},a_4=\frac{1513751180\ 9264254261\ 577}{4973625898\ 1948852651\ 52},\cdots,$$

and

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}\left(1+\frac{1}{x^6}\frac{b_1}{x+\frac{b_2}{x+\frac{b_3}{x+\ddots}}}\right), \tag{1.10}$$

where

$$b_1=-\frac{2117}{5080320}, b_2=\frac{1892069}{978054}, b_3=\frac{4064269668\,5}{9288453088\,2}, b_4=\frac{8499178650\,3793092231}{1599494758\,6610983120}, \cdots.$$

In this paper, based on continued fractions, we provide a new continued fraction approximation and continued fraction bounds for the psi function.

The rest of this paper is arranged as follows.

In Sect. 2, some useful lemmas are given. In Sect. 3, a new continued fraction approximation and bounds for the psi function are provided. In the last section, the conclusions are given.

## 2. Lemmas

In this section, we present a main method to construct continued fraction based on a given power series using Euler connection.

The Euler connection states the connection between series and continued fractions as follows.

**Lemma 2.1.**(The Euler connection [11, p.19, Eq. (1.7.1, 1.7.2)]) Let $\{c_k\}$ be a sequence in $\mathbb{C}\setminus\{0\}$ and

$$f_n=\sum_{k=0}^{n} c_k\,, \quad n\in\mathbb{N}_0. \tag{2.1}$$

Since $f_0\neq\infty$, $f_n\neq f_{n-1}$, $n\in\mathbb{N}$, there exists a continued fraction $b_0+K(a_m/b_m)$ with $n^{th}$ approximant $f_n$ for all $n$. This continued fraction is given by

$$c_0+\frac{c_1}{1}\,\frac{-c_2/c_1}{+1+c_2/c_1}\,+\cdots+\frac{-c_m/c_{m-1}}{1+c_m/c_{m-1}}\,+\cdots. \tag{2.2}$$

The following lemma states our main method.

**Lemma 2.2.** Let $\{c_k\}$ be a sequence in $\mathbb{R}\setminus\{0\}$. Then for every $x\neq 0$,

$$\sum_{i=1}^{n}\frac{c_{2i}}{x^{2i}}=\frac{1}{x^2}\mathop{K}_{i=1}^{n}\frac{a_i x^2}{x^2+b_i}=\frac{1}{x^2}\,\frac{a_1x^2}{x^2+\mathop{K}\limits_{i=2}^{n}\dfrac{a_i x^2}{x^2-a_i}}, \qquad n\in\mathbb{N}, \tag{2.3}$$

where

$$a_1=c_2,\ b_1=0,$$
$$a_i=-\frac{c_{2i}}{c_{2(i-1)}},\ b_i=-a_i,\ i=2,3,\cdots,n.$$

**Proof.** Assume that

$$f_0(x)\neq\infty,\quad f_n(x)=\sum_{i=1}^{n}\frac{c_{2i}}{x^{2i}},\quad \mathrm{n}\in\mathbb{N},\quad x\neq 0. \tag{2.4}$$

The left-side of (2.3) is equal to $f_n(x)$ $(\mathrm{n}\in\mathbb{N})$.

Since

$$f_0(x)\neq\infty,\quad f_n(x)\neq f_{n-1}(x),\quad \mathrm{n}\in\mathbb{N},$$

using Lemma 2.1,

$$
\begin{aligned}
f_n(x) &= \sum_{i=1}^{n}\frac{c_{2i}}{x^{2i}} \\
&= \frac{\frac{c_2}{x^2}}{1} + \frac{-\frac{c_4}{c_2x^2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2}{1} + \frac{-\frac{c_4}{c_2x^2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}} + \frac{-\frac{c_6}{c_4}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}} + \frac{-\frac{c_6}{c_4}x^2}{x^2+\frac{c_6}{c_4}} + \frac{-\frac{c_8}{c_6}}{1+\frac{c_8}{c_6x^2}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \cdots\ \ \cdots\ \ \cdots \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}} + \frac{-\frac{c_6}{c_4}x^2}{x^2+\frac{c_6}{c_4}} + \frac{-\frac{c_8}{c_6}x^2}{x^2+\frac{c_8}{c_6}} +\cdots+ \frac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}} +\cdots+ \frac{-\frac{c_{2n}}{c_{2(n-1)}}x^2}{x^2+\frac{c_{2n}}{c_{2(n-1)}}}
\end{aligned}
$$

$$=\frac{1}{x^2}\cfrac{c_2x^2}{x^2+\mathop{K}\limits_{i=2}^{n}\cfrac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}}}=\frac{1}{x^2}\cfrac{c_2x^2}{x^2+0+\mathop{K}\limits_{i=2}^{n}\cfrac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}}}\quad. \tag{2.5}$$

The middle expression of (2.3) is equal to

$$\frac{1}{x^2}\mathop{K}\limits_{i=1}^{n}\frac{a_ix^2}{x^2+b_i}=\frac{1}{x^2}\cfrac{a_1x^2}{x^2+b_1+\mathop{K}\limits_{i=2}^{n}\frac{a_ix^2}{x^2+b_i}},\qquad x\neq 0. \tag{2.6}$$

Thus,

$$a_1=c_2,\ b_1=0,$$
$$a_i=-\frac{c_{2i}}{c_{2(i-1)}},b_i=\frac{c_{2i}}{c_{2(i-1)}}=-a_i,\ \ i=2,3,\cdots,n.$$

Then, it is obviously true that

$$\frac{1}{x^2}\mathop{K}\limits_{i=1}^{n}\frac{a_ix^2}{x^2+b_i}=\frac{1}{x^2}\cfrac{a_1x^2}{x^2+\mathop{K}\limits_{i=2}^{n}\frac{a_ix^2}{x^2-a_i}},\qquad x\neq 0. \tag{2.7}$$

The proof of Lemma 2.2 is complete.

**Lemma 2.3.** The psi function $\psi$ has the asymptotic formulas as follows;

$$\psi(x)\sim \ln x-\frac{1}{2x}-\sum_{i=1}^{\infty}\frac{B_{2i}}{2ix^{2i}}\quad,\quad x\to\infty \tag{2.8}$$

$$\psi\left(x+\frac{1}{2}\right)\sim \ln x-\sum_{i=1}^{\infty}\frac{B_{2i}(1/2)}{2ix^{2i}}=\ln x+\sum_{i=1}^{\infty}\frac{B_{2i}(1-2^{1-2i})}{2ix^{2i}}\quad,\qquad x\to\infty \tag{2.9}$$

where $B_n (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$ denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\ \ |z|<2\pi, \tag{2.10}$$

then the first few terms of $B_n$ are as follows.

$$B_{2n+1}=0, n\geq 1,$$
$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{10}=\frac{5}{66},\cdots$$

We can find the expressions above by differentiating expressions (3.14) and (5.4) in [8].

**Lemma 2.4.** (see [15])**.** For $x>0$,

$$\ln x+\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}}<\psi\left(x+\frac{1}{2}\right)<\ln x+\sum_{i=1}^{2n+1}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}}, \qquad n\in\mathbb{N}_0. \qquad (2.11)$$

## 3. Main results

In this section, we present a new continued fraction approximation and bounds for the psi function using our main method and two remarks.

**Theorem 3.1.** We have a new continued fraction approximation for the psi function:

$$\psi\left(x+\frac{1}{2}\right)\sim\ln x+\frac{1}{x^2}\mathop{K}_{i=1}^{\infty}\frac{a_i x^2}{x^2+b_i}=\ln x+\frac{1}{x^2}\cfrac{a_1x^2}{x^2+b_1+\cfrac{a_2x^2}{x^2+b_2+\cfrac{a_3x^2}{x^2+b_3+\ddots}}}, \qquad x\to\infty \qquad (3.1)$$

where

$$a_1=\frac{B_2}{4},\ b_1=0,$$
$$a_i=-\frac{(i-1)(4^i-2)B_{2i}}{i(4^i-8)B_{2(i-1)}},\ b_i=-a_i,\ i=2,3,\cdots.$$

**Proof.** Let

$$c_{2i}=\frac{B_{2i}\left(1-2^{1-2i}\right)}{2i}, \qquad i=1,2,3,\cdots. \qquad (3.2)$$

From (3.2) and Lemma 2.2,

$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i}}=\sum_{i=1}^{\infty}\frac{B_{2i}\left(1-2^{1-2i}\right)}{2ix^{2i}}=\frac{1}{x^2}\mathop{K}_{i=1}^{\infty}\frac{a_ix^2}{x^2+b_i}, \qquad (3.3)$$

where

$$a_1=c_2=\frac{B_2}{4},\ b_1=0,$$
$$a_i=-\frac{c_{2i}}{c_{2(i-1)}}=-\frac{(i-1)(4^i-2)B_{2i}}{i(4^i-8)B_{2(i-1)}},\ b_i=\frac{c_{2i}}{c_{2(i-1)}}=-a_i,\ i=2,3,\cdots.$$

According to (2.9) and (3.3),

$$\psi\left(x+\frac{1}{2}\right)\sim\ln x+\frac{1}{x^2}\mathop{K}_{i=1}^{\infty}\frac{a_ix^2}{x^2+b_i}. \qquad (3.4)$$

Thus, our new continued fraction approximation can be obtained.

**Remark 3.1.** As you can see, our new continued fraction approximation for the psi function is equal to (2.9) but the expression is totally different.

From (2.7), we have another expression of (3.4) as follows:

$$\psi\left(x+\frac{1}{2}\right) \approx \ln x + \frac{a_1 x^2}{x^2 + \underset{i=2}{\overset{n}{K}} \frac{a_i x^2}{x^2 - a_i}} = \ln x + \frac{1}{x^2}\cfrac{a_1 x^2}{x^2 + \cfrac{a_2 x^2}{x^2 - a_2 + \cfrac{a_3 x^2}{x^2 - a_3 + \ddots}}}, \tag{3.5}$$

where

$$a_1 = \frac{1}{24}, a_2 = \frac{7}{40}, a_3 = \frac{155}{294}, a_4 = \frac{2667}{2480}, a_5 = \frac{2555}{1397}, a_6 = \frac{15559247}{5580120}, a_7 = \frac{11180715}{2828954}, \cdots.$$

For the convenience of readers, we rewrite.

$$\psi\left(x+\frac{1}{2}\right) \sim \ln x + \frac{1}{x^2}\cfrac{\frac{1}{24}x^2}{x^2 + \cfrac{\frac{7}{40}x^2}{x^2 - \frac{7}{40} + \cfrac{\frac{155}{294}x^2}{x^2 - \frac{155}{294} + \cfrac{\frac{2667}{2480}x^2}{x^2 - \frac{2667}{2480} + \cfrac{\frac{2555}{1397}x^2}{x^2 - \frac{2555}{1397} + \ddots}}}}} \tag{3.6}$$

**Theorem 3.2.** For $x > 0$,

$$\ln x + \frac{1}{x^2}\underset{i=1}{\overset{2n}{K}} \frac{a_i x^2}{x^2 + b_i} < \psi\left(x+\frac{1}{2}\right) < \ln x + \frac{1}{x^2}\underset{i=1}{\overset{2n+1}{K}} \frac{a_i x^2}{x^2 + b_i}, \quad n \in \mathbb{N}_0 \tag{3.7}$$

where

$$a_1 = \frac{B_2}{4},\ b_1 = 0,$$

$$a_i = -\frac{(i-1)(4^i - 2)B_{2i}}{i(4^i - 8)B_{2(i-1)}}, b_i = -a_i,\ \ i = 2, 3, \cdots 2n+1$$

**Proof**. Let

$$c_{2i} = \frac{(1-2^{1-2i})B_{2i}}{2i}, \qquad i = 1, 2, 3, \cdots 2n+1 \tag{3.8}$$

From (3.8) and Lemma 2.2,

$$\sum_{i=1}^{2n}\frac{c_{2i}}{x^{2i}}=\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}}=\frac{1}{x^2}\mathop{K}_{i=1}^{2n}\frac{a_ix^2}{x^2+b_i},$$
$$\sum_{i=1}^{2n+1}\frac{c_{2i}}{x^{2i}}=\sum_{i=1}^{2n+1}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}}=\frac{1}{x^2}\mathop{K}_{i=1}^{2n+1}\frac{a_ix^2}{x^2+b_i}, \tag{3.9}$$

where

$$a_1=c_2=\frac{B_2}{4},\ b_1=0,$$
$$a_i=-\frac{c_{2i}}{c_{2(i-1)}}=-\frac{(i-1)(4^i-2)B_{2i}}{i(4^i-8)B_{2(i-1)}},b_i=\frac{c_{2i}}{c_{2(i-1)}}=-a_i,\quad i=2,3,\cdots 2n+1.$$

From Lemma 2.4 and (3.9), it's clear that

$$\ln x+\frac{1}{x^2}\mathop{K}_{i=1}^{2n}\frac{a_ix^2}{x^2+b_i}<\psi\left(x+\frac{1}{2}\right)<\ln x+\frac{1}{x^2}\mathop{K}_{i=1}^{2n+1}\frac{a_ix^2}{x^2+b_i} \tag{3.10}$$

Thus, our new continued fraction bounds for the psi function are obtained.

**Remark 3.2.** For the convenience of readers, we take $n=2$ and the following result are derived.

$$\frac{1}{x^2}\cfrac{\frac{1}{24}x^2}{x^2+\cfrac{\frac{7}{40}x^2}{x^2-\frac{7}{40}+\cfrac{\frac{155}{294}x^2}{x^2-\frac{155}{294}+\cfrac{\frac{2667}{2480}x^2}{x^2-\frac{2667}{2480}}}}}<\psi\left(x+\frac{1}{2}\right)-\ln x<\frac{1}{x^2}\cfrac{\frac{1}{24}x^2}{x^2+\cfrac{\frac{7}{40}x^2}{x^2-\frac{7}{40}+\cfrac{\frac{155}{294}x^2}{x^2-\frac{155}{294}+\cfrac{\frac{2667}{2480}x^2}{x^2-\frac{2667}{2480}+\cfrac{\frac{2555}{1397}x^2}{x^2-\frac{2555}{1397}}}}}} \tag{3.11}$$

## 4. Conclusion

As mentioned above, in our investigation, we present a main method to construct continued fraction based on a given power series using Euler connection. Then we establish a new continued fraction approximation and bounds for the psi function. Especially, we analytically determine all parameters of the continued fraction by Bernoulli numbers.

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