Proof of the Collatz Conjecture Using a Laplacian Matrix

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Abstract.

We prove that the second smallest Laplacian eigenvalue of a Collatz graph is greater than zero. Thus the Collatz graph is connected.

Introduction.

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……The inverted Collatz sequences can be represented as an undirected Collatz graph [1]. In order to prove the Collatz conjecture, one must demonstrate that this graph is connected.

Laplacian of a Graph.

Let G(V,E) be a Collatz graph with a set of node V and a set of edge E. Let A and D be the adjacency matrix and the degree matrix of the Collatz graph, respectively. The Laplacian matrix (L) is defined as

$$L= D-A$$

Since L is symmetric, its eigenvalues λ_i are real ; assume them ordered in an increasing sequence [2]:

$$0 = \lambda_1 \le \lambda_2 \le \ldots\ldots\ldots\ldots \le \lambda_\infty$$

Let $x \in R^\infty$ be a function on the vertices of G(V,E). Then

$$x^T L x = \sum_{\{i,j\} \in E} (x_i - x_j)^2$$

Let $\quad R(x) = \frac{x^T Lx}{<x,x>}$

From the Rayleigh 's principle [3],

$$\lambda_1 = min_{z\neq 0} \ R(z)$$

$$\lambda_2 = max_{z\neq 0} \ \min_{<x.z>=0} R(x)$$

According to Collatz's rules, let a variable x_i represents a node i then

$$x^T Lx = \sum_{i=1..\infty}(x_i - x_{2i})^2 + (x_{2i+1} - x_{3i+2})^2$$

Let z be the first eigenvector of L

$$z = [1\ 1\ 1\ 1 \ldots\ldots\ldots\ldots 1].$$

λ_2 can be obtained from

$$\lambda_2 = \text{Min } R(x) \ ; \quad x \neq 0,$$

with a constraint

$$x_1 + x_2 + \text{-----------} + x_\infty = 0. \qquad (1)$$

If R(x) = 0 then

$$x_i - x_{2i} = 0 \qquad (2)$$

$$x_{2i+1} - x_{3i+2} = 0 \qquad (3)$$

for i = 1,2, 3,---------,∞.

The solution x for (1), (2), and (3) is the zeros vector, x = [0, 0,--------,0] which is not an eigenvector. It means that $\lambda_2 > 0$ then the Collatz graph is connected.

References

[1] W. Homsup and N. Homsup (2024). " Proof of the Collatz Conjecture". Vixra.org , 2311.0105v2.

[2] M. Tulsiani. " Lecture 10: July 15, 2013". UChicago REU 2013 Apprentice Program, Summer 2013.

[3] https://ecroot.math.gatech.edu/notes_linear.pdf