Quantum Fidelity

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Abstract

In this paper, quantum fidelity is introduced, starting with the Loschmidt echo. The quantum fidelity of teleportation is discussed and computed in an example of teleportation through a nonmaximally entangled channel.

Keywords: Loschmidt echo; Quantum fidelity ; Teleportation

1 Introduction

Understanding macroscopic irreversibility from reversible microscopic evolution laws is one of the fundamental controversies of statistical mechanics [1]. The problem has been reformulated in terms of Loschmidt echo, mainly in the context of quantum chaos and quantum information, and later in the context of classical dynamics. Echoes arise when through suitable manipulations in a system the dynamics is reversed and a more or less complete recovery of the initial state is achieved. Some examples of echoes include acoustical echoes which arise from reflections of sound at walls, spin echoes which arise from reversals of magnetic fields [2, 3] current echoes arise through a sequence of suitable electromagnetic pulses [4].

The Loschmidt echo arises from a reversal of momenta in a Hamiltonian system. It is a measure of the revival occurring when an imperfect time-reversal procedure is applied to a complex quantum system. Peres considered the Loschmidt echo as a measure of the sensibility and reversibility of quantum evolution [5],

$$M(t) = |\langle\psi_0| \, e^{i\frac{H_2 t}{\hbar}} e^{-i\frac{H_1 t}{\hbar}} \, |\psi_0\rangle|^2 \, , \tag{1}$$

where $|\psi_0\rangle$ is the state of the system at time $t = 0$, H_1 the Hamiltonian governing the forward evolution and H_2 is the Hamiltonian governing the backward evolution. The parameter t is the instant at which the reversal takes place. The Loschmidt echo $M(t)$ quantifies the degree of irreversibility. Alternatively, the equation of the echo can be interpreted as the overlap at time t of two states evolved from $|\psi_0\rangle$ under the action of the Hamiltonian operators H_1 and H_2. The quantity $M(t)$ is usually referred to as fidelity. Broad interest of the Loschmidt echo include quantum chaos, or the quantum theory of classically chaotic systems, decoherence or the emergence of classical world (open quantum systems), quantum computation and quantum information, spin echo in nuclear magnetic resonance, linear waves (elastic waves, microwaves time-reversal mirrors), nonlinear waves, statistical mechanics of small systems, quantum chemistry and molecular dynamics, quantum phase transition and quantum criticality, mathematical aspects of the Loschmidt echo [1, 2, 3, 4, 5, 6, 7, 8, 9].

The notion of quantum fidelity, also called Loschmidt echo, amounts to considering the behavior in time of the overlap of two quantum states: one evolved according to a given dynamics, and the other one evolved by a slight perturbation of it, but starting from the same initial state at time zero. The overlap of the two quantum states equals 1 at time $t = 0$ but starts decreasing as time evolves, although rather slightly if the size of the perturbation is small. This phenomenon should be described as sensitivity to perturbations instead of the fidelity adopted in the literature. The efficiency of teleportation is determined by measuring fidelity, which gives a measure of the quality of the teleported state. What is teleportation? The term teleportation comes from science fiction, meaning to make a person or object disappear while an exact replica appears somewhere else. Quantum teleportation was suggested by Bennett et al., where the process, unlike some science fiction versions, defies no physical laws. In particular, it cannot take place instantaneously or over a spacelike interval because it requires, among other things, sending a classical message from a sender to a receiver [10]. Quantum teleportation is recognized nowadays as an important application of quantum entanglement in correlation with quantum information processing.

Quantum teleportation involves transferring complete information of one quantum state from one location to another location with the aid of long-range EPR correlations in an entangled state. This entangled-state is shared between the two parties, which are known as the sender and the receiver. At first, the sender makes some measurements with the information state and her or his shared part of the entangled state. In this process, the information state disappears at the sender's end and instantly appears at the receiver's end. This is obtained when the receiver makes some unitary transformation that depends on some result of the sender's measurement, which is received through some classical channel. The transmission of quantum states can be accomplished without using any entanglement, whether it is only classical communication or by the transmission of classical bits where the sender and the receiver share entanglement. We are interested in the second situation.

The paper is organized as follows. The section 2 is about a concrete example on how to compute the fidelity and the section 3 is about the conclusion.

2 Quantum teleportation fidelity

2.1 Definition

In quantum teleportation, the fidelity, denoted F, is determined as follows:

$$F = \langle\psi|\rho|\psi\rangle \,, \tag{2}$$

that is the overlap of the input information state $|\psi\rangle$ with the normalized output teleported state ρ.

$$F = \mathrm{Tr}[\rho_{\mathrm{out}}|\phi\rangle\langle\phi|], \tag{3}$$

where $|\phi\rangle$ is the input information state. If $|I\rangle$ represents the information state, to be teleported, and $|T\rangle$ represents the teleported copy of the initial information state that the receiver has after application of the unitary transformation, then fidelity of the teleported state is calculated by using $\rho_{\mathrm{out}} = |T\rangle\langle T|$ which is

$$F = |\langle T|I\rangle|^2. \tag{4}$$

$$F = \sum_{i=1}^{4} P_i \,|\, \langle I \,|\, \zeta_i\rangle \,|^2, \tag{5}$$

$|I\rangle$ is the input state and $P_i = \mathrm{Tr}(\langle\Omega|M_i|\Omega\rangle)$ with $|\Omega\rangle = |I\rangle \otimes |\psi_{channel}\rangle$ $M_i = |\psi_i\rangle\langle\psi_i|$ is a measurement operator on a Bell basis. $|\zeta_i\rangle$ is the teleported state corresponding to i^{th} projective measurement on a Bell basis, where F depends on the parameters of the state to be teleported. In some references, $|\langle I \mid \zeta_i\rangle|^2$ is considered as fidelity and F as average fidelity [11].

The quality of the teleportation can also be studied through other measures such as the minimum assured fidelity (MASF), the average teleportation fidelity ($F_{\rm ave}$), the optimal fidelity (f). The fidelity is state dependent, and $0 \leq \mathrm{F} \leq 1$. If the output state is exactly the same as the input information, then the fidelity of the teleportation is equal to unity, F = 1). If the output state is an imprecise copy of the input information, then the fidelity is smaller than 1, F < 1. If the output state is completely orthogonal to the input state, then the fidelity is zero, F = 0 and the teleportation is not possible. The states that are less entangled reduce the fidelity of teleportation [11, 12, 13].

2.2 Teleportation through a nonmaximally entangled channel

The teleportation protocol begins with a qubit $|\psi\rangle_C$, in Amy's possession, that she wants to send to Bouba. This is

$$|\psi\rangle_C = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\exp{(i\phi)}|1\rangle. \tag{6}$$

In the protocol, Amy and Bouba share this nonmaximal entangled channel

$$|\psi\rangle_{AB} = \alpha|00\rangle + \beta|11\rangle, \tag{7}$$

where we assume that α and β are real and that $\alpha \leq \beta$. The subscripts A and B in the entangled state refer to Amy's or Bouba's particle, while the subscript C refers to the initial state in Amy's possession. Since the entangled channel shared by Amy and Bouba is nonmaximal, it is not possible to perform this qubit teleportation perfectly, so we would like to compute the fidelity. Amy obtains one of the particles in the pair, with the other going to Bouba. At this point, Amy has two particles: the one she wants to teleport, and A, one of the entangled pairs, and Bouba has one particle, B. In the total system, the state of the three particles is given by

$$\begin{aligned} |\psi\rangle_C \otimes |\psi\rangle_{AB} &= (\cos\frac{\theta}{2}|0\rangle_C + \sin\frac{\theta}{2}\exp{(i\phi)}|1\rangle_C) \\ &\otimes (\alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B). \end{aligned} \tag{8}$$

The initial state of three particle is then

$$\begin{aligned} |\psi\rangle_{CAB} &= \alpha\cos\frac{\theta}{2}\,|0\rangle_C|0\rangle_A|0\rangle_B + \beta\cos\frac{\theta}{2}\,|0\rangle_C|1\rangle_A|1\rangle_B \\ &+ \alpha\sin\frac{\theta}{2}\,e^{i\phi}\,|1\rangle_C|0\rangle_A|0\rangle_B + \beta\sin\frac{\theta}{2}\,e^{i\phi}\,|1\rangle_C|1\rangle_A|1\rangle_B. \end{aligned} \tag{9}$$

Amy makes a local measurement on the Bell basis on the two particles in her possession. In order to make the result of her measurement clear, it is best to write the state of Amy's two qubits as superposition of the Bell basis. This is possible by using the following general identities:

$$|0\rangle|0\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle); \tag{10}$$

$$|0\rangle|1\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle); \quad (11)$$

$$|1\rangle|0\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle); \quad (12)$$

$$|1\rangle|1\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle). \quad (13)$$

Applying the general identities to the qubits with A and C subscripts, we have

$$|0\rangle_C|0\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+\rangle_{CA} + |\phi^-\rangle_{CA}); \quad (14)$$

$$|0\rangle_C|1\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{CA} + |\psi^-\rangle_{CA}); \quad (15)$$

$$|1\rangle_C|0\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{CA} - |\psi^-\rangle_{CA}); \quad (16)$$

$$|1\rangle_C|1\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+\rangle_{CA} - |\phi^-\rangle_{CA}). \quad (17)$$

The total three particle state of A, B and C together form the following four-term superposition.

$$\begin{aligned}
|\psi\rangle_{CAB} &= \frac{1}{\sqrt{2}}\Bigg[|\phi^+\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}\,|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right) \\
&+ |\phi^-\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}\,|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right) \\
&+ |\psi^+\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}\,|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right) \\
&+ |\psi^-\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}\,|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right)\Bigg]. \quad (18)
\end{aligned}$$

All the three particles are still in the same total state, since no operations have been performed. The above equation is just a change of basis on Amy's part of the system. This change has moved the entanglement from particles A and B to particles C and A. The teleportation occurs when Amy measures her two qubits in the Bell basis: $|\phi^+\rangle_{CA}$, $|\phi^-\rangle_{CA}$, $|\psi^+\rangle_{CA}$, $|\psi^-\rangle_{CA}$. The result of Amy's local measurement is a collection of two classical bits 00, 01, 10, 11 related to one of the four states after the three-particle state has collapsed into one of the states:

$$|\phi^+\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}\,|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \quad (19)$$

$$|\phi^-\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}\,|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \quad (20)$$

$$|\psi^+\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}\,|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right); \quad (21)$$

$$|\psi^-\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}\,|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right). \quad (22)$$

Amy's two particles are now entangled to each other in one of the four Bell states, and the entanglement originally shared between Amy's and Bouba's particles is now broken. Bouba's particle takes on one of the four superposition states:

$$\alpha \cos\frac{\theta}{2}\,|0\rangle_B + \beta \sin\frac{\theta}{2} e^{i\phi}|1\rangle_B; \tag{23}$$

$$\alpha \cos\frac{\theta}{2}\,|0\rangle_B - \beta \sin\frac{\theta}{2} e^{i\phi}|1\rangle_B; \tag{24}$$

$$\beta \cos\frac{\theta}{2}\,|1\rangle_B + \alpha \sin\frac{\theta}{2} e^{i\phi}|0\rangle_B; \tag{25}$$

$$\beta \cos\frac{\theta}{2}\,|1\rangle_B - \alpha \sin\frac{\theta}{2} e^{i\phi}|0\rangle_B. \tag{26}$$

Bouba's qubit is now in a state that resembles the state to be teleported. The four possible states for Bouba's qubit are unitary images of the state to be teleported. After Bouba receives the message from Amy, he guesses which of the four states his qubit is in. Using this information, Bouba accordingly chooses one of the unitary transformation $\{\mathbb{I}, \sigma_x,\ i\sigma_y,\ \sigma_z\}$, to perform his part of the channel. Here

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

$\mathbb{I}$ represents the identity operator, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli operators. The correspondence between the measurement outcomes and the unitary operations are

$$|\phi^+\rangle_{CA} \Rightarrow \mathbb{I}; \quad |\phi^-\rangle_{CA} \Rightarrow \sigma_z; \quad |\psi^+\rangle_{CA} \Rightarrow \sigma_x; \quad |\psi^-\rangle_{CA} \Rightarrow i\sigma_y\,. \tag{28}$$

The teleportation is achieved, and to measure the efficiency of the teleportation protocol, we compute the fidelity of this teleportation

$$F(|\psi\rangle_C) = \sum_{i=1}^{4} P_i\,|\langle\psi_C|\zeta_i\rangle|^2\,, \tag{29}$$

where $P_i = \mathrm{Tr}({}_{CAB}\langle\psi|M_i|\psi\rangle_{CAB})$, $M_i = |\phi_i\rangle\langle\phi_i|$ the measurement operator on the quasi-Bell basis $|\phi_i\rangle \in \{|\phi^+\rangle_{CA},\ |\phi^-\rangle_{CA},\ |\psi^+\rangle_{CA},\ |\psi^-\rangle_{CA}\}$, and $|\zeta_i\rangle$ Bouba's normalized and corrected outcome given the measurement result in i.

The probabilities of the measurement are explicitly the following:

$$\begin{aligned}
P_1 &= \mathrm{Tr}({}_{CAB}\langle\psi|\phi_1\rangle\langle\phi_1|\psi\rangle_{CAB}) = \mathrm{Tr}({}_{CAB}\langle\psi|\phi^+_{CA}\rangle\langle\phi^+_{CA}|\psi\rangle_{CAB});\\
P_2 &= \mathrm{Tr}({}_{CAB}\langle\psi|\phi_2\rangle\langle\phi_2|\psi\rangle_{CAB}) = \mathrm{Tr}({}_{CAB}\langle\psi|\phi^-_{CA}\rangle\langle\phi^-_{CA}|\psi\rangle_{CAB});\\
P_3 &= \mathrm{Tr}({}_{CAB}\langle\psi|\phi_3\langle\phi_3|\psi\rangle_{CAB}) = \mathrm{Tr}({}_{CAB}\langle\psi|\psi^+_{CA}\langle\psi^+_{CA}|\psi\rangle_{CAB});\\
P_4 &= \mathrm{Tr}({}_{CAB}\langle\psi|\phi_4\rangle\langle\phi_4|\psi\rangle_{CAB}) = \mathrm{Tr}({}_{CAB}\langle\psi|\psi^-_{CA}\rangle\langle\psi^-_{CA}|\psi\rangle_{CAB}).
\end{aligned}$$

The probability of Amy measuring $|\phi^+\rangle_{CA}$ or $|\phi^-\rangle_{CA}$ is

$$P_1 = P_2 = \frac{1}{2}\left(\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}\right); \tag{30}$$

The probability of Amy measuring $|\psi^+\rangle_{CA}$ or $|\psi^-\rangle_{CA}$ is

$$P_3 = P_4 = \frac{1}{2}\left(\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}\right). \tag{31}$$

The normalized qubits of Bouba after each measurement are given as follows:

- for the mesurement of $|\phi^+\rangle_{CA}$, Bouba applies the unit operator to $\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B$ and normalizes to obtain

$$|\zeta_1\rangle = \frac{1}{\sqrt{\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}}}\left(\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \tag{32}$$

- for the mesurement of $|\phi^-\rangle_{CA}$, Bouba applies the operator σ_x on $\alpha\cos\frac{\theta}{2}|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B$ and after normalization the outcome is

$$|\zeta_2\rangle = \frac{1}{\sqrt{\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}}}\left(-\beta\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B + \alpha\cos\frac{\theta}{2}|1\rangle_B\right); \tag{33}$$

- for the mesurement of $|\psi^+\rangle_{CA}$, Bouba applies the operator $i\sigma_y$ on $\beta\cos\frac{\theta}{2}|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B$; and after normalization the outcome is

$$|\zeta_3\rangle = \frac{1}{\sqrt{\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}}}\left(\beta\cos\frac{\theta}{2}|0\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \tag{34}$$

- for the mesurement of $|\psi^-\rangle_{CA}$, Bouba applies the operator σ_z on $\beta\cos\frac{\theta}{2}|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B$.; and after normalization the outcome is

$$|\zeta_4\rangle = \frac{1}{\sqrt{\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}}}\left(-\alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B - \beta\cos\frac{\theta}{2}|1\rangle_B\right). \tag{35}$$

Using the equations in (30), (31), and the equations in (32), (33), (34), (35) in the equation (29), the fidelity is given by

$$F(|\psi\rangle_C) = \cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} + \alpha\beta\sin^2\theta. \tag{36}$$

3 Conclusion

In this paper, we explicitly computed the fidelity in the teleportation of a qubit through a nonmaximally entangled channel. The motivation for such calculations is to furnish a clear understanding of the concept and the way it is computed as these information are lacking in the papers we have studied about the topic. For a nonmaximally entangled channel, it is known that there is no perfect teleportation and therefore fidelity is needed to determine the efficiency of the teleportation. This work follows from our recent work on the teleportation of a qubit using quasi-Bell states [16]. In both cases, the fidelity of teleportation depends on the parameters of the initial state to be teleported. For the protocol, the unitary operators used by the receiver are the Pauli matrices. It may be interesting to find the convenient unitary operators which give a perfect teleportation.

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