A heuristic for Skolems solution for linear-recurrences, based on the lower bound of the non-zero magnitude of the discrete sine function

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Abstract: - Our previous paper derived a result that an eventual non-zero lower bound of minimum(absolute(sin(mπθ)), over positive integers m less than n), for large positive integers n, is (2 / (sqrt(5) n)). In this paper, we use this result to develop a heuristic to estimate the upper bound of the last non-periodic zero of any given homogenous linear recurrence with integer coefficients and integer starting points.

1. Introduction

Given integer constants $\{b_1, b_2, ... b_L, a_0, a_1, a_2 ... a_{L-1}\}$, we denote our degree-L homogenous linear recurrence (HLR) sequence f_n in the same way as in our previous paper [1], as follows:

1. $f_n = 0$, for integers $n < 0$
2. $f_n = a_n$, for integers $0 \leq n \leq (L-1)$
3. $f_n = b_1 f_{n-1} + b_2 f_{n-2} + ... + b_L f_{n-L}$, for integers $n \geq L$

The reader is referred to previous papers [1][2][3] for an introduction to the Skolem-Mahler-Lech (SML) Theorem and to Skolem's still-open problem on the HLR, which is to find an effective upper bound (denoted as U in this paper) for the last non-periodic zero of the HLR. There are numerous results obtaining U for special cases of f_n [1][2][3]. For example, when the number of dominant roots (i.e. roots of maximum modulus in the characteristic polynomial of f_n) is limited to 2, U can be obtained effectively [2]. The difficulty tends to arise when the number of dominant roots exceeds two. Our paper [1] gives a deterministic algorithm for finding U, which can be applied to all instances of f_n, except when f_n has multiple dominant roots with incommensurable arguments.

In [1], we derived an important result on the eventual non-zero lower bound of the magnitude of the discrete sine function. In this paper, we use that result to develop a heuristic that can be applied to reasonably estimate U for all instances of f_n. This means that one can conclude with reasonable probability that f_n will not have a non-periodic zero beyond U.

2. The Heuristic

In Theorem 1 of our paper [1], we wrote that f_n = SUM(SUM(($n^j r_k^n s_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), where each $s_{n,j,k}$ = SUM(($d_{j,k} \cos(n\theta_k+\beta_{j,k})$), over all integers k in [1,L]). Here, each of $\{r_k, d_{j,k}, \cos(\theta_k), \cos(\beta_{j,k})\}$ is a real algebraic number, where each r_k is non-negative. Without loss of generality, denote $r_1 > r_2 > ... > r_L$. So a necessary and sufficient condition for U is that for all n > U: (r_1^n minimum($s_{m,j,1}$, over positive integers m < n)) > SUM(($n^L r_k^n$), over integers k in [2,L]).

There are two steps to satisfy the above condition:

1. Find a good choice of $s_{n,j,1}$, whose non-zero minimum absolute value (hereafter denoted as B_n) diminishes as rapidly as possible. We are currently unable to prove that it is the best, hence till then we can only call it a heuristic.
2. For the above good choice of B_n , find U such that ($r_1^n B_n$) > SUM(($n^L r_k^n$), over integers k in [2,L])..

So our heuristic to estimate U may be described as follows.

1. Write $s_{n,j,1} = c_0 \cos(2\pi n) + c_1 \cos(4\pi(1g)n) + c_2 \cos(4\pi(2g)n) + c_3 \cos(4\pi(3g)n) + ... + c_L \cos(4\pi(Lg)n)$, where g is the golden ratio.
2. For each integer k in [1,L], write the product term $c_k \cos(4\pi(kg)n)$ as a k-degree univariate polynomial of $\cos(4\pi(g)n)$, using the well-known polynomial expansions of cos(nx) [].
3. Since $\cos(4\pi(g)n) = (1 - 2\sin^2(2\pi(g)n))$, replace each occurrence of $\cos(4\pi gn)$ with $(1-\Delta_n)$, and expand.
4. We then are able to write $s_{n,j,1}$ as a L degree univariate polynomial in Δ_n, which we denote as $P(\Delta_n)$.
5. In the polynomial expression of $P(\Delta_n)$, equate the coefficients of Δ_n^i to 0 for all integers i in [0,L-1], and equate the coefficient of Δ_n^i to 1. Thus, equate $P(\Delta_n)$ to B_n.

6. We are now able to find rational solutions to $\{c_0, c_1, c_2, ... ,c_L\}$, such that $B_n = \Delta_n^L$.
7. Since $\Delta_n = 2(2 / (sqrt(5)\ n))^2$, therefore $B_n = K/n^{2L}$, where $K = 1.4^L$.
8. Find U such that for all integers n>U, $((r_1^n B_n) > SUM((n^L r_k^n)$, over integers k in [2,L])).

The following example illustrates the application of our heuristic for L=3..

Example:

Consider a HLR of degree L=3. Then the following steps may be used to derive U.

1. Write:
   $s_{n,j,1}= c_0 \cos(2\pi n) + c_1 \cos(4\pi(1g)n) + c_2 \cos(4\pi(2g)n) + c_3 \cos(4\pi(3g)n)$.
2. Write:
   $\cos(4\pi(2g)n) = 2 \cos^2(2\pi gn) - 1$.
   $\cos(4\pi(3g)n) = 4 \cos^3(2\pi gn) - 3 \cos(2\pi gn)$.
3. Write:
   $\cos(4\pi(1g)n) = (1 - \Delta_n)$
   $\cos(4\pi(2g)n) = 2 (1 - \Delta_n)^2 - 1 = 1 - 4\Delta_n + 2\Delta_n^2$
   $\cos(4\pi(3g)n) = 4 (1 - \Delta_n)^3 - 3 (1 - \Delta_n) = 1 - 9\Delta_n + 12 \Delta_n^2 - 4 \Delta_n^3$
4. Write:
   $s_{n,j,1} = c_0 \cos(2\pi n) + c_1 \cos(4\pi(1g)n) + c_2 \cos(4\pi(2g)n) + c_3 \cos(4\pi(3g)n)$
   $= c_0 + c_1 (1 - \Delta_n) + c_2 (1 - 4\Delta_n + 2\Delta_n^2) + c_3 (1 - 9\Delta_n + 12 \Delta_n^2 - 4 \Delta_n^3)$
   $= (c_0 + c_1 + c_2 + c_3) - \Delta_n (c_1 + 4c_2 + 9c_3) + \Delta_n^2 (2c_2 + 12c_3) - \Delta_n^3 (4c_3)$
5. Write:
   $(c_0 + c_1 + c_2 + c_3) = 0$
   $(c_1 + 4c_2 + 9c_3) = 0$
   $(2c_2 + 12c_3) = 0$
   $-(4c_3) = 1$
   Starting from the last equation and proceeding to the first in sequence, find $c_3 = -1/4$, $c_2 = 3/2$, $c_1 = -15/4$ and $c_0 = 5/2$.
6. Write $B_n = \Delta_n^3$, since there exists the rational vector $<c_3, c_2, c_1, c_0>$ that allows this.
7. Write $B_n = K/n^6$.
8. Find U such that for all integers n>U, $(r_1^n B_n) > (n^3 (r_2^n + r_3^n))$. This happens when $K > (n^9((r_2/r_1)^n + (r_3/r_1)^n))$.

3. Conclusion and Future Work

We presented a heuristic for obtaining the upper bound of the non-periodic zeros of a given integer HLR. The upper bound obtained by our heuristic grows exponentially with respect to the degree of the HLR. One immediate line of future research will be to improve the accuracy of our heuristic by trying to describe $s_{n,j,1}$ as a sum of cosine functions with larger powers, so that B_n can be inversely proportional to a higher power of n. The challenge will, of course, be to find the rational vector $<c_0, c_1, c_2, ...>$ that allows this.

References

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About the author

I, Deepak Ponvel Chermakani, wrote this paper, which is original to the best of my knowledge, out of my own interest and initiative during my spare time. I completed a fulltime two-year Master of Science Degree in Electrical Engineering from University of Hawaii at Manoa USA (www.hawaii.edu) in Aug 2015, a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization from University of Edinburgh UK (www.ed.ac.uk) in Sep 2010, a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg) in Jul 2003, and fulltime high schooling from National Public School Indiranagar in Bangalore in India in Jul 1999. I am most grateful to my parents (my mother Mrs. Kanaga Rathinam Chermakani and my father Mr. T. Chermakani) for their sacrifices in educating me and bringing me up.