THE DIRAC EQUATION AND THE WHEELER-FEYNMAN TIME-SYMMETRIC THEORY

JAMES CONOR O'BRIEN

ORCID iD 0009-0000-8895-2345

(27/09/24)

Keywords: Dirac's Equation, Gamma Matrices, Spherical Electrons, Wheeler-Feynman Time-Symmetric Theory, longitudinal electromagnetic scalar potential waves, Wheeler-Feynman electromagnetic four-vector potential time symmetric theory (WFEFVPTST), Parallelogram Law.

Abstract:

In previous papers [1][2][3][4]–specifically *A Dynamical Theory of the Electromagnetic Potential* [2018]–electrons were modelled as longitudinal electromagnetic scalar potentials which generate both electromagnetic waves travelling in $\mathbb{R}^{1,3}$ as well as perfectly spherical charged particles in $\mathbb{R}^3$ by satisfying Maxwell's equations [5] under the Wheeler-Feynman time symmetric theory[6][7]. In this present paper that model is extended by giving derivations of the Klein-Gordon equation, the Gamma Matrices and the Dirac equation through an examination of the Wheeler-Feynman Time Symmetric theory through an examination of the Parallelogram Law and the Inner Product of the four-vector potentials.

§1 Introduction

This paper attempts to show the Wheeler-Feynman time symmetric theory (WFTST) provides a foundation for the Dirac equation and thereby most of the principles of Quantum Electrodynamics. Discussing various methods to construct the WFTST for the Parallelogram law; the radius of the electron; quantum wavefunctions; the Dirac Gamma matrices; the Klein-Gordon Equation; and finally the Dirac equation is derived from first principles through an examination of the Wheeler-Feynman summation.

§2 Wheeler-Feynman Summation and the Parallelogram Law

It was shown[1][2][3][4] from a study of the Wheeler-Feynman summation [5][6] of the wavefunctions we can express the phases of the wavefunction for an electron using the four-vector potentials. Where the ψ_{int} is the intrinsic (int) wavefunction in the local frame, whereas the four-potentials $A_{\mu}^{adv,ret}$ are the advanced (adv) or retarded (ret) phases of the wavefunctions in the non-local frames of the future and past.

$$\psi_{int} = \exp(-\tfrac{ie}{2\hbar} \int A_{\mu}^{\mathrm{ret}} - A_{\mu}^{\mathrm{adv}} + A_{\mu}^{\mathrm{adv}} + A_{\mu}^{\mathrm{ret}}\, \mathrm{dx}^{\mu}) \tag{1}$$

This notation is simplified by writing $A_{\mu} \equiv A_{\mu}^{ret}$, $A_{\nu} \equiv A_{\nu}^{adv}$ and $A_{\eta} \equiv A_{\eta}^{int}$ for the intrinsic potential by assuming *ret*, *adv* and *int* are implied for μ, ν and η respectfully such that,

$$(A_{\mu}^{\mathrm{adv}}, A_{\nu}^{\mathrm{ret}}, A_{\eta}^{\mathrm{int}}) \to (A_{\mu}, A_{\nu}, A_{\eta}) \tag{2}$$

The method of construction of equation (1) differs from the Wheeler-Feynman field approach where they tied their particles to all the free particles in the universe, where given Wheeler-Feynman assumptions,

$$\mathbf{E}_{total} = \mathbf{E}_{total} + \mathbf{E}_{free} = \mathbf{E}_{ret} \tag{3}$$

such that

$$\begin{aligned}\mathbf{E}_{total} + \mathbf{E}_{free} &= \sum_n \frac{1}{2}\left(\mathbf{E}_n^{ret} + \mathbf{E}_n^{adv}\right) + \sum_n \frac{1}{2}\left(\mathbf{E}_n^{ret} - \mathbf{E}_n^{adv}\right) \\ &= \sum_n \mathbf{E}_n^{ret}\end{aligned} \tag{4}$$

In the W.F.E.F.V.P.T.S.T. method the free particles are simply equated to the *epoch angles*, of the advanced and retarded particles, and because this does not tie the particle to the entire universe this greatly simplifies the construction while returning the same result.

Now we can show the Parallelogram Law when applied to (1) also obeys the Wheeler-Feynman summation, in effect, we are looking for the probability density amplitude of the Wheeler-Feynman sum of the phases,

$$\tfrac{1}{2}\left[(A_\mu + A_\nu)^2 + (A_\mu - A_\nu)^2\right] = A_{int}^2 = A_\eta^2 \tag{5}$$

We begin by taking the expansion of the first term, $(A_\mu + A_\nu)^2$

$$(A_\mu + A_\nu)^2 = A_\mu^2 + 2A_\mu A_\nu + A_\nu^2 \tag{6}$$

and then the second $(A_\mu - A_\nu)^2$,

$$(A_\mu - A_\nu)^2 = A_\mu^2 - 2A_\mu A_\nu + A_\nu^2 \tag{7}$$

adding these expansions we obtain,

$$\begin{aligned}(A_\mu + A_\nu)^2 + (A_\mu - A_\nu)^2 &= \left(A_\mu^2 + 2A_\mu A_\nu + A_\nu^2\right) + \left(A_\mu^2 - 2A_\mu A_\nu + A_\nu^2\right) \\ &= A_\mu^2 + 2A_\mu A_\nu + A_\nu^2 + A_\mu^2 - 2A_\mu A_\nu + A_\nu^2 \\ &= A_\mu^2 + A_\mu^2 + A_\nu^2 + A_\nu^2 \\ &= 2A_\mu^2 + 2A_\nu^2\end{aligned} \tag{8}$$

We find this is the Parallelogram Law, which we can simplify by introducing a new potential A_η in the local frame of reference,

$$A_\mu^2 + A_\nu^2 = A_\eta^2 \tag{9}$$

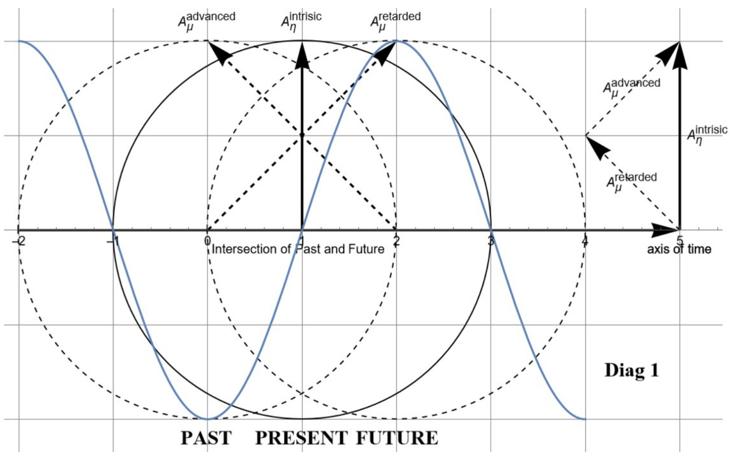
$$\tfrac{1}{2}\left[(A_\mu + A_\nu)^2 + (A_\mu - A_\nu)^2\right] = A_\mu^2 + A_\nu^2 = A_\eta^2 \tag{10}$$

Previously [1][2][3][4] this was labelled the intrinsic potential A_{int} as it exists in the local frame of reference relative to the non-local frames of the advanced and retarded potentials. This is new potential is necessary to satisfy Maxwell's equation in the Coulomb gauge in the local frame, and in effect we have shown the Wheeler-Feynman summation is an instance of the parallelogram law.

$$\frac{1}{2}\left[(A_\mu + A_\nu)^2 + (A_\mu - A_\nu)^2\right] = A_{int}^2 \tag{11}$$

It also means that if we apply this result to the wavefunction of equation (1) we see the Wheeler-Feynman summation also applies to the Parallelogram Law space of the wavefunctions via their phases as given by their four-vector potentials.

In the two dimensional projection of diagram(1) we see the radius (and the amplitude as this is a spherical particle) of the resultant four-vector is equal in magnitude of the sum of the A_μ and A_ν where they intersect in the present frame, effectively the potentials of the past and the future frames of reference are swallowed up into the local frame.



The two four-dimensional waveforms intersect in a single three-dimensional waveform where normal to every point on the surface of a sphere having the A_η projecting outwards.

§3 Radius of the Intrinsic Particle

Next we consider the radius of the resultant intrinsic particle by noting the Wheeler-Feynman summation was justified on the premise that since

$$A_\mu = (\varphi/c, \mathbf{A}) \tag{12}$$

the magnetic vector potential vanishes along the axis of time (c.f. page 441 of [8]), such that,

$$A_{\mu,\nu} = (\varphi/c, \mathbf{A}) \rightarrow A_\mu = (\varphi_{\mu,\nu}/c, 0) \tag{13}$$

this expresses the phases of the wavefunctions as,

$$\tfrac{1}{2}\left[(A_\mu + A_\nu) + (A_\mu - A_\nu)\right] = A_\mu \tag{14}$$

which simplifies in the Coulomb gauge to,

$$\textstyle\sum_n \tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} + \frac{q}{R_{\text{adv}}}\right) + \sum_n \tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} - \frac{q}{R_{\text{adv}}}\right) = \frac{q}{R_{int}} \tag{15}$$

This also can be simplified to,

$$\tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} + \frac{q}{R_{\text{adv}}}\right) + \tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} - \frac{q}{R_{\text{adv}}}\right) \tag{16}$$

on expanding this,

$$\tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} + \frac{q}{R_{\text{adv}}}\right) + \tfrac{1}{2}\left(\frac{q}{R_{\text{ret}}} - \frac{q}{R_{\text{adv}}}\right) = \tfrac{1}{2}\cdot\frac{q}{R_{\text{ret}}} + \tfrac{1}{2}\cdot\frac{q}{R_{\text{adv}}} + \tfrac{1}{2}\cdot\frac{q}{R_{\text{ret}}} - \tfrac{1}{2}\cdot\frac{q}{R_{\text{adv}}} \tag{17}$$

the terms involving $\frac{q}{R_{\text{adv}}}$ cancel out,

$$\tfrac{1}{2}\cdot\frac{q}{R_{\text{ret}}} + \tfrac{1}{2}\cdot\frac{q}{R_{\text{ret}}} = \frac{q}{R_{\text{ret}}} \tag{18}$$

reducing to,

$$\frac{q}{R_{\text{ret}}} \tag{19}$$

Implying

$$\frac{q}{R_{\text{ret}}} = \frac{q}{R_{int}} \tag{20}$$

therefore

$$R_{\text{ret}} = R_{int} \tag{21}$$

This is the radius of the electron in the local frame, in effect the radius of the local particle must equal the radii of the non-local particles. We can now express the radii in terms of the parallelogram law, after first assuming the rule,

$$\frac{1}{2}\left[(R_\mu + R_\nu)^2 + (R_\mu - R_\nu)^2\right] = R_{int}^2 \tag{22}$$

To show this expand both terms inside the brackets of (22)

$$(R_\mu + R_\nu)^2 = R_\mu^2 + 2R_\mu R_\nu + R_\nu^2 \tag{23}$$

$$(R_\mu - R_\nu)^2 = R_\mu^2 - 2R_\mu R_\nu + R_\nu^2 \tag{24}$$

add (23) and (24),

$$(R_\mu + R_\nu)^2 + (R_\mu - R_\nu)^2 = (R_\mu^2 + 2R_\mu R_\nu + R_\nu^2) + (R_\mu^2 - 2R_\mu R_\nu + R_\nu^2) \tag{25}$$

after cancelling out.

$$(R_\mu + R_\nu)^2 + (R_\mu - R_\nu)^2 = 2R_\mu^2 + 2R_\nu^2 \tag{26}$$

This is identical to the Parallelogram Law for the Parallelogram Law of two vectors of a normed vector space.

$$||x + y||^2 + ||x - y||^2 = 2\,||x||^2 + 2\,||y||^2 \tag{27}$$

on dividing (26) by 2.

$$\frac{1}{2}\left[(R_\mu + R_\nu)^2 + (R_\mu - R_\nu)^2\right] = \left[R_\mu^2 + R_\nu^2\right] \tag{28}$$

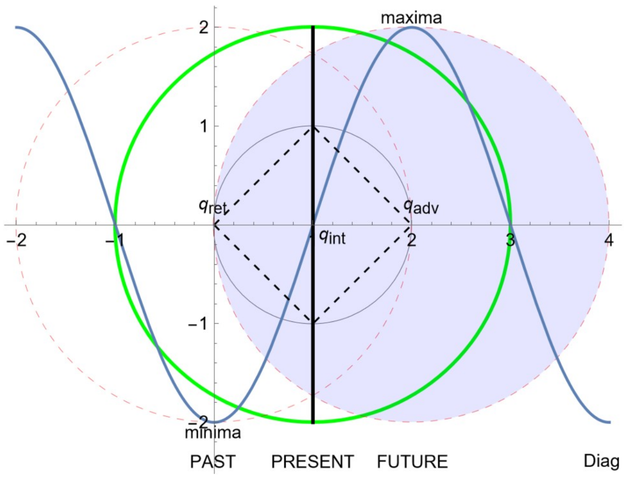
After the Wheeler-Feynman summation the A_ν vanishes and so must the φ_ν leaving us with the parallelogram law for the intrinsic radius R_ν of the potentials in equation (12),

$$\frac{1}{2}\left[(R_\mu + R_\nu)^2 + (R_\mu - R_\nu)^2\right] = R_{int}^2 = R_\eta^2 \tag{29}$$

We can picture this as the intersection of two circles similar to the Vesica Pisces (see diagram 2) where the A_μ and A_ν pass through each other halfway across the radius of the intrinsic A_{int},

$$A_{int} = A_\mu + A_\nu = A_\eta \tag{30}$$

The envelope of the particles movement contains the advanced, retarded and intrinsic four-vector potentials along the axis of time, where to conserve phase the angle at which the A_μ and A_ν make to each other must be 90°, and where the sum of these distances of these potentials gives an intrinsic radius R_{int} equal to the R_{adv} and R_{ret} particles.

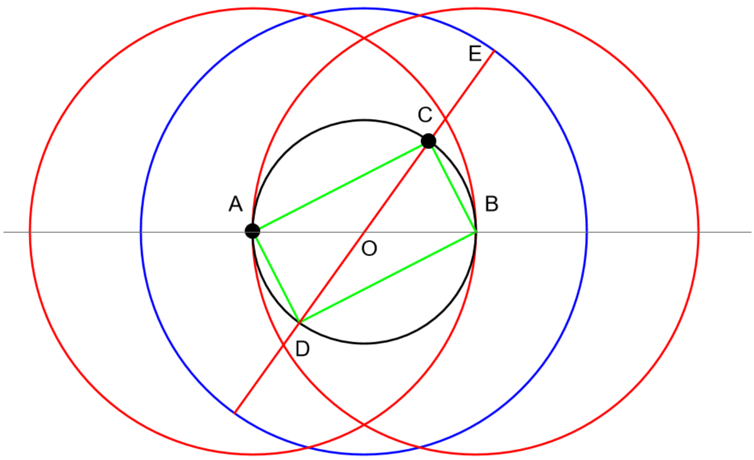


Diag 2

Given R_{int}, R_{adv} and R_{ret} are all constant radii we can construct an envelope of spheres along the axis of time, and it has been shown previously [1][2][3][4] this has wavelength identically equal to the reduced Compton wavelength and the electron diameter.

Note the charges of the advanced and retarded particles correspond to the maxima of the negative and positive amplitudes of the wavefunction, this corresponds to the negative and positive energies of the adv and ret particles.

We can also show the radius of the particle in the present is identical to the radii of the advanced and retarded particles as a geometric construction, from Diagram (3) (Surprisingly, this geometric construction could be lifted straight from Book 1 Proposition 1 of Euclid's elements.),



O is the centre of the intrinsic electron, OE the radius, OB and AO and the radii of the advanced and retarded particles, such that $AC \perp CB$ and $AD \perp DB$ where given,

$$|AO| = |OB| = |DO| = |OC| \tag{32}$$

and

$$|AC| = |BC| \quad and \quad |DB| = |AC| \tag{32}$$

then

$$AC^2 + BC^2 = AB^2 \tag{33}$$
$$DC^2 = DB^2 + BC^2 \tag{34}$$

Therefore the radius of the present particle equals the radius of the retarded and future particles.

$$OE = AB \tag{35}$$

This is a surprising result that even Euclid could have understood–that in itself it is not significant, all that really matters is that the internal structure of matter can be understood as a geometric shape based on electromagnetism–yet it is a nice idea about the continuity of ideas.

§4 The Wheeler-Feynman Summation and Wavefunctions

This section is an elaboration upon section §1 where we seek to show the Wheeler-Feynman sum

$$\tfrac{1}{2}[(\psi_{ret} + \psi_{adv})] + \tfrac{1}{2}[(\psi_{ret} - \psi_{adv})] = \psi_{ret} \tag{38}$$

can also be expressed as a Probability Density Function,

$$\tfrac{1}{2}\left(|\psi_{ret}|^2 + |\psi_{adv}|^2\right) = |\psi_{ret}|^2 \tag{39}$$

The total wavefunction Ψ_{total} has two components– the retarded and advanced,

$$\Psi_{total} = \psi_{ret} + \psi_{adv} \tag{36}$$

similarly for the free wavefunction Ψ_{free} there are two adv and ret components,

$$\Psi_{free} = \psi_{ret} - \psi_{adv} \tag{37}$$

To express the Wheeler-Feynman summation $|\Psi_{total}|^2 + |\Psi_{free}|^2$ first expand $|\Psi_{\mathrm{total}}|^2$,

$$\begin{aligned} |\Psi_{total}|^2 &= |\psi_{ret} + \psi_{adv}|^2 \\ &= (\psi_{ret} + \psi_{adv})^*(\psi_{ret} + \psi_{adv}) \\ &= (\psi^*_{ret} + \psi^*_{adv})(\psi_{ret} + \psi_{adv} \\ &= \psi^*_{ret}\psi_{ret} + \psi^*_{ret}\psi_{adv} + \psi^*_{adv}\psi_{ret} + \psi^*_{adv}\psi_{adv} \\ &= |\psi_{ret}|^2 + \psi^*_{ret}\psi_{adv} + \psi^*_{adv}\psi_{ret} + |\psi_{adv}|^2 \end{aligned} \tag{40}$$

similarly expand, $|\Psi_{free}|^2$

$$\begin{aligned} |\Psi_{free}|^2 &= |\psi_{ret} - \psi_{adv}|^2 \\ &= (\psi_{ret} - \psi_{adv})^*(\psi_{ret} - \psi_{adv}) \\ &= (\psi^*_{ret} - \psi^*_{adv})(\psi_{ret} - \psi_{adv} \\ &= \psi^*_{ret}\psi_{ret} - \psi^*_{ret}\psi_{adv} - \psi^*_{adv}\psi_{ret} + \psi^*_{adv}\psi_{adv} \\ &= |\psi_{ret}|^2 - \psi^*_{ret}\psi_{adv} - \psi^*_{adv}\psi_{ret} + |\psi_{adv}|^2 \end{aligned} \tag{41}$$

adding $|\Psi_{\mathrm{total}}|^2$ and $|\Psi_{\mathrm{free}}|^2$,

$$
\begin{aligned}
|\Psi_{total}|^2 + |\Psi_{free}|^2 = |\psi_{ret}|^2 + \psi^*_{ret}\psi_{adv} + \psi^*_{adv}\psi_{ret} + |\psi_{adv}|^2 \\
+ |\psi_{ret}|^2 - \psi^*_{ret}\psi_{adv} - \psi^*_{adv}\psi_{ret} + |\psi_{adv}|^2
\end{aligned}
\tag{42}
$$

$$|\Psi_{total}|^2 + |\Psi_{free}|^2 = |\psi_{ret}|^2 + |\psi_{adv}|^2 + |\psi_{ret}|^2 + |\psi_{adv}|^2 \tag{43}$$

$$|\Psi_{total}|^2 + |\Psi_{free}|^2 = 2|\psi_{ret}|^2 + 2|\psi_{adv}|^2 \tag{44}$$

divide by two,

$$\tfrac{1}{2}(|\psi_{ret}|^2 + |\psi_{adv}|^2) = |\psi_1|^2 + |\psi_2|^2 \tag{45}$$

substituting the original expressions for,

$$\tfrac{1}{2}\left(|\psi_{ret} + \psi_{adv}|^2 + |\psi_{ret} - \psi_{adv}|^2\right) = |\psi_{ret}|^2 + |\psi_{adv}|^2 \tag{46}$$

Since we know A_{adv} vanishes after the Wheeler-Feynman summation, it follows that $|\psi_{adv}|^2$ also vanishes, and we are left with the wavefunction formulation of the Wheeler-Feynman time symmetric theory,

$$\tfrac{1}{2}\left(|\psi_{ret} + \psi_{adv}|^2 + |\psi_{ret} - \psi_{adv}|^2\right) = |\psi_{ret}|^2 \tag{47}$$

substituting the $|\psi_{int}|^2$ the probability of the intrinsic particle in the local frame,

$$\tfrac{1}{2}\left(|\psi_{ret} + \psi_{adv}|^2 + |\psi_{ret} - \psi^{adv}|^2\right) = |\psi_{int}|^2 \tag{48}$$

We see the Wheeler-Feynman sum still applies under the probability density function, and in effect we have derived the inner product of Born's probability density function for the advanced and retarded waveforms. The resultant intrinsic waveforms are identical to the extrinsic advanced and retarded waveforms, with identical wavelength, radius, charge and potential, and it is seen the Wheeler-Feynman summation for the wavefunction is once more a variation on the parallelogram law.

§5 Wheeler-Feynman Summation and the Dirac Gamma Matrices

We can now apply the Wheeler-Feynman summation directly to the gamma matrices of the Dirac equation, this is a necessary step to show the Dirac equation is form-invariant under the Wheeler-Feynman summation.

To derive this first apply the parallelogram law to the gammas, (again note the similarity to the Wheeler-Feynman sum),

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 \tag{49}$$

yielding.

$$\left[(\gamma^\mu + \gamma^\nu)^2 + (\gamma^\mu - \gamma^\nu)^2\right] = 2\gamma^\mu\gamma^\mu + 2\gamma^\nu\gamma^\nu \tag{50}$$

Expanding the left hand terms,

$$(\gamma^\mu + \gamma^\nu)^2 = \gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\nu \tag{51}$$

and

$$(\gamma^\mu - \gamma^\nu)^2 = \gamma^\mu\gamma^\mu - \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\nu \tag{52}$$

add them.

$$\begin{aligned}(\gamma^\mu + \gamma^\nu)^2 + (\gamma^\mu - \gamma^\nu)^2 = {} & (\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\nu) \\ & + (\gamma^\mu\gamma^\mu - \gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\nu)\end{aligned} \tag{53}$$

Cancel out the $\gamma^\mu\gamma^\nu$ and $\gamma^\nu\gamma^\mu$ terms.

$$(\gamma^\mu + \gamma^\nu)^2 + (\gamma^\mu - \gamma^\nu)^2 = 2\gamma^\mu\gamma^\mu + 2\gamma^\nu\gamma^\nu \tag{54}$$

Next we can show the gamma matrices are determined by the Clifford algebra structure in flat spacetime by introducing the Minkowski metric,

$$2\gamma^\mu\gamma^\mu = 2\eta^{\mu\mu}\mathbf{I}_4$$

where for four dimensions the trace of $\eta^{\mu\mu} = 2$,

$$\gamma^{\mu}\gamma^{\mu} = 4\mathbf{I}_4$$

substituting into the R.H.S. of (54)

$$2\gamma^{\mu}\gamma^{\mu} + 2\gamma^{\nu}\gamma^{\nu} = 8\mathbf{I}_4 \tag{55}$$

yielding,

$$(\gamma^{\mu} + \gamma^{\nu})^2 + (\gamma^{\mu} - \gamma^{\nu})^2 = 2\gamma^{\mu}\gamma^{\mu} + 2\gamma^{\nu}\gamma^{\nu} = 8\mathbf{I}_4 \tag{56}$$

after rearranging this gives us the Wheeler-Feynman summation on the left hand side and the Clifford algebra on the right hand side.

$$\tfrac{1}{2}[(\gamma^{\mu} + \gamma^{\nu})^2 + (\gamma^{\mu} - \gamma^{\nu})^2] = \gamma^{\mu}\gamma^{\mu} + \gamma^{\nu}\gamma^{\nu} = 4\mathbf{I}_4 \tag{57}$$

For the flat space Minkowski space the gamma matrices are Lorentz covariant and obey a Lorentz transformation Λ,

$$\gamma'^{\mu} = S(\Lambda)\gamma^{\nu}S^{-1}(\Lambda) \tag{58}$$

Where $S(\Lambda)$ is a spinor representation of the Lorentz transformation ensuring the Dirac matrices remain form-invariant.

This demonstrates the gamma matrices are form-invariant under the Wheeler-Feynman summation while maintaining the required Clifford algebra necessary for the spinors of the Dirac equation and maintaining Lorentz invariance under the sum.

§6 Wheeler-Feynman Summation and the Klein-Gordon Equation

Before we can derive the Dirac equation we must first derive the Klein-Gordon equation. First multiplying the charge e across equation (10),

$$\tfrac{1}{2}\left[\left(eA_\mu + eA_\nu\right)^2 + \left(eA_\mu - eA_\nu\right)^2\right] = e^2 A_\mu A^\mu + e^2 A_\nu A^\nu \tag{59}$$

expand the square $\left(qA_\mu + qA_\nu\right)^2$,

$$\begin{aligned}\left(eA_\mu + eA_\nu\right)^2 &= \left(eA_\mu\right)^2 + 2\left(eA_\mu\right)\cdot(eA_\nu) + (eA_\nu)^2 \\ &= e^2 A_\mu A^\mu + 2e^2 A_\mu A_\nu + e^2 A_\nu A^\nu\end{aligned} \tag{60}$$

Similarly, $\left(eA_\mu - eA_\nu\right)^2$,

$$\begin{aligned}\left(eA_\mu - eA_\nu\right)^2 &= \left(eA_\mu\right)^2 - 2\left(eA_\mu\right)\cdot(eA_\nu) + (eA_\nu)^2 \\ &= e^2 A_\mu A^\mu + 2e^2 A_\mu A_\nu + e^2 A_\nu A^\nu\end{aligned} \tag{61}$$

Add the two expansions.

$$\begin{aligned}\left(eA_\mu + eA_\nu\right)^2 + \left(eA_\mu - eA_\nu\right)^2 &= \left(e^2 A_\mu A^\mu + 2e^2 A_\mu A_\nu + e^2 A_\nu A^\nu\right) \\ &\quad + \left(e^2 A_\mu A^\mu - 2e^2 A_\mu A_\nu + e^2 A_\nu A^\nu\right) \\ &= e^2 A_\mu A^\mu + e^2 A_\nu A^\nu\end{aligned} \tag{62}$$

Introduce the $\frac{1}{2}$ factor.

$$\tfrac{1}{2}\left[\left(eA_\mu + eA_\nu\right)^2 + \left(eA_\mu - eA_\nu\right)^2\right] = \tfrac{1}{2}\left[e^2 A_\mu A^\mu + e^2 A_\nu A^\nu\right] \tag{63}$$

Now we can express this in the energy-momentum relation, where p_μ is the four-momentum of the particle, assume $p_\mu = eA_\mu$, where e is the charge of the particle and A_μ is the four-potential, we obtain,

$$p_\mu p^\mu = e^2 A_\mu A^\mu \tag{64}$$

$$p_\nu p^\nu = e^2 A_\nu A^\nu \tag{65}$$

(and you can see why we left the charge in the equations) substituting p_μ and p_ν

$$\tfrac{1}{2}\left[p_\mu p^\mu + p_\nu p^\nu\right] = \tfrac{1}{2}\left[e^2 A_\mu A^\mu + e^2 A_\nu A^\nu\right] \tag{66}$$

For a free particle, the relation for energy and momentum is:

$$p^\mu p_\mu = -m^2 c^2 \tag{67}$$

In terms of the four-momentum p_μ,

$$p^2 = -m^2 c^2 \tag{68}$$

On writing the energy-momentum relation as

$$E^2 = (pc)^2 + (mc^2)^2 \tag{69}$$

From the four-momentum $p^\mu = (E/c, \mathbf{p})$, we have,

$$p^\mu p_\mu = -E^2/c^2 + \mathbf{p}\cdot\mathbf{p} \tag{70}$$

$$-E^2/c^2 + \mathbf{p}^2 = -m^2 c^2 \tag{71}$$

or

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4 \tag{72}$$

It can be seen that from the original equation of applying the Parallelogram Law lead to the Wheeler-Feynman summation of eA_μ,

$$\tfrac{1}{2}\left[\left(eA_\mu + eA_\nu\right)^2 + \left(eA_\mu - eA_\nu\right)^2\right] = e^2 A_\mu A^\mu + e^2 A_\nu A^\nu \tag{73}$$

we are able to derive the Energy-Momentum relation.

$$E^2 = (pc)^2 + (mc^2)^2 \tag{74}$$

Therefore from the Wheeler-Feynman summation we are able to derive the Energy-Momentum relation and from this the Klein-Gordon equation follows, and now in a position to *derive Dirac's equation from first principles.*

§7 Wheeler-Feynman Summation and the Dirac Equation

This section is an attempt to apply the Wheeler-Feynman model to the Dirac equation.

$$(\imath\hbar\gamma^{\mu}\partial_{\mu}-mc)\psi = 0 \tag{75}$$

A central feature of this paper is the observation that the inner product of the wavefunctions is equivalent summation to the parallelogram law of the potentials under the Wheeler-Feynman, this occurs as we consider the system either from the four-dimensional space $R^{1,3}$ of the waveform or the three-dimensional space R^{3} of the spherical particle respectfully.

We first note that in flat spacetime the gamma matrices are not functions of spacetime coordinates but are constants, so without loss of generality we can move them across the derivative,

$$\partial_{\mu}(\gamma^{\mu}\psi) = (\partial_{\mu}\gamma^{\mu})\psi + \gamma^{\mu}(\partial_{\mu}\psi) \tag{76}$$

Since $\partial_{\mu}\gamma^{\mu} = 0$ we are left with,

$$\gamma^{\mu}(\partial_{\mu}\psi) = \partial_{\mu}(\gamma^{\mu}\psi) \tag{77}$$

and now we can apply the Wheeler-Feynman summation directly to the Dirac equation by applying the Inner Product to the waveforms and introduce the Gamma matrices as structure constants for flat spacetime.

$$\tfrac{1}{2}\left[(\gamma^{\mu}\psi_{\mu} + \gamma^{\nu}\psi_{\nu})^{2} + (\gamma^{\mu}\psi_{\mu} - \gamma^{\nu}\psi_{\nu})^{2}\right] \tag{78}$$

Expanding the first square,

$$\begin{aligned}(\gamma^{\mu}\psi_{\mu} + \gamma^{\nu}\psi_{\nu})^{2} = {} & (\gamma^{\mu}\psi_{\mu})(\gamma^{\mu}\psi_{\mu}) + (\gamma^{\mu}\psi_{\mu})(\gamma^{\nu}\psi_{\nu}) \\ & + (\gamma^{\nu}\psi_{\nu})(\gamma^{\mu}\psi_{\mu}) + (\gamma^{\nu}\psi_{\nu})(\gamma^{\nu}\psi_{\nu})\end{aligned} \tag{79}$$

yielding,

$$(\gamma^{\mu}\gamma^{\mu})(\psi_{\mu}\psi_{\mu}) + 2(\gamma^{\mu}\gamma^{\nu})(\psi_{\mu}\psi_{\nu}) + (\gamma^{\nu}\gamma^{\nu})(\psi_{\nu}\psi_{\nu}) \tag{80}$$

Expanding the second square,

$$(\gamma^\mu\psi_\mu + \gamma^\nu\psi_\nu)^2 = (\gamma^\mu\psi_\mu)(\gamma^\mu\psi_\mu) - (\gamma^\mu\psi_\mu)(\gamma^\nu\psi_\nu) \\ - (\gamma^\nu\psi_\nu)(\gamma^\mu\psi_\mu) + (\gamma^\nu\psi_\nu)(\gamma^\nu\psi_\nu) \tag{81}$$

yielding,

$$(\gamma^\mu\gamma^\mu)(\psi_\mu\psi_\mu) - 2(\gamma^\mu\gamma^\nu)(\psi_\mu\psi_\nu) + (\gamma^\nu\gamma^\nu)(\psi_\nu\psi_\nu) \tag{82}$$

adding both squares

$$2(\gamma^\mu\gamma^\mu)(\psi_\mu\psi_\mu) + 2(\gamma^\nu\gamma^\nu)(\psi_\nu\psi_\nu) \tag{83}$$

noting that in flat spacetime,

$$(\gamma^\mu\gamma^\mu) = 4\mathbf{I}_4 \\ (\gamma^\nu\gamma^\nu) = 4\mathbf{I}_4 \tag{84}$$

we can substitute,

$$(\gamma^\mu\gamma^\mu)\psi_\mu\psi_\mu = 4I_4\psi_\mu\psi^\mu \\ (\gamma^\nu\gamma^\nu)\psi_\nu\psi_\nu = 4I_4\psi_\nu\psi^\nu \tag{85}$$

then add to give the R.H.S. of (83)

$$2(\gamma^\mu\gamma^\mu)(\psi_\mu\psi_\mu) + 2(\gamma^\nu\gamma^\nu)(\psi_\nu\psi_\nu) = 8I_4(\psi_\mu\psi^\mu + \psi_\nu\psi^\nu) \tag{86}$$

then substituting into the original equation [78],

$$\tfrac{1}{2}\left[(\gamma^\mu\psi_\mu + \gamma^\nu\psi_\nu)^2 + (\gamma^\mu\psi_\mu - \gamma^\nu\psi_\nu)^2\right] = 4I_4(\psi_\mu\psi^\mu + \psi_\nu\psi^\nu) \tag{87}$$

and since we know the Dirac matrices observe the Clifford algebra under the anticommutator,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \tag{88}$$

it follows the Wheeler-Feynman model must in turn obey the Clifford algebra, and since under contraction the advanced wavefunction vanishes under the

Wheeler-Feynman summation, such that the identities hold,

$$\psi_\mu \psi^\mu = \psi_\nu \psi^\nu = \psi_\eta \psi^\eta = \psi^2 \tag{89}$$

it follows,

$$\tfrac{1}{2}\left[(\gamma^\mu \psi_\mu + \gamma^\nu \psi_\nu)^2 + (\gamma^\mu \psi_\mu - \gamma^\nu \psi_\nu)^2\right] = 4I_4\psi^2 \tag{90}$$

this means we can develop the model from the Wheeler-Feynman summation,

$$\frac{1}{2}\left[(\psi_\mu + \psi_\nu) + (\psi_\mu - \psi_\nu)\right] = \psi_\eta \tag{91}$$

to the expansion under the inner product,

$$\tfrac{1}{2}\left[(\psi_\mu + \psi_\nu)^2 + (\psi_\mu - \psi_\nu)^2\right] = \psi^2 \tag{92}$$

Which means in flat spacetime we can multiply the ψ with γ^μ to yield the Dirac equation as we derive it from the Klein-Gordon equation of the previous section,

$$(\imath\hbar\gamma^\mu \partial_\mu - mc)\psi = 0 \tag{93}$$

Thus it is demonstrated from first principles that not only can the electron's radius and wavelength, but also wavefunctions, the Klein-Gordon equation, the gamma matrices, and the Dirac equation are consistent with the Wheeler-Feynman time symmetric model, and therefore an argument can be made that the Wheeler-Feynman summation forms a basis for Quantum Electrodynamics.

This should be not surprising as we know that under the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

the electron and photon are both electromagnetic waveforms connected through the interaction term $e\bar{\psi}\gamma^\mu A_\mu \psi$, and so it follows that at some fundamental level they should arise from the same physical model. Therefore an argument is made for building a framework of Quantum Electrodynamics based on Maxwell's equations and the Wheeler-Feynman time symmetric theory.

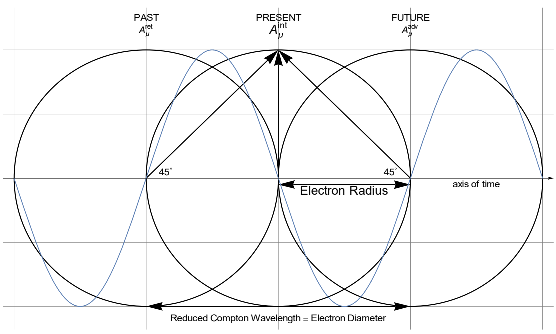
Corrections

This section contains corrections to previous papers[1][2][3][4], including the position of the advanced and retarded particles with respect to the intrinsic particle; the velocity of that intrinsic particle; and importantly its magnetic field and spin.

Correction 1: The diagrams previously published were incorrect as the position of the advanced and retarded charges are not on the leading and trailing edges of the waveform but at the maxima and minima of the wavelength, and this conforms with the maximal and minimal strength of the advanced and retarded charges.

Where previously I suggested that

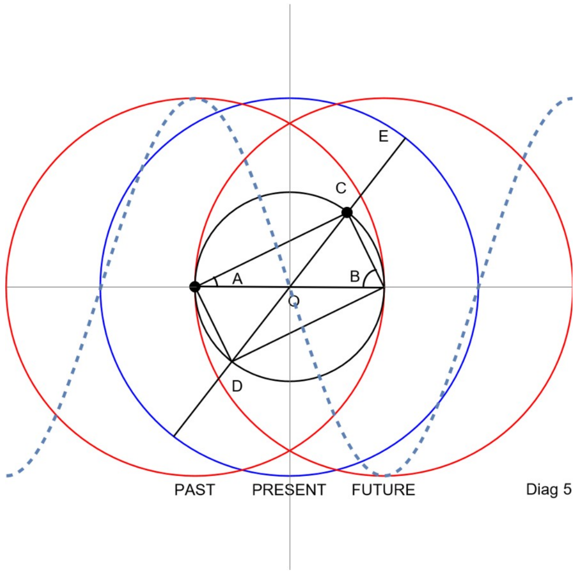
> "*By conservation of energy the angle between the direction of motion of the particle in space-like coördinates and the* ***E*** *and* ***B*** *fields in time-like coördinates must be at 45°, where the* ***E*** *and* ***B*** *fields travel as an electromagnetic wave at the speed of light along the axis of light (R) in* $\mathbb{R}^{1,3}$"



This was erroneous, for if we take into account the internal structure of the parallelogram law of the Wheeler-Feynman summation, whereby the conservation of energy the A_μ and A_ν potentials must always intersect at right angles yielding a right-angle triangle, we find the only angle the potentials can make with each other is 90° degrees,

$$\measuredangle A + \measuredangle B = 90^\circ \tag{94}$$

This results in a Parallelogram Law for the corrected diagram (5) where the potentials originate at the quarter-wavelengths and the angles of $\measuredangle A$ and $\measuredangle B$ range from 0 to 90 −which is certainly different from the original model,



Correction 2: Previously[1][2][3][4], the velocity of the resultant particle was given assuming fields **E** and **B** existed on the envelope of the particle, a much simpler technique is given in terms of the wavefunctions, by equating the advanced and retarded waveforms,

$$\mathrm{e}^{\mathbf{i}(\mathbf{k}_{adv}\cdot\mathbf{r}-\omega t)} = e^{\mathbf{i}(\mathbf{k}_{ret}\cdot\mathbf{r}-\omega t)} \tag{95}$$

$$\mathbf{k}_{adv} = \mathbf{k}_{ret} \tag{96}$$

for $\mathbf{p} = \hbar\mathbf{k}$,

$$\mathbf{p}_{adv} = \mathbf{p}_{ret} \tag{97}$$

and since $m_{adv} = m_{ret}$ and $\mathbf{v} = \frac{\mathbf{p}}{m}$, then the velocities must equal,

$$\mathbf{v}_{adv} = \mathbf{v}_{ret} \tag{98}$$

and it follows the intrinsic velocity must also be same

$$\mathbf{v}_{adv} = \mathbf{v}_{int} = \mathbf{v}_{ret} \tag{99}$$

Correction 3:

This last correction follows on from this present paper where given the radius as determined by section §3, now describes a loci of points over a sphere from the Parallelogram Law of A_μ and A_ν which is constant from point O (the centre of the sphere), as seen in diagram (5) by the radius OE.

This gives us the $\varphi(R)$ as parametrized by time along the axis of time, this requires us to balance Maxwell's equation by reintroducing the resultant intrinsic magnetic vector potential $\mathbf{A}_{int}$ as,

$$\mathbf{A}_{int} \cdot \hat{\mathbf{n}} = -\frac{1}{c}\frac{\partial \varphi_{ext}}{\partial t} \tag{100}$$

where φ_{ext} is the Wheeler-Feynman summation of the external potentials.

$$\varphi_{ext} = \frac{1}{2}(\varphi_{ret} + \varphi_{adv}) + \frac{1}{2}(\varphi_{ret} + \varphi_{adv}) = \varphi_{int} \tag{101}$$

In turn the appearance of $\mathbf{A}_{int}$ gives rise to $\mathbf{B}_{int} = \nabla \times \mathbf{A}_{int}$. which is a circulating **B** on the surface of the sphere, this is not a magnetic multipole as the Heisenberg Uncertainty Principle only allows the observation of a single pole on a single axis.

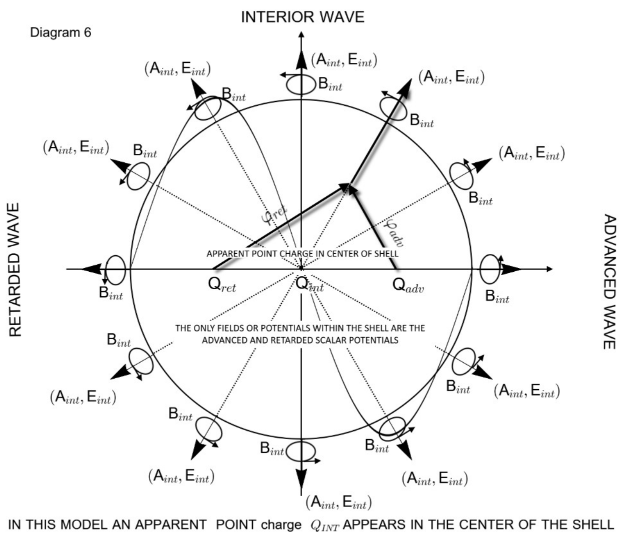
Implicit with this result $\mathbf{B}_{int} = \nabla \times \mathbf{A}_{int}$ is the observation there is a rotating charged mass with a non-zero radius, this overcomes the problem of the Standard Model where fundamental particles are treated as "point-like", as any spin model based on mass rotation must have a non-zero radius.

Finally, in the frame of reference of the resultant particle the scalar potential is given in the Coulomb gauge, and the fields (**E**, **B**) reappear along with the electron current flow $\mathbf{J}$ completing Maxwell's equations in potential form as,

$$\begin{aligned} \mathbf{A} \cdot \hat{\mathbf{n}} &= -\frac{1}{c}\frac{\partial \varphi_{int}}{\partial t} \\ \mathbf{E} &= -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \eta_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{102}$$

All of these fields and potentials are derived from the intrinsic potentials on the surface of a spherical charged particle, we can portray this as an "apparent intrinsic charge" in the centre of the local particle while the advanced and retarded charges occur at the energy maxima and minima of the retarded and advanced particles respectfully. All the while the resultant electromagnetic wave with localised electric and magnetic fields appear on the surface of the hollow sphere, where $\mathbf{A}_{int}$, $\mathbf{E}_{int}$, $\mathbf{B}_{int}$ can only occur at a distance R from the intrinsic charge Q_{int}, they cannot appear within the radius of the particle, and this neatly side-steps the problem of infinite self-energy.

We can see this in the following diagram (6) (which is also a correction on previous versions which had the sources at the leading edges of the particle, rather than dividing it into the quarters which you would expect for the Wheeler-Feynman summation).



The result of satisfying Maxwell's equations in the local frame under the Coulomb gauge results in a new four-vector potential A_η, which in turn becomes a new source for the wave progression.

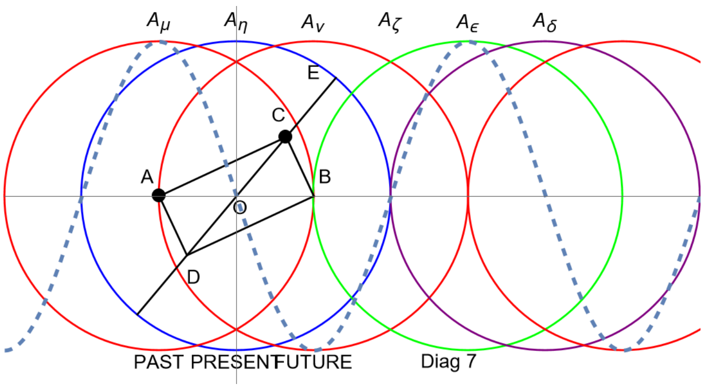
$$\tfrac{1}{2}[(A_\eta + A_\zeta) + (A_\eta - A_\zeta)] = A_\eta \tag{103}$$

This yields a continuous source of charge spheres along the axis of time as the particle transits along the Lagrangian, assuming a four-vector velocity $u^u = \frac{dx^u}{d\tau}$ as A_η progressively evolves forward in time under the Wheeler-Feynman summation,

$$\mathcal{L}(x^\mu) = -mc^2\sqrt{1 - \tfrac{v^2}{c^2}} - qA_\mu u^\mu \tag{104}$$

where the charged spheres form a continuous progression over time

$$A_\mu \to A_\eta \to A_v \to A_\zeta \to A_\delta \to etc... \tag{105}$$



Thus Matter is shown to be a longitudinal electromagnetic wave evolving forward through time as a series of hollow, charged, spinning spheres, just as Maxwell was able to demonstrate Light is a transverse electromagnetic wave from Maxwell's equations.

Conclusions

In this paper it is shown the Wheeler-Feynman Time Symmetric theory is consistent with both the Parallelogram Law and Born's Probability Density Function. This makes it possible to give a complete derivation of the Gamma Matrices, the Klein-Gordon equation, and the Dirac equation from first principles. This model (WFEFVPTST) proposes the electron is a hollow charged rotating spherical particle in $\mathbb{R}^3$ travelling via Dirac's equation in $\mathbb{R}^{1,3}$, and it is strongly suggested this electromagnetic wave for matter is equivalent to James Clerk Maxwell's electromagnetic wave for light.

REFERENCES

[1] O'Brien, J.C., "A Dynamical Theory of the Electromagnetic Potential". viXra:1912.0173 submitted 2018.

[2] O'Brien, J.C., "The Revolution of Matter" (2nd ed.). Amazon Press, (September 1, 2022).

[3] O'Brien, J.C., "Spherical Electrons and the Wheeler-Feynman Time Symmetric Theory", viXra:2407.0162 submitted 2024.

[4] O'Brien, J.C., "The Arrow of Time and the Wheeler-Feynman Time-Symmetric Theory", viXra:2312.0029 submitted on 2023-12-05.

[5] Maxwell, James Clerk., "A Dynamical Theory of the Electromagnetic Field". Philosophical Transactions of the Royal Society of London. 155: 459–512. (Paper read at a meeting of the Royal Society on 8 December 1864).

[6] Wheeler, J.A.; Feynman, R.P., "Interaction with the Absorber as the Mechanism of Radiation", Reviews of Modern Physics. 17 (2-3): 157-181, (April 1945).

[7] Wheeler, J. A.; Feynman, R. P., "Classical Electrodynamics in Terms of Direct Interparticle Action", Reviews of Modern Physics. 21 (3): 425-433. (July 1949).

[8] Griffiths, D.J., "Introduction to Electrodynamics" (4th ed.). Cambridge University Press. (2017).