The Proof of the Riemann conjecture

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Abstract:

In order to strictly prove the conjecture in Riemann's 1859 paper on the Number of prime Numbers Not Greater than x from a purely mathematical point of view, and strictly prove the correctness of Riemann's conjecture, this paper uses Euler's formula to prove that if the independent variables of ζ(s) function are conjugate, then the values of ζ(s) function are also conjugate, thus obtaining that the independent variables of ζ(s) function are also conjugate at zero. And using the conjugation of the zeros of the Riemann ζ(s) function and the zeros of ζ(s)=0 and the zeros of ζ(1-s)=0, s and 1-s must also be conjugated, The nontrivial zero of Riemann function ζ(s) must meet $s=\frac{1}{2}+ti(t\in R \text{ and } t\neq 0)$ and $s=\frac{1}{2}-ti(t\in R \text{ and } t\neq 0)$. And the symmetry of the zeros of Riemann ζ(s) function is the necessary condition that the nontrivial zeros of Riemann ζ(s) function are located on the critical boundary. According to the symmetry property of the zeros of Riemann ζ (s) function s and the zeros of Riemann ζ(s) function 1-s, combined with the conjugated property of the zeros of Riemann ζ(s) function s and Riemann ζ(s) function 1-s, It is shown that the real part of the nontrivial zero of the ζ(s) function must only be equal to $\frac{1}{2}$. And by Riemann set $s=\frac{1}{2}+ti(t\in C \text{ and } t\neq 0)$ and auxiliary function $\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)(s\in C \text{ and } s\neq 1)$, Get $\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)=0$, combining the nontrivial zeros of Riemann function ζ(s) must meet $s=\frac{1}{2}+ti$ $(t\in R \text{ and } t\neq 0)$ and $s=\frac{1}{2}-ti$ $(t\in R \text{ and } t\neq 0)$, Thus it is proved equivalently that the zeros of the Riemann ξ(t) function must all be non-zero real numbers, and the Riemannian conjecture is completely correct.

Key words:

Euler's formula, Riemann ζ(s) function, Riemann function ξ(t), Riemann conjecture, symmetric zeros, conjugate zeros, uniqueness.

I. Introduction

The Riemann hypothesis and the Riemann conjecture is an important and famous mathematical problem left by Riemann in his 1859 paper "On the Number of primes not greater than x ", which is of great significance to the study of the distribution of prime numbers and is known as the greatest unsolved mystery in mathematics. After years of hard work, I solved this problem and rigorously proved that both the Riemann conjecture and the generalized Riemann conjecture are completely correct. The Polignac conjecture, the twin prime conjecture, and Goldbach's conjecture are also completely correct. It would be nice if you understood Riemann's conjecture thoroughly from the outset of his paper "On Prime Numbers not Greater than x" and were completely convinced of the logical reasoning behind it. You need to do this before you read my paper. The following is about the first half of Riemann's paper "On the Number of primes not Greater than x ", which I have explained and derived, which is the premise and basis for your understanding of Riemann's conjecture.In 1859, Riemann was admitted to the Berlin Academy of Sciences as a corresponding member, and in order to express his gratitude for the honor, he thought it would be best to use the permission he received immediately to inform the Berlin Academy of a study on the density of the distribution of prime numbers, a subject in which Gauss and Dirichlet had long been interested. It does not seem entirely unworthy of a report of this nature. Riemann used Euler's discovery of the following equation as his starting point:

$$\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Where p on the left side of the equation takes all prime numbers, n on the right side takes all natural numbers, and the function of the complex variable s represented by the two series above (when they converge) is denoted by ζ(s). That is, to define a function of complex variables:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})。$$

The two series above converge only if the real part of s is greater than 1,is also say when

Re(s)>1,then $\sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})$ converge only.if s=1,then $\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$, It's called a harmonic series, and it diverges.If Re(s)<1, $\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+...$, it's more divergent.Because if Re(s)<1,then $\frac{1}{1^s}=\frac{1}{1},\frac{1}{2^s}>\frac{1}{2},\frac{1}{3^s}>\frac{1}{3},\frac{1}{4^s}>\frac{1}{4},$ ... , .But if s is a negative number, for example s= -1,then it does not satisfy the condition that Re(s)>1. So you need to find an expression for ζ(s) function a that is always valid for any s. In modern mathematical language, that is, to carry out an analytical extension of a complex function ζ(s), and the best way to analyze the extension is to find a more extensive and effective representation of the function such as an integral representation or an appropriate function representation.Therefore, we want to define a new function, this new function also ζ(s) to represent, this new function of the independent variable s is not only full Re(s)>1, but also satisfy Re(s)≤1(s≠1), and the function image is smooth, every point on the function image can find its tangent slope, that is, the function everywhere can find the derivative. However, it is no longer called the Euler zeta function, but the Riemann ζ function. Riemann used the integral to express the function ζ(s). In this paper, I have added another complex variable to express the Riemann function ζ(s).

Because Π(s) = Γ(s +1) = sΓ(s) , where Π(s) is the factorial function , Γ(s) is the Euler gamma function,$\Gamma(s)=\int_0^{\infty} x^{s-1} e^{-x} dx$, Let the variable $x \to nx$ ($n\in Z^{+}$) in the integral symbol,then

$\int_0^{\infty}(nx)^{s-1} e^{-nx}d(nx)$ $=n\int_0^{\infty} e^{-nx}$ $n^{s-1}x^{s-1}$=$n^s\int_0^{\infty} e^{-nx}$ x^{s-1}=$\Gamma(s)$= Π(s-1),so

$$\int_0^{\infty} e^{-nx} x^{s-1} = \frac{\Pi(s-1)}{n^s}$$

That's exactly what Riemann says in his paper, he says he's going to use

$$\int_0^{\infty} e^{-nx} x^{s-1} = \frac{\Pi(s-1)}{n^s}$$

Since n is all positive integers, we need to assign $\sum$ to e^{-nx} and $\frac{1}{n^s}$on both sides of the equation, so

$\sum_{n=1}^{\infty} e^{-nx} = 1+\sum_{n=1}^{\infty} e^{-nx} - 1 = (1+e^{-x}+e^{-2x}+e^{-3x}+\ldots) - 1 = \frac{1}{1-e^{-x}}$ $-1=\frac{e^{-x}}{1-e^{-x}}=\frac{1}{e^{x}-1}$,

The common ratio q satisfies $0<q=|e^{-x}|<=1(0<=x \to +\infty)$, $\frac{\Pi(s-1)}{n^s} = \frac{\Pi(s-1)}{1^s+2^s+3^s+4^s+5^s+\cdots}$,

and $\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1^s+2^s+3^s+4^s+5^s+\cdots} = \zeta(s)$, so according

$$\int_0^{\infty} e^{-nx} x^{s-1} = \frac{\Pi(s-1)}{n^s}$$

,can get $\Pi(s-1)\zeta(s)=\int_0^{\infty}\frac{x^{s-1}dx}{e^{x}-1}$, this is exactly what Riemann found in his paper.

Now consider the following integral

$$\int \frac{(-x)^{s-1}dx}{e^{x}-1}$$

According to modern mathematical notation, the integral should be denoted as $\int_C \frac{(-x)^{s-1}dx}{e^{x}-1}$, or considering that the complex number is generally represented by Z, the integral should be denoted as $\int_C \frac{(-Z)^{s-1}dZ}{e^{Z}-1}$, Its integral path proceeds from $+\infty$ to $+\infty$ on the forward boundary of a region containing the value 0 but not any other singularities of the integrable function, where the integral path C is shown in Figure 1 below.

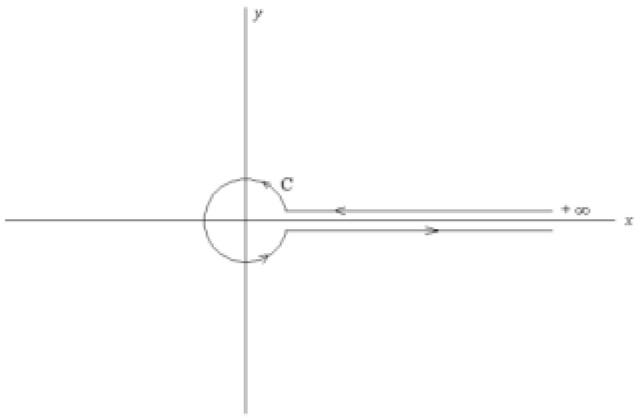


Figure 1

The forward boundary of a region that contains the value 0 but does not contain any other singularities of the integrand (such as s=1). Easy the integral value is:

$(e^{-\pi si} - e^{\pi si})\int_0^\infty \frac{x^{s-1}dx}{e^x-1}$,

Where we agree that in the many-valued function $(-x)^{s-1}$, the value of ln$(-x)$ is real for negative x, thus obtaining $2\sin(\pi s)\,\Pi(s-1)\zeta(s) = i\int_\infty^\infty \frac{(-x\ )^{s-1}dx}{e^x-1}$ $(x \in R)$.This equation now gives the value of the function $\zeta(s)$ for any complex variable s, and shows that it is single-valued analytic, and takes a finite value for all finite s except 1, and zero when s is equal to a negative even number.The right side of the above equation is an integral function, so the left side is also an integral function , $\Pi(s-1) = \Gamma(s)$,and the first-order poles of $\Gamma(s)$ at s = 0,–1,–2,–3,... cancels out $\sin(\pi s)$'s zero. Riemann zeta function $\zeta(s)$ is a series expression of $\sum_{n=1}^{\infty}\frac{1}{n^s}(s \in C \text{ and } Re(s) > 1, \textit{iterate over all positive integers})$, on the complex plane analytical continuation. The reason for the analytical extension of the above series expression is that this expression only applies to the region of the complex plane where the real part of s Re(s)>1 (otherwise the series does not converge). Riemann found an analytical continuation of this expression (of course Riemann did not use the modern term "analytic continuation" in complex function theory). Using the circumchannel integral, the analytically extended Riemann zeta function can be expressed as:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_\infty^\infty \frac{(-z)^s}{e^Z-1}\frac{dz}{z}$$

The integral in the above formula is actually a circumchannel integral around the positive real axis (that is, starting from +∞, integrating above the real axis to near the origin, integrating around the origin to below the real axis, and then integrating below the real axis to +∞ - the distance from the real axis and the radius around the origin are all approaching 0); The Γ function Γ(s) in the equation is an analytical extension of the factorial function in the complex plane, for positive integers s>1:Γ(s)=(s-1)! . It can be shown that the integral expression for ζ(s) above resolves everywhere over the entire complex plane except for a simple pole at s=1. Such an expression is an example of a so-called meromorphic function - that is, a function that resolves everywhere over the entire complex plane except for the existence of poles on an isolated set of points. This is the complete definition of the Riemann ζ function.

To obtain the value of this integral, we assume that there is a complex number of arbitrarily small moduli δ, and that the moduli $\left|\delta\right|$ of δ,|δ|→0,Because $(-Z)^s = e^{s\ln(-Z)}$,and $\ln(-Z)=\ln(Z)+\pi i$ or $\ln(-Z) = \ln(Z) - \pi i$, so

$$\int_C \frac{(-Z)^{s-1}dZ}{e^Z-1} = \int_\infty^\delta \frac{(-Z)^{s-1}dZ}{e^Z-1} + \int_\delta^\infty \frac{(-Z)^{s-1}dZ}{e^Z-1} + k\int_{|\delta|\to 0}^\infty \frac{(-Z)^{s-1}dZ}{e^Z-1} = \int_{+\infty}^\delta \frac{(-Z)^s dZ}{(e^Z-1)Z} + \int_\delta^{+\infty} \frac{(-Z)^s dZ}{(e^Z-1)Z}$$

$$+k\int_{|\delta|\to 0} \frac{(-Z)^s dZ}{(e^Z-1)Z} = (e^{\pi si}-e^{-\pi si})\int_\delta^\infty \frac{e^{s\ln(Z)}dZ}{(e^Z-1)Z} + k\int_{|\delta|\to 0} \frac{(-Z)^s dZ}{(e^Z-1)Z}$$, k is a constant.

The definition of trigonometric functions of complex variables is given by Euler's formula

$\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}$, if z=πs ,then $\sin(\pi s)=\frac{e^{\pi si}-e^{-\pi si}}{2i}$.so $e^{\pi si}-e^{-\pi si}=2i\sin(\pi s)$, $i=\frac{e^{\pi si}-e^{-\pi si}}{2\sin(\pi s)}$ .so $\int_C \frac{(-Z)^{s-1}dZ}{e^Z-1} = (e^{\pi si} - e^{-\pi si})\int_\delta^\infty \frac{e^{s\ln(Z)}dZ}{e^Z-1} + k\int_{|\delta|\to 0} \frac{(-Z)^s dZ}{(e^Z-1)Z}$, if δ is a real number and the absolute value |δ| of δ, |δ| →0,

then $\int_{|\delta|\to 0} \frac{(-Z)^s dZ}{(e^Z-1)Z} = 0$ then $\int_C \frac{(-Z)^{s-1}dZ}{e^Z-1} = 2i\sin(\pi s)\int_0^\infty \frac{x^{s-1}dx}{e^x-1}$ $(x \in R)$.then

$\frac{1}{2i\sin(\pi s)}\int_C \frac{(-Z)^{s-1}dZ}{e^Z-1} = \int_0^\infty \frac{x^{s-1}dx}{e^x-1}$ $(x \in R)$.We got

$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}dx}{e^x-1} (x \in R)$ before, so $2\sin(\pi s)\,\Pi(s-1)\zeta(s) = i\int_0^\infty \frac{x^{s-1}dx}{e^x-1}$ .When the real part of s is negative, the above integral can be performed not along the region positively surrounding the given value, but along the region negatively containing all the remaining complex values.See Figure 2 below, where the radius of the great circle C' approaches infinity and thus contains all poles of the integrand, i.e. , all zeros of the denominator $e^x - 1$, nπi (n is an integer), and the following calculation applies Cauchy's residue theorem.

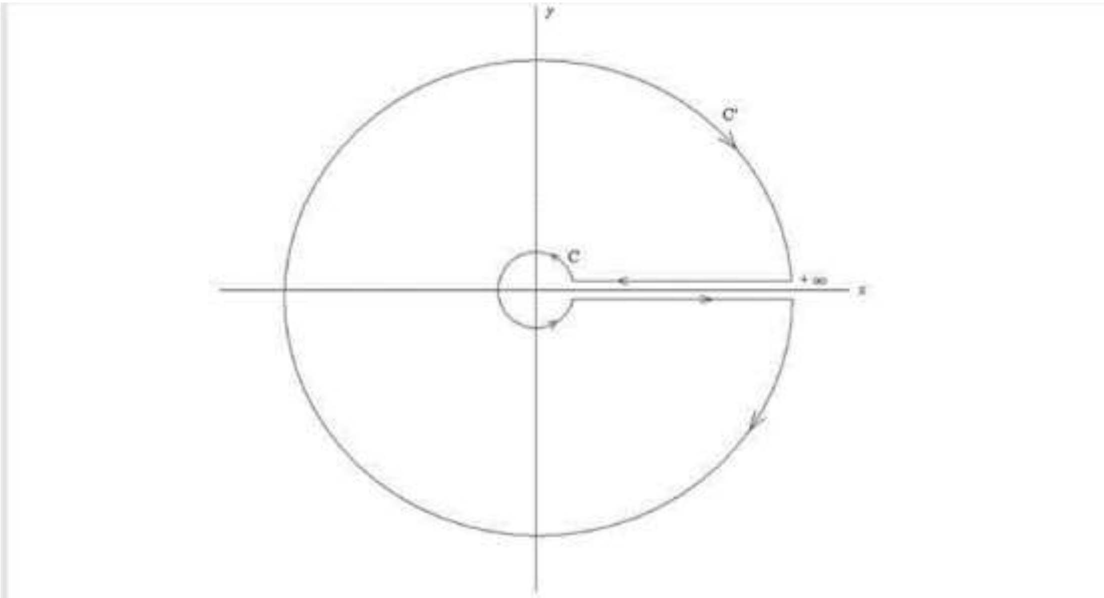


Figure 2

Since the value of the integral is infinitesimal for modular infinite complex numbers, and in this region the integrand has a singularity only if x is equal to an integral multiple of $2\pi i$, the integral is equal to the sum of the integrals negatively around these values, but the integral around the value $n2\pi i(n \in R^+)$ is equal to $(-n2\pi i)^{s-1}(-2\pi i)(n \in R^+)$.The residue of the integrand $\frac{(-x)^{s-1}}{(e^x-1)}$ at $n2\pi i(n \neq 0)$ is equal to

$[\frac{(-x)^{s-1}}{(e^x-1)'}]_{x=n2\pi i}=[\frac{(-x)^{s-1}}{e^x}]_{=n2\pi i}=(n2\pi i)^{s-1}(n \neq 0)$.

So we get

$2\sin(\pi s)\prod(s-1)\zeta(s)=(2\pi)^s\sum n^{s-1}((-i)^{s-1}+i^{s-1})$ [1] (Formula 3),

It reveals a relationship between ζ(s) and ζ(1-s), using known properties of the function Π(s), that is, using the coelements formula of the gamma function Γ(s) and Legendre's formula. It can also be expressed as: $\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s⟶1-s.

based on euler's $e^{ix}=\cos(x)+i\sin(x)\ (x \in R)$, can get

$e^{i(-\frac{\pi}{2})}=\cos(\frac{-\pi}{2})+i\sin(\frac{-\pi}{2})=0-i=-i$,

$e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})=0+i=i$,

then

$(-i)^{s-1}+i^{s-1}=(-i)^{-1}(-i)^s+(i)^{-1}(i)^s=(-i)^{-1}e^{i\left(-\frac{\pi}{2}\right)s}+i^{(-1)}e^{i\left(\frac{\pi}{2}\right)s}=$

$ie^{i\left(-\frac{\pi}{2}\right)s}-ie^{i\left(\frac{\pi}{2}\right)s}=i(\cos\frac{-\pi s}{2}+i\sin\frac{-\pi s}{2})-i(\cos\frac{\pi s}{2}+i\sin\frac{\pi s}{2})=i\cos(\frac{\pi s}{2})-i\cos(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})$

$=2\sin(\frac{\pi s}{2})$ (Formula 4)

According to the property of Π(s-1)=Γ(s) of the gamma function,and

$\sum_{n=1}^{\infty} n^{s-1}=\zeta(1-s)$($n \in Z^+$ and n traves all positive integer, $s \in C$, and $s \neq 1$), Substitute the above (Formula 4) into the above (Formula 3), will get

$2\sin(\pi s)\Gamma(s)\zeta(s)=(2\pi)^s\zeta(1-s)2\sin\frac{\pi s}{2}$ (Formula 5),

according to the double Angle formula $\sin(\pi s)=2\mathrm{Sin}(\frac{\pi s}{2})\cos(\frac{\pi s}{2})$, we Will get

$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s\in C$ and $s\neq 1$) (Formula 6),

Substituting s→1-s, that is taking s as 1-s into Formula 6, we will get

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ and $s\neq 1$) (Formula 7),

This is the functional equation for ζ(s) ($s\in C$ and $s\neq 1$). To rewrite it in a symmetric form, use the residual formula of the gamma function

$\Gamma(Z)\Gamma(1-Z)=\frac{\pi}{\sin(\pi Z)}$ (Formula 8)

and Legendre's formula $\Gamma(\frac{Z}{2})\Gamma(\frac{Z}{2}+\frac{1}{2})=2^{1-Z}\pi^{\frac{1}{2}}\Gamma(Z)$ (Formula 9) ,

Take $z=\frac{s}{2}$ in (Formula 8) and substitute it to get

$\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (Formula 10) ,

In (Formula 9), let z=1-s and substitute it in to get

$\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (Formula 11)

By substituting $\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (Formula 10) and $\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (Formula 11)

into $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ and $s\neq 1$) (Formula 7), can get

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$($s\in C$ and $s\neq 1$) (Formula 12),

Also

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s→1-s,

And that's exactly what Riemann said in his paper.That is to say:

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s→1-s ,

Also

$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1-s)$($s\in C$ and $s\neq 1$),

or

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$($s\in C$ and $s\neq 1$)(Formula 2),

Then $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ and $s\neq 1$)(Formula 7) .

This property of the function induces me to introduce $\Pi(\frac{s}{2}-1)$ instead of $\Pi(s-1)$ into the general term of the series $\sum_{n=1}^{\infty}\frac{1}{n^s}$, from which we obtain the function a very convenient expression forζ(s), which we actually have

$$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx\ .$$

To derive the above equation, let's look at $\Pi(\frac{s}{2}-1)=\Gamma(\frac{s}{2})=\int_0^{\infty}x^{\frac{s}{2}-1}e^{-x}dx$, in

$\Pi(\frac{s}{2}-1)=\Gamma(s)=\int_0^{\infty}x^{\frac{s}{2}-1}e^{-x}dx$, replace $x\to n^2\pi x$ as follows, then

$\Pi(\frac{s}{2}-1)=\Gamma(s)=\int_0^{\infty}(n^2\pi x)^{\frac{s}{2}-1}e^{-n^2\pi x}dx=n^s.n^{-2}.\pi^{\frac{s}{2}}.\pi^{-1}\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}d(n^2\pi x)=$

$n^s.n^{-2}.\pi^{\frac{s}{2}}.\pi^{-1}.n^2.\pi\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx=n^s.\pi^{\frac{s}{2}}\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx$, so

$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx$.

So, if we call $\sum_{n=1}^{\infty}e^{-n^2\pi x}=\psi(x)$, get immediately

$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx=\int_0^{\infty}(\sum_{n=1}^{\infty}e^{-n^2\pi x})x^{-\frac{s}{2}}dx=\int_0^{\infty}\psi(x)x^{-\frac{s}{2}}dx.$

According to the Jacobi theta function

$\theta(x)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=e^{-0^2\pi x}+2\sum_{n=1}^{\infty}e^{-n^2\pi x}=1+2(e^{-\pi x}+e^{-4\pi x}+e^{-9\pi x}+e^{-16\pi x}+\cdots)$,

Easy to see $\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}=\frac{\theta(x)-1}{2}$.

The transformation formula of theta function is derived as follows: $\theta(\frac{1}{x})=\sqrt{x}\ \theta(x)$.

Let the first class of complete elliptic integrals k,k' is called modulus and complement of Jacobi elliptic functions or elliptic integrals, respectively.

$k=k(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$,

$k'=k(k')=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-k'^2\sin^2\theta)}}$,

let $\tau=k'/k$,then get

$\sqrt{\frac{2k}{\pi}}=\theta(\tau)=1+2(e^{-\pi\tau}+e^{-4\pi\tau}+e^{-9\pi\tau}+e^{-16\pi\tau}+\cdots)$,

The modulo k and the complement k' are interchangeable

$\sqrt{\frac{2k'}{\pi}}=\theta\left(\frac{1}{\tau}\right)=1+2(e^{-\pi/\tau}+e^{-4\pi/\tau}+e^{-9\pi/\tau}+e^{-16\pi/\tau}+\cdots)$,

Compare the two formulas to obtain $\theta(\frac{1}{\tau})=\sqrt{\tau}\theta(\tau)$. It was first obtained by Cauchy using Fourier analysis, and later proved by Jacobi using elliptic functions.

Apply the integral expression above

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-z)^s}{e^z-1}\frac{dz}{z}$$

can also prove Riemannζ function satisfy the above algebraic equation - also called zeta function equation $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ and $s\neq 1)$ (Formula 7), is not hard to find in this relation, The Riemann ζ function takes zero at s=-2n (n is a positive integer) , because $\sin(\frac{\pi s}{2})$ is zero. The point on the complex plane where the value of the Riemann ζ function is zero is called the zero of the Riemann ζ function.So s=-2n (n is a positive integer) is the zero of the Riemann zeta function. These zeros have a simple and orderly distribution and are called trivial zeros of the Riemann ζ function. In addition to these trivial zeros, the Riemann ζ function has many other zeros whose properties are far more complex than those trivial zeros, and are rightly called nontrivial zeros.

Riemann described it in his paper as follows:

$$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^{\infty}\psi(x)\ x^{\frac{s}{2}-1}dx+\int_1^{\infty}\psi(\tfrac{1}{x})\ x^{\frac{s-3}{2}}dx+\frac{1}{2}\int_0^1(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1})dx$$

$$=\frac{1}{s(s-1)}+\int_1^{\infty}\psi(x)\,(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx\ ,$$

Let's look at the last part of the equation, if s⟶1-s, then

$$\frac{1}{s(s-1)}=\frac{1}{(1-s)(1-s-1)}=\frac{1}{(1-s)(-s)}\frac{1}{(s-1)s}\ ,$$

$$x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}}=x^{\frac{1-s}{2}-1}+x^{-\frac{1+(1-s)}{2}}=x^{\frac{-1-s}{2}}+x^{-\frac{2-s}{2}}=x^{-\frac{1+s}{2}}+x^{\frac{s}{2}-1},\text{so}$$

$\prod\left(\frac{s}{2}-1\right)x^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s⟶1-s.

Riemann then derived the function equation for $\zeta(s)$ again, which is simpler than the previous derivation using the circum-channel integral and residue theorems.

If we introduce auxiliary function function $\Phi(s)=\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)$.

This can be succinctly written as Φ(s) =Φ(1– s), But it is more convenient to add the factor s(s –1) to Φ(s), which is what Riemann does next, i.e. (To keep with Riemann's notation, the number factor $\frac{1}{2}$ is introduced): $\zeta(s)=\frac{1}{2}s(s-1)\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$.

Because factor (s –1) cancels out the pole of ζ(s) at s=1, factor s cancels out the pole of $\Gamma(\frac{s}{2})$ at s = 0, and ζ(s)'s trivial zeros -2, -4, -6,... cancel out the rest of the poles of $\Gamma(\frac{s}{2})$, so ζ(s) is an integral function and is zero only at the nonnormal zero points of ζ(s). Note that since sub s(s –1) obviously does not change under s →1– s, there is a function equation ξ(s) = ξ(1– s). The zeros of $\zeta(s)$ are all zeros of ξ(s) except the trivial zero s=-2n(n is a natural number), which, since it happens to be the pole of $\Gamma(\frac{s}{2}+1)$ in $\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$, is not the zero of ξ(s), and thus the zeros of ξ(s) coincide with the nontrivial zeros of the Riemann ζ function. In other words, ξ(s) separates the nontrivial zeros of the Riemann ζ(s) function from the total zeros.

Now Riemann suppose s= $\frac{1}{2}$ +ti (t ∈ C and t ≠ 0) , $\prod\left(\frac{s}{2}\right)$ (s–1) $\pi^{-\frac{s}{2}}\zeta(s) = \Gamma(\frac{s}{2}+1)$ (s –1)$\pi^{-\frac{s}{2}}\zeta(s)$=ξ(t), thus get ξ(t) =$\frac{1}{2}$ -($t^2+\frac{1}{4}$)$\int_1^{\infty}\psi(x)\,x^{-\frac{3}{4}}\cos\left(\frac{1}{2}t\ln x\right)$dx

Or

ξ(t) = $4\int_1^{\infty}\frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx$.

The function $\prod\left(\frac{s}{2}\right)$(s–1)$\pi^{-\frac{s}{2}}\zeta(s)$=ξ(t) defined by Riemann is essentially the same as the function ξ(s)= $\frac{1}{2}$s(s –1)$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$commonly used today. Because

$\prod\left(\frac{s}{2}\right)$=$\Gamma(\frac{s}{2}+1)$= $\frac{s}{2}\Gamma(\frac{s}{2})$,so $\prod\left(\frac{s}{2}\right)$(s–1)$\pi^{-\frac{s}{2}}\zeta(s)$=$\frac{s}{2}\Gamma(\frac{s}{2})$(s–1)$\pi^{-\frac{s}{2}}\zeta(s)$=$\frac{1}{2}$s(s –1)$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$=ξ(s)。

The only difference is that Riemann takes t as the independent variable, while ζ(s), which is now commonly used, still takes s as the independent variable, and s and t differ by a linear transformation: s=$\frac{1}{2}$+ti, that's a 90 degree rotation plus a translation of $\frac{1}{2}$. This means that the complex number t is rotated by 90 degrees counterclockwise and shifted by in the positive direction of the real number line, which is $t(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})+\frac{1}{2}$. In this way, the line Re(s)= $\frac{1}{2}$ in the complex plane of s corresponds to the real axis in the t plane, and the real part of the zeros of the Riemann ζ(s)(s∈C and s≠1) function on the critical line Re(s)= $\frac{1}{2}$ corresponds to the real root of ξ(t). Note that in Riemann's notation, the functional equation ξ(s) = ξ(1– s) becomes ξ(t) = ξ(–t), that is, ξ(t) is an even function, so its power series expansion is only an even power, and the zeros are symmetrically distributed with respect to t = 0.In addition, it is also clear from the above two integral representations that ξ(t) is an even function, since $\cos(\frac{1}{2}t\ln x)$ is an even function of t.

For all finite t, function ξ(t)=$\frac{1}{2}$ -($t^2+\frac{1}{4}$)$\int_1^{\infty}\psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)dx$ or function ξ(t) = $4\int_1^{\infty}\frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx$ is finite in value,

And can be expanded to a power of t^2 as a rapidly convergent series, because for an s value with a real part greater than 1, the value of $\ln\zeta(s) = -\sum\ln(1-p^{-s})$ is also finite.It is same true for the logarithm of the other factors of ξ(t), so the function ξ(t) can take zero only if the imaginary part of t lies between $\frac{1}{2}$ and –$\frac{1}{2}$i. That is, A can take a zero value only if the real part of s lies between 0 and 1. The number of roots of the real part of the equation ξ(t) between 0 and T is approximately equal to N(T)= $\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}+O(\ln T)$, approximately to ($\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$)(this result of Riemann's estimate of the number of zeros was not strictly proved until 1859 by Mangoldt).This is because the value of the integral $\int d\ln\xi(t)$ (after omitting small quantities of order $\frac{1}{T}$) approximately equal to ($T\ln\frac{T}{2\pi}-T$)i. The value of this integral is equal to the number of roots of the equation in this region multiplied by 2πi(this is the application of the amplitude Angle principle).In fact, Riemann found that the number of real roots in this region is approximately equal to this number, and it is highly likely that all the roots are real. Riemann naturally hoped for a rigorous proof of this, but after some hasty and unsuccessful initial attempts, Riemann temporarily set aside the search for proof because it was not necessary for the purposes of Riemann's subsequent studies. What Riemann wrote down is the famous Riemann conjecture, the most famous conjecture in mathematics!

According to Riemann's assumption in the paper : s=$\frac{1}{2}$+ti($t \in C$ and $t \neq 0$), then the Riemann conjecture is equivalent to that for ζ(s)=0, its complex roots s (except for negative even numbers) must all be complex numbers satisfying only s=$\frac{1}{2}$+ti ($t \in R$ and $t \neq 0$)and $s=\frac{1}{2}-ti(t \in$ R and t≠0, and they all lie on the critical boundary of the vertical real number axis satisfying Re(s)=$\frac{1}{2}$. These complex roots s (except negative even numbers) are called nontrivial zeros of Riemannn $\zeta(s)(s \in C$ and $s \neq 1$ and $s \neq -2n, n \in R^{+})$ functions.

The study of the non-trivial zeros of the Riemann ζ function constitutes one of the most difficult subjects in modern mathematics. The Riemann conjecture that we are going to discuss is a conjecture about these nontrivial zeros. Here we first describe its content, and then describe its context. Riemann conjecture: All nontrivial zeros of the Riemann ζ function lie on the line Re(s)= $\frac{1}{2}$. In the study of the Riemann conjecture, mathematicians call the line Re(s)= $\frac{1}{2}$ in the complex plane a critical boundary. Using this term, the Riemann conjecture can also be expressed as: all non-trivial zeros of the Riemann ζ function lie on the critical boundary. This is the content of the Riemann conjecture, which Riemann proposed in 1859 in his paper "On the Number of Prime Numbers Not Greater than x." In its formulation, the Riemann conjecture appears to be a purely complex function proposition, but as we shall soon see, it is in fact a mysterious piece of music about the distribution of prime numbers.

How can the distribution of nontrivial zeros of a function over a complex number field, the Riemann zeta function, which we sometimes refer to simply as zeros if there is no ambiguity, be related to the distribution of prime numbers in the seemingly unrelated natural numbers (which in this book refer to positive integers)? It starts with what's called the Euler product formula. We know that as early as the ancient Greeks, Euclid proved with a wonderful proof by contradiction that there are infinitely many prime numbers. With the deepening of the study of number

theory,
people are naturally more and more interested in the distribution of prime numbers on the set of natural numbers. In 1737, the mathematician Euler published a very important formula at the St. Petersburg Academy of Sciences in Russia, which laid the foundation for mathematicians to study the law of the distribution of prime numbers. This formula is the Euler product formula, which is $\sum_n n^{-s}=\prod_p(1-p^{-s})^{-1}$,The sum on the left of this formula is performed on all natural numbers, and the continued product on the right is performed on all prime numbers. It can be shown that this formula holds for all complex numbers s with Re(s)>1. The left side of this formula is the series expression of the Riemann ζ function for Re(s)>1, which we have described above, and the right side is an expression purely concerning prime numbers (and containing all prime numbers), which is a sign of the relationship between the Riemannζ function and the distribution of prime numbers. So what does this formula tell us about the distribution of prime numbers? How does the zero of the Riemann zeta function appear in this relation?
Euler himself was the first to study the information contained in this formula. He noticed that at s=1, the left-hand side of the formula

$$\sum_n n^{-1}$$

is a divergent series (this is a famous divergent series, called a harmonic series), which diverges logarithmically. None of this was new to Euler. To deal with the continued product on the right side of the formula, he took the logarithm of both sides of the formula, so that the continued product became a sum, from which he obtained:

$\ln(\sum_n n^{-1})=\sum_p(p^{-1}+\frac{p^{-2}}{2}+\frac{p^{-3}}{3}+\cdots)$,

Or, rather,

$\sum_{p<N} p^{-1} \sim \ln\ln(N)$,

This result, which diverges in the form of lnln (N), is another important research result on prime numbers since Euclid proved that there are infinitely many primes. It is also a novel proof of the proposition that there are infinitely many prime numbers (because if there are only finite numbers of prime numbers, then the sum has only a finite number and cannot diverge). But this new proof by Euler contains much more than Euclid's proof, because it shows that prime numbers are not only infinitely many, but that their distribution is much denser than that of many sequences that also contain infinitely many elements, such as { n } sequences (because the sum of the reciprocal convergences of the latter).
Moreover, if we further note that the right end of $\sum_{p<N} p^{-1} \sim \ln\ln(N)$ can be rewritten as an integral expression:

$\ln\ln(N) \sim \int^{N} \frac{x^{-1}}{\ln(x)} dx$,

By introducing a density function $\rho(x)$ for the distribution of prime numbers, which gives the probability of finding prime numbers in the unit interval near x, the left end of $\sum_{p<N} p^{-1} \sim \ln\ln(N)$ can also be rewritten as an integral expression:

$\sum_{p<N} p^{-1} \sim \int^{N} x^{-1}\,\rho(x)\mathrm{d}x$,

Comparing these two integral expressions, it is not difficult to guess that the distribution density of the prime numbers is $\rho(x) \sim 1/\ln x$, so that the number of prime numbers within x, usually represented by $\pi(x)$, is

$\pi(x) \sim Li(x)$,

among

$Li(x)=\int \frac{1}{\ln x}\mathrm{d}x$,

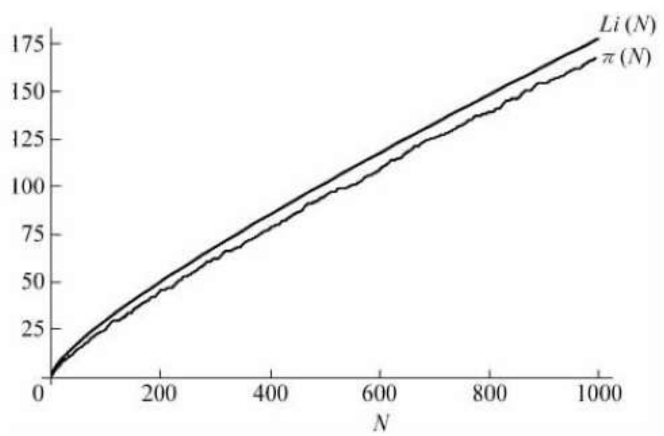
It's a logarithmic integral function.This result is the famous prime number theorem - although this crude reasoning does not constitute a proof of the prime number theorem. So this result that Euler discovered is a secret door to the prime number theorem. Unfortunately, Euler himself did not follow this line of thinking and missed this secret door, and the time for mathematicians to develop the prime number theorem was delayed by several decades.

The credit for developing the prime number theorem eventually fell to two other mathematicians: the German Friedrich Gauss (1777-1855) and the French Adrien-Marie Legendre (1752-1833).Gauss's work on the distribution of prime numbers began between 1792 and 1793, when he was only fifteen years old. During that time, whenever he was "doing nothing," the precocious genius mathematician would pick a few natural number intervals of length 1,000, count the number of primes in these intervals, and compare them. After doing a lot of calculations and comparisons, Gauss discovered that the density of the prime distribution can be approximately described by the reciprocal of the logarithmic function, $\rho(x) \sim 1/\ln x$, which is the main content of the prime number theorem mentioned above. But Gauss did not publish the results. Gauss was a mathematician who pursued perfection, and he rarely published results that he thought were not perfect, and his mathematical ideas and inspiration were like a vast and surging river, which often made him start a new research topic before he had time to beautify a research result. As a result, Gauss did far more mathematical research in his lifetime than he officially published. On the other hand, Gauss often revealed some of his unpublished work through other means, such as letters, which caused considerable embarrassment to some of his contemporaries. One of the hardest hit was Legendre. The French mathematician was the first to publish the least square method for linear fitting in 1806, but Gauss mentioned in a work published in 1809 that he had discovered the same method in 1794 (that is, 12 years before Legendre), much to Legendre's dismay.

As the saying goes, friends don't get together. In the formulation of the prime number theorem, poor Legendre once again had the misfortune to collide with the mathematical giant Gauss. Legendre published his research on the distribution of prime numbers in 1798, which is the earliest document on the theorem of prime numbers in the history of mathematics. Since Gauss did not publish his results, Legendre was the rightful author of the prime number theorem. Legendre maintained this priority for a total of 51 years. But in 1849, Gauss, in a letter to the German astronomer Johann Encke (1791-1865), mentioned his work on the distribution of prime numbers in 1792-93, thus taking the half-century-old priority out of Legendre's pocket. On top of his already bulging pockets.

Fortunately, by the time Gauss wrote to Enke, Legendre had been dead for 16 years, and he had

avoided another cruel blow in the most helpless way.
Both Gauss's and Legendre's studies of the distribution of prime numbers were presented in the form of guesses (Legendre's study had a certain element of inference, but it was still far from proving). Therefore, to be sure, the prime number theorem was at that time only a conjecture, that is, the prime number conjecture, and what we mean by the formulation of the prime number theorem is only the formulation of the prime number conjecture. The mathematical proof of the prime number theorem was not given until a century later, in 1896, by the French mathematician Jacques Hadamard (1865-1963) and the Belgian mathematician Charlesde la Vallee-Poussin (1866-1962), independently of each other. Their proof has a deep connection with the Riemann conjecture, and the timing and occasion of Hadamard's proof are dramatic, as we shall describe later.
The prime number theorem is concise and elegant, but its description of the distribution of prime numbers is still relatively rough, it gives only an asymptotic form of the distribution of prime numbers - the distribution of primes less than N as N approaches infinity. From the distribution of prime numbers and the prime number theorem, we can also see that there is a deviation between $\pi(x)$ and $\mathrm{Li}(x)$, and the absolute value of this deviation seems to continue to increase with the increase of x (fortunately, the increase of this deviation is still negligible compared to the increase of $\pi(x)$ and $\mathrm{Li}(x)$ itself - otherwise the prime number theorem would not hold).Is there a formula that describes the distribution of prime numbers more accurately than the prime number theorem? This was the question that Riemann set out to answer in 1859. That year, five years after Gauss's death, Riemann, 32, succeeded the German mathematician Johann Dirichlet (1805-1859) as Gauss's successor at the University of Gottingen. On 11 August of the same year, he was elected a corresponding member of the Academy of Sciences in Berlin. In return for this high honor, Riemann submitted a paper to the Berlin Academy of Sciences - a short eight-page paper entitled: On the Number of primes Less than a Given Value. It was this paper that deciphered the information contained in Euler's product formula, and it was this paper that linked the distribution of zeros of the Riemann zeta function to the distribution of prime numbers.



(The above diagram shows the distribution of prime numbers and the prime number theorem).

This paper pushed the study of the distribution of prime numbers to a magnificent peak, and left a great mystery for later generations of mathematicians.

According to Euler's formula $\zeta(s) \equiv \sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})$, this is the basis for studying the distribution of prime numbers. Riemann's research also takes this formula as a starting point. In order to eliminate the continued product on the right side of this formula, Euler took the logarithm of both sides of the formula, and Riemann did the same (even the product is really something that one wants to divide quickly), thus obtaining $\ln\zeta(s) \equiv \sum_p \ln(1-p^{-s})$=$\sum_p\sum_n\frac{p^{-s}}{n}$, but after this step, Riemann and euler parted ways: euler proved that sounded after prime Numbers have an infinite number not quit; Riemann, on the other hand, continued to walk along a thorny road and came out of a new world of prime number research.

It can be shown that the double summation to the right of the given $\ln\zeta(s) \equiv \sum_p \ln(1-$ p–s)=pnp−sn is absolutely converges in the region Re(s)>1 on the complex plane, and can be rewritten as the Stielchers integral:

$$\ln\zeta(s)=\int_0^{\infty} x^s d\, J(x),$$

Where $J(x)$ is a special step function that takes a value of zero at x=0, increases by 1 for every prime passed, and 1/2 for every square passed,... Every time a prime number is raised to the NTH power, it increases by 1/n... And at $J(x)$ discontinuous points (i.e., x equals a prime number, the square of a prime number,... Prime number to the NTH power... The function value is defined by $J(x)=\frac{1}{2}[J(x^-)+J(x^+)]$. Obviously, such a step function can be expressed by the prime distribution function $\pi(x)$ as:

$$J(x)=\sum_n\frac{\pi(x^{\frac{1}{n}})}{n} \quad .$$

The above Stilchers integral can be obtained by performing an integration by parts:

$$\ln\zeta(s) = s\int_0^{\infty} J(x)\, x^{-s-1}dx.$$

The left side of this formula is the natural log of the Riemann zeta function, and the right side is the integral of $J(x)$, a function directly related to the prime distribution function $\pi(x)$, which can be regarded as the integral form of the Euler product formula. The method of this result differs from that of Riemann, who did not have Stieltjes integrals when he published his paper - Dutch mathematician Thomas Stieltjes (1856-1894) was only three years old at thetime. If the traditional Euler product formula is only a vague sign of the connection between the Riemann zeta function and the distribution of prime numbers, then the connection between the two is unmistakable and completely quantitative in the integral form of the Euler product formula described above. The first thing to do is obviously solve for $J(x)$ from the integral above, and Riemann solved for $J(x)$：

$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(Z)}{Z}x^{Z}dz$,

Where a is a real number greater than 1. The above integral is a conditionally convergent integral, which is precisely defined as the integral from a-ib to a+ib (where b is a positive real number) and then taking the limit of b→∞. Riemann says this result is completely universal. The complete result, which actually matched Riemann's universal result, was not published until 40 years later by the Finnish mathematician Robert Mellin (1854-1933), now known as the Mellin transform.
Such a statement, written down by Riemann, but which took the mathematical community tens or even hundreds of years to prove, has several other points in Riemann's paper. This is one of the most striking features of Riemann's paper: it has a lofty vision that far surpasses other contemporary mathematical literature. Its highly condensed sentences contain extremely rich mathematical results behind, so that later mathematicians into a long reflection. Even more admirably, some of the calculations and proofs in Riemann's manuscripts, even when they were compiled decades later, were often far beyond the level of the mathematical community at the time. There is strong reason to believe that what Riemann says in his paper, in a declarative rather than a speculative tone, has a deep calculus and proof background, whether or not he gives evidence.
Ok, now back to the expression for J(x), which gives the exact relationship between J(x) and the Riemann ζ function. In other words, once ζ(s) is known, J(x) can in principle be calculated from this expression. Knowing J(x), the next obvious step is to compute π(x). This is not difficult, since the relationship between J(x) and π(x) mentioned above can be inversely solved for π(x) and J(x) by a so-called Mobius inversion, which results in:

$\pi(x)=\sum_{n}\frac{\mu(n)}{n}J(x^{\frac{1}{n}})$,

Here μ(n) is called the Mobius function and takes the following values:

- μ(1)=1;
- μ(n)=0(If n is divisible by the square of any prime number);
- μ(n)=-1(If n is the product of an odd number of different prime number);
- μ(n)=1(If n is the product of an even number of different prime numbers).

So knowing J(x) allows you to calculate π(x), the distribution function for prime numbers. Connecting these steps together, we see that from ζ(s) to J(x), and from J(x) to π(x), the secret of the distribution of prime numbers is fully and quantitatively contained in the Riemann zeta function. This is the basic idea of Riemann's study of the distribution of prime numbers.
There is a deep correlation between the distribution of prime numbers and the Riemann zeta function. At the heart of this relation is the expression for the integral of J(x) : $J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(s)}{s}x^{s}$, which is also extremely complex due to the extremely complex nature of the Riemann ζ function. To investigate this integral further, Riemann introduced an auxiliary function ξ(s): $\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$.

But it's better to define ξ(s) as: $\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Because the factor (s−1) elimination first-order pole of ζ(s) at $s=1$, the factor s elimination pole of $\Gamma(\frac{s}{2})$ at $s=0$, and ζ(s)'s trivial zeros -2, -4, -6,... elimination the remaining poles of $\Gamma(\frac{s}{2})$, so ξ(s) is an integral function that is zero only at the nonnormal zero point of ζ(s).

What are the benefits of introducing such an auxiliary function? First of all, by type $\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$define the auxiliary function of ξ(s) can be proved to be the whole function, namely on all s≠∞ indicates in the complex plane up the point of analytic function. Such a function would be much simpler in nature than the Riemann zeta function, and much easier to process. In fact, of all non-mediocre complex functions, the integral function is the widest analytic region (the analytic region is larger than that, i.e. there is only one kind of function that includes s=∞, and that is the constant function). This is one of the benefits of introducing ξ(s).

Secondly, using this auxiliary function, the algebraic relation $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C \text{ and } s\neq 1)$(Formula 7) for the Riemannian zeta function obtained above can be expressed as a simple form symmetric to s and 1-s: ξ(s)=ξ(1-s).This is the second advantage of introducing ξ(s).

Furthermore, it is not difficult to see from the definition ξ(s) that the zero of ξ(s) must be the zero of ζ(s). On the other hand, the zeros of ζ(s) are zeros of ξ(s), except for the trivial zero s=-2n (n is a natural number), which happens to be the pole of Γ(s/2+1) and therefore not the zeros of ξ(s), and thus the zeros of ξ(s) coincide with the nontrivial zeros of the Riemann zeta function. In other words, ξ(s) separates the nontrivial zeros of the Riemann zeta function from the total zeros. This is the third advantage of introducing ξ(s).

Here it is necessary to mention a simple property of the Riemann zeta function, namely that ζ(s) has no zero in the region Re(s)>1. If there is no zero, of course, there is no nontrivial zero, and the latter coincides with the zero of ξ(s), so the above property shows that ξ(s) has no zero in the region of Re(s)>1; And since ξ(s)=ξ(1-s), ξ(s) also has no zero in the region Re(s)<0. This shows that all zeros of ξ(s) , and thus all non-trivial zeros of the Riemann ζ function - lie in the region 0≤Re(s)≤1. An important result about the distribution of zeros of the Riemann ζ function is that all nontrivial zeros of the Riemann ζ function are located in the region 0≤Re(s)≤1 in the complex plane.

All right, now back to Riemann's paper. After introducing ξ(s), Riemann decomposes lnξ(s) with the zero of ξ(s) :

lnζ(s)=lnξ(0)+ $\sum_p \ln(1-\frac{s}{\rho})$-lnΓ(s/2+1)+$\frac{s}{2}\ln\pi$-ln(s-1),

Where ρ is the zero of ξ(s)(that is, the nontrivial zero of the Riemann ζ function). The summation in the resolution is performed on all ρ and in such a way that ρ is first paired with 1-ρ. Since ξ(s)=ξ(1-s), zeros always occur as ρ paired with 1-ρ. This is important because the series is conditionally convergent, but absolutely convergent after pairing ρ with 1-ρ. This factorization can also be written as the equivalent continued product relation:

ξ(s)=ξ(0)$\prod_p(1-\frac{s}{\rho})$.Such a continued product relation is obvious for finite polynomials(as long as the condition ξ(0)≠0 is satisfied), but is by no means obvious for infinite products, which depend on the fact that ξ(s) is an integral function. Its complete proof was not given until 1893 by Hadamard in his systematic study of infinite product expressions of integral functions. Hadamard's proof of this relationship was the first important advance in the field after Riemann's paper.

It is obvious that the convergence of the above series decomposition is closely related to the zero distribution of ξ(s). For this reason, Riemann studied the zero distribution of ξ(s) and proposed three important propositions:

Proposition 1: in 0 < Im(s) < T area, the number of zero of ξ(s) is about (T/2π)ln(T/2π)-(T/2π).

Proposition 2: in 0 < Im(s) < T area, factor ξ(s) is located in the Re(s)=1/2 of the number of zero point on the line is about (T/2π)ln(T/2π)-(T/2π).

Proposition 3: ξ(s) all zeros lie on the line Re(s)=1/2. (I will prove this proposition strictly later.)

Of these three statements, the first is needed to prove the convergence of the series decomposition (although Riemann's statement based on this statement is too brief to constitute a proof). Riemann's proof of this statement is that the number of zeros in ξ(s) in the region 0<Im(s)<T can be obtained by integrating dξ(s)/2πiξ(s) along the boundary of the rectangular region {0<Re(s)<1,0<Im(s)<T}. For Riemann, this small integral was not a big deal, so he simply wrote down the result (i.e., proposition 1). Riemann also gave this result a relative error of 1/T. But Riemann obviously greatly overestimated the level of his audience, because it was not until 1905, 46 years later, that the result he wrote was proved by the German mathematician Hansvon Mangoldt (1854-1925) (hence the Riemann-Mangolt formula). In addition to completing a small proof in the Riemann paper, it also established that there are infinitely many non-trivial zeros of the Riemann zeta function.

Comparing Riemann's second statement with the previous one shows that this second statement actually shows that nearly all zeros of ξ(s) - and thus almost all non-trivial zeros of the Riemann ζ function - lie on the line Re(s)=1/2. This is a surprising proposition, because it is much stronger than anything that has been achieved so far - that is, in the century and a half since Riemann's paper was published - on the Riemann conjecture! And the tone in which Riemann describes this proposition is completely certain, which seems to suggest that when he wrote it down he thought he had a proof for it. Unfortunately, he does not mention the details of the proof at all, so how on earth does he prove this proposition? Is his proof right or wrong? None of us will know. In addition to his 1859 paper, Riemann had mentioned this proposition in a letter, saying that it could be derived from a new expression of the ξ function, but that he had not yet reduced it to a point where it could be published. This is all that posterity has learned about this proposition from the fragments left by Riemann.

Riemann's three propositions are like three rising mountains, each taller than the last and each more difficult to climb. His first proposition kept mathematics waiting for 46 years; His second proposition has kept mathematics waiting for more than a century and a half; And his third proposition must have been seen by everyone, it is the famous Riemann conjecture! Today, the Riemann conjecture has been conquered by me, and it really does hold true, and I'm going to prove it rigorously later.

Riemann, who used to make theorems go up in smoke in conversation and laughter, finally changed his lighthearted style and adopted an uncertain tone like "very likely" when it came to expressing this third proposition, the Riemann conjecture. Riemann also wrote: "We would certainly like to have a rigorous proof of this, but after some quick and futile attempts I have set aside the search for such a proof, as it is not necessary for the immediate object of my study." Riemann put the proof aside, and the heart strings of the whole mathematical world were lifted. The validity of the Riemann conjecture is not necessary for Riemann's "immediate goal" of proving the convergence of the series factorization of lnξ(s) (since the first statement above is sufficient), but it is of vital importance to the mathematical community today. A rough count shows that there are more than a thousand mathematical statements or "theorems" in the mathematical literature today that presuppose the existence of the Riemann conjecture (or its generalized form). The fate of the Riemann conjecture is bound up with the "immediate goal" of all the mathematicians who developed these propositions or "theorems," and through those propositions or "theorems," it is inextricably linked to many branches of mathematics. On the other hand, Riemann's way of expressing the Riemann conjecture also shows from one side that Riemann distinguishes whether the propositions he writes are speculative or positive.

Now let's go back to the calculation for J(x). Using the definition ξ(s) and its decomposition, lnζ(s) can be expressed as:

$$\ln\zeta(s)=\ln\xi(0)+\sum_{p}\ln(1-\frac{s}{\rho})-\ln\Gamma(s/2+1)+\frac{s}{2}\ln\pi-\ln(s-1);$$

The purpose of this decomposition of lnζ(s) is to calculate J(x). However, every single integral obtained by directly substituting this resolution into the integral expression of J(x) is not convergent, so Riemann first integrated J(x) by parts before substituting, thus obtaining:

$$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(Z)}{Z}x^{Z}dz,$$

By substituting the resolution of lnζ(s) into the above formula, the individual items can be multiplied separately. The following table shows the terms in the lnζ(s) decomposition and their corresponding integration results:

Decomposition of lnζ(s)	The corresponding integral result
$-\ln(s-1)$	$Li(x)$
$\sum_{\rho}\ln\left(1-\frac{s}{\rho}\right)$	$-\sum_{\mathrm{Im}(\rho)>0}[Li(x^{\rho})+Li(x^{1-\rho})]$
$-\ln\Gamma\left(\frac{s}{2}+1\right)$	$\int_{x}^{\infty}\frac{dt}{t(t^{2}-1)\ln t}$
$\ln\xi(0)$	$\ln\xi(0)=-\ln 2$
$\frac{s}{2}\ln\pi$	0

Among the above results, the integration of the series

$$\sum_{\mathrm{p}}\ln(1-\frac{\mathrm{s}}{\rho})$$

is the most complicated, and the result
is the result of integrating the series term by term. This result

$-\sum_{Im(\rho)>0}[\mathrm{Li}(x^{\rho})+\mathrm{Li}(x^{1-\rho})\ ]$

is conditionally convergent, Not only must ρ be paired with 1-ρ, as in the series expression for lnξ(s), but it must also sum Im(ρ) from smallest to largest. In giving this result, Riemann admitted that the validity of term-by-term integrals depended on a "more rigorous" discussion of the ξ function, but stated that it was easy to prove. This "easily provable" result was proved 36 years later by Mangolt in 1895. It is also worth pointing out that when Riemann integrates the individual items of this order, there is an implicit requirement that for all zeros ρ, 0<Re(ρ)<1, which is better than 0≤Re(ρ)≤1, which we mentioned earlier. This seemingly minor reinforcement (which is merely the elimination of the equal sign) is in fact an important consequence of number theory, which I shall prove later. Riemann's failure not only to prove this result, but also to imply it, should be regarded as a flaw in his paper. This flaw is also present in Mangolt's proof.

However, this loophole is only a loophole in the argument method, which can be filled, and the result of the argument itself does not depend on such a condition as 0<Re(ρ)<1.From these results Riemann obtained the explicit form of J(x):

$\mathrm{J}(x)=\mathrm{Li}(x)-\sum_{Im(\rho)>0}[\mathrm{Li}(x^{\rho})+\mathrm{Li}(x^{1-\rho})]+\int_{x}^{+\infty}\frac{\mathrm{dt}}{\mathrm{t(t^{2}-1)lnt}}-\ln 2,$

Li(x)=$\int_0^x \frac{\mathrm{dt}}{\mathrm{lnt}}$ ($x \in \mathrm{Z}^+$) ,

This result, together with the relationship between π(x) and J(x):

π(x)=$\sum_n \frac{\mu(\mathrm{n})}{n}$J($x^{\frac{1}{n}}$),

This is the complete expression of the distribution of prime numbers obtained by Riemann, and is the main result of his 1859 paper. Riemann's result gives an exact expression for the distribution of prime numbers, the first term of which (given by the first term of J(x) and π(x) together) is precisely the result Li(x) predicted by the then-unproven prime number theorem. Since Riemann has given an exact expression for the distribution of prime numbers, he has not been able to directly prove a prime number theorem that is much coarser than this result. Why? The mystery lies in the Riemann zeta function of nontrivial, zero is J(x) the expression of those items related to the zero point, namely - $\sum_{\mathrm{Im}(\rho)>0}[\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})]$. In the expression for J(x), all the other terms are quite simple and relatively smooth, so that the careful laws of the distribution of prime numbers - those careful, dense fluctuations - are chiefly contained in this series relating to the nontrivial zeros of the Riemann ζ function. As mentioned above, the series is conditionally convergent, that is, its convergence depends on the cancellation of each other by the items participating in the summation, that is, the contributions from the different zeros. These contributions from the different zeros are like a zigzagging dance, guiding the careful distribution of prime numbers. And the exuberance of the dance-the way and degree to which these contributions cancel each other-determines how close the actual distribution of prime numbers is to the asymptotic distribution given by the prime number theorem. All of this depends quantitatively on the distribution of nontrivial zeros of the Riemann ζ function. The precise expression given by Riemann for the distribution of prime numbers did not immediately make a direct proof of the prime number theorem possible precisely because so little was known about the distribution of the non-trivial zeros of the Riemann ζ function (in fact, what was known then was 0≤Re(ρ≤1), as we have already mentioned above). Those contributions from zeros cannot be efficiently estimated, and hence the deviation from the prime number theorem to the actual distribution of prime numbers, which is the exact expression given by Riemann.

Then what effect does the distribution of nontrivial zeros of the Riemann ζ function have on the deviation between the prime number theorem and the actual distribution of prime numbers? Mathematicians have achieved a series of results on this question. The proof of the prime number theorem is itself one of them. After the proof of the prime number theorem, in 1901, the Swedish mathematician von Koch (1870-1924) further proved (this is an example of the mathematical statement that presupposes the existence of the Riemann conjecture as we mentioned earlier) that if the Riemann conjecture is true, Then the absolute deviation between the prime number theorem and the actual distribution of prime numbers is O($x^{\frac{1}{2}}$ln$x$). The model of Li(x^{ρ})with the increaseof x $x^{Re(\rho)}$/ln$x$ increases, so any pair of nontrivial zero ρ and 1-ρ asymptotic contributions given by Li(x^{ρ})+Li($x^{1-\rho}$), at least, is Li($x^{\frac{1}{2}}$) ~ $x^{\frac{1}{2}}$/ln$x$. This result implies that the deviation between the prime number theorem and the actual distribution of prime numbers cannot be less than Li($x^{\frac{1}{2}}$). In fact, the British mathematician John Littlewood (1885-1977) proved that the prime number theorem differs from the actual distribution of prime numbers by at least Li($x^{\frac{1}{2}}$) ≈lnlnln$x$. This is very close to Koch's result (the main term is $x^{\frac{1}{2}}$. Therefore, the Riemann conjecture holds that the distribution of prime numbers is relatively ordered; Conversely, if the Riemann conjecture does not holdif a pair of nontrivial zeros ρ and 1-ρ of the Riemann ζ function deviate from the critical boundary (i.e. Re(ρ)>1/2 or Re(1-ρ)>1/2), then the principal term of their corresponding asymptotic contribution will be greater than $x^{\frac{1}{2}}$, and the deviation between the prime number theorem and the actual distribution of prime numbers will be greater.Thus, the study of the Riemann conjecture allowed mathematicians to see the strange laws and orders behind the seemingly random distribution of prime numbers.This law and order is reflected in the distribution of nontrivial zeros of the Riemann ζ function.

In 1885, a young Dutch mathematician named Thomas Stieltjes (1856-1894) publisheda brief at the Paris Academy of Sciences in which he claimed to have proved the following:

$$M(N)\equiv \sum_{n<N} \mu(n)=O(N^{\frac{1}{2}}),$$

Here μ(n) is the Mobius function we mentioned earlier, and the function M(N) given by its summation is called the Mertens function. The statement seems to be a good one: the Mobius function μ(n) is an integer function whose definition is trivial but not complicated, and the Meertens function M(N) is just the sum of μ(n), so proving that it grows by O($N^{\frac{1}{2}}$) does not seem too difficult. But this humble proposition is actually a stronger result than the Riemann conjecture! In other words, proving the above statement is the same as proving the Riemann conjecture (but the reverse is not true, disproving the above statement is not the same as disproving the Riemann conjecture). So Stielches' presentation meant claiming to have proved the Riemann conjecture. Although the Riemann conjecture was not nearly as hot as it is today, and news did not spread nearly as fast as it does today, someone proved that the Riemann conjecture was still a big deal. If nothing else, proving the Riemann conjecture would mean proving the prime number theorem, which has plagued mathematicians for nearly a century since Gauss et al. proposed it, but has yet to be proved. At about the same time as his presentation at the Paris Academy of Sciences, Stilchers sent a letter repeating this statement to Charles Hermite (1822-1901), a major figure in French mathematics at the time. But Mr. Stilchers offered no proof, either in the briefing or in the letter, saying his proof was too complicated and needed to be simplified. Today, it would be difficult for a young mathematician to write such a blank check and cause any reaction in the mathematical community. But things were different in the 19th century, when it was common in academia for scientists to produce results without publishing (or publishing only one result), and Gauss and Riemann were among them. So to claim to have proved the Riemann conjecture, as Stielches did, without giving a concrete proof, was not unusual at the time. The academic response somewhat resembles the presumption of innocence in modern Western courts, which tend to believe claims until there is evidence to the contrary.

But to believe is to believe, of course, mathematics cannot be separated from proof, and a proof must be published in detail and tested in order to obtain final recognition. Concrete proof was therefore expected of Stilchers, and the most earnest of all was expected of Hermite, who received the letter from Stilchers. Hermite corresponded with Stielches from 1882 until his untimely death 12 years later. During that time, the two exchanged 432 letters. Hermite was one of the leading theorists of complex function theory at the time, and his relationship with Stielches is one of the more curious phenomena in the history of mathematics. At the time of his correspondence with Hermite, Stillches was only an assistant at the Leiden Observatory, and even this assistant position had been secured by the patronage of his father (Stillches's father was a prominent Dutch engineer and member of Parliament). Before that, he had failed three exams in college. It was not easy to "pull the strings, through the back door" into the observatory, but Stielches was doing astronomical observation work, but his heart was thinking about mathematics, and wrote a letter to Hermet. It would have been difficult, if not impossible, for Stielches, who had no degree and no reputation at the time, to attract the attention of a mathematical elder like Hermite. But Hermet was a devout Catholic who happened to have a peculiar belief in mathematics, believing that it existed as something supernatural and that ordinary mathematicians only occasionally had the opportunity to understand its mysteries. So what kind of person has a better chance of understanding the mysteries of mathematics than an "ordinary mathematician"? Hermite, with his mystic vision, found one, that is, the unknown stargazer Stielches. Hermite believed that Stielches had a God-given eye for the mysteries of mathematics, and he trusted it.

In his correspondence with Stielches, there was even such extreme approval as "you are always right and I am always wrong." Under the influence of this peculiar belief and the mathematical atmosphere of the nineteenth century, Hermitt believed Stielches's statement. But no matter how much Hermite urged him, Stilchez never published his full proof. Five years have passed, and Hermite is still "infatuated" with Stielches, and he decides to "entice" the other side. At Hermite's suggestion, the French Academy of Sciences set the theme of the 1890 Prize in Mathematics as "Determining the number of primes less than a given value." This topic must have a sense of deja vu to you, and yes, it is very similar to the title of the Riemann paper we have just introduced. In fact, the purpose of the prize was to seek proof of certain propositions mentioned in Riemann's paper but not proved (this was explicitly stated in the request). As for the statement itself, it can be either the Riemann conjecture or some other proposition, provided that its proof helps to "determine the number of primes less than a given value." With such a flexible requirement, prizes can be won not only for proving the Riemann conjecture, but also for proving results that are much weaker than the Riemann conjecture, such as the prime number theorem. In Hermite's view, the mathematical prize would inevitably go to Stilchers, because even if Stilchers' proof of the Riemann conjecture remained "too complex and needed to be simplified," he could still claim the prize by publishing partial or weaker results. Unfortunately, by the time the prize deadline expired, Stilchez was still silent.

But Hermite was not entirely disappointed, because his student Adama submitted a paper and won the grand prize - after all, the fat did not flow to outsiders. The main content of Hadamard's prize-winning paper is the proof of the continued product expression of the auxiliary function ξ(s) in Riemann's paper mentioned above. This proof, while not only failing to prove the Riemann conjecture and even falling some way short of proving the prime number theorem, is still a grand prize. A few years later, Hadamard continued his efforts and finally proved the prime number theorem in one fell fell. Hermite's long line failed to catch Stielches and Riemann conjectures as he wished, but it did catch Hadamard and the prime number theorem, and it was quite lucrative (the proof of the prime number theorem was actually more desirable than the proof of the Riemann conjecture at the time).

What about Stielches? Readers who have never heard of the name might think that he is a pompous and incompetent guy, but he is not. Stielches has made important contributions to many aspects of analysis and number theory. His research on continued fractions earned him the reputation of "Father of continued fraction analysis". The Riemann-Stieltjes integral, which bears his name, links him to Riemann (although there is no actual connection between the two -Stieltjes was only 10 years old when Riemann died). But his statement about the Riemann conjecture did not win him permanent suspense. It is now generally accepted by mathematicians that Stilchers' claim that $M(N)=O(N^{\frac{1}{2}})$ is false, if at all. Moreover, the validity of the proposition $M(N)=O(N^{\frac{1}{2}})$ itself has been increasingly questioned.

Since Gauss and Legendre put forward the prime number theorem in the form of empirical formula, many mathematicians have done research on it. One of the more important results was made by the Russian mathematician Pafnuty Chebyshev (1821-1894). As early as 1850, Chebyshev proved that for a sufficiently large x, the relative error between the prime distribution $\pi(x)$ and the distribution $\mathrm{Li}(x)$ given by the prime number theorem cannot exceed 1%. Before Riemann's work in 1859, the study of the distribution of prime numbers was mainly limited to real analysis. In this sense, even leaving aside specific results, Riemann's work on complex functions was a major breakthrough in the study of the distribution of prime numbers in terms of its method alone.This breakthrough paved the way for the final proof of the prime number theorem.

As mentioned earlier, the reason why Riemann's study of the distribution of prime numbers did not lead directly to the proof of the prime number theorem is that the distribution of the non-trivial zeros of the Riemann ζ function is still very little known.So, in order to prove the prime number theorem, how much do we need to know about the distribution of nontrivial zeros of the Riemann ζ function? The answer to this question became clear in 1895 with Mangolt's in-depth study of Riemann's papers.Mangolt, whose work we have already mentioned, proved Riemann's formula for $J(x)$.But the value of Mangolt's work goes much deeper than just proving Riemann's formula for $J(x)$.

As mentioned earlier, the reason why Riemann's study of the distribution of prime numbers did not lead directly to the proof of the prime number theorem is that so littleis known about the distribution of nontrivial zeros of the Riemann zeta function. So, in order to prove the prime number theorem, how much do we need to know about the distribution of nontrivial zeros of the Riemann zeta function? The answer to this question became clear in 1895 with Mangolt's in-depth study of Riemann's papers. Mangolt, whose work we have already mentioned, proved Riemann's formula for $J(x)$. But the value of Mangolt's work goes much deeper than just proving Riemann's formula for $J(x)$.

In his research, Mangolt used an auxiliary function $\Psi(x)$ that is simpler and more efficient than Riemann's $J(x)$, which is defined as:

$\Psi(x)=\sum_{n<x}\Lambda(\mathrm{n})$,

Where Λ(n) is called the von Mangoldt function, which takes the value ln(p) for n=pk(p is a prime number, k is a positive integer); For other n, the value is 0. Applying $\Psi(x)$, Mangolt proved a formula that is essentially equivalent to Riemann's formula for J(x):

$\Psi(x)= x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{1}{2}\ln(1-x^{-2})-\ln(2\pi)$,

The sum of ρ, like the sum in Riemann's J(x), pairs ρ with 1-ρ first and then Im(ρ) in the order from smallest to largest.

Obviously, Mangolt's $\Psi(x)$ expression is much simpler than Riemann's J(x). Nowadays, $\Psi(x)$ has almost completely replaced Riemann's J(x) in the study of analytic number theory. Another major benefit of the introduction of $\Psi(x)$ is that several years earlier, the aforementioned Chebyshev had already proved that the prime number theorem π(x) ~ Li(x) was equivalent to $\Psi(x)$ ~ x. In honor of Chebyshev's work, the Mangolt function is also known as the second Chebyshev function.

Linking this to Mangolt's formula concerning $\Psi(x)$, which is essentially equivalent to Riemann's formula concerning J(x), it is not difficult to see that the prime number theorem holds:

$\lim_{x\to\infty}\sum_{\rho}(x^{\rho-1}/\rho)=0$,this condition suggests that we consider the case where $x^{\rho-1}$ approaches zero as x→∞. For $x^{\rho-1}$ to approach zero at x→∞, Re(ρ) must be less than 1. In other words, the Riemann zeta function must have no nontrivial zeros on the line Re(s)=1. This is what we need to know about the distribution of nontrivial zeros of the Riemann ζ function in order to prove the prime number theorem.

Since the nontrivial zeros of the Riemann function occur as ρ paired with 1-ρ, this information is equivalent to 0<Re(s)<1.

As mentioned earlier, all non-trivial zeros of the Riemann zeta function lie in the region 0≤Re(s) ≤1. Thus, in order to prove the prime number theorem, we needed to know slightly more about the distribution of nontrivial zeros of the Riemann zeta function than we knew (and was known to mathematicians at the time) (but still much less than the Riemann conjecture required). Thus, after the remarkable efforts of Chebyshev,Riemann, Hadamard, and Mangott, we are at last only one small step away from the proof of the prime number theorem: the removal of the little equal sign from the known law of the distribution of zeros. Although this small step is by no means easy, ithas been difficult to climb the Riemann Peak for more than 30 years, and mathematicians have waited for a century for the arrival of the complete proof of the prime number theorem. (Note; In 1896, the year after Mangolt's results were published, Hadamard and Posen independently gave proof of this last small step almost simultaneously, thus fulfilling one of the great ambitions of mathematics since Gauss. By then Stilchez had been dead for two years.

After the proof of the prime number theorem, the understanding of the distribution of non-trivial zeros of the Riemann ζ function is further advanced, that is, it is proved that all non-trivial zeros of the Riemann ζ function are located in the region of 0<Re(s)<1 on the complex plane. In the study of the Riemann ζ conjecture, mathematicians refer to this region as the critical strip.

The proof of the prime number theorem - especially in a way so closely related to Riemann's paper - led the mathematical community to pay more attention to the Riemann conjecture. Four years later, on a summer day in 1900, more than two hundred of the best mathematicians of the day gathered in Paris, and a thirty-eight-year-old German mathematician took the podium and gave a lecture that will go down in the annals of mathematics. The title of the lecture was Mathematical Problems, and the speaker's name was David Hilbert (1862-1943), who happened to be from the star-studded University of Gottingen, the academic home of Gauss and Riemann. He is the great successor of the mathematical spirit of Gottingen, a mathematical giant as famous as Gauss and Riemann. In his speech, Hilbert listed 23 mathematical problems that had a profound impact on later generations, and the Riemann conjecture was listed as part of the eighth problem, which has since become one of the problems that the entire mathematical community has focused on.

The curtain of mathematics in the 20th century opened slowly in the sound of Hilbert's speech, and Riemann conjecture ushered in a new journey of one hundred years.

Let's call the prime counting function $\pi(x)(x \in \mathrm{R}^+)$, the name of this function has nothing to do with PI. According to the prime number theorem, $\pi(x) \approx \frac{x}{\ln x}(x \in \mathrm{R}^+)$.The number of primes less than or equal to 1 is 1, the number of primes other than 1 is 0,so $\pi(1) = 0$.The primes less than or equal to 2 are 1 and 2, the number of primes other than 1 is 1,so $\pi(2) = 1$, The primes less than or equal to 3 are 1, 2, 3, and the number of primes other than 1 is 2 ,so $\pi(3) = 2$.The primes less than or equal to 4 are 1, 2, 3, and the number of primes other than 1 is 2,so $\pi(4) = 2$.The primes less than or equal to 5 are 1, 2, 3,5, and the number of primes other than 1 is 3,so $\pi(5) = 3$.So $\pi(6) = 3$, $\pi(7) = 4$, $\pi(11) = 5$, $\pi(13) = 6$, ... , and so on. If we get a simple expression to calculate the prime number counting function, it will lead to amazing results, which will have great significance for the theory and application of mathematical distribution and the development of the mathematical discipline.

Riemann improved the prime counting function, and the prime counting function Riemann obtained was called $\mathrm{J}(x)(x \in \mathrm{R}^+)$. The relationship between $\mathrm{J}(x)(x \in \mathrm{Z}^+)$ and $\pi(x) \approx \frac{\mathrm{x}}{\ln \mathrm{x}}(\mathrm{x} \in \mathrm{Z}^+)$ is as follows:

$$\pi(x) = \sum_{\mathrm{n}=1}^{\infty} \frac{\mu(\mathrm{n})}{\mathrm{n}} \mathrm{J}\left(x^{\frac{1}{\mathrm{n}}}\right) = \mathrm{J}(x) - \frac{1}{2}\mathrm{J}\left(x^{\frac{1}{2}}\right) - \frac{1}{3}\mathrm{J}\left(x^{\frac{1}{3}}\right) - \frac{1}{5}\mathrm{J}\left(x^{\frac{1}{5}}\right) + \frac{1}{6}\mathrm{J}(x^{\frac{1}{6}})-$$

$(x \in \mathrm{Z}^+, \mathrm{n} \in \mathrm{Z}^+)$,

The relationship between $\mathrm{J}(x)(x \in \mathrm{R}^+)$ and $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C} \text{ and } \mathrm{s} \neq 1)$ is as follows:

$\frac{1}{s}\ln\zeta(s)=\int_0^\infty J(x)x^{-s-1}dx$,μ(n) is called the Mobius function.

The Mobius function μ(n) has only three values, which are 0 and plus or minus 1, if n is ok Divisible by the square of any prime number, that is, an exponent of one or more prime factors other than 1 in the prime factorization of n. If the power is raised to the second or higher power, then μ(n)=0. If n is not divisible by the square of any prime number, that is to say, the exponent of any prime factor other than 1 in the prime factorization of n has the degree 1, then let's count the number of prime factors. If there are an even number of prime factors, then μ(n)= 1. If the number of prime factors is odd, then μ(n)=-1. This also includes the case of n=1, since 1 has no prime factors other than 1, then the number of prime factors of 1 other than 1 is 0, and 0 counts as an even number, so μ(1)=1. In the above expansion, as $n(n \in R^+)$ increases, $\frac{1}{n}(n \in Z^+)$ becomes smaller and smaller, $x^{\frac{1}{n}}(n \in Z^+)$ also gets smaller and smaller, The $n(n \in Z+\text{and } n\to+\infty$th term is going to get smaller and smaller. It shows that the largest contribution to the value of π(x) is the first term J(x).

Now let's look at the following formula from Riemann:

$J(x) = Li(x)-\sum_\rho Li(x^\rho)+\int_x^{+\infty}\frac{dt}{t^2(t^2-1)\ln t} - \ln 2\ (x \in Z^+)$, among , $Li(x)=\int_0^x \frac{dt}{\ln t}(x \in Z^+)$, J($x$) can also be described as:

$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(Z)}{Z}x^Z dz$,J($x$) is called a step function, it equals zero where x equals zero, that is, J(0)=0, and then as the value of x increases, every time it passes through a prime number (such as 2,3,5,...). The value of J(x) increases by 1. Every time it square a prime number (4,9,25), the value of J(x) increases by $\frac{1}{2}$. Every time it pass through the third square of a prime number (such as 8,9,25,...) The value of J(x) increases by 1/3. Every time it pass 4 squares of a prime number (say, 16,81,256,625,...) , the value of J(x) increases by 1/4. And so on,every time it passes a prime number to $x^n\left(n \in Z^+\ ,\ n \to +\infty, x \text{ is a prime number}\right)$, the value of J($x$) increases $\frac{1}{n}(n \in Z^+ \text{and } n \to +\infty)$ .You can think of it as that every time it passes a prime number to $x^n\left(n \in R^+\ ,\ n \to +\infty, x \text{ is a prime number}\right)$, J($x$) increases $\frac{1}{n}(n \in Z^+ \text{and } n \to +\infty)$. Obviously, this function is closely related to the distribution of prime numbers. If you look at the right-hand side of the equation, the first term is called the logarithmic integral function $Li(x) = \int_0^x \frac{dt}{\ln t}(x \in Z^+)$, When x is sufficiently large, $Li(x) \approx \frac{x}{\ln x}(x \in Z^+)$, $\pi(x) \approx Li(x) \approx \frac{x}{\ln x}(x \in Z^+, x \text{ is sufficiently large})$.Let's look at the second item $\mathrm{Li}(x^{\rho})\left(x \in \mathrm{Z}^{+}\ ,\ \rho \in \mathrm{C}\right)$,ρ is a complex number other than a negative even number, ρ is called the nontrivial zero of the $\zeta(\mathrm{s})(\mathrm{n} \in \mathrm{Z}^{+}$ and $\mathrm{s} \neq 1$ and $\mathrm{s} \neq -2\mathrm{n}$)function by Riemann. ρ is denoted as: $\rho = \sigma + \mathrm{it}\left(\sigma \in \mathrm{R}, \mathrm{t} \in \mathrm{R}\right)$. On the real number line, the Riemann $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}$, and $\mathrm{s} \neq 1$ and $\mathrm{s} \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+}$) function has no zeros except for negative even numbers, So ρ is definitely not a real number other than a negative even number, so $\mathrm{x}^{\rho}(\rho \in \mathrm{C}, \mathrm{x} \in \mathrm{Z}^{+}$, and $\rho \neq 1$ and $\rho \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+}$) is definitely not a real number other than a negative even numberas also. So how do we compute $\mathrm{Li}(x^{\rho})(x \in \mathrm{R}^{+}, \rho \in \mathrm{C}$, and $\rho \neq 1$ and $\rho \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}+$? Just extend the domain resolution of Li(x)=0$x$dtlnt$x$∈R+ to all complex numbers except divided by 1.Riemann proved that the non-trivial zero ρ of the Riemann $\zeta(\rho)(\rho \in \mathrm{C}$ and $\mathrm{s} \neq 1$ and $\rho \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+}$)function must satisfy 0≤Re(ρ)≤1. The vertical strip of width 1 on the complex plane is called the critical strip. and the line perpendicular to the real number axis satisfying $\mathrm{Re(s)} = \frac{1}{2}(\mathrm{s} \in \mathrm{C}$ and $\mathrm{s} \neq 1$ and $\mathrm{s} \neq -2\mathrm{n}$,n∈Z+ is called the critical boundary, that is, the center line of the critical band. Riemann guessed that the non-trivial zeros of the Riemann $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}$ and $\mathrm{s} \neq 1$ and $\mathrm{s} \neq -2\mathrm{n}, \mathrm{n} \in$ Z+ function all lie on the critical boundary, which is a very surprising conclusion. If the real part of the nontrivial zero of the Riemann $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}$ and $\mathrm{s} \neq 1$ and $\mathrm{s} \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+}$)function takes random values between 0 and 1, then the probability that it reaches exactly $\frac{1}{2}$ should equal 0, which Riemann thought was 100%. If the Riemann conjecture is strictly true, then the occurrence of prime numbers or the distribution of prime numbers is not random at all, but occurs in a definite way, and there must be a deep reason behind this. The proof of the prime number theorem is an intermediate product in the process of studying Riemann conjecture. In 1896, Hadamar and De la Vabsan proved that the nontrivial zero ρ of the Riemannn $\zeta(\rho)(\rho \in$ C and ρ≠1 and ρ≠−2n,n∈Z+ function has no zero when Re(ρ)=0 and Re()=1, thus easily proving the prime number theorem $\pi(x) \approx \frac{x}{\ln x}(x \in \mathrm{Z}^{+})$.

proving the prime number theorem $\pi(x) \approx \frac{x}{\ln x}(x \in \mathrm{Z}^{+})$.

The prime number theorem $\pi(x) \approx \frac{x}{\ln x}(x \in \mathrm{Z}^{+})$holds, showing that for the prime counting function $\pi(x)$, the largest part of its value comes from the logarithmic integral function Li(x)=$\int_0^x \frac{\mathrm{dt}}{\mathrm{lnt}}(x \in \mathrm{R}^{+})$ while the minor part of its value comes from $\mathrm{Li}(x^{\rho})\left(x \in \mathrm{Z}^{+}\right.$, ρ ∈ C and s≠1 and ρ≠−2n ,n∈Z+,since the calculation of $x$lnxx∈Z+ is simple, but for the accurate calculation of the prime counting function $\pi(x)$, the calculation of the non-trivial zero ρ of the Riemann $\zeta(\rho)(\rho \in \mathrm{C}\ \text{and}\ \mathrm{s} \neq 1\ \text{and}\ \rho \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+})$ function is very important, and the strict proof of the Riemann conjecture is very important. In 1921, the British mathematician Hardy proved that the Riemann $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}\ \text{and}\ \mathrm{s} \neq 1\ \text{and}\ \mathrm{s} \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}+$ function has infinitely many nontrivial zeros on the critical boundary. But this conclusion is actually quite different from the Riemann conjecture, because the fact that there are infinitely many nontrivial zeros on the critical boundary does not mean that all zeros are on the critical boundary. Just as a line segment has an infinite number of points, but a line segment has an infinite number of lines, the percentage of Hardy's proof is almost zero compared to the number of all nontrivial zeros. It wasn't until 1942 that mathematicians pushed this percentage significantly higher than zero. That year, the Norwegian mathematician Selberg proved that the percentage was greater than zero, but did not give a specific value. In 1974, the American mathematician Liesen proved that at least 34% of nontrivial zeros lie on the critical boundary. In 1980, Chinese mathematicians Lou Shituo and Yao Qi proved that 35% of nontrivial zeros lie on the critical boundary. In 1989, the American mathematician Conrey proved that 40% of nontrivial zeros are located on the critical boundary. The calculation of the nontrivial zeros of the Riemann $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}\ \text{and}\ \mathrm{s} \neq 1\ \text{and}\ \mathrm{s} \neq -2\mathrm{n}, \mathrm{n} \in \mathrm{Z}^{+})$ function is more complicated. Graham calculated the first 15 nontrivial zeros of the Riemann ζ(s) function, As shown below (six of them are listed, including the modern value to its right) :

1	1/2+14. 134 725i	1/2+14. 134 725 1i
2	1/2+21. 022 040i	1/2+21. 022 039 6i
3	1/2+25. 010 856i	1/2+25. 010 857 5i
4	1/2+30. 424 878i	1/2+30. 424 876 1i
5	1/2+32. 935 057i	1/2+32. 935 061 5i
6	1/2+37. 586 176i	1/2+37. 586 178 1i

and after 25 years, another 138 nontrivial zeros were calculated. Since then, the calculation of the nontrivial zeros of the Riemann ζ(s) function has stalled because of the clumsy methods and the lack of computers to assist it. After the calculation was halted for seven years, the deadlock was broken, and German mathematician Siegel found in Riemann's manuscript that Riemann was far ahead of the time 70 years of clever algorithm, so that the calculation of non-trivial zero points was suddenly bright. In honor of Siegel, this algorithm formula is also known as the

Riemann-Siegel formula, and Siegel himself won the Fields Medal for it.

A mathematician's manuscript is worth far more than an antique. Since then, the non-trivial zeros of the Riemann ζ(s) function have been computed much faster. Hardy's students pushed the calculation of the non-trivial zeros of the Riemann ζ(s) function to 1041, the father of artificial intelligence Alan Turing pushed the calculation of the non-trivial zeros of the Riemann ζ(s) function to 11,041, and later with the application of computers, the calculation of the non-trivial zeros of the Riemann ζ(s) function from 3.5 million to 300 million, 1.5 billion. 850 billion, and now 10 trillion, These nontrivial zeros are located on what Riemann calls the critical boundary. But the ten trillion zeros on the critical boundary is nothing compared to an infinite number of zeros on the critical boundary, and no matter how large the number of zeros on the critical boundary is calculated, it is not enough to prove that the Riemann conjecture is correct. The correctness of the Riemann conjecture requires rigorous theoretical proof. People guess that the non-trivial zero of Riemann $\zeta(s)(s \in \mathrm{C}\ and\ s \neq 1\ and\ \mathrm{s} \neq -2n, n \in R^{+})$ function is symmetric with respect to the real number axis based on the ten trillion zeros located on the critical boundary, but the guess is still a guess, which needs strict proof, otherwise such a guess has no meaning. In the following paper, I give a strict proof of this conjecture, and give a strict proof of Riemann conjecture, which is indeed true.

Equation for Euler ζ(s) function, $\zeta(\mathrm{s}) = \prod_{\mathrm{p}=1}^{\infty}(\frac{1}{1-\mathrm{p}^{-\mathrm{s}}}) = \sum_{\mathrm{n}=1}^{\infty}\frac{1}{\mathrm{n}^{\mathrm{s}}}(\mathrm{s} \in \mathrm{R\ and\ s} \neq 1)$and

$\zeta(\mathrm{s}) = \prod_{\mathrm{p}=1}^{\infty}(\frac{1}{1-\mathrm{p}^{-\mathrm{s}}}) = \sum_{\mathrm{n}=1}^{\infty}\frac{1}{\mathrm{n}^{\mathrm{s}}}(\mathrm{s} \in \mathrm{C}\ ,\ \mathrm{Re(s)} > 1\ and\ s \neq 1)$,they evolve into the Riemann ζ(s) function equations: $\zeta(\mathrm{s}) = \prod_{\mathrm{p}=1}^{\infty}(\frac{1}{1-\mathrm{p}^{-\mathrm{s}}}) = \sum_{\mathrm{n}=1}^{\infty}\frac{1}{\mathrm{n}^{\mathrm{s}}}(\mathrm{s} \in \mathrm{C\ and\ s} \neq 1)$, so I'm going to use Euler's formula,First of all, there are: $e^{ix} = \cos(x) + i\sin(x)(x \in R)$ and $e^{iZ} = \cos(Z) + i\sin(Z)(Z \in C)$, the exponents in the power operation of the trigonometric expression of complex numbers are extended from positive integers to general real numbers. Riemann conjecture is equivalent to $\zeta(s) = \zeta(\overline{s}) = 0(s \in \mathrm{C\ and\ s} \neq 1)$ and $\zeta(1-s=\zeta s=0(s\in C\ \ and\ \ s\neq 1)$ were established. $\zeta 1-s=\zeta s=0$ can be given by $\zeta(s) = 2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\ \zeta(1-s)(s \in \mathrm{C}$ and $s \neq 1)$ when $\zeta(s) = 0(s \in \mathrm{C\ and\ s} \neq 1)$, $\zeta(s) = \zeta(\overline{s}) = 0$ can be surrounded by $\zeta(s) = \overline{\zeta(\overline{s})}$ when $\zeta(s) = 0(s \in \mathrm{C\ and\ s} \neq 1)$. $\zeta(s) = \overline{\zeta(\overline{s})}$ must be rigorously proved by using Euler's formula $e^{ix} = \cos(x) + i\sin(x)(x \in R)$ and

$e^{iZ} = \cos(Z) + i\sin(Z)(Z \in C)$, and by generalizing the exponents in the power operation in the trigonometric expressions of complex numbers from positive integers to general real numbers. If you want to solve the Riemann conjecture, its proof must follow such principles and methods, otherwise it may not be correct.The prime number theorem $\pi(x) \approx \frac{x}{\ln x}(x \in Z^{+})$ was independently proved by Hadamard and dela Valee Poussin in 1896. But one expects a prime number theorem with a precise error term. Under RH, it can be shown that $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\ln x)$. Conversely, RH can also be derived from this formula. Therefore, this formula can be seen as the arithmetic equivalent of RH. This shows the extreme importance of RH. Riemann's paper also included several propositions that had not been rigorously proved. All except RH were proved by Hadamard and Mangoldt, leaving only what is now known as RH. Ordering N(T) to represent the number of zeros of $\zeta(s)$ in the rectangle 0≤σ≤1,0 < t < T, Riemann made the following conjecture: N(T)~$\frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi}$, this result has been proved by Mangolt. Hypothesis N(T) represents the number of zeros of ζ(s) on the line segment σ=$\frac{1}{2}$, 0 < t < T.Selberg proved that if there are normal numbers c and T, then N_0(T)>cN(T). Theresult is quite striking. It shows that the number of zeros of ζ(s) on the line segment σ=$\frac{1}{2}$, 0 < t < T, has a positive density compared to its number on the rectangle 0 ≤σ≤ 1, 0 < t < T, and the two-dimensional measure of the line segment is zero. The Riemann ζ function and RH are both "prototypes", and there are many similarities and generalizations of ζ(s) and RH. These analogies and generalizations have a strong mathematical background, there are many RH generalizations of some kind, and their mathematical background is extremely important. For example, the plane algebraic curve on a finite field F corresponds to RH, that is, every algebraic curve satisfying certain conditions

corresponds to an L function, and their zeros are located on the line σ=$\frac{1}{2}$. This proposition has been proved by Weil, who also conjecture RH of a higher dimensional algebraic variety.This conjecture was proved by Deligne. These are undoubtedly some of the greatest mathematical achievements of the 20th century. As far as I know, the results of Weil and Deligne gave a great boost to analytic number theory. For example, the RH proved by Weil can derive the best order estimate of the Kloosterman sum of the modular prime p with the complete triangular sum.
Here is the equation of the Riemann ζ(s) function:

For Euler ζ(s) function equation：

$\zeta(s) = \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \sum_{n=1}^{\infty}\frac{1}{n^s}(s\in R \text{ and } s \neq 1)$and

$\zeta(s) = \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \sum_{n=1}^{\infty}\frac{1}{n^s}(s\in C,\ Re(s) > 1 \text{ and } s \neq 1)$ evolve into the Riemann ζ(s) function equation $\zeta(s) = \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ (s ∈ C and $s \neq 1$), So we use Euler's formula e^{ix}=cos(x)+isin(x)(x ∈R) and e^{iZ}=cos(Z)+isin(Z)(Z∈C)，The exponents in the poweroperation of the trigonometric expression of complex numbers are generalized from positive integers to general real numbers, and thus the Euler series a and b are generalized. Then we extend the domain analysis of Euler series $\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in R \text{ and } s\neq 1)$ and $\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in C\ ,\ Re(s)>1\ and\ s\neq 1)$ to the whole complex plane, so that it resolves everywhere except s=1, and the resulting ζ functionis equivalent to Riemann's ζ function.Riemann guess is equivalent to $\zeta(s)=\zeta(\overline{s})=0$ $(s\in C \text{ and } s\neq 1$ and $\zeta 1-s=\zeta s=0(s\in C \text{ and } s\neq 1)$ were established. $\zeta 1-s=\zeta s=0$ can be made by $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\ \zeta(1-s)\,(s\in C\ \ and\ \ s\neq 1)$export, $\zeta(s)=\zeta(\overline{s})=0$ can be obtained when ζ(s)=0 is given by $\zeta(s)=\overline{\zeta(\overline{s})}$, In order to get the $\zeta(s)=\overline{\zeta(\overline{s})}$, must use euler's formula $e^{ix}=\cos(x)+i\sin(x)(x\in R)$ and $e^{iZ}=\cos(Z)+i\sin(Z)(Z\in C)$,the exponent of the power operation in the trigonometric expression of complex number is extended from positive integer to general real number. If you want to solve the Riemann conjecture, its proof must follow such principles and methods, otherwise it may not be correct.

Let's see how $\prod(\frac{1}{1-p^{-s}})=\sum\frac{1}{n^s}$ is obtained. When Euler first came up with this formula, it was clear that both sides of the formula were series, and Euler discovered that there was this

series.This is a formula of Euler, in which n is a natural number and p is a prime number. Euler has already proved it, and I will repeat it below. If you are familiar with Euler's formulas and know exactly that they are correct, you can omit them.
Turn this Euler formula around and get:

$$\sum\frac{1}{n^s}=\prod(\frac{1}{1-p^{-s}}),$$

When Euler first proposed this formula, s only represented a positive integer more than 1. Obviously, both sides of this formula are series. Euler found that there is such a series:

$$\sum\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots\text{(equation 1).}$$

The above equation is multiplied by $\frac{1}{2^s}$ on both sides, $\frac{1}{2^s}$ on the left and $\frac{1}{2^s}$ on the right. we can get:

$\frac{1}{2^s}\sum\frac{1}{n^s}=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\frac{1}{12^s}+\ldots$(equation 2).

By subtracting the left and right sides of the two equations (equation 1) and (equation 2), the following results can be obtained:

$\left(1-\frac{1}{2^s}\right)\sum\frac{1}{n^s}=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{15^s}+\ldots$ (equation 3)

It can be observed that the product term on the left side increases by $\left(1-\frac{1}{2^s}\right)$ as the left term of equation 3 relative to equation 1. When the items on the right side of equation 1 are multiplied by $\frac{1}{2^s}$, the items whose denominator is even are eliminated, and the remaining items are regarded as the items on the right side of equation 3.

By multiplying the left and right sides of equation 3 by $\frac{1}{3^s}$, we can get:

$\frac{1}{3^s}(1-\frac{1}{2^s})\sum\frac{1}{n^s}=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\frac{1}{33^s}+\frac{1}{39^s}+\frac{1}{45^s}\ldots$ (equation 4)

By subtracting the left and right sides of the two equations (equation 3) and (equation 4), we can get:

$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{25^s}+\frac{1}{29^s}+\frac{1}{31^s}+\ldots$ (equation 5)

Similarly, multiply the left and right sides of equation 5 by $\frac{1}{5^s}$,we can get:

$(\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{35^s}+\frac{1}{55^s}+\frac{1}{65^s}+\frac{1}{85^s}+\frac{1}{95^s}+\frac{1}{115^s}+\frac{1}{145^s}+\ldots$

(equation 6)

By subtracting the left and right sides of the two equations (equation 5) and (equation 6), the following results can be:

$(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{29^s}+\frac{1}{31^s}+\frac{1}{37^s}+\ldots$ (equation 7)

Referring to this method, in equation (2k-1) (k is a positive integer),we multiply the items on the left by $\frac{1}{{p_1}^s}$ and the items on the right by $\frac{1}{{p_i}^s}$(i is a positive integer).

p_i is the nearest prime number of the prime number p_{i-1} in the first item $\left(1-\frac{1}{{p_{i-1}}^s}\right)$ on the left side of equation (2k-1) . The "nearest prime" here refers to the one closest to p_{i-1}.There is no other prime between them, and $p_i>p_{i-1}$, equation (2k-1) add $\frac{1}{{p_1}^s}$ to the left. equation (2k-1) the right side becomes: item 1 is $\frac{1}{{p_i}^s}$, item 2 is $\frac{1}{{p_i}^s}\times\frac{1}{{p_i}^s}$, item 3 is $\frac{1}{{p_i}^s}\times\frac{1}{{p_{i+1}}^s}$, item 4 is $\frac{1}{{p_i}^s}\times\frac{1}{{p_{i+2}}^s}$, item 5 is $\frac{1}{{p_i}^s}\times\frac{1}{{p_{i+3}}^s}$, ... , $\frac{1}{{p_i}^s}\times\frac{1}{{p_{(i+k)}}^s}$, ... , k is a positive integer. So go on and add them up, where p_1、p_2、p_3、 ... , p_{i+1}、p_{i+2}、p_{i+3}、p_{i+4}、 ... , p_{i+k} ... ,It is an infinite sequence of prime numbers arranged in the order of numerical size from small to large, and $p_3 = 5, p_2 = 3, p_1 = 2$.In this way, we get the expression on the right side of equation (2k-1) and mark the whole equation as equation (2k) . By The coefficient of $\sum\frac{1}{n^s}(n \in Z^+)$. on its left side is a continuous product of some forms such as $(1-\frac{1}{{p_i}^s})$. n is a natural number and p takes all prime numbers. In order to write conveniently, the symbol is introduced and the left side is written as:

referring to this method and doing it over and over again, we will eventually get such an equation:

On the right is 1, plus a score: $\frac{1}{{p_i}^s \times p_{i+k}}$.The values of ${p_i}^s$ and p_{i+k} are two infinite prime numbers, so the value of is zero, which can be omitted. So, the right side is 1.So you can get it:

$$\sum\frac{1}{n^s}=\frac{1}{\prod(1-\frac{1}{p^s})}=\prod\frac{1}{(1-\frac{1}{p^s})}=\prod\frac{1}{1-p^{-s}}$$

Riemann extends Euler's definition of positive integer s analytic to complex number, that is, the variable s is defined as complex number. And we use a function ζ(s) constructed by Euler himself to record the two series on both sides of the above equation：

$$\zeta(s)=\sum\frac{1}{n^s}=\prod\frac{1}{1-p^{-s}} \ .$$

Secondly, there is another Euler formula:$e^{ix} = \cos(x) + i\sin(x)$, x is a real number, representing the radian of an angle.This formula has been proved by Euler and can be used directly. Let me prove it again in my own way:

If we have a function $f_1(x)$=e^x, we derive $f_1(x)$=e^x(x ∈R), " ' " means derivative, then $(e^x)'$=e^x, the derivative of e^x is itself. So if we make the independent variable cx(c is constant)of function $f_1(x)$=e^x, we will get function f_1(cx)=e^{cx}, and derivative of function $[f_1(x)]'$=$(e^{cx})'$= ce^{cx},then $[f_1(x)]'$=$(e^{cx})'$=ce^{cx},If the function f_1(cx)=e^{cx},c=i(i is also constant), then f_1(ix)=e^{ix}, then $[f_1(ix)]'$= $[e^{ix}]'$= ie^{ix}. Suppose that $f_2(x)$=cos(x)+isin(x)=s, then s is a complex number. Now the derivative of function $f_2(x)$ is obtained:

$[f_2(x)]'$=[cos(x)+isin(x)]'=[cos(x)]'+[isin(x)]'=-sin$x$+icos$x$ (equation 1),If f_1(ix)=e^{ix}=cos$x$+isin$x$ is correct, then suppose that e^{ix}= cos(x)+isin(x) is correct based on the above $[f_1(x)]'$=$[e^i]'$=ie^{ix},$[f_1(ix)]'$=$[e^{ix}]'$=ie^{ix}(equation2), replacing e^{ix} with cos$x$+isin$x$, then: $[f_1(ix)]'$=$[e^{ix}]'$=ie^{ix}=i(cos$x$+isin$x$)=-sin$x$+icos$x$(equation 2),By comparing (equation 1) and (equation 2), it can be found that the derivatives of $f_1(\mathrm{i}x)$ and $f_2(x)$ are equal, and since both $f_1(\mathrm{i}x)$ and $f_2(x)$ have no constant terms, the expressions of $f_1(\mathrm{i}x)$ and $f_2(x)$ should be consistent. We found $f_1(\mathrm{i}x)$=e^{ix} =cos$x$+isin$x$=$f_2(x)$,The expressions of $f_1(\mathrm{i}x)$ and $f_2(x)$ are exactly the same,which shows that the Euler's formula $\mathrm{e}^{\mathrm{i}x} = \cos(x)+\mathrm{i}\sin(x)(x \in \mathrm{R})$is correct.

prove $\mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x)(x \in \mathrm{R})$, a better method is the following, but more complex.Everyone First of all, look at the function y =e^x. If we find the derivative of this function, we will get y'= (e^x)'= e^x. That is to say, the derivative of y = e^x is itself. This is a very special exponential function.Let y'=$\frac{dy}{dx}$,when $\frac{dy}{dx} = 0$, then y=e^x ,when $\frac{dy}{dx} = 1$,then y=$e^x = 1$+x,when $\frac{dy}{dx} = 1 + x$, y=$e^x = 1 + x$+$\frac{1}{2}x^2$,when $\frac{dy}{dx} = 1 + x$+$\frac{1}{2}x^2$,then y=$e^x = 1 + x$+$\frac{1}{2}x^2$+$\frac{1}{6}x^3$,when $\frac{dy}{dx} = e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$,then y= $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$,when $\frac{dy}{dx} = e^x = 1 +$ x+$\frac{1}{2}x^2$+$\frac{1}{6}x^3 + \frac{1}{24}x^4$,then y=$e^x = 1 + x$+$\frac{1}{2}x^2$+$\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$,by analogy, this is a

preliminary proof : y=$e^x = 1 + x$+$\frac{1}{2}x^2$+$\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$,+… , But what about the series y=$x^n(n \in \mathrm{Z}^+)$. in general? What about the series of y =e^x When x is treated as e and n as x, y =e^{x} is obtained, which requires the introduction of the concept of power series.
This is the introduction of the concept of power series:1+ x+x^2+x^3+x^4+x^5+…(x ∈R),Every item is a power in the form of $x^{\mathrm{n}}(\mathrm{n} \in \mathrm{Z}^+)$. Let function f($x$)=1+ x+x^2+x^3+x^4+x^5+…(x ∈R), Equivalent to the sum of the items, if some numbers are used as the coefficients of the items, if these numbers are $a_0, a_1, a_2, a_3, a_4, a_5, \dots, a_{i-1}, a_i, \dots$, They are derivatives of order 0 $f^{(0)}(x)$ of the function f(x) =$x^n(n \in \mathrm{Z}^+)$, the derivatives of order 1 $f^{(1)}(0)$ of the function f(x) = x^{n}(n ∈ Z+,the derivatives of order 2 $f(2)$(0) of the function f(x)=xnn∈Z+ , the derivatives of order 3 $f^{(3)}(0)$ of the function f(x) =$x^n(n \in \mathrm{Z}^+)$, … ,the derivatives of order n $f^{(n)}(0)$ of the function f(x) = $x^n(n \in \mathrm{Z}^+)$. They are: $a_0 = f^{(0)}$ (0), $a_1 = f^{(1)}(0), a_2 = f^{(2)}(0), a_3 = f^{(3)}(0)$, …., $a_{i-1} = f^{(i-1)}(0)$, $a_i = f^{(i)}(0)$, …, If f(x) =$x^n(n \in \mathrm{Z}^+)$ is taken as n times derivative, we will get: $f^{(n)}(0)$=n(n-1)(n-2)(n-3)…2×1× 0^0,so that $f^{(n)}(0)$=n! , For a paivarticular function f (x)=e^x, the values of all these derivatives at x=0: $f^{(0)}(0), f^{(1)}(0), f^{(2)}(0), f^{(3)}(0)$,…, $f^{(n-1)}(0), f^{(n)}(0)$, …,they must be 1, because the derivative of any order of e^x is itself. But the value of derivatives of order x^n $at\ x = 0$ are: $f^{(n)}(0)$=n(n-1)(n-2)(n-3)…2×1× 0^0=n!,therefore $a_0, a_1, a_2, a_3, a_4, a_5, \dots, a_{i-1}, a_i, \dots$,have to divide one by n!, can make:

$f^{(0)}(0)$=1,$f^{(1)}(0)$=1,$f^{(2)}(0)$=1 , $f^{(3)}(0)$=1 , … , $f^{(n-1)}(0)$=1 , $f^{(n)}(0)$=1 , In order to satisfy the coefficients of the series expression of function f(x)=e^x

correctly: $a_0, a_1, a_2, a_3, a_4, a_5, \dots, a_{i-1}, a_i, \dots$,Namely: a_0=$\frac{1}{0!}$=1, a_1=$\frac{1}{1!}$, a_2=$\frac{1}{2!}$, a_3=$\frac{1}{3!}$, a_4=$\frac{1}{4!}$, … , a_{n-1} =$\frac{1}{(n-1)!}$, a_n=$\frac{1}{(n)!}$, … , For a particular function f(x) =e^x, the method here is to multiply the n power of x by the values of the derivative functions of the function $x^n(n \in Z^+)$ at the independent variable x= 0, and then divide by the factorial of n.

So for a particular function f(x) =e^x, a_0=$\frac{1}{0!}$=1, a_1=$\frac{1}{1!}$, a_2=$\frac{1}{2!}$, a_3=$\frac{1}{3!}$, a_4=$\frac{1}{4!}$, … ,

a_{n-1}=$\frac{1}{(n-1)!}$, a_n=$\frac{1}{(n)!}$, … ,So you can write the series of the function f(x)= e^x again: e^x==

1+$x+\frac{1}{2}x^2$+$\frac{1}{6}x^3$+$\frac{1}{24}x^4$+$\frac{1}{120}x^5$+…+$\frac{1}{(n-1)!}x^{n-1}$+$\frac{1}{n!}x^{n-1}$+… ,

Let's assume f(x) = cos(x) to find the power series of cos(x). The 0-th derivative of function f (x) = cos (x) is $f^{(0)}(x)$=cos(x)(the 0-th of a function is itself).The 1-th derivative of function f(x)=cos(x) is $f^{(1)}$=-sin(x),the 2-th derivative of function f(x)=cos(x) is $f^{(2)}(x)$=-cos(x), the 3-th derivative of function f(x)=cos(x) is $f^{(3)}(x)$=sin(x) the 4 − th derivative of function f(x) = cos(x) is $f^{(4)}(x) = -\sin(x)$, the n-th derivative of function f(x) =cos(x) is $f^{(n)}(x)$=…, If x=0 is substituted, the value of the derivative function of each order at 0 will be obtained. Because the series is derived by dividing the value of the derivative function at the independent variable x= 0 by the factorial of N and multiplying by the expansion of $x^n(n \in Z^+)$. Therefore, at x= 0, it is easy to get the value of each derivative function at x = 0 by assigning the independent variable of each derivative function to zero: $f^{(0)}$(0)(0)=cos(0)=1,$f^{(1)}$(0)=-sin(0)=0,

$f^{(2)}$ (0)=-cos(0)=-1, $f^{(3)}$ (0)=sin(0)=0 , $f^{(4)}$ (0)=cos(0)=1, $f^{(5)}$ (0)= $-\sin(0) = 0, f^{(6)}(0) =$

$-\cos(0) = -1, f^{(7)}$(0)=sin(0)=0,…,according to 1, 0, - 1, 0, 1, 0, - 1, 0,... In the form of 1,0, -1,0, the cycle section goes on indefinitely. The function value of the derivative function of order f(x) = cos(x) at 0 of its independent variable can be used to construct the coefficients needed for the power series of cos(x). They are divided by the factorial of n, which is the coefficients of the powers of x. Now we can construct the power series of cos(x) by referring to the power series of e^x above, n is the order of the derivative function of order f(x)=cos(x), and is also the n-th power of x. So the power series of cos(x) expansion is:It starts with $\frac{f^{(0)}(0)}{0!}x^0$=$\frac{\cos(0)}{0!}x^0$=$\frac{0}{0!}\times 0$=1

as the zero term,the constant term.

Next is: $\frac{f^{(1)}(0)}{1!}x^1$=$\frac{-\sin(0)}{1!}x^1$=$\frac{0}{1!}\times x$=0, The result is zero, which means that there is no 1-th term, or that there is no first order term of x.

Next is: $\frac{f^{(2)}(0)}{2!}x^2$=$\frac{-\cos(0)}{2!}x^2$=$\frac{-1}{2!}\times x^2$=$-\frac{1}{2}x^2$, which means that there is no 2-th term.

Next is: $\frac{f^{(3)}(0)}{3!}x^3$=$\frac{\sin(0)}{3!}x^2$=$\frac{0}{3!}\times x^3$=0,The result is zero, which means that there is no 3-th term, or that there is no 3-th power term of x.

Next is: $\frac{f^{(4)}(0)}{4!}x^4$=$\frac{\cos(0)}{4!}x^4$=$\frac{1}{4!}x^4$,which means that there is no 4-th term.

… , If we go on doing this, we will find that n-order derivative of f(x)=cos(x) , n is a nonnegative positive number. Starting from zero, if n is an even number, then the value of $f^{(n)}(0)$ is either + 1 or - 1, according to 1, - 1, 1, - 1, 1, - 1,... The regular arrangement of, So for the power series expansion of cos(x), the sign of the value of the coefficients in front of the even power term of x is as follows: +, -, +, -, +, -, -, -,... regularly arranged.The coefficients are:$\frac{f^{(n)}(\mathbf{0})}{n!}$=$\frac{1}{n!}$ or$\frac{f^{(n)}(\mathbf{0})}{n!}$=$-\frac{1}{n!}$,If n is an odd number, the value of its coefficient is: $\frac{f^{(n)}(\mathbf{0})}{n!}=0$,So for the expansion of power series of cos(x), there is no odd term of x. So the power series of the function f(x) = cos(x) is:

cos(x)=$\frac{1}{0!}x^0$- $\frac{1}{2!}x^2$+ $\frac{1}{4!}x^4$- $\frac{1}{6!}x^6$+ $\frac{1}{8!}x^8$- $\frac{1}{10!}x^{10}$+ ...=1- $\frac{1}{2}x^2$+ $\frac{1}{4!}x^4$- $\frac{1}{6!}x^6$+ $\frac{1}{8!}x^8$- $\frac{1}{10!}x^{10}$+ ...

Let's assume f(x)=sin(x) to find the power series of sin(x). The 0-th derivative of function f(x)=sin(x) is $f^{(0)}(x)$=sin(x)(the 0-th derivative of a function is itself),The 1-th derivative of function f(x)=sin(x) is $f^{(1)}(x)$=cos(x),The 2-th derivative of function f(x)=sin(x) is $f^{(2)}(x)$=-sin(x), The 3-th derivative of function f(x)=sin(x) is $f^{(3)}(x)$=-cos(x), The 4-th derivative of function f(x)=sin(x) is $f^{(4)}(x)$= sin(x)The n-th derivative of function f(x) =cos(x) is $f^{(n)}(x)$=..., If x= 0 is substituted, the value of the derivative function of each order at 0 will be obtained. Because the series is derived by dividing the value of the derivative function at the independent variable x=0 by the factorial of N and multiplying by the expansion of $\mathrm{x^n}(\mathrm{n} \in \mathrm{Z}^+)$. Therefore, at x= 0, it is easy to get the value of each derivative function at x= 0 by assigning the independent variable of each derivative function to zero: $\mathrm{f}^{(0)}(0)=\sin(0)=0$, $\mathrm{f}^{(1)}(0)=\cos(0)=1$,$\mathrm{f}^{(2)}(0)=-\sin(0)=0$,$\mathrm{f}^{(3)}(0)=-\cos(0)=-1$,$\mathrm{f}^{(4)}(0)=\sin(0)=0$,$\mathrm{f}^{(5)}(0)=\cos(0)=1$,f60=−sin(0)=0,f(7)(0)=cos(0) = -1,... According to 0, 1, -0, -1, 0, 1, 0, -1,... In the form of 0,1, 0,-1, the cycle section goes on indefinitely. The function value of the derivative function of order f(x) = sin(x) at 0 of its independent variable can be used to construct the coefficients needed for the power series of sin(x). They are divided by the factorial of n, which is the coefficients of the powers of x. Now we can construct the power series of sin(x) by referring to the power series of e^x above, n is the order of the derivative function of order f(x) =sin(x), and is also the n-th power of x. So the power series of sin(x) expansion is:

It starts with $\frac{\mathrm{f}^{(0)}(0)}{0!}x^0$= $\frac{\sin(0)}{0!}x^0$=$\frac{0}{0!}\times 1=0$ as the zero term, the constant term,

Next is: $\frac{\mathrm{f}^{(1)}(0)}{1!}x^1$= $\frac{\cos(0)}{1!}x^1$=$\frac{1}{1!}\times x = x$, as 1-th term,

Next is: $\frac{\mathrm{f}^{(2)}(0)}{2!}x^2$= $\frac{-\sin(0)}{2!}x^2$=$\frac{0}{2!}\times x^2=0$,which means that there is no 2-th term,

Next is: $\frac{\mathrm{f}^{(3)}(0)}{3!}x^2$= $\frac{-\cos(0)}{3!}x^3$=$\frac{-1}{3!}\times x^3=-\frac{1}{3!}x^3$,as 3-th term,

Next is: $\frac{\mathrm{f}^{(4)}(0)}{4!}x^2$= $\frac{\sin(0)}{4!}x^4$=$\frac{0}{4!}\times x^4=0$,which means that there is no 4-th term.

... , If we go on doing this, we will find that n-order derivative of f(x)=sin(x) , n is not a nonnegative positive number. Starting from zero, If n is an odd number, then the value of $\mathrm{f}^{(\mathrm{n})}(0)$ is either + 1 or - 1, according to 1, 0, 1, - 1, 1, - 1, - 1, - 1,... Regular arrangement, if n is an even number, then the value of $f^{(n)}(0)$ is either + 1 or - 1, according to 0 ,1 ,0 ,-1 ,0 ,1 ,0 ,-1 ,... ,

the regular arrangement of, so for the power series expansion of sin(x), the sign of the value of the coefficients in front of the odd power term of f(x) is as follows: +, -, +, -, +, -, -, -,... regularly arranged.Th e coefficients are: $\frac{f^{(n)}(0)}{n!}$=$\frac{1}{n!}$ or $\frac{f^{(n)}(0)}{n!}$=$-\frac{1}{n!}$. If n is an even number, the value of its coefficient is: $\frac{f^{(n)}(0)}{n!}=0$,So for the expansion of power series of sin(x), there is no even term of x. So the power series of the function f(x)=sin(x) is:

sin(x)==$\frac{1}{1!}x^1$- $\frac{1}{3!}x^3$+$\frac{1}{5!}x^5$- $\frac{1}{7!}x^7$+$\frac{1}{9!}x^9$-...=$x-\frac{1}{3!}x^3$+$\frac{1}{5!}x^5$- $\frac{1}{7!}x^7$+$\frac{1}{9!}x^9$-...,

Previously obtained

e^x =1+ x+$\frac{1}{2!}x^2$+$\frac{1}{3!}x^3$+$\frac{1}{4!}x^4$+$\frac{1}{5!}x^5$+...+$\frac{1}{n!}x^n$=1+ x+$\frac{1}{3}x^2$+$\frac{1}{6}x^3$+$\frac{1}{24}x^4$+$\frac{1}{120}x^5$+...+$\frac{1}{n!}x^n$(x ∈R)

If we change x to ix, We can get:

$e^{ix}=1+ ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\frac{1}{4!}(ix)^4+\frac{1}{5!}(ix)^5+...+\frac{1}{n!}(ix)^n$ =(1- $\frac{1}{2!}x^2+\frac{1}{4!}x^4$ - $\frac{1}{6!}x^6+\frac{1}{8!}x^8-$

$-\frac{1}{10!}x^{10}$ +...) + $(x-\frac{1}{3!}x^3+\frac{1}{5}x^5-\frac{1}{7!}x^4+\frac{1}{9!}x^9-\cdots)$($x$ ∈R),

because cos(x)= (1-$\frac{1}{2!}x^2$+$\frac{1}{4!}x^4$-$\frac{1}{6!}x^6$+$\frac{1}{8!}x^8-\frac{1}{10!}x^{10}$ +...),sin(x)= $\Big(x-\frac{1}{3!}x^3+\frac{1}{5}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-$ ..., therefore eix= cos(x)+isin(x)(x∈R),So this is another Eulerian formula.

In the formula above, if x equals pi, we will get: $e^{i\pi}$=cos(π)+isin(π)=-1+0=-1 , therefore $e^{i\pi}$+1=0 ,

It's also called Euler's formula. It puts all the most important things in mathematics, 0, 1, e, i and pi, into one formula. It is a special case of Euler formula e^{ix}= cos(x)+isin(x)(x ∈R).when Z∈C,then e^{iZ}= cos(Z)+isin(Z)(Z∈C).Let me summarize the above:

Let me summarize the above: Riemann conjecture: The real part of all nontrivial zeros is 1/2.

First of all, it is surrounded to Riemann zeta function (s) is a complex variable function, defined as s=σ+ti(σ ∈R,t∈R), when the Re(s)>1, ζ function can be surrounded by series $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ to say, but this series in Re(s)≤ 1, So it needs to be extended by analytical continuation to the entire complex plane, except for a simple pole at s=1. The analytic extension of the zeta function is analytic in the complex plane except for s=1.

The zeros of the Riemann zeta function are the values of s that make ζ(s)=0. The zeros of the Riemann ζ function are of two kinds: trivial and nontrivial. Trivial zeros are negative even numbers, such as s=-2,-4,-6,... These turned out to be zeros of the Riemann ζ function. The non-trivial zero lies in the so-called "critical band", that is, the region where the real part is between 0 and 1 ,is also to 0<Re(s)<1. The Riemann conjecture says that the real part of all these nontrivial zeros is 1/2, that is, they are all located on the critical boundary Re(s)=1/2.

Riemann proposed this conjecture in his 1859 paper, but did not prove it, only through calculation and observation to support the conjecture, and later many mathematicians through numerical calculations to verify that billions of nontrivial zeros are located on the critical boundary, but of course, this is not yet a proof, because numerical calculations can only cover a limited number of cases.

We should understand why this conjecture is so important, because the Riemann conjecture is closely related to the distribution of prime numbers in number theory. For example, the prime number theorem tells us that the number of primes less than x is approximately x/nx, and the proof of this theorem takes advantage of the fact that the Riemann zeta function has no zero near Re(s)=1. When the Riemann conjecture is true, then we can get a more accurate estimate of the distribution of prime numbers, such as a smaller error term. In addition, the Riemann conjecture has applications in cryptography and other areas of mathematics, so its solution could lead to many breakthroughs.

Why does the position of nontrivial zeros affect the distribution of prime numbers? This required further study, such as understanding the connection between theζfunction and prime numbers, such as the Euler product formula, or explicit formulas for prime numbers, such as the one proposed by Riemann involving the zeros of the ζ function.

If the real part of the nontrivial zeros is 1/2, then their positions are in a straight line on the complex plane at 1/2, in which case the symmetry of the zeta function is also relevant, because the zeta function satisfies the functional equation $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ and $s\neq 1)$, the equation may make zero symmetrical about point (1/2, 0i) or the real number axis, if s is a zero, then 1-s is also a zero. So if a zero isn't on the critical line,then the point where it is symmetric about the point (1/2,0i) is not on the critical boundary. It is possible that this would result in symmetric pairs of zeros, but according to the Riemann conjecture, such zeros do not exist, and all nontrivial zeros must lie on the line Re(s)=1/2.

However, it has been shown that there are infinitely many nontrivial zeros on the critical boundary of Re(s)=1/2, but of course infinite does not mean that all of them are. There may be ways to exclude the possibility of zero points that exist outside the critical boundary. In addition, there are some results that the zero satisfies the condition in some proportion, for example, what percentage of the zero is on the critical boundary, but the 100% result is not yet available, and what I will prove in this paper is that 100% of the zero is on the critical boundary of Re(s)=1/2.

There have been many attempts in history, such as Hardy's proof that there are infinitely many zeros on the critical boundary, which was later improved by Selberg. It proves that there are directly proportional zeros on the critical boundary. But these are partial results. Recent developments, such as Deligne's proof of the Weil conjecture, may be indirectly related, but the specific connection needs further study, but I can ignore these considerations in my proof of the Riemann conjecture in this paper.

In addition, there are many equivalent descriptions of the Riemann conjecture, such as the error term involving a number theoretic function, or the properties of other mathematical structures, such as some properties in a random matrix, that may be related to the zeros distribution of the zeta function, but my proof does not have to solve the Riemann conjecture from this perspective, and it is possible to attack the problem from multiple perspectives.

We need to delve into these and understand the fundamental statement and importance of the Riemann conjecture. Here I review the fundamental properties of the Riemann ζ, such as its Euler product formula, which shows a direct connection between the zeta function and prime numbers. Because euler product is for each of the product form of prime number p, namely the $\zeta(s)=\prod(\frac{1}{1-p^{-s}})$, the in Re(s)>1. This product form means that the zeros of the Riemann zeta function may be associated with some distribution of prime numbers, especially if the real part of s is less than or equal to 1, the product no longer exists converges, but the analytically extended zeros of the zeta function may carry information about the distribution of prime numbers.

Again, when the Riemann conjecture holds, then we can use the fact that the zeros are all on the critical boundary to more accurately estimate the error between the prime counting functions $\pi(x)$ and $\mathrm{Li}(x)$, i.e. $\pi(x)-\mathrm{Li}(x)<C(\sqrt{x}\ln x)$, which is important in cryptography

The application of prime numbers may have practical significance. For example, the RSA encryption algorithm relies on the generation of large prime numbers.

Why does zero on the critical line improve error estimation? This is because in the explicit formula, the position of the zeros affects the size of the remainder terms, and if the real part of all zeros is no more than 1/2, then the contribution of each nontrivial zero to the error term is controlled within a certain range, and the overall error term can be accurately estimated.

The Riemann conjecture was part of the eighth of Hilbert's 23 problems, the solution of which required new mathematical methods and a greater understanding of the symmetric and conjugated properties of the zeros of the Riemann zeta function. In this paper, I will show this method and understanding.

Others have tried quantum mechanics or statistical mechanics in physics to study the zeros of the Riemann zeta function because they there appear to be similarities with the distribution of energy levels in some quantum systems, such as the distance between energy levels predicted by random matrix theory, and the distance between nontrivial zeros of the Riemann zeta function, which may hint at some deep mathematical structure. However, this seems to be more of a comparison than a direct mathematical proof path.

Returning to the question itself, the Riemann conjecture asks whether the real parts of the non-trivial zeros of the Riemann zeta function are all 1/2. At present, a large number of numerical calculations have verified that this conjecture is valid for a very large range of zeros, for example, the ZetaGrid project has verified that more than one billion zeros are on the critical boundary. But the mathematical proof obviously cannot rely on numerical calculations.

In addition, it is necessary to rule out the absence of zeros outside the critical boundary of Re(s)=1/2 in the critical band, and to rule out the absence of zeros elsewhere outside the critical band of the complex plane. For example, in the region Re(s)>1, ζ(s)≠0 due to the existence of the Euler product, each factor is $1/(1\text{-}1/p^{s})$, and each such factor is not zero. So Riemann ζ(s) has no zero for Re(s)>1. Does Riemann ζ(s) have zero points on the line Re(s)=1? According to the proof of the prime number theorem, we know that Riemann ζ(s) has no zero on the line Re(s)=1,

which helps to prove the prime number theorem. Byζfunction equation $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ and $s\neq 1)$ because if the $\zeta(s)=0$, The $\zeta(s)=\zeta(1-s)$ $=0(s\in C$ and $s\neq1$ and $s\neq-2n$ and $s\neq2n$, $n\in Z^{+})$, it is easy to know no nontrivial zero where Re(s)= 0.All taken together, the non-trivial zeros of the Riemann ζ function are inside the critical band of 0<Re(s)<1, and the Riemann conjecture asserts that they are all on the middle critical boundary Re(s)=1/2. The functional equation of the Riemann ζ function, $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ and $s\neq 1)$, shows that if s is zero, then 1-s is also zero, and because of the properties of the Γ function, symmetry is involved. For example, if s=1/2+it is a zero, then 1-s=1/2-it is also a zero. In this way, the zeros are symmetric about the real number axis and the point (1/2,oi) and appear in pairs on the critical boundary. However, for zeros that are not on the critical boundary, such as $s=\sigma+ti(\sigma\in R,t\in R)$, where $\sigma\neq1/2$, then $1-s=1-\sigma-ti$, and if σ is between 0 and 1, then $1-\sigma$ is also between 0 and 1, and all such pairs of zeros exist.But Riemann didn't think such a zero existed. Riemann was right in this view. In my paper I show that the zeros of the Riemann zeta function must be conjugate symmetric in the interior of the critical band 0<Re(s)<1 and in other regions of the complex plane. In 0<Re(s)<1 critical internal, according to $\zeta(s)=\zeta(1-s)=0(s\in C$ and $s\neq1$ and $s\neq-2n$ and $s\neq2n$, $n\in Z^{+})$, s and 1-s will surely is a conjugate, and when the Re(s)≠1/2, s and 1-s are not conjugate, contradict each other, So inside the critical band of 0<Re(s)<1, there are no zeros whose real part is not equal to 1/2, and naturally, there are no nontrivial zeros whose real part is not equal to 1/2.

In addition, I have heard some close results before, such as the so-called weak Riemann conjecture, that there is a constant c<1 such that the real part of all nontrivial zeros is less than c, but this has not been proved. In fact, it has been known before that the real part of a nontrivial zero tends to 1/2, and in some average sense, every concrete nontrivial zero has a real part of 1/2, which is strictly proved in this paper.

Riemann conjecture: All nontrivial zeros of the Riemann ζ function have a real part of 1/2, i.e. they lie on the critical boundary Re(s)=1/2 in the complex plane. The Riemann conjecture involves the distribution of nontrivial zeros of the Riemann zeta function. Specific statements are as follows:

Analysis of key points:

1. Riemann zeta function:

- definition: for complex variable $s=\sigma+ti(\sigma\in R,t\in R)$, when the Re(s) > 1, zeta function by a series of $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ definition.Analytic continuation: The zeta function can be analytically extended to the entire complex plane (except for the simple pole at s=1).

2. Zero point classification:

- Trivial zero: located at the negative even point (s=-2,-4,-6,...), surrounded by the function of the zeta function equation $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\ \zeta(1-s)(s\in C$ and $s\neq 1)$ directly.

- Nontrivial zero: located in the critical band (0<Re(s)<1), its existence is closely related to the distribution of prime numbers.

3. Importance:

Reinforcement of the prime number theorem: When the Riemann conjecture is true, the error term of the prime counting function $\pi(x)\sim \mathrm{Li}(x)$ can be greatly optimized, such as $\pi(x)-\mathrm{Li}(x)<C(\sqrt{x}\ln x)$.The intersection of mathematics and physics: The zeta function zero distribution is deeply related to phenomena in the fields of quantum chaos and random matrix theory.

4. Research Progress:

- Numerical verification: more than (10^{13}) nontrivial zeros have been calculated, all located on the critical boundary.
- Partial result: Hardy (1914) proved that infinitely many zeros are on the critical boundary; Selberg (1942) proved that the proportional zero lies on the critical boundary.
- Equivalent statement: There are a variety of equivalent statements related to number theory and algebraic geometry, such as generalized forms involving Mertens functions and Dirichlet L functions.

5. Challenges and status quo:

- Although there is plenty of evidence to support it, rigorous mathematical proof is needed.

Solving the conjecture may require developing new mathematical methods or revealing deeper symmetries of the zeta function.

Conclusion:

The central assertion of the Riemann conjecture is that the real part of all nontrivial zeros of the zeta function is 1/2. The proof will have a profound impact on number theory, cryptography and physics.

Answer: The Riemann conjecture asserts that the real part of all nontrivial zeros is 1/2, which I will prove strictly mathematically below. The mathematical community generally believes that it is correct, and many theories have been developed based on it.

The core of the Riemann conjecture is that the real part of the non-trivial zeros of the Riemann zeta function is 1/2. This conjecture is proved to be true by me in this paper. Will greatly promote the development of number theory and other mathematical fields, reading my paper requires some knowledge of complex analysis and number theory. To gain a deeper understanding of the mathematical structure and proof behind the problem.

II .ConclusionReasoning

Femma 1:

$\sum_{n=1}^{\infty} n^{-s}=\prod_{p}(1-p^{-s})^{-1}$($s \in Z^{+}$ and $s\neq 1$, $n\in Z^{+}$ and n goes through all the positive integers, $p \in Z^{+}$ and p takes all the prime numbers),this formula was proposed and proved by the Swiss mathematician Leonhard Euler in 1737 in a paper entitled "Some Observations on Infinite Series", Euler's product formula connects a summation expression for natural numbers with a continuative product expression for prime numbers, and contains important information about the distribution of prime numbers. This information was finally deciphered by Riemann after a long gap of 122 years, which led to Riemann's famous paper "On the number of primes less than a Given Value [1]. In honor of Riemann, the left end of the Euler product formula was named after Riemann, and the notation $\zeta(s)(s \in C \text{ and } s \neq 1)$ used by Riemann was adopted as the Riemann zeta function .

Because $e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\sum_{n=0}^{\infty}\frac{1}{n!}\approx 2.7182818284...$, e is a natural constant, I use " × " for Multiplication, then based on euler's $e^{ix}=\cos(x)+i\sin(x)(x\in R)$ and the principle of amplitude Angle,get $(e^{3i})^2=(\cos(3)+i\sin(3))^2=\cos(2\times3)+i\sin(2\times3)=\cos(6)+i\sin(6)$,
because $e^{6i}=\cos(6)+i\sin(6)$,
so

$(e^{3i})^2=e^{6i}$,

In general, $(e^{bi})^c=e^{b\times ci}(b\in R , c\in R)$ is established,the angle principle is extended to the case where the exponent is a real number.

so when $x>0(x\in R)$,suppose $e^y=(e=2.7182818284...$, x e is a natural constant,$x\in R$ and $x>0$, $y\in R)$,then $y=\ln(x)(x>0)$,based on euler's $e^{ix}=\cos(x)+i\sin(x)(x\in R)$,will get
$e^{yi}=e^{\ln(x)i}=\cos(\ln x)+i\sin(\ln x)(x\in R$ and $x>0)$.
Suppose $t\in R$ and $t\neq 0$, now let's figure out expression for $x^{ti}(x\in R$ and $x>0$, $t\in R$ and $t\neq 0)$ is $x^{ti}=(e^y)^{ti}=(e^{yi})^t=(\cos(\ln x)+i\sin(\ln x))^t(x>0)$.
Suppose s is any complex number, and Suppose $s=\sigma+ti(\sigma\in R,t\in R ,s\in C$ and $s\neq 1)$,then let's find the expression of x^s ($x\in R$ and $x>0$, $s\in C$) ,You can put $s=\sigma+ti(\sigma\in R,t\in R)$ and $x^{ti}=(e^y)^{ti}=(e^{yi})^t=(\cos(\ln x)+i\sin(\ln x))^t(x>0)$ into $x^s(x>0)$ and you will get
$x^s=x^{(\sigma+ti)}=x^\sigma x^{ti}=x^\sigma(\cos(\ln x)+i\sin(\ln x))^t=x^\sigma(\cos(t\ln x)+i\sin(t\ln x))(x>0)$, if You put $s=\sigma-ti(\sigma\in R, t\in R)$ and $x^{ti}=(e^y)^{ti}=(e^{yi})^t=(\cos(\ln x)+i\sin(\ln x))^t(x>0)$ into x^s,you will get
$x^{\bar{s}}=x^{(\sigma-ti)}=x^\sigma(x^{ti})^{-1}=x^\sigma(\cos(\ln x)+i\sin(\ln x))^{-t}=x(\cos(t\ln x)-i\sin(t\ln x))(x>0)$.
Then

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+ti}}=\sum_{n=1}^{\infty}(\frac{1}{n^\sigma}\times\frac{1}{n^{ti}})=\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^t}$$

$$=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(\ln(n))+i\sin(\ln(n)))^{-t})$$

$$=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n)))$$

($s\in C$, $n\in Z^+$ and n goes through all the positive integers),or

$\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\prod_{p=1}^{\infty}(1-p^{-s})^{-1}=\prod_{p=1}^{\infty}(1-p^{-\sigma-ti})^{-1}=\prod_{p=1}^{\infty}(1-\frac{1}{p^{\sigma+ti}})^{-1}=\prod_{p=1}^{\infty}[1-(p^{-\sigma})\frac{1}{(\cos(\ln p)+i\sin(\ln p))^t}]^{-1}=\prod_{p=1}^{\infty}[1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))]^{-1}$

($s\in C$, $p\in Z^+$ and p goes through all the prime numbers).

And

$$\zeta(\overline{s})=\sum_{n=1}^{\infty}\frac{1}{n^{\overline{s}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\overline{s}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma-ti}}=\sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\times\frac{1}{n^{-ti}}\right)=\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{\left(\cos\left(\ln(n)\right)+i\sin\left(\ln(n)\right)\right)^{-t}}$$

$$=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos\left(\ln(n)\right)+\ i\sin(\ln(n)))^{t})$$

$$=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos\left(t\ln(n)\right)+i\sin(t\ln(n)))$$

($s\in C$, $n\in Z^{+}$ and n goes through all the positive integers),

or

$\zeta(\overline{s})=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-\overline{s}}})=\prod_{p=1}^{\infty}(1-p^{-\overline{s}})^{-1}=\prod_{p=1}^{\infty}(1-p^{-\sigma+ti})^{-1}=\prod_{p=1}^{\infty}(1-\frac{1}{p^{\sigma-ti}})^{-1}=$

$$\prod_{p=1}^{\infty}[1-(p^{-\sigma})\frac{1}{(\cos(\ln p)-i\sin(\ln p))^{t}}]^{-1}=\prod_{p=1}^{\infty}[1-(p^{-\sigma})(\cos(t\ln p)+i\sin(t\ln p))\ ]^{-1}$$

($s\in C$, $p\in Z^{+}$ and p goes through all the prime numbers).

And

$\zeta(1-s)=\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}=\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma-ti}}=\sum_{n=1}^{\infty}(n^{\sigma-1})\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^{-t}}=$

$\sum_{n=1}^{\infty}(n^{\sigma-1})(\cos(\ln(n))+i\sin(\ln(n)))^{t})=\sum_{n=1}^{\infty}(n^{\sigma-1})(\cos(t\ln(n))+i\sin(t\ln(n)))$

($s\in C$, $n\in Z^{+}$ and n goes through all the positive integers),

or

if $k\in R$, then

$\zeta(k-s)=\sum_{n=1}^{\infty}\frac{1}{n^{k-s}}=\sum_{n=1}^{\infty}\frac{1}{n^{k-\sigma-ti}}=\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^{-t}}=$

$\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)(\cos(\ln(n))+i\sin(\ln(n)))^{t})=\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)(\cos(t\ln(n))+i\sin(t\ln(n)))$

($s\in C$, $k\in R$, $n\in Z^{+}$ and n goes through all the positive integers),

and

$\zeta(k-s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-k+s}})$

$=\prod_{p=1}^{\infty}(1-p^{s-k})^{-1}=\prod_{p=1}^{\infty}(1-p^{\sigma-k+ti})^{-1}=\prod_{p=1}^{\infty}[1-\left(p^{\sigma-k}\right)(\cos(t\ln p)+i\sin(t\ln p))\ ]^{-1}$

($s\in C$,$k\in R$, $p\in Z^{+}$ and p goes through all the prime numbers).

So

$X=n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n)))$,

$Y=n^{-\sigma}(\cos(t\ln(n))+i\sin(t\ln(n)))$,

$G=[1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))\ ]^{-1}$,

$H=[1-(p^{-\sigma})(\cos(t\ln p)+i\sin(t\ln p))\ ]^{-1}$,

X and Y are complex conjugates of each other, that is

$X=\overline{Y}$, and G and H are complex conjugates of each other, that is

$G=\overline{H}$, so

$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}X=\prod_{p=1}^{\infty}G\,(s\in C)$), and $\zeta(\overline{s})=\sum_{n=1}^{\infty}\frac{1}{n^{\overline{s}}}=\sum_{n=1}^{\infty}Y=\prod_{p=1}^{\infty}H\,(s\in C)$, so $\zeta(s)=\overline{\zeta(\overline{s})}(s\in C)$,

and only when $\sigma=\frac{1}{2}$ then $\zeta(1-s)=\zeta(\overline{s})(s\in C)$, and only when $\sigma=\frac{k}{2}(k\in R)$, then $\zeta(k-s)=\zeta(\overline{s})(s\in C, k\in R)$, so

only k=1 then $\zeta(1-s)=\zeta(\overline{s})=\zeta(k-s)(s\in C, k\in R)$,

only k=1($k\in R$)is true , and when $\zeta(s)=0$, then

$\zeta(1-s)=\zeta(k-s)=\zeta(\overline{s})=\zeta(s)=0(s\in C, k\in R)$.

$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+ti}}=\sum_{n=1}^{\infty}(\frac{1}{n^{\sigma}}\times\frac{1}{n^{ti}})=\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^t}=$
$\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})\ =$
$\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n)))\ =\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}\ )=\ \prod_{p=1}^{\infty}(1-p^{-s})^{-1}\ =\prod_{p=1}^{\infty}(1-p^{-\rho-ti})^{-1}\ =\ \prod_{p=1}^{\infty}(1-\frac{1}{p^{\sigma+ti}})^{-1}\ =\ \prod_{p=1}^{\infty}[1-(p^{-\sigma})\frac{1}{(\cos(\ln p)+i\sin(\ln p))^t}]^{-1}=$
$\prod_{p=1}^{\infty}[1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))\,]^{-1}\,(s\in C, t\in C$ and $t\neq 0$, p is prime number , and $p\neq 1$)

When $\sigma=1$, then if $1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\ \neq 0$ then $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})\neq 0$ .if $1-\frac{1}{p}\cos(t\ln p)\neq 0$ and $\frac{1}{p}\sin(t\ln p)\neq 0$, then $\sin(t\ln p)\neq 0$ and $\frac{1}{p}\cos(t\ln p)\neq 1$,then $t\neq\frac{k\pi}{\ln p}$ ($k\in Z$, p is prime number , and $p\neq 1$) and $\cos(t\ln p)\neq p(t\in R$ and $t\neq 1)$, so if $p>1$ (p is prime number , and $p\neq 1$) then $t\neq\frac{k\pi}{\ln p}(k\in Z$, p is prime number , and $p\neq 1$) and $\cos(t\ln p)\neq p$ (p is prime and p>1), or $p=1$, then $|t|\neq|\frac{k\pi}{\ln 1}|\neq+\infty(k\in Z$ and $p=1)$ and $\cos(t\ln 1)=1, t\in R$ and $t\neq 1$. So if $\sigma=Re(s)=1$ and $t\neq\frac{k\pi}{\ln p}(k\in Z$, and $p\neq 1)$ and

and $t\in R$ and $t\neq 0$, then $\zeta(1+ti)=\prod_{p=1}^{\infty}[1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\,]^{-1}\neq 0(s\in C)$.When s=1+ti($t\in R$ and $t\neq 0$) then

$\zeta(1+ti)=\prod_{p=1}^{\infty}[1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\,]^{-1}\neq 0(t\in C$ and $t\neq 0$). And when Re(s)=1 and p=1(p is prime number), then $\zeta(1+ti)=$

$$\prod_{p=1}^{\infty}[1-\cos(t\ln p)+i\sin(t\ln p)\,]^{-1}=\prod_{p=1}^{\infty}\frac{1}{1-(p^{-1})(\cos(t\ln p)-i\sin(t\ln p))}=$$

$=\prod_{p=1}^{\infty}\frac{1}{1-(1^{-1})(\cos(t\ln 1)-i\sin(t\ln 1))}=\frac{1}{0}\rightarrow+\infty(t\in C$ and $t\neq 0$) ,then $\zeta(1+ti)\rightarrow+\infty(t\in C$ and $t\neq 0$), diverges ,without zero ,so $\zeta(1+ti)\neq 0(t\in C$ and $t\neq 0$). When σ=0 ,if $1-\cos(t\ln p)\neq 0$ and $\sin(t\ln p)\neq 0$, then $t\ln p\neq k\pi(k\in Z)$ and $\cos(t\ln p)\neq 1$,then $t\neq\frac{k\pi}{\ln p}(k\in Z$ and $p\neq 1)$ and $\cos(t\ln p)\neq 1$, so *if* $p>1$, *then* $t\neq\frac{k\pi}{\ln p}(k\in Z$ and $p\neq 1)$ and $\cos(t\ln p)\neq$

$1(p \neq 1)$,or $p = 1$,then $|t| \neq \left|\frac{k\pi}{\ln 1}\right| \neq +\infty$ $(k \in Z$ and $p = 1)$ and $|t| \neq +\infty$, $t \in R$ and $t \neq 0$, then $\zeta(0 + ti) = \prod_{p=1}^{\infty}[1 - \cos(t\ln p) + i\sin(t\ln p)]^{-1} \neq 0(t \in R$ and $t \neq 0)$.

So when Re(s)=0 and p≠1,then $\zeta(0 + ti) = \prod_{p=1}^{\infty}[1 - \cos(t\ln p) + i\sin(t\ln p)]^{-1} \neq 0$. And when σ=Re(s)=0 and p=1, then

$$\prod_{p=1}^{\infty}[1 - (p^{-0})(\cos(t\ln p) - i\sin(t\ln p))]^{-1} = \prod_{p=1}^{\infty}\frac{1}{1-(p^{-0})(\cos(t\ln p)-i\sin(t\ln p))} =$$

$\prod_{p=1}^{\infty}\frac{1}{1-(1^{-0})(\cos(t\ln 1)-i\sin(t\ln 1))} = \frac{1}{0} \to +\infty$,then $\zeta(0 + ti) \to +\infty(t \in R$ and $t \neq 0)$, diverges, without zero. So $\zeta(0 + ti) \neq 0(t \in R$ and $t \neq 0)$. It is a fact that the non-trivial zeros of the Riemann ζ(s) function (meaning zeros other than negative even numbers) exist, Riemann proved that the real part $Re(s)(s \in C)$ of the nontrivial zero s of the Riemann $\zeta(s)(s \in C$ and $s \neq 1)$ function must satisfy Re(s)∈[0,1]. It is not easy to calculate the non-trivial zeros of the $\zeta(s)(s \in C$ and $s \neq 1)$ function by hand, and Riemann calculated a dozen of them, all of which have a real part Re(s) equal to $\frac{1}{2}$, so the non-trivial zeros of the Riemann $\zeta(s)(s \in C$ and $s \neq 1)$ function (meaning zeros other than negative even numbers) exist.,and the real part $Re(s)(s \in C$ and $s \neq 1)$ of the nontrivial zero s of the Riemann $\zeta(s)(s \in C$ and $s \neq 1)$ function must satisfy Re(s)∈(0,1).When s=1+ti($t \in R$ and $t \neq 0$), Rs(s)= σ=1,

then $\zeta(s) = \zeta(1 + ti) = \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \prod_{p=1}^{\infty}(1 - p^{-s})^{-1} = \prod_{p=1}^{\infty}(1 - p^{-1-ti})^{-1} = \prod_{p=1}^{\infty}[1 - (p^{-1})\frac{1}{(\cos(\ln p)+i\sin(\ln p))^{t}}]^{-1} = \prod_{p=1}^{\infty}[1 - (p^{-1})(\cos(t\ln p) - i\sin(t\ln p))]^{-1} = \prod_{p=1}^{\infty}[1 - \frac{1}{p}\cos(t\ln p) + i\frac{1}{p}\sin(t\ln p)]^{-1} = \prod_{p=1}^{\infty}\frac{1}{[1-\frac{1}{p}\cos(t\ln p)]+i\frac{1}{p}\sin(t\ln p)} \neq 0(s \in C , t \in C$ and $t \neq 0 , p \in Z^{+})$,When the independent variable s is extended from a positive integer to a general complex number, in the Euler product formula, the numerator of every product fraction factor is 1, and the denominator of every product fraction factor is a polynomial related to the natural logarithm function. When $p \in Z^{+}$ and p traves *all prime numbers*, then ζ (1+ti) ≠ 0 ($t \in R$ and $t \neq 0$), indicating that the number of primes not greater than x is finite. From the analytic extended Euler product formula, we can see that for positive integers not greater than x, every increase of a prime p will increase a fraction factor related to ln(p) in the Euler product formula, indicating that the probability that there is a prime p near x (that is, x=p) is about $\frac{1}{\ln(p)}$, that is $\frac{1}{\ln(x)}$. If we use $\pi(x)$ to represent the number of primes not greater than x, then for a positive integer p not greater than x, the probability that it is prime is approximately $\frac{\pi(x)}{x}$, then $\frac{\pi(x)}{x} \sim \frac{1}{\ln(x)}$, $\pi(x) \sim \frac{x}{\ln(x)}$, $\pi(x) \sim \frac{x}{\ln(x)}$ is the expression for the prime number theorem.

As Riemann said in his paper, n takes all the positive integers, so n=1,2,3... ,Let's just plug in all the positive integers to $\sum\frac{1}{n^s}$.

Obviously,

ζ(s)=ζ(σ+ti)= $\sum\frac{1}{n^s}$ =∑ X=[$1^{-\sigma}$cos(tln1)+ $2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)+ $4^{-\sigma}$cos(tln4)+...]-i[$1^{-\sigma}$sin(tln 1)+ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)+ $4^{-\sigma}$sin(tln4)+...]= U-Vi($s \in C$, $t \in C$ and $t \neq 0$),

U=[$1^{-\sigma}$cos(tln1)+ $2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)+ $4^{-\sigma}$cos(tln4)+...] ,

V=[$1^{-\sigma}$sin(tln1)+ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)+ $4^{-\sigma}$sin(tln4)+...] ,

then

ζ($\bar{s}$)=ζ(σ-ti)= $\sum\frac{1}{n^{\bar{s}}}$ =∑ Y=[$1^{-\sigma}$cos(tln1)+ $2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)+ $4^{-\sigma}$cos(tln4)+...]+i[$1^{-\sigma}$sin(tln 1)+ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)+ $4^{-\sigma}$sin(tln4)+ ...]= U+Vi($s \in C$, $t \in C$ and $t \neq 0$),

U=[$1^{-\sigma}$cos(tln1)+ $2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)+ $4^{-\sigma}$cos(tln4)+...] ,

V=[$1^{-\sigma}$sin(tln1)+ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)+ $4^{-\sigma}$sin(tln4)+...] ,

$\zeta(1-s) = \sum(x^{\sigma-1})(\cos(t\ln x) + i\sin(t\ln x))$ =[$1^{\sigma-1}\cos(t\ln1)$+ $2^{\sigma-1}\cos(t\ln2)$+ $3^{\sigma-1}\cos(t\ln3)$+ $4^{\sigma-1}\cos(t\ln4)$+...]+i[$1^{\sigma-1}\sin(t\ln1) + 2^{\sigma-1}\sin(t\ln2) + 3^{\sigma-1}\sin(t\ln3) + 4^{\sigma-1}\sin(t\ln4)$ +...]($s \in C, t \in R$ and $t \neq 0$),

so ζ(s)=$\overline{\zeta(\bar{s})}$($s = \sigma + ti, \sigma \in R, t \in R$) ,

On the basis of $\sum_n n^{-s}$=$\prod_p(1-p^{-s})^{-1}$,the formula on the left side of the sum of all the natural Numbers, the right of the product is for all the prime Numbers. This formula holds for all complex numbers s with Re(s)>1. The left side of this formula is the series expression of the Riemann zeta function for Re(s)>1,which we have described above, and the right side is an expression purely concerning prime numbers (and containing all prime numbers), which is a sign of the relationship between the Riemann Zeta function and the distribution of prime numbers. So I'm going to assume that Re(s)>1.

Because when Re(s)>1 Euler ζ function is equivalent to the Riemann ζ function,so ζ(s)= $\overline{\zeta(\bar{s})}$($s = \sigma + ti, Re(s) > 1, \sigma \in R, t \in R$) is true. According to the Euler product formula $\prod_p(1-p^{-s})^{-1}$, when Re(s)>1, since every product factor : $(1-p^{-s})^{-1}$ in the Euler product formula is not equal to zero, so when Re(s)>1,ζ(s) is not equal to zero, and according to ζ(s)=$2^s\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)(s∈C and $s \neq 1$) (Formula 7),so the positive even number 2n($n \in Z^+$) can make sin($\frac{\pi s}{2}$) =0, but it is not the zero of Riemann ζ(s).

For any complex number s, when $Rs(s) > 0$ and $s \neq 1$, and if s=σ+ti(σ ∈R,t∈R and $t \neq 0$,s∈C), then according to Dirichlet η(s), then the relationship between the Riemannn ζ(s)(s∈C and Rs(s)>0 and s≠1) function and the Dirichlet η(s)(s∈C and Rs(s)>0 and s≠1) function is :

because $\eta(s)=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\ldots(s\in C \text{ and } Rs(s)>0 \text{ and } s\neq 1)$,

$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots(s\in C \text{ and } Rs(s)>0 \text{ and } s\neq 1)$, so

$\eta(s)-\zeta(s)=$

$-(\frac{2}{2^s}+\frac{2}{4^s}+\frac{2}{6^s}+\cdots)=-\frac{2}{2^s}(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots)=-\frac{2}{2^s}\zeta(s)(s\in C \text{ and } Rs(s)>0 \text{ and } s\neq$

1 , then ηs=1−22sζs=(1−21−s)ζss∈C and Rss>0 and s≠1, then

$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}(s\in C \text{ and } Rs(s)>0 \text{ and } s\neq 1)$ and $\eta(s)=(1-2^{1-s})\zeta(s)(s\in C \text{ and } Rs(s)>$ 0 and s≠1, ζsis the Riemann Zeta function, η(s) is the Dirichlet η(s) function,

so Riemann $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$ (s ∈ C and Rs(s) > 0 and s ≠ 1, n∈ Z^+, p ∈ Z^+, s ∈C , n goes through all the positive integers, p goes through all the prime numbers).

When Rs(s)>0 and s≠1) ,Let's prove that ζ(s) and $\zeta((\bar{s})$ are complex conjugations of each other.

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$=[$1^{-\sigma}$cos(tln1)− $2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)−$4^{-\sigma}$cos(tln4)-...]-i[$1^{-\sigma}$sin(tln1)− $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3) − $4^{-\sigma}$sin(tln4)+...]= U-Vi,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$=[$1^{-\sigma}$cos(tln1)−$2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)−$4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$sin(tln1)− $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)−$4^{-\sigma}$sin(tln4)+...]= U+Vi,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$=[$1^{\sigma-1}$ cos(tln1) − $2^{\sigma-1}$ cos(tln2)+ $3^{\sigma-1}$ cos(tln3) −$4^{-\sigma}$ cos(tln4)-...]+i[$1^{-\sigma}$ sin(tln1) − $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)− $4^{-\sigma}$sin(tln4)+...],

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$=[$1^{\sigma-k}$ cos(tln1) − $2^{\sigma-k}$ cos(tln2)+ $3^{\sigma-k}$ cos(tln3) −$4^{\sigma-k}$ cos(tln4)-...]+i[$1^{\sigma-k}$ sin(tln − $2^{\sigma-k}$sin(tln2)+ $3^{\sigma-k}$sin(tln3) − $4^{\sigma-k}$sin(tln4)+...],

$(s\in C \text{ and } s\neq 1, n\in Z^+$ and n traves all positive integer, $k\in R$),

because ,

$\frac{(-1)^{n-1}}{(1-2^{1-s})}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}}$,

$\prod_p(1-p^{-s})^{-1}=\overline{\prod_p(1-p^{-\bar{s}})^{-1}}$

($s \in C$ and $s \neq 1, p \in Z^+$ and p traves all prime numbers),
so

$\frac{(-1)^{n-1}}{(1-2^{1-s})}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}}$,

so

$\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\overline{\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}}$,

$\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}}$,

$\zeta(s)=\frac{1}{(1-2^{1-s})}\sum_1^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$,

$\zeta(\bar{s})=\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}=\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}$ ($s \in C$ and $s \neq 1, n \in Z^+$ and n traves all positive integer, $p \in Z^+$ and p traves all prime numbers),
so

only $\zeta(s)=\overline{\zeta(\bar{s})}$ ($s \in C$, $Rs(s) > 0$ and $s \neq 1$), so

$p^{1-s}=p^{(1-\sigma-ti)}=p^{1-\sigma}p^{-ti}=p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}=p^{1-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$,
$p^{1-\bar{s}}=p^{(1-\sigma+ti)}=p^{1-\sigma}p^{ti}=p^{1-\sigma}(p^{ti})=p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^t=(p^{1-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$
($s \in C$, $Rs(s) > 0$ and $s \neq 1, t \in C$ and $t \neq 0, p \in Z^+$, and p traves all prime numbers),
then

$p^{-(1-s)}=p^{(-1+\sigma+ti)}=p^{\sigma-1}p^{ti}=p^{\sigma-1}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}=(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$,

$p^{-(\bar{s})}=p^{-(\sigma-ti)}=p^{-\sigma}p^{ti}=(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$
($s \in C$, $Rs(s) > 0$ and $s \neq 1, t \in C$ and $s \neq 0, p \in Z^+$),
so
$(1-p^{-(1-s)})=1-(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))=1-p^{\sigma-1}\cos(t\ln p)-ip^{\sigma-1}\sin(t\ln p)$,

$(1-p^{-(\bar{s})})=1-(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))=1-p^{-\sigma}\cos(t\ln p)-ip^{-\sigma}\sin(t\ln p)$,
($s \in C$, $Rs(s) > 0$ and $s \neq 1, t \in C$ and $t \neq 0, p \in Z^+$),

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}=[1^{\sigma-1}\cos(t\ln 1)-2^{\sigma-1}\cos(t\ln 2)+3^{\sigma-1}\cos(t\ln 3)-4^{\sigma-1}\cos(t\ln 4)-\ldots]+i[1^{\sigma-1}\sin(t\ln 1)-2^{\sigma-1}\sin(t\ln 2)+3^{\sigma-1}\sin(t\ln 3)-4^{\sigma-1}\sin(t\ln 4)+\ldots]$,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}=[1^{-\sigma}\cos(t\ln 1)-2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)-4^{-\sigma}\cos(t\ln 4)-\ldots]+i[1^{-\sigma}\sin(t\ln 1)-2^{-\sigma}\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)-4^{-\sigma}\sin(t\ln 4)+\ldots]$
($s \in C$, $Rs(s) > 0$ and $s \neq 1$),

when $\sigma=\frac{1}{2}$,then

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$ ($s \in C$ and $s \neq 1, n \in Z^+$ and n traves all positive integer, $k \in R$),

$(1-p^{-(1-s)})=(1-p^{-\bar{s}})$($s \in C$, $Rs(s) > 0$ and $s \neq 1, p \in Z^+$), and

$(1-p^{-(1-s)})^{-1}$=$(1-p^{-\overline{s}})^{-1}$$(s \in C, Rs(s) > 0 \text{ and } s \neq 1, p \in Z^{+})$,

$\prod_p(1-p^{-(1-s)})^{-1}$=$\prod_p(1-p^{-\overline{s}})^{-1}$ $(s \in C \text{ and } s \neq 1, \text{ and } p \text{ traves all prime numbers}, k \in R)$,

$\prod_p(1-p^{-(1-s)})^{-1}$ = $\prod_p(1-p^{-\overline{s}})^{-1}$ $(s \in C, Rs(s) > 0 \text{ and } s \neq 1, p \in Z^{+} \text{ and } p \text{ traves all prime numbers}, k \in R)$,

and

$\frac{(-1)^{n-1}}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}} = \frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$,

$\frac{(-1)^{n-1}}{(1-2^{s})}\prod_p(1-p^{-(1-s)})^{-1} = \frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p(1-p^{-\overline{s}})^{-1}$

$(s \in C, Rs(s) > 0 \text{ and } s \neq 1, \text{and n traverses all positive integers}, p \in Z^{+} \text{ and } p \text{ traves all prime numbers})$,

And

$\zeta(1-s)$=$\frac{(-1)^{n-1}}{(1-2^{s})}\prod_p(1-p^{-(1-s)})^{-1}$,

$\zeta(\overline{s})$=$\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p(1-p^{-\overline{s}})^{-1}$,

$\zeta(1-s)$=$\frac{1}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$,

$\zeta(\overline{s})$=$\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$

$(s \in C, Rs(s) > 0 \text{ and } s \neq 1, p \in Z^{+} \text{ and } p \text{ traves all prime numbers}, n \in Z^{+} \text{ and } n \text{ traves all positive integer})$,

so when σ=$\frac{1}{2}$, then only $\zeta(1-s)$=$\zeta(\overline{s})$$(s \in C, Rs(s) > 0 \text{ and } s \neq 1)$must be true.

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$=[$1^{\sigma-k}$cos(tln1)$-$ $2^{\sigma-k}$cos(tln2)+ $3^{\sigma-k}$cos(tln3) $-4^{\sigma-k}$cos(tln4)-...]+i[$1^{\sigma-k}$sin(tln1) $-$ $2^{\sigma-k}$sin(tln2)+ $3^{\sigma-k}$sin(tln3) $-$ $4^{\sigma-k}$sin(tln4)+...],

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$=[$1^{-\sigma}$cos(tln1)$-2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)$-4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$Sin(tln1)$-$ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)$-4^{-\sigma}$sin(tln4)+...],

p^{k-s}=$p^{(k-\sigma-ti)}$=$p^{k-\sigma}p^{-ti}$=$p^{k-\sigma}(\cos(\ln p) + i\sin(\ln p))^{-t}$=$p^{k-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$,

$p^{1-\overline{s}} = p^{(1-\sigma+ti)} = p^{1-\sigma}p^{ti} = p^{1-\sigma}(p^{ti}) = p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p))^{t} = (p^{1-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,

$(s \in C, Rs(s) > 0 \text{ and } s \neq 1, p \in Z^{+}, \text{and } p \text{ traves all prime numbers}, n \in Z^{+} \text{ and } n \text{ traves all positive integer}, k \in R)$,

then

$p^{-(k-s)}$=$p^{(-k+\sigma+ti)}$=$p^{\sigma-k}p^{ti} = p^{\sigma-k}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$ =$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(\overline{s})}$=$p^{-(\sigma-ti)}$=$p^{-\sigma}p^{ti} = (p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(k-s)}$=$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$(s \in C, Rs(s) > 0 \text{ and } s \neq 1, p \in Z^{+} \text{ and and } p \text{ is a prime number } k \in R)$,

So

$(1-p^{-(k-s)})$=1-$(p^{\sigma-k}(\cos(tlnp)+i\sin(tlnp))$ =$1-p^{\sigma-k}\cos(tlnp)-ip^{\sigma-k}\sin(tlnp)$,
$(1-p^{-\overline{s}})$=1-$(p^{-\sigma}(\cos(tlnp)+i\sin(tlnp))$ =$1-p^{-\sigma}\cos(tlnp)-ip^{-\sigma}\sin(tlnp)$,
($s\in C$, $Rs(s)>0$ and $s\neq 1$, $p\in Z^{+}$ and and p is a prime number $k\in R$),

So when $\sigma=\frac{k}{2}$(k∈R) then

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-k+s}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ ($s\in C$, $Rs(s)>0$ and $s\neq 1$, $k\in R$, $n\in Z^{+}$),

$(1-p^{-(k-s)})$ =$(1-p^{-\overline{s}})$ ($s\in C$, $Rs(s)>0$ and $s\neq 1$, $k\in R$, p is a prime number),

and $(1-p^{-(k-s)})^{-1}$=$(1-p^{-\overline{s}})^{-1}$(s∈C , $Rs(s)>0$ and $s\neq 1$,k∈R, p∈Z^{+} and p is a prime number),

$\prod_p(1-p^{-(k-s)})^{-1}$=$\prod_p(1-p^{-\overline{s}})^{-1}$, ($s\in C$ and $s\neq 1$, $p\in Z^{+}$ and p traves all prime numbers, $n\in Z^{+}$ and n traves all positive integer, $k\in R$),
and

$$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}=\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$$

($s\in C$, $Rs(s)>0$ and $s\neq 1$, $p\in Z^{+}$ and p traves all prime numbers, $n\in Z^{+}$ and n traves all positive integer, $k\in R$),
and

$\zeta(k-s)$=$\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\prod_p(1-p^{-(k-s)})^{-1}$,

$\zeta(\overline{s})$=$\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p(1-p^{-\overline{s}})^{-1}$,

$\zeta(k-s)$=$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$ (s∈C, $Rs(s)>0$and s≠ 1, k ∈R),

$\zeta(\overline{s})$=$\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ (s∈C , $Rs(s)>0$ and s≠ 1),

($s\in C$, $Rs(s)>0$ and $s\neq 1$, $p\in Z^{+}$ and p traves all prime numbers, $n\in Z^{+}$ and n traves all positive integer, $k\in R$),

so when $\sigma=\frac{k}{2}$(k∈R) then only $\zeta(k-s)$=$\zeta(\overline{s})$($s\in C$, $Rs(s)>0$ and $s\neq 1$, $k\in R$).

According the equation ζ(1-s)=$2^{1-s}\pi^{-s}$cos($\frac{\pi s}{2}$)Γ(s)ζ(s)($s\in C$, $Rs(s)>0$and $s\neq 1$) obtained by Riemann,since Riemann has shown that the Riemann ζ(s) function has zero, that is, in

ζ(1-s)= $2^{1-s}\pi^{-s}$ cos($\frac{\pi s}{2}$)Γ(s)ζ(s) ($s\in C$, $Rs(s)>0$ and $s\neq 1$) (Formula 6), ζ(s)=0 ($s\in C$, $Rs(s)>0$ and $s\neq 1$) is true.
When $\zeta(s)$ =0 ($s\in C$, $Re(s)>0$ and $s\neq 1$) , then only $\zeta(k-\overline{s})$ = $\zeta(s)$ =0 ($s\in C$, $Rs(s)>0$ and $s\neq 1$), and
When $\zeta(\overline{s})$=0($s\in C$, $Re(s)>0$ and $s\neq 1$),then $\zeta(k-s)$=$\zeta(\overline{s})$=0($s\in C$, $Re(s)>0$and $s\neq 1$).
And because

when $\zeta(\overline{s})=0(s \in C, Re(s) > 0 \text{ and } s \neq 1)$, then only $\zeta(1-s)=\zeta(\overline{s})=0(s \in C, Re(s) > 0 \text{ and } s \neq 1)$, which is $\zeta(k-s) = \zeta(\overline{s})(s \in C, Re(s) > 0 \text{ and } s \neq 1, k \in R)$,so only k=1 be true, so only

$\zeta(s)=\overline{\zeta(\overline{s})}$ $(s \in C, Re(s) > 0 \text{ and } s \neq 1)$is true.

The reasoning in Riemann's paper goes like:
$2\sin(\pi s)\prod(s-1)\zeta(s)=(2\pi)^s \sum n^{s-1}((-i)^{s-1}+i^{s-1})$ (Formula 3),
based on euler's $e^{ix}=\cos(x) + i\sin(x)\ (x \in R)$ can get

$e^{i(-\frac{\pi}{2})}=\cos(\frac{-\pi}{2}) +i\sin(\frac{-\pi}{2}) =0-i= -i$,

$e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})=0+i=i$,

then

$(-i)^{s-1} + i^{s-1} = (-i)^{-1}(-i)^s+(i)^{-1}(i)^s=(-i)^{-1}e^{i\left(-\frac{\pi}{2}\right)s} + i^{(-1)}e^{i\left(\frac{\pi}{2}\right)s}=$

$ie^{i\left(-\frac{\pi}{2}\right)s}-ie^{i\left(\frac{\pi}{2}\right)s} =i(\cos\frac{-\pi s}{2}+i\sin\frac{-\pi s}{2})-i(\cos\frac{\pi s}{2}+i\sin\frac{\pi s}{2})=i\cos(\frac{\pi s}{2})-i\cos(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})$

$=2\sin(\frac{\pi s}{2})$ (Formula 4).

According to the property of $\Pi(s-1)=\Gamma(s)$ of the gamma function,and
$\sum_{n=1}^{\infty} n^{s-1}=\zeta(1-s)(\ n \in Z^+$ and n traves all positive integer, $s \in C$, and $s \neq 1)$,
Substitute the above (Formula 4) into the above (Formula 3), will get

$2\sin(\pi s)\Gamma(s)\zeta(s)=(2\pi\ )^s\zeta(1-s)2\sin\frac{\pi s}{2}$ (Formula 5),

If I substitute it into (Formula5), according to the double Angle formula

$\sin(\pi s)=2\mathrm{Sin}(\frac{\pi s}{2})\cos(\frac{\pi s}{2})$,

we Will get $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ and $s\neq 1)$(Formula 6),because $\pi^{-\frac{1-s}{2}} \neq 0 \neq 0$ and $\Gamma(\frac{1-s}{2}) \neq 0$,so when $\zeta(s)=0(s \in C \text{ and } s \neq 1)$, then $\zeta(1\text{-}s)=0(s \in C \text{ and } s \neq 1)$,

Substituting s→1-s, that is taking s as 1-s into Formula 6, we will get

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)(s\in C$ and $s\neq 1)$(Formula 7),

This is the functional equation for $\zeta(s)(s \in C \text{ and } s \neq 1)$. To rewrite it in a symmetric form, use the residual formula of the gamma function [3]

$\Gamma(Z)\Gamma(1\text{-}Z)= \frac{\pi}{\sin(\pi Z)}$ (Formula 8)

and Legendre's formula

$\Gamma(\frac{Z}{2})\Gamma(\frac{Z}{2}+\frac{1}{2})=2^{1-Z}\pi^{\frac{1}{2}}\Gamma(Z)$ (Formula 9) ,

Take $z= \frac{s}{2}$ in (Formula 8) and substitute it to get

$\sin(\frac{\pi s}{2})= \frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (Formula 10) , In (Formula 9), let z=1-s and substitute it in to get

$\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (Formula 11)

By substituting (Formula 10) and (Formula 11) into (Formula 7), we get

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$(s∈C and $s\neq 1$), also

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s→1-s,

And that's exactly what Riemann said in his paper.

That is to say:

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is invariant under the transformation s→1-s ,

also

$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1-s)$(s∈C and $s\neq 1$),

or

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$($s\in C$ and $s\neq 1$)(Formula 2),

Then $\zeta(s)=2^{s}\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$(s∈C and $s\neq 1$) (Formula 7) ,

under the transformation s→1-s ,will get

$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C and $s\neq 1$) (Formula 6). Then

$\zeta(1-s)=\frac{\zeta(s)}{2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$(s∈C and $s\neq 1$), when $\zeta(s)=0$ and $s\neq 2n(n\in Z^{+})$, then if $\zeta(1-s)=\frac{\zeta(s)}{2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$ (s∈C and $s\neq 1$) is going to make sense, then the denominator $2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\neq 0$, Clearly indicates $2^{s}\neq 0$(s∈C and $s\neq 1$), $\pi^{s-1}\neq 0$(s∈C and $s\neq 1$), $\Gamma(1-s)\neq 0$(s∈C and $s\neq 1$), so $\sin(\frac{\pi s}{2})$ can not equal to zero, so $\sin(\frac{\pi s}{2})\neq 0$(s∈C and $s\neq 1$), so So when $\zeta(s)=0$ and $s\neq 2n(n\in Z^{+})$,

then $\zeta(1-s)=\zeta(s)=0$(s∈C and $s\neq 1$,and $s\neq -2n$, $n\in Z^{+}$).

According to the property that Gamma function Γ(s) and exponential function are nonzero, is also that $\Gamma(\frac{1-s}{2})\neq 0$, and $\pi^{-\frac{1-s}{2}}\neq 0$, according to $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$(s∈C and $s\neq 1$)(Formula 12), According the equation $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C and $s\neq 1$) obtained by Riemann,since Riemann has shown that the Riemann ζ(s)(s∈C and $s\neq 1$) (Formula 6) function has zero, that is, in $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s∈C and $s\neq 1$)

(Formula 6), so ζ(s)=0(s∈C and s≠ 1) is true. According to the property that Gamma function Γ(s) and exponential function are nonzero, is also that $\Gamma(\frac{1-s}{2}) \neq 0$, and $\pi^{-\frac{1-s}{2}} \neq 0$,

So when ζ(s)=0(s∈C and s≠ 1), then ζ(1-s)=0(s∈C and s≠ 1), also must ζ(s)=ζ(1-s)=0(s∈C and s≠ 1).

Because $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \sum_{n=0}^{\infty}\frac{1}{n!} \approx 2.7182818284...$,

and because $\sin(Z)=\frac{e^{iZ}-e^{-iZ}}{2i}$, Suppose Z=s= σ+ti (σ ∈R,t∈R and t ≠ 0), then

$$\sin(s)=\frac{e^{is}-e^{-is}}{2i} = \frac{e^{i(\sigma+ti)}-e^{-i(\sigma+ti)}}{2i},$$

$$\sin(\overline{s})=\frac{e^{i\overline{s}}-e^{-i\overline{s}}}{2i} = \frac{e^{i(\sigma-ti)}-e^{-i(\sigma-ti)}}{2i},$$

according $x^s = x^{(\sigma+ti)} = x^\sigma x^{ti} = x^\sigma(\cos(\ln x) + i\sin(\ln x))^t = x^\sigma(\cos(t\ln x) + i\sin(t\ln x))(x > 0)$, then

$e^s=e^{(\sigma+ti)}=e^\sigma e^{ti}=e^\sigma(\cos(t) + i\sin(t)) = e^\sigma(\cos(t) + i\sin(t))$,

$e^{is}=e^{i(\sigma+ti)}=e^{\sigma i}(\cos(it) + i\sin(it)) = (\cos(\sigma) + i\sin(\sigma))(\cos(it) + i\sin(it))$,

$e^{i\overline{s}} = e^{i(\sigma-ti)}=e^{\sigma i}(\cos(-it) + i\sin(-it)) = (\cos(\sigma) + i\sin(\sigma))(\cos(it) - i\sin(it))$,

$e^{-is}=e^{-i(\sigma+ti)}=e^{-\sigma i}(\cos(-it) + i\sin(-it)) = (\cos(\sigma) - i\sin(\sigma))(\cos(it) - i\sin(it))$,

$e^{-i\overline{s}} = e^{-i(\sigma-ti)}=e^{-\sigma i}(\cos(it) + i\sin(it)) = (\cos(\sigma) - i\sin(\sigma))(\cos(it) + i\sin(it))$,

$2^s=2^{(\sigma+ti)}=2^\sigma 2^{ti}=2^\sigma(\cos(\ln 2) + i\sin(\ln 2))^t=2^\sigma(\cos(t\ln 2) + i\sin(t\ln 2))$,

$2^{\overline{s}}=2^{(\rho-ti)}=2^\sigma 2^{-ti}=2^\sigma(\cos(\ln 2) + i\sin(\ln 2))^{-t} =2^\sigma(\cos(t\ln 2) - i\sin(t\ln 2))$,

$\pi^{s-1}=\pi^{(\sigma-1+ti)}=\pi^{\sigma-1}\pi^{ti}=\pi^{\rho-1}(\cos(\ln\pi) + i\sin(\ln\pi))^t=\pi^{\sigma-1}(\cos(t\ln\pi) + i\sin(t\ln\pi))$,

$\pi^{\overline{s}-1}=\pi^{(\sigma-1-ti)}=\pi^{\sigma-1}\pi^{-ti}=\pi^{\sigma-1}(\cos(\ln\pi) + i\sin(\ln\pi))^{-t}=2^{\sigma-1}(\cos(t\ln\pi) - i\sin(t\ln\pi))$,

So

$2^s=\overline{2^{\overline{s}}}$, $\pi^{s-1}=\overline{\pi^{\overline{s}-1}}$,

and

$\frac{e^{is}-e^{-is}}{2i} = \frac{\overline{e^{i\overline{s}}-e^{-i\overline{s}}}}{2i}$,

So

$\sin(s)=\overline{\sin(\overline{s})}$,

and

$\sin(\frac{\pi s}{2})=\overline{\sin(\frac{\pi\overline{s}}{2})}$.

And the gamma function on the complex field is defined as:

$\Gamma(s)=\int_0^{+\infty} t^{s-1} e^{-t}dt$,

Among Re(s)>0, this definition can be extended by the analytical continuation principle to the entire field of complex numbers except for positive integers (zero and negative integers).

So

$\Gamma(s)=\overline{\Gamma(\overline{s})}$,

and

$\Gamma(1\text{-}s)=\overline{\Gamma(1-\overline{s})}$ .When $\zeta(1\text{-}\overline{s})=\overline{\zeta(1-\overline{s})}=0=\zeta(s)=\zeta(1\text{-}s)=0(s\in C, Re(s)>0$and $s\neq 1)$, and according

$\zeta(s)=2^s\pi^{s-1}\text{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)(s\in C, Re(s)>0$and $s\neq 1)$, then $\zeta(s)=\overline{\zeta(\overline{s})}=0(s\in C, Re(s)>0$and $s\neq 1)$,is also say $\zeta(s)=\zeta(\overline{s})=\zeta(1\text{-}\overline{s})=0(s\in C, Re(s)>0$ and $s\neq 1)$. so only $\zeta(\sigma+ti)=\zeta(\sigma\text{-}ti)=0$ is true. According the equation

$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)\,(s\in C \text{ and } s\neq 1)$ obtained by Riemann,since Riemann has shown that the Riemann $\zeta(s)$ $(s\in C \text{ and } s\neq 1)$ function has zero, that is, in $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C, \text{and } s\neq 1)$(Formula 7), $\zeta(s)=0(s\in C, \text{and } s\neq 1)$ is true, so when $\zeta(s)=0(s\in C, \text{and } s\neq 1)$, then only $\zeta(s)=\zeta(1\text{-}s)=0(s\in C, \text{and } s\neq 1)$ is true.

If $\zeta(s)=0(s=\sigma+ti, \sigma\in R, t\in R, Re(s)>0 \text{ and } s\neq 1)$, then $\zeta(s)=\zeta(\overline{s})=0(s=\sigma+ti, \sigma\in R, t\in R, Res>0$ and $s\neq 1$, it shows that the zeros of the Riemann $\zeta(s)$ function must be conjugate, then there must be $\zeta(s)=\zeta(\overline{s})=0$, indicating that the zeros of the Riemannian $\zeta(s)$ function must be conjugate , and in the critical band of $Re(s)\in(0,1)$, there are no non-conjugate zeros.According $\zeta(s)=\zeta(\overline{s})=0$, if $s=\overline{s}$, then $s\in R$, because $s=-2n(n\in Z^+)$make he function $\zeta(s)(s\in C$ and $s\neq 1)$ has the value zero in $2\sin(\pi s)\Pi(s-1)\zeta(s)=i\int_0^\infty \frac{x^{s-1}dx}{x-1}$ and $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s\zeta 1-s s\in C$ and $s\neq 1$((Formula 7), so a negative even number can be the zero of Riemann $\zeta(s)(s\in C$ and $s\neq 1)$. If $s\neq\overline{s}$, then s and $\overline{s}$ are not both real numbers but both imaginary numbers, $t\in R$ and $t\neq 0$. And according to $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(s\in C \text{ and } s\neq 1$(Formula 7), if the $\zeta(s)=0$ $(s\in C$ and $s\neq 1)$ was established, then $\zeta(1\text{-}s)=\zeta(s)=0 s\in C$ and $s\neq 1$ must be true , so only when $\sigma=\frac{1}{2}$ and $\zeta(s)=0\,(s\in C \text{ and } s\neq 1)$,then it must be true that $\zeta(1\text{-}s)=\zeta(s)=0$ $(s\in C \text{ and } s\neq 1)$.$\zeta(s)$ $(s\in C, Re(s)>0 \text{ and } s\neq 1)$ and $\zeta(\overline{s})(s\in C, Re(s)>0 \text{ and } s\neq 1)$ are complex conjugates of each other,that is $\zeta(s)=\overline{\zeta(\overline{s})}$ $(s\in C, Re(s)>0 \text{ and } s\neq 1)$, if $\zeta(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1)$, then must $\zeta(\overline{s})=0(s\in C, Re(s)>0 \text{ and } s\neq 1)$, and so if $\zeta(s)=0\,(s\in C, Re(s)>0 \text{ and } s\neq 1)$, then it must be true that $\zeta(s)=\zeta(\overline{s})=0\,(s\in C, Re(s)>0 \text{ and } s\neq 1)$.

According to Riemann's paper "On the Number of primes not Greater than x", we can obtain an expression $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)$ in relation to the Riemann $\zeta(s)$ $(s\in C \text{ and } s\neq 1)$function, which has long been known to modern mathematicians, and which I derive later. According ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s ∈ C and s ≠ 1)(Formula 6) obtained by Riemann,so when ζ(s)=0 then ζ(1-s)=ζ(s)=0(s∈C and s≠1), and sin($\frac{\pi s}{2}$ = 0), then only s=$\overline{s}$ or s=1-s or $\overline{s}$=1-s , and sin($\frac{\pi s}{2}$ = 0),so s∈R and s=-2n(n ∈ Z^+),drop s=2n(n ∈ Z^+),so only when σ=$\frac{1}{2}$, the next three equations ζ(σ+ti)=0, ζ(1-σ-ti)=0, and ζ(σ-ti)=0 are all true,so only s=$\frac{1}{2}$+ti (t∈R and t≠0) and s=$\frac{1}{2}$-ti(t∈R and t≠0) is true. And when ζ(s)=0 then according ζ(1-s)＝ζ(s)=0 and ζ(s)=$\overline{\zeta(\overline{s})}$=0(s ∈ C , Re(s) > 0 and s ≠ 1),is also say ζ(s)=ζ($\overline{s}$)=0(s ∈ C, Re(s) > 0 and s ≠ 1) and ζ(1-s)=ζ($\overline{s}$)=0(s ∈ C, Re(s) > 0 and s ≠ 1),then only ζ(σ+ti)=ζ(σ-ti)= 0 is true.Since Riemann has shown that the Riemann ζ(s) (s ∈ C and s ≠ 1) function has zero, that is, in ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s ∈ C and s ≠ 1)(Formula 7) , ζ(s)=0(s ∈ C and s ≠ 1) is true, so when ζ(s)=0,In the process of the Riemann conjecture proved about ζ(s)=ζ($\overline{s}$)=0(s ∈ C , Re(s) > 0 and s ≠ 1) and ζ(1-s)=ζ($\overline{s}$) =0(s ∈ C , Re(s) > 0 and s ≠ 1), is refers to the ζ(s)(s ∈ C and s ≠ 1) is a functional numbe. Since Riemann has shown that the Riemann ζ(s)(s ∈ C and s ≠ 1) function has zero, that is, in ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s ∈ C and s ≠ 1)(Formula 6), ζ(s)=0(s ∈ C and s ≠ 1) is true.

According ζ(s)=ζ(1-s) =0 (s ∈ C and s ≠ 1) and ζ(s)=ζ($\overline{s}$)=ζ(1-$\overline{s}$)=0(s ∈ C, (s ∈ C , Re(s) >0 and s ≠ 1) and s ≠ 1),then s=$\overline{s}$ or s=1-s or $\overline{s}$=1-s , and sin($\frac{\pi s}{2}$) = 0,so s∈R and s=-2n(n ∈ Z^+),drop s=2n(n ∈ Z^+),or σ+ti=1-σ-ti ,or σ-ti=1-σ-ti, so s ∈ R and s = −2n(n ∈ Z^+), or σ=$\frac{1}{2}$ and t=0, or $\sigma = \frac{1}{2}$ and t ∈R and t≠ 0, so t ∈ R, or s=$\frac{1}{2}$+0i ,or s=$\frac{1}{2}$+ti(t ∈R and t ≠ 0) and s=$\frac{1}{2}$-ti(t ∈R and t ≠ 0), because $\zeta\left(\frac{1}{2}\right) \to +\infty, \zeta(1) \to +\infty$, ζ(1) is divergent, ζ($\frac{1}{2}$) is more divergent,so drop them.Beacause only when σ=$\frac{1}{2}$,the next three equations, ζ(σ + ti)= 0, ζ(1 − σ − ti)=0, and ζ(σ-ti)= 0 are all true, because $\zeta\left(\frac{1}{2}\right) \to +\infty, \zeta(1) \to +\infty$, ζ(1) is divergent, ζ($\frac{1}{2}$) is more divergent, so drop s=1 and s=$\frac{1}{2}$, so only s=$\frac{1}{2}$+ti(t∈R and t ≠ 0,s∈C) is true.

If ζ(s)=0(s = σ + ti, σ ∈ R, t ∈ R and s ≠ 1), then ζ(s)= ζ($\overline{s}$)=0, it shows that the zeros of the Riemann ζ(s) function must be conjugate, then there must be ζ(s)=ζ($\overline{s}$)=0, indicating that the zeros of the Riemannian ζ(s) function must be conjugate , and in the critical band of Re(s)∈(0,1), there are no non-conjugate zeros.According ζ(s)= ζ($\overline{s}$)=0 , if s=$\overline{s}$, then $s \in R$, because

$s=-2n(n\in Z^{+})$make he function ζ(s)(s∈C and s≠1) has the value zero in 2sin(πs)Π(s−1ζs=i0∞𝑥s−1d𝑥𝑥−1 and ζ(s)=2sπs−1sinπs2Γ1− sζ1− ss∈C and s≠1((Formula 7), so a negative even number can be the zero of Riemann ζ(s)(s∈C and s≠1). If $s\neq\bar{s}$, then s and $\bar{s}$ are not both real numbers but both imaginary numbers, t∈R and t≠0. And according to $\zeta(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(s\in C\ and\ s\neq1)$(Formula 7), if the ζ(s) = 0 (s ∈ C and s≠1) was established, then ζ(1-s)=ζ(s)=0(s ∈ C and s ≠ 1) must be true , because ζ(s)=$\overline{\zeta(\bar{s})}$ (s = σ + ti, σ∈R,t∈R ,s∈C, *Res>0* and s≠1 , so when ζ(s)=0s=σ+ti, σ∈R,t∈R,s∈C, *Res>0* and s≠1 and s≠1,then ζ(s)= ζ(s)=0s=σ+ti, σ∈R,t∈R ,s∈C, *Res>0 and* s≠1, so the two zeros s and 1-s of Riemann ζ(s)(s∈C, $Re(s)>0$ *and* s≠1) must also be conjugate. If either of s and 1-s are real numbers other than negative even numbers, since s and 1-s are conjugate, then s=1-s, then s=$\frac{1}{2}$. Since $\sin\left(\frac{\pi s}{2}\right)=\sin\left(\frac{\pi}{2}\times\frac{1}{2}\right)=\sin\left(\frac{\pi}{4}\right)\neq0$, and because ζ($\frac{1}{2}$) diverge, then

neither s nor 1-s are zeros of Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1), that is, Riemann ζ(s)(s∈C , $Re(s)>0$ and s≠1) has no real zeros other than negative even numbers. If Re(s)=1, then Re(1-s)=0, then s and 1-s are not conjugate, so Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1) has no zeros with real parts of 1 or 0. If Re(s)>1, then Re(1-s)<0, then s and 1-s are not conjugate, and because if Re(s)> 1, then Riemann ζ(s)(s∈C and s≠1) has no zero, and according to ζ(s)=ζ(1-s)=0(s∈C and s≠1) was established, then when Re(s)<0, the ζ(s)(s ∈C and s≠1) is not equal to zero. Because when ζ(s)(s ∈C and s≠1), if the Re(s)=1, the Re(1-s)=0, then s and 1 s not conjugate, and according to ζ(s)=ζ(1-s)=0(s ∈C and s≠1) was established, so if Re(s)=0 or Re(s)=1, then ζ(s)(s∈C and s≠1) has no zero. Therefore, in addition to negative even numbers, Riemann ζ(s)+(s∈C and s≠1) has a zero if the value of Re(s) is in the interval (0,1).So in addition to negative even numbers, so the real part of Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1) zero s must be 0<Re(s)<1, that is, Re(s)∈(0,1), which shows that the prime number theorem holds. When 0<Re(s)<1,if s and 1-s are both real and imaginary, then s and 1-s are not conjugated, then s and

1-s cannot both be zeros of Riemann ζ(s)(s∈C, $0 < Re(s) < 1$,and s≠1), so 1-s and s can only be both imaginary and conjugate, and s cannot be pure imaginary, because if s is pure imaginary, then 1-s and s are not conjugated. So ζ(s)(s∈C, $0 < Re(s) < 1$ and s≠1) has no pure imaginary.And if Re(s)≠$\frac{1}{2}$, then Re(s)≠Re(1-s), then 1-s and s are not conjugate, so Re(s)≠$\frac{1}{2}$ cannot be true. So only 1-s=$\overline{s}$ is true, that is, only 1-σ-ti=σ-ti is true, so only σ=$\frac{1}{2}$, t∈R and t≠0, so the real part of the non-real zeros of Riemann ζ(s)(s∈C ,$0 < Re(s) < 1$) can only be , that is, only Re(s)=$\frac{1}{2}$ is true, Equivalent to $\xi(s) = 0\left(s = \frac{1}{2} + ti \text{ or } s = \frac{1}{2} - ti, t \in R \text{ and } t \neq 0\right.$, s ∈ C and s≠1or ξ12+ti=0t∈R and t≠0 and ξ12−ti=0t∈R and t≠0. Therefore, in the critical band of Re(s) ∈(0,1), Re(s)≠ $\frac{1}{2}$ is impossible, and there is no zero whose real part is not equal to $\frac{1}{2}$, so the Riemann conjecture holds. The symmetries of zeros s and zeros 1-s are not sufficient to prove that the nontrivial zeros of the Riemannn ζ(s)(s∈C, s≠1) function are on the critical line, and zeros s and zeros 1-s are symmetric only about the point ($\frac{1}{2}$,i) on the critical line. The conjugacy of s and 1-s is the fundamental reason why the nontrivial zeros of Riemann ζ(s)(s∈C, s ≠1) are all *located* on the critical line. According to ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C and s≠ 1) (Formula 6),so when ζ(s)=0, then ζ(s)=ζ(1-s)=0 is true. Because $\zeta(s) = \overline{\zeta(\overline{s})}$(s ∈ C and s ≠ 1, then when ζs=0 or ζs=0, then it must be true that ζs=ζs=0. So when Riemann ζ(s)=0, then s and 1-s must also be conjugate. From this we get s=$\frac{1}{2}$+ti(t∈ R and t ≠ 0), or s=$\frac{1}{2}$-ti(t∈ R and t≠ 0).According to the Euler product formula, when Re(s)>1, since every product factor in the Euler product formula is not equal to zero, so when Re(s)>1, Euler ζ function is equivalent to the Riemann ζ function,Since each of the product factors in the Euler product formula is not equal to zero, When Re(s)>1, ζ(s) is not equal to zero, so the positive even number 2n(n ∈ Z^+) can make sin($\frac{\pi s}{2}$) =0, but it is not the zero of Riemann ζ(s). If s is any real number other than negative and positive even, and if it is the zero of the Riemann ζ(s) function, then s and 1-s must be conjugate, for real numbers other than negative and positive even numbers, in addition to not making sin($\frac{\pi s}{2}$)=0, it must satisfy that s=1-s, then s=$\frac{1}{2}$, and function ζ($\frac{1}{2}$) diverge, so real numbers other than negative even numbers are not zeros of Riemann ζ(s). It holds that ζ(s)= ζ(1-s)=0($s \in \mathrm{C}$ and $s \neq 1$), and we know that the zero of ζ(s)(s∈C and $0 < Re(s) < 1$) is symmetric with respect to the point ($\frac{1}{2}$,0i). But is it possible to determine that the nontrivial zeros of the Riemann ζ(s) function are all on the critical boundary where the real part is equal to $\frac{1}{2}$, just because the zeros of ζ(s) are symmetric with respect to the point ($\frac{1}{2}$,0i)? Obviously not, when Re(s)∈(0,1), example s=0.54+ti($t \in \mathrm{R}$), Re(s)=0.54,then Re(1-s)=0.46,and 1-s are symmetric about the point ($\frac{1}{2}$,0i), but Riemann argued that such a complex number is not the zero of Riemann ζ(s).Riemann was right, and it is clear that when Re(s) is not equal to $\frac{1}{2}$, then s and 1-s must not be conjugate, and according to the zeros of the ζ(s) function must be conjugate, then if Re(s) is not equal to $\frac{1}{2}$, then it must not be the zero of the ζ(s) function. To sum up, the non-trivial zeros of the Riemann ζ(s) function must all lie on the critical boundary where the real part of the complex plane is equal to $\frac{1}{2}$, and the Riemann conjecture must be true.So only when σ=$\frac{1}{2}$ and ζ(s)=0($s \in \mathrm{C}$ and $s \neq 1$),then it must be true that ζ(1-s)=ζ(s)=0($s \in \mathrm{C}$ and $s \neq 1$).ζ(s)($s \in \mathrm{C}$ and $s \neq 1$) and ζ($\overline{s}$)($s \in \mathrm{C}$ and $s \neq 1$) are complex conjugates of each other,that is ζ(s)=$\overline{\zeta(\overline{s})}$ ($s \in \mathrm{C}$, $\mathrm{Re(s)} > 0$ and $s \neq 1$), if ζ(s)=0($s \in \mathrm{C}$, $\mathrm{vRe(s)} > Re(s) > 1$ and $s \neq 1$), then must ζ($\overline{s}$)=0($s \in \mathrm{C}$, $\mathrm{Re(s)} > 0$ and $s \neq 1$), and so if ζ(s)=0($s \in \mathrm{C}$, $\mathrm{Re(s)} > 0$ and $s \neq 1$), then it must be true that ζ(s)=ζ($\overline{s}$)=0($s \in \mathrm{C}$, $\mathrm{Re(s)} > 0$ and $s \neq 1$).

According to Riemann's paper "On the Number of primes not Greater than x", we can obtain an expression ζ(1-s)=$2^{1-s}\pi^{-s}$cos($\frac{\pi s}{2}$)Γ(s)ζ(s)($s \in \mathrm{C}$ and $s \neq 1$) in relation to the Riemann ζ(s)($s \in \mathrm{C}$ and $s \neq 1$)function, which has long been known to modern mathematicians, and which I derive later. According ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)$(Formula 6) obtained by Riemann,so when ζ(s)=0 then ζ(1-s)=ζ(s)=0(s∈C and s≠1), and $\sin(\frac{\pi s}{2}=0)$, then only s=$\bar{s}$ or s=1-s or $\bar{s}$=1-s , and $\sin(\frac{\pi s}{2}=0)$,so s∈R and s=-2n($n\in Z^+$),drop s=2n($n\in Z^+$),so only when σ=$\frac{1}{2}$, the next three equations ζ(σ+ti)=0, ζ(1-σ-ti)=0, and ζ(σ-ti)=0 are all true,so only s=$\frac{1}{2}$+ti (t∈R and t≠0) and s=$\frac{1}{2}$-ti(t∈R and t≠0) is true.And when ζ(s)=0 then according ζ(1-s)=ζ(s)and ζ(s)=$\overline{\zeta(\bar{s})}$=0($s\in C, \mathrm{Re}(s)>0 \text{ and } s\neq 1$),is also say ζ(s)=ζ($\bar{s}$)=0 and ζ(1-s)=ζ($\bar{s}$) =0($s\in C, \mathrm{Re}(s)>0 \text{ and } s\neq 1$),then only ζ(σ+ti)=ζ(σ-ti)= 0 is true.Since Riemann has shown that the Riemann ζ(s)($s\in C \text{ and } s\neq 1$) function has zero, that is, in ζ(1-s)= $2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ ($s\in C \text{ and } s\neq 1$)(Formula 7) , ζ(s)=0($s\in C \text{ and } s\neq 1$) is true, so when ζ(s)=0($s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$),In the process of the Riemann conjecture proved about ζ(s)=ζ($\bar{s}$)=0($s\in C, \mathrm{Re}(s)>0 \text{ and } s\neq 1$) and ζ(1-s)=ζ($\bar{s}$) =0($s\in C, \mathrm{Re}(s)>0 \text{ and } s\neq 1$), is refers to the ζ(s)($s\in C \text{ and } s\neq 1$) is a functional numbe.In the process of the Riemann hypothesis proved about ζ(s)=ζ(1-s)=0($s\in C \text{ and } s\neq 1$), ζ($\bar{s}$)=0($s\in C, \mathrm{Re}(s)>0 \text{ and } s\neq 1$), is refers to the ζ(s) is a functional number? It's not. Does ζ(s)=ζ(1-s)=ζ($\bar{s}$)($s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$) mean the symmetry of the ζ(s) function equation? Does that mean the symmetry of the equation s=$\bar{s}$=1-s? Not really. In my analyst,

ζ(s)、ζ(1-s) and ζ($\bar{s}$) function expression are $\sum_{n=1}^{\infty} n^{-s}$(n∈ Z^+ and n traves all positive integer, $s\in C, \text{and } s\neq 1$), so according to $\sum_{n=1}^{\infty} n^{-s}$ ($n\in Z^+$ and n traves all positive integer, $s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$),ζ(s)($s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$) function of the independent variable s, follows: According ζ(s)=ζ(1-s)=0($s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$) and ζ(s)=ζ($\bar{s}$) =0($s\in C, \mathrm{Re}(s)>1 \text{ and } s\neq 1$),so s and 1-s are also conjugate,then only s=$\bar{s}$ or s=1-s or $\bar{s}$=1-s , and $\sin(\frac{\pi s}{2}=0)$,so s∈R, and s=-2n(n $n\in Z^+$), $drop\ s=2n(n\ n\in Z^+)$, or σ+ti=1-σ-ti ,or σ-ti=1-σ-ti, so $s\in R \text{ and } s=-2n(n\in Z^+)$, or σ = $\frac{1}{2}$ and t=0,or $\sigma=\frac{1}{2}$ and t ∈ R and t ≠ 0 ,so s ∈ R, or s=$\frac{1}{2}$+oi ,or s=$\frac{1}{2}$+ti(t ∈R and t ≠ 0) and s=$\frac{1}{2}$-ti(t ∈R and t ≠ 0),because $\zeta\left(\frac{1}{2}\right)\to+\infty, \zeta(1)\to+\infty$, ζ(1) is divergent, ζ($\frac{1}{2}$) is more divergent,so drop them.Beacause only when ρ=$\frac{1}{2}$,the next three equations, $\zeta(\sigma+\mathrm{ti})=0$, $\zeta(1-\sigma-\mathrm{ti})=0$, and ζ($\sigma$-ti)=0 are all true, because $\zeta\left(\frac{1}{2}\right)\to+\infty, \zeta(1)\to+\infty$, ζ(1) is divergent, ζ($\frac{1}{2}$) is more divergent,so only s=$\frac{1}{2}$+ti(t∈R and t ≠ 0) and s=$\frac{1}{2}$-ti(t∈R and t ≠ 0) are true, or say only s=$\frac{1}{2}$+ti(t∈R and t ≠ 0) and s=$\frac{1}{2}$-ti(t∈R and t ≠ 0)are true.Since Riemann has shown that the Riemann ζ(s) ($s\in C \text{ and } s\neq 1$) function has zero, that is, in ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)$(Formula 7), ζ(s)=0($s\in C, \text{and } s\neq 1$)is true.

According the equation $\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)(s\in C, \text{and } s\neq 1, \text{and } s\neq -2n, n\in Z^{+})$ obtained by Riemann, so $\xi(s)=\xi(1-s)(s\in C, \text{and } s\neq 1 \text{and } s\neq -2n, n\in Z^{+})$, because $\Gamma(\frac{s}{2})=\overline{\Gamma(\frac{\bar{s}}{2})}$, and $\pi^{-\frac{s}{2}}=\overline{\pi^{-\frac{\bar{s}}{2}}}$, and because $\zeta(s)=\overline{\zeta(\bar{s})}(s\in C, Re(s)>0 \text{ and } s\neq 1)$, so $\xi(s)=\overline{\xi(\bar{s})}(s\in C, Re(s)>1, \text{and } s\neq 1 \text{ and } s\neq -2n, n\in Z^{+}, \text{and n traverses all positive integers})$, So when $\zeta(s)=0$ $(s\in C, Re(s)>0 \text{ and } s\neq 1 \text{ and } s\neq -2n, n\in Z^{+})$, then $\zeta(s)=\zeta(1-s)=\zeta(\bar{s})=0(s\in C \text{ and } s\neq 1 \text{and } s\neq -2n, n\in Z^{+})$ and $\xi(s)=\xi(1-s)=\xi(\bar{s})=0$ $(s\in C, Re(s)>1 \text{ and } s\neq 1 \text{and } s\neq -2n, n\in Z^{+})$ must be true, so the zeros of the Riemann$\xi(s)$ function and the nontrivial zeros of the Riemann $\zeta(s)(s\in C, \text{and } s\neq 1)$ function are identical, so the complex root of Riemann $\xi(s)=0(s\in C, \text{and } s\neq 1)$ satisfies $s=\frac{1}{2}+ti$ ($t\in R$ and $t\neq 0$) and $s=\frac{1}{2}-ti(t\in R \text{ and } t\neq 0)$.According to the Riemann function $\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)(t\in C \text{ and } t\neq 0, s\in C, Re(s)>0 \text{ and } s\neq 1)$ and he Riemann definded $s=\frac{1}{2}+ti(t\in C \text{ and } t\neq 0)$, because $s\neq 1$ and and $s\neq -2n(n\in Z^{+})$, and $\prod\frac{s}{2}\neq o$, $\pi^{-\frac{s}{2}}\neq 0$, so $\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\neq o$, and when $\xi(t)=0$, then

$\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0$, and

$\zeta(\frac{1}{2}+ti)=\frac{\xi(t)}{\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}}=\frac{0}{\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}}=0$, so $t\in R$ and $t\neq 0$. So the root t of the equations

$\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0$ and $4\int_{1}^{\infty}\frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx=$

$\xi(t)=0$ $(t\in C \text{ and } t\neq 0)$ and $\xi(t)=\frac{1}{2}-(t^{2}+\frac{1}{4})\int_{1}^{\infty}\Psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)=0(t\in C \text{ and } t\neq 0)$ must be real and $t\neq 0$. If $Re(s)=\frac{k}{2}(k\in R)$, then $\zeta(k\text{-}s)=2^{k-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ $(s\in C \text{ and } s\neq 1 \text{ and } s\neq -2n, n\in Z^{+}, k\in R)$ and $\xi(k-s)=\frac{1}{2}s(s\text{-}k)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)(s\in C, Re(s)>0 \text{ and } s\neq 1 \text{and } s\neq -2n, n\in Z^{+}, s\in C, \text{and } s\neq 1, \text{and } s\neq -2n, \text{n traverses all positive integers}, k\in R)$are true. So when $\zeta(s)=0(s\in C, \text{and } s\neq 1 \text{ and } s\neq -2n, n\in Z^{+})$,then $\zeta(s)=\zeta(k-s)=\zeta(\bar{s})=0(s\in C, Re(s)>1 \text{ and } s\neq 1, \text{and } s\neq -2n, n\in Z^{+}, \text{n traverses all positive integers}, k\in R)$ and $\xi(s)=\xi(k-s)=\xi(\bar{s})=0$ $(s\in C \text{ and } s\neq 1, \text{and } s\neq -2n, \text{n traverses all positive integers}, k\in R)$ must be true, and $s=\frac{k}{2}+ti$ ($k\in R, t\in R$ and $t\neq 0$)must be true, then $\prod\frac{s}{2}(s\text{-}k)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0$ $(k\in R, t\in C \text{ and } t\neq 0, s\in C, \text{and } s\neq 1, \text{and } s\neq -2n, n\in Z^{+}, \text{n traverses all positive integers})$,

$\zeta(\frac{k}{2}+ti)=\frac{\xi(t)}{\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=\frac{0}{\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=0(k\in R, t\in C \text{ and } t\neq 0)$,so $t\in R$ and $t\neq 0$. So the root of the equations $\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0(k\in R, t\in C \text{ and } t\neq 0)$ must be real and $t\neq 0$. But the Riemann ζ(s) function only satisfies $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)$ and $\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$

$(s\in C, Re(s)>0 \text{ and } s\neq 1, s\neq -2n, n \text{ traverses all positive integers})$, is also say that only

$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)$ (Formula 7) is true , so only $Re(s)=\frac{k}{2}=\frac{1}{2}(k\in R)$ is true, so only k=1 is true.The Riemann conjecture must satisfy the properties of the Riemann $\zeta(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1)$ function and the Riemann $\xi(s)=0(s\in C, Re(s)>1 \text{ and } s\neq 1 \text{ and } s\neq -2n, n \text{ traverses all positive integers})$ function, The properties of the Riemann $\zeta(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1, s\neq -2n, n\in Z^+)$ function and the Riemann $\xi(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1 \text{ and } s\neq -2n, n \text{ traverses all positive integers})$ function are fundamental, the Riemann conjecture must be correct to reflect the properties of the Riemann $\zeta(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1, s\neq -2n, n \text{ traverses all positive integers})$ function and the Riemann $\xi(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1 \text{ and } s\neq -2n, n \text{ traverses all positive integers})$function, that is, when $\zeta(s)=0(s\in C, Re(s)>0 \text{ and } s\neq 1 \text{ and } s\neq -2n, n\in Z^+)$, the roots of the Riemann $\xi(t)(t\in C \text{ and } t\neq 0)$ function can only be real, that is, Re(s) can only be equal to $\frac{1}{2}$, and Im(s) must be real, and Im(s) is not equal to zero.So the Riemann conjecture must be correct.Riemann found in his paper that

$$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^\infty \psi(x)\, x^{\frac{s}{2}-1}dx+\int_1^\infty \psi(\frac{1}{x})\, x^{\frac{s-3}{2}}dx+\frac{1}{2}\int_0^1(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1})dx$$

$=\frac{1}{s(s-1)}+\int_1^\infty \psi(x)\,(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx\ (s\in C \text{ and } s\neq 1)\ (s\in C \text{ and } s\neq 1)$, Because $\frac{1}{s(s-1)}$ and $\int_1^\infty \psi(x)(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx$ are all invariant under the transformation s→1-s If I introduce the auxiliary function $\psi(s)=\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)(s\in C, \text{and } s\neq 1 \text{ and } s\neq -2n, n\in Z^+)$,So I can just write it as ψ(s)=ψ(1-s). But it would be more convenient to add the factor $s(s-1)$to ψ(s) and introduce the coefficient $\frac{1}{2}$, which is exactly what Riemann did, is that to take $\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)(s\in C \text{ and } s\neq -2n, n\in Z^+, \text{and } s\neq 1)$.Because the factor (s-1) cancels out the first pole of $\zeta(s)$ at s=1, And the factor s cancels out the pole of $\Gamma\left(\frac{s}{2}\right)$ at s=0, and s is equal to -2, -4, -6,...,the rest of the poles of $\Gamma\left(\frac{s}{2}\right)$ cancel out . So $\xi(s)$is an integral function.And the factor $s(s-1)$ obviously doesn't change under the transformation $s\to 1-s$,so we also have the function $\xi(s)=\xi(1-s)=0(s\in C, \text{and } s\neq 1 \text{ and } s\neq -2n, n \text{ traverses all positive integers})$, base on $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C \text{ and } s\neq 1)(Formula\ 7)$. At the same time, according to $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s \in C \text{ and } s \neq 1)$, if ζ(s)=0 (s∈C and s≠ 1),then must ζ(1-s)=0 (s∈C and s≠ 1), is that to say ζ(s)=ζ(1-s)=0(s∈C and s≠ 1). According to Riemann definded $s=\frac{1}{2}+ti(t\in C \text{ and } t \neq 0)$, s and t differ by a linear transformation . It's a 90 degree rotation plus a translation of $\frac{1}{2}$. So line Re(s)=$\frac{1}{2}$ in the s plane corresponds to the real number line in the t plane,the zero of Riemann $\zeta(s)(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+}, \text{n traverses all positive integers})$ on the critical line Re(s)=$\frac{1}{2}$ corresponds to the real root of $\xi(t)(t\in C \text{ and } t \neq 0)$. In Riemann function $\xi(t)(t \in C \text{ and } t \neq 0)$, the function equation $\xi(s)=\xi(1-s)(s \in C, \text{and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+})$becomes equation $\xi(t)=\xi(-t)(t \in C \text{ and } t \neq 0)$ is an even function, an even function is a symmetric function, it's zeros are distributed symmetrically with respect to t=0 .The function $\xi(t)(t \in C\text{and } t \neq 0)$ designed by Riemann and $s=\frac{1}{2}+ti(t \in C \text{ and } t \neq 0)$ definded by Riemann and $\xi(s)=\xi(1-s)(s \in C, \text{and } s \neq 1\text{and } s \neq -2n, n \in Z^{+}, \text{n traverses all positive integers})$are equivalent to $\xi(t)=\xi(-t)(t \in C \text{ and } t \neq 0)$.So the function $\xi(s)(s \in C, \text{and } s \neq 1 \text{ and } s \neq -2n, \text{n traverses all positive integers})$ is also an even function.The zero points on the graph of an even function $\xi(s)(s \in C, \text{and } s \neq 1\text{and } s \neq -2n, \text{n traverses all positive integers})$ with respect to the

coordinates of its argument on the real number line equal to some value are symmetrically distributed on the line perpendicular to the real number line of the complex plane. When $\xi(t)=0(t \in C \text{ and } t \neq 0)$, is also that $\xi(t)=\xi(-t)=0(t \in C \text{ and } t \neq 0)$,the zeros of $\xi(t)(t \in C \text{ and } t \neq 0)$are symmetrically distributed with respect to t equals 0.When $\xi(s)=0(s \in C \text{ and } s \neq 1\text{and } s \neq -2n, n \in Z^{+},, \text{n traverses all positive integers})$,is also that $\xi(s)=\xi(1-s)=0(s \in C , \text{and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+})$,the zeros of $\xi(s)(s \in C , \text{and } s \neq 1\text{and } s \neq -2n, n \in Z^{+},, \text{n traverses all positive integers})$are symmetrically distributed with respect to point ($\frac{1}{2}$,0i) on a line perpendicular to the real number line of the complex plane.So when $\xi(s)=\xi(1-s)=0(s \in C, \text{and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+},, \text{n traverses all positive integers})$, s and 1-s are pair of zeros of the function $\xi(s)(s \in C, \text{and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+})$ symmetrically distributed in the complex plane with respect to point ($\frac{1}{2}$, 0i) on a line perpendicular to the real number line of the complex plane.When ζ(s)=0(s∈C and s≠ 1 $\text{and } s \neq -2n, n \in Z^{+}, \text{n traverses all positive}$ integers), then ζ(1-s)=0(s∈C and s≠ 1) , is aslo that ζ(s)=ζ(1-s)=0$(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n$,n∈Z+,n traverses all positive integers. We find ζ(s)=ζ(1-s)=0s∈C and s≠1and s≠−2n,n∈Z+,n traverses all positive integers and ξs=ξ1−s=0s∈C,and s≠1 and s≠−2n,n traverses all positive integers are just the name of the function is idifferent,the independent variable s is equal to $\frac{1}{2}$+ti(t∈C and $t \neq 0$),that means that the zero arguments of

function $\zeta(s)$ $\left(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, n \in Z^{+}\right)$ and function $\xi(s)(s \in C, \text{and } s \neq 1$ and $s \neq -2n$,,n traverses all positive integers are exactly the same,so the zeros of the $\zeta(s)(s \in C, \text{ and } s \neq 1, \text{and } s \neq -2n, n \in Z^{+},$
n traverses all positive integers function in the complex plane also correspond to the symmetric distribution of point ($\frac{1}{2}$, 0i) on a line perpendicular to the real number line in the complex plane, so When $\zeta(s) = \zeta(1 - s) = 0(s \in C, \text{and } s \neq 1 \text{and } s \neq -2n,$
$n \in Z^{+}$, n traverses all positive integers),s and 1-s are pair of zeros of the function $\zeta(s)(s \in$ C,and $s \neq 1$and $s \neq -2n, n \in Z+$,n traverses all positive integers symmetrically

distributed in the complex plane with respect to point ($\frac{1}{2}$, 0i) on a line perpendicular to the real number line of the complex plane.We got $\overline{\zeta(s)}$=$\zeta(\overline{s})$(s=σ+ti, σ ∈ R, t ∈R and $t \neq 0, \text{Re}(s) > 0$ and $s \neq 1$and $s \neq -2n$, n traverses all positive integers) before,When t in Riemann definded

s=$\frac{1}{2}$+ti(t∈C and $t \neq 0$) is a complex number, then s in $\overline{\zeta(s)}$=$\zeta(\overline{s})$(s=σ+ti, σ ∈ R, t ∈R and $t \neq 0, \text{Re}(s) > 0$ and $s \neq 1$and $s \neq -2n$, n traverses all positive integers) are consistent with s in Riemann's appoint s=$\frac{1}{2}$+ti(t∈C and $t \neq 0$) and s=$\frac{1}{2}$-ti(t∈C and $t \neq 0$).If $\zeta(s)=\zeta(\overline{s})=0(s \in$

$C, \text{Re}(s) > 0$ and $s \neq 1$and $s \neq -2n, n \in Z^{+}$),Since s and $\overline{s}$ are a pair of conjugate complex numbers,So s and $\overline{s}$ must be a pair of zeros of the function $\zeta(s)\left(s \in C, 0 < \text{Re}(s) < 1\text{and } s \neq 1\text{and } s \neq -2n, n \in Z^{+}\right)$ in the complex plane with respect to point (σ,0i) on a line perpendicular to the real number line.s is a conjugate and symmetric zero of 1-s, and a conjugate and symmetric zero of $\overline{s}$. By the definition of complex numbers, how can a symmetric zero of the same function $\zeta(s)\left(s \in C, 0 < \text{Re}(s) < 1 \text{ and } s \neq 1\text{and } s \neq -2n, n \in Z^{+}\right)$ of the same zero independent variable s on a line perpendicular to the real number axis of the complex plane be both a conjugate and symmetric zero of 1-s on a line perpendicular to the real number axis of the complex plane with respect to point ($\frac{1}{2}$, 0i) and a conjugate and symmetric zero of $\overline{s}$ on a line perpendicular to the real number axis of the complex plane with respect to point (σ, 0i)? Unless σ and $\frac{1}{2}$ are the same value, is also that $\sigma = \frac{1}{2}$, and only 1-s=$\overline{s}$ is true, and 1-s=s is wrong.Otherwise it's impossible,this is determined by the uniqueness of the zero of the function $\zeta(s)\left(s \in C, \text{and } s \neq 1\text{and } s \neq -2n, n \in Z^{+}\right)$ on the line passing through that point perpendicular to the real number axis of the complex plane with respect to the vertical foot symmetric distribution of the zero of the line and the real number axis of the complex plane,only one line can be drawn perpendicular from the zero independent variable s of the function $\zeta(s)\left(s \in C, \text{and } s \neq 1\text{and } s \neq -2n, n \in Z^{+}\right)$ to the real number line of the complex plane, the vertical line has only one point of intersection with the real number axis of the complex plane. In the same complex plane, the same zero point of the function $\zeta(s)\left(s \in C, 0 < \text{Re}(s) < 1 \text{ and } s \neq 1\right.$and $s \neq -2n, n \in Z+$ on the line passing through that point perpendicular to the real number line of the complex plane there will be only one zero point about the vertical foot symmetric distribution of the line and the real number line of the complex plane.Because $\overline{\zeta(s)}$=$\zeta(\overline{s})$($s \in C, \text{Re}(s) > 1$ and $s \neq 1$and $s \neq -2n, n \in Z^{+}$), then if$\zeta(\sigma + ti)$=0, then $\zeta(\sigma - ti)$=0,and because

$\zeta(s)=\zeta(1-s)=0\ (s\in C,\text{and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$,and $\sin(\frac{\pi s}{2}=0)$,so $s\in R$, and s=-2n($n\in Z^{+}$), $drop\ s=2n(n\ n\in Z^{+})$, then except s=-2n($n\in Z^{+}$) , the next three equations, $\zeta(\sigma+ti)$=0, $\zeta(\sigma-ti)$=0, and ζ(1-σ-ti)=0, are all true, so except s=-2n($n\in Z^{+}$) , only 1-σ=σ is true, then except s=-2n($n\in Z^{+}$),only s=$\frac{1}{2}$+ti(t∈R and $t\neq 0$) and s=$\frac{1}{2}$-ti(t∈R and $t\neq 0$)are true.Since the harmonic series $\zeta(1)$ diverges, it has been proved by the late medieval French scholar Orem (1323-1382).The Riemann hypothesis and the Riemann conjecture must satisfy the properties of the Riemann ζ(s)$(s\in C,\text{and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ function and the Riemann ξ(s)$(s\in C,\text{and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ function, The properties of the Riemann ζ(s)$=0(s\in C, Re(s)>0\text{ and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ function and the Riemann $\xi(s)=0$(s∈C , and s≠ 1) function are fundamental, the Riemann conjecture must be correct to reflect the properties of the Riemann ζ(s)$(s\in C,\text{and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ function and the Riemann $\xi(s)(s\in C\text{ and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ function, that is, the roots of the Riemann ξ(t)(t∈C and t≠ 0) function must only be real, that is, Re(s) can only be equal to $\frac{1}{2}$, and Im(s) must be real, and Im(s) is not equal to zero.So the Riemann hypothesis and the Riemann conjecture must be correct. Riemann got $\prod\frac{s}{2}$(s-1)$\pi^{-\frac{s}{2}}$ζ(s)=ξ(t)$(t\in R\text{ and } t\neq 0,\ s\in C,\text{and } s\neq 1, s\neq -2n, n\in Z^{+})$,and ξ(t)=$\frac{1}{2}$ -$(t^{2}+\frac{1}{4})\int_{1}^{\infty}\Psi(x)\,x^{-\frac{3}{4}}\cos\left(\frac{1}{2}t\ln x\right)dx$(t∈R and $t\neq 0$) in his paper, or

$\prod\frac{s}{2}$(s-1)$\pi^{-\frac{s}{2}}$ζ(s) $=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)$=ξ(t)$(t\in R\text{ and } t\neq 0,\ s\in C,\text{and } s\neq 1\text{and } s\neq -2n, n\in Z^{+})$ and ξ(t)=4 $\int_{1}^{\infty}\frac{d\left(x^{\frac{3}{2}}\Psi'(x)\right)}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)\ dx(t\in C\text{ and } t\neq 0, s\in C\text{ and } s\neq 1\text{ and } s\neq -2n, n\in Z^{+})$. Because ζ($\frac{1}{2}$+ti)=0(t∈C and $t\neq 0$), so the roots of

$\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)$=ξ(t)= 0(t∈C and $t\neq 0$) and $4\int_{1}^{\infty}\frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx$=ξ(t)= 0 (t∈C and $t\neq 0$) and ξ(t)=$\frac{1}{2}$ -$(t^{2}+\frac{1}{4})\int_{1}^{\infty}\Psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)$=0(t∈C and $t\neq 0$) must all be real numbers. According to the $2\sin(\pi s)\prod(s-1)\zeta(s)=\int_{\infty}^{\infty}\frac{x^{s-1}dx}{e^{x}-1}$ Riemann got in his paper and the $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s ∈ C and $s\neq 1$), We know that the Riemann ζ(s)(s∈C and s≠1) function is a two-to-one mapping, or even a many-to-one mapping deterministic universal function, or a one-to-two mapping, or even a one-to- many mapping deterministic universal function. If we consider the Riemann ζ(s)(s∈C and s ≠1) function as a general complex number whose domain includes real numbers, then s=-2n(n is a positive integer) is the only class of real zeros of the Riemann ζ(s)(s∈C and s≠1) function at the root, If we consider the Riemann ζ(s)(s∈C and s≠1) function as a general complex number whose domain does not include real numbers, then s=$\frac{1}{2}$+ti(t∈R and t≠0) is the only class of complex zeros of the Riemann ζ(s)(s∈C and s≠1) function at the root, so the zero real root of the Landau-Siegel function L(β, ✗(n))(β ∈R, ✗(n)=1) does not exist.

When $\zeta(s)=0(s\in C, \text{and } s\neq 1 \text{ and } s\neq -2n, n\in Z^{+})$ and ξ(t)=0($t\in C$ and $t\neq 0$), the real part of the equation ξ(t)=0(t∈C and $t\neq 0$) must be real between 0 and T. Because the real part of the equation ξ(t)=0 has the number of complex roots between 0 and T approximately equal to $\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$ [1] ,This result of Riemann's estimate of the number of zeros was rigorously proved by Mangoldt in 1895. Then,when $\zeta(s)=0$(s∈C and s≠ 1 and $s\neq -2n, n\in Z^{+}$) and ξ(t)=0($t\in C$ and $t\neq 0$), the number of real roots of the real part of the equation ξ(t)=0($t\in C$ and $t\neq 0$) between 0 and T must be approximately equal to $\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$,so when the Riemann $\zeta(s)$(s∈C and s≠ 1 and $s\neq -2n, n\in Z^{+}$) function has nontrivial zeroes, then and the Riemann conjecture are perfectly valid. N=$\lim_{T\to+\infty}(\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi})\to\infty$, so the Riemann ζ(s)(s∈C and s≠ 1 and $s\neq -2n, n\in Z^{+}$) function in Re(s)=$\frac{1}{2}$ nontrivial critical line zero have an infinite number, 1921, The British mathematician Hardy has proved that the Riemann ζ(s)(s∈C and s≠ 1 and $s\neq -2n, n\in Z^{+}$) function has an infinite number of non-trivial zeros on the critical line Re(s)= $\frac{1}{2}$, but he did not prove that the non-trivial zeros of the Riemann ζ(s)(s∈C and s≠ 1 and $s\neq -2n, n\in Z^{+}$) function are all on the critical line Re(s)= $\frac{1}{2}$.

Because the number of roots t of $\zeta\left(\frac{1}{2}+it\right)=\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(t\ln(n))-i\sin(t\ln(n)))=\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(\ln(n^{t}))-i\sin(\ln(n^{t})))=0$ is the number of roots of ξ(t)=$\frac{1}{2}$-$(t^2+\frac{1}{4})\int_1^{\infty}\Psi(x)\, x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)$=0. Because when t=0, then $\zeta\left(\frac{1}{2}\right)$ is divergent, when $\ln(n^{t})\in[0,2\pi]$, the numbers of the root t of

$\zeta\left(\frac{1}{2}+it\right)=\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(t\ln(n))-i\sin(t\ln(n)))=$

$\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(\ln(n^{t}))-i\sin(\ln(n^{t})))=0$ is $\ln\frac{T}{2\pi}-1$, so when t∈(0, T] , the numbers of the roots of

$\zeta\left(\frac{1}{2}+it\right)=\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(t\ln(n))-i\sin(t\ln(n)))=\sum_{n=1}^{\infty}(n^{-\frac{1}{2}}(\cos(\ln(n^{t}))-i\sin(\ln(n^{t})))=0$

is $N=n_1\times n_2=\frac{T}{2\pi}\times(\ln\frac{T}{2\pi}-1)$.

Definition: Assuming that a(n) is a uniproduct function, then the Dirichlet series $\sum_{n=1}^{\infty}a(n)n^{-s}$ ($s\in C$ and $s\neq 1, n\in Z^{+}$ and n goes through all the positive numbers) is equal to the Euler product

$\prod_p P(p，s)$ (s ∈ C and s ≠ 1, p ∈ Z^+ and p goes through all the prime numbers) .Where the product is applied to all prime numbers p, it can be expressed as: 1+$a(p)p^{-s}$+$a(p^2)p^{-2s}$+... , this can be seen as a formal generating function, where the existence of a formal Euler product expansion and a(n) being a product function are mutually sufficient and necessary conditions. When a(n) is a completely integrative function, an important special case is obtained,where $P(p，s)$(s ∈ C and s ≠ 1, p ∈ Z^+ and p goes through all the prime numbers) is a geometric series, and

$P(p,s) = \frac{1}{1-a(p)p^{-s}}$ (s ∈ C and s ≠ 1, p ∈ Z^+ and p goes through all the prime numbers) .When a(n)=1,it is the Riemann zeta function, and more generally the Dirichlet feature.

Euler's product formula: for any complex number s, $Rs(s) > 1$ and $s \neq 1$, then $\sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}$(s ∈ C and s ≠ 1, p ∈ Z^+ and p goes through all the prime numbers, n ∈ Z^+ and n goes through all positive numbers) , and when $Rs(s) >$ 1 Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$=$\prod_p (1-p^{-s})^{-1}$(s ∈ C and Rs(s) > 0 and s ≠ 1, n∈ Z^+, p ∈ Z^+, s ∈C， n goes through all the positive numbers, p goes through all the prime numbers).

Riemann ζfunction expression:

ζ(s)=$1/1^s+1/2^s+1/3^s+...+1/m^s$ (m tends to infinity, and m is always even).

(1) Multiply both sides of the expression by ($1/2^s$),

$(1/2^s)\zeta(s)=1/1^s(1/2^s)+1/2^s(1/2^s)+1/3^s(1/2^s)+...+1/m^s(1/2^s)=1/2^s+1/4^s+1/6^s+...+1/(2m)^s$

This is given by (1) - (2)

$\zeta(s)-(1/2^s)\zeta(s)=1/1^s+1/2^s+1/3^s+...+1/m^s-[1/2^s+1/4^s+1/6^s+...+1/(2m)^s]$

The derivation of Euler product formula is as follows:

$\zeta(s)-(1/2^s)\zeta(s)=1/1^s+1/3^s+1/5^s+...+1/(m-1)^s$.

Generalized Euler product formula:

Suppose f(n) is a functionthat satisfies $f(n_1)f(n_2)=f(n_1n_2)$ and $\sum_n |f(n)| < +\infty$ (n_1 and n_2 are both natural numbers), then$\sum_n f(n)$=$\prod_p[1+f(p)+f(p^2)+f(p^3)+\dots]$.

Proof:

The proof of Euler product formula is very simple, the only caution is to deal with infinite series and infinite products, can not arbitrarily use the properties of finite series and finite products. What I prove below is a more general result, and the Euler product formula will appear as a special case of this result.

Due to $\sum_{n=1}^{\infty} |f(n)| < +\infty$, so $1+f(p)+f(p^2)+f(p^3)+\dots$ absolute convergence.Consider the part of p<N in the continued product (finite product),Since the series is absolutely convergent and the product has only finite terms, the same associative and distributive laws can be used as ordinary finite summations and products.

Using the product property of f(n), we can obtain:

$\prod_{p<N}[1+f(p)+f(p^2)+f(p^3)+\ldots]$=$\sum f(n)$.The right end of the summation is performed on all natural numbers with only prime factors below N (each such natural number occurs only once in the summation, because the prime factorization of the natural numbers is unique).Since all natural numbers that are themselves below N obviously contain only prime factors below N, So

Σ'f(n) = $\sum_{n<N} f(n)$ + R(N),Where R(N) is the result of summing all natural numbers that are greater than or equal to N but contain only prime factors below N.From this we get: $\prod_{p<N}[1+$ f(p)+f(p2)+f(p3)+...]=n<N f(n) + R(N).For the generalized Euler product formula to hold, it is only necessary to prove $\lim_{n\to\infty} R(N)$ =0,and this is obvious,because |R(N)| $\leq \sum_{n\geq N} |f(n)|$,and $\sum_n |f(n)| < +\infty$ sign of
$\lim_{n\to\infty}\sum_{n\geq N} |F(n)|$ =0,thus $\lim_{n\to\infty} R(N)$ =0.Beacuse $1+f(p)+f(p^2)+f(p^3)+..$.= $1+$ f(p)+f(p)^2+f(p)^3+...=$[1-f(p)]^{-1}$, so the generalized Euler product formula can also be written as:
$\sum_n f(n)=\prod_p[1-f(p)]^{-1}$.In the generalized Euler product formula, take f(n)=n^{-s},Then

obviously $\sum_n |f(n)| < +\infty$ corresponds to the condition Rs(s)>1 in the Euler product formula, and the generalized Euler product formula is reduced to the Euler product formula.
From the above proof, we can see that the key to the Euler product formula is the basic property that every natural number has a unique prime factorization, that is, the so-called fundamental theorem of arithmetic.
For any complex number s, ✗(n) is the Dirichlet characteristic and satisfies the following properties:
1: There exists a positive integer q such that ✗(n+q)=✗(n);
2: when n and q are not mutual prime, ✗(n)=0;
3: ✗(a). ✗(b)=✗(ab) for any integer a and b;
Reasoning 3:
If $\mathrm{Re}(s)>1$,then

L(s, ✗(n))=$\sum_{n=1}^{\infty}\frac{✗(n)}{n^s}$ (n∈ Z_+, $p\in Z_+$, s ∈C and s≠ 1 , n goes through all the positive numbers, p goes through all the prime numbers, ✗(n)∈R

and (✗(n) ≠ 0),a(n) = a(p)=✗(n)),P(p , s)=$\frac{1}{1-a(p)p^{-s}}$),

If $\mathrm{Rs}(s)>0$ and s ≠ 1,then

ζ(s) is the Riemann $\zeta(s) = \frac{\eta(s)}{(1-2^{1-s})} = \frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$ (s ∈ C and $\mathrm{Rs}(s)>0$ and s ≠ 1 , n∈ Z^+ and n goes through all the positive integers, $p\in Z^+$ and p goes through all the prime numbers).
So If $\mathrm{Re}(s)>1$,then

GRH(s, ✗(n))=L(s, ✗(n))= $\sum_{n=1}^{\infty}\frac{✗(n)}{n^s}=\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_p P(p,s)=\prod_p(\frac{1}{1-a(p)p^{-s}})$ (n ∈ Z^+, $p\in Z^+$,s∈C and $\mathrm{Re}(s)>1$, n goes through all the positive integers, p goes through all prime numbers, $\chi(n) \in R$ and ($\chi(n) \neq 0$), $a(n) = a(p) = \chi(n)$), $P(p, s) = \frac{1}{1-a(p)p^{-s}}$).

$a(p)p^{-s} = a(p)p^{-\sigma}\frac{1}{(\cos(t\ln p)+i\sin(t\ln p))} = a(p)(p^{-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$($s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0$) ,

$(1 - a(p)p^{-s})$ =1- $a(p)(p^{-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$ = $1 - a(p)p^{-\sigma}\cos(t\ln p) + ia(p)p^{-\sigma}\sin(t\ln p)$($s \in C$ and $s \neq 1, t \in C$ and $t \neq 0$) ,

$a(p)p^{-\bar{s}} = a(p)p^{-\sigma}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))} = a(p)(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$($s \in C$, $Re(s) > 1, t \in C$ and $t \neq 0$) ,

$(1 - a(p)p^{-\bar{s}})$ =1- $a(p)p^{-\sigma}\cos(t\ln p) - ia(p)p^{-\sigma}\sin(t\ln p)$($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0$) ,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$ =[$1^{\sigma-1}\cos(t\ln 1) - 2^{\sigma-1}\cos(t\ln 2) + 3^{\sigma-1}\cos(t\ln 3) - 4^{\sigma-1}\cos(t\ln 4)$-...]+i[$1^{\sigma-1}\sin(t\ln 1) - 2^{\sigma-1}\sin(t\ln 2) + 3^{\sigma-1}\sin(t\ln 3) - 4^{\sigma-1}\sin(t\ln 4)$+...] ,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$=[$1^{-\sigma}\cos(t\ln 1) - 2^{-\sigma}\cos(t\ln 2) + 3^{-\sigma}\cos(t\ln 3) - 4^{-\sigma}\cos(t\ln 4)$-...]+i[$1^{-\sigma}\sin(t\ln 1) - 2^{-\sigma}\sin(t\ln 2) + 3^{-\sigma}\sin(t\ln 3) - 4^{-\sigma}\sin(t\ln 4)$+...]

($s \in C$ and $Re(s) > 1$, and n traverses all positive integers),

because

$a(p)p^{-s}$)=$\overline{1 - a(p)p^{-\bar{s}}}$ ($s \in C$ and $Re(s) > 1, p \in Z^+$ and p is a prime integer),

so

$(1 - a(p)p^{-s})^{-1}$=$\overline{(1 - a(p)p^{-\bar{s}})^{-1}}$ ($s \in C$ and $Re(s) > 1, p \in Z^+$ and p is a prime number),

So

$\prod_p(1 - a(p)p^{-s})^{-1}$=$\overline{\prod_p(1 - a(p)p^{-\bar{s}})^{-1}}$

($s \in C$ and $Re(s) > 1, p \in Z^+$ and p goes through all the prime numbers)) .

becuse $L(s, \chi(n)) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p(1 - a(p)p^{-s})^{-1}$ ($s \in C$ and $Re(s) > 1$)and
$L(\bar{s}, \chi(n)) = \sum_{n=1}^{\infty} a(n)n^{-\bar{s}} = \prod_p(1 - a(p)p^{-\bar{s}})^{-1}$ ($s \in C$ and $Re(s) > 1$),
($s \in C$ and $Re(s) > 1, n \in Z^+$ and n goes through all the positive integers, $p \in Z^+$ and p goes through all the prime numbers)).

For the Generalized Riemann function $L(s, \chi(n)) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p \frac{1}{1-a(p)p^{-s}}$

($\chi(n) \in R$ and ($\chi(n) \neq 0$, $a(n) = a(p) = \chi(n)$), $P(p, s) = \frac{1}{1-a(p)p^{-s}}$, $s \in C$ and $Re(s) > 1, n \in Z^+$ and n goes through all the positive integers, $p \in Z^+$ and p goes through all the prime numbers)) .so $L(s, \chi(n)) = \overline{L(\bar{s}, \chi(n))}$

($s \in C$ and $Re(s) > 1, n \in Z^+$ and n goes through all the positive integers).

$a(p)p^{1-s} = a(p)p^{(1-\sigma-ti)} = a(p)p^{1-\sigma}x^{-ti} = a(p)p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p)) - t$=a(p)p1−σ(costlnp−isin(tlnp))(s∈C and Res>1,t∈C and t≠0)

($s \in C$ and $s \neq 1, t \in C$ and $t \neq 0$, , $p \in Z^+$ and p goes through all the prime numbers),

$a(p)p^{1-\bar{s}} = a(p)p^{(1-\sigma+ti)} = a(p)p^{1-\sigma}p^{ti} = a(p)p^{1-\sigma}(p^{ti}) = a(p)p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p))^{t} = a(p)p^{1-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$ ($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0, p \in Z^{+}$ and p goes through all the prime numbers),
then

$a(p)p^{-(1-s)} = a(p)p^{\sigma-1}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))} = a(p)(p^{\sigma-1}(\cos(t\ln p) + i\sin(t\ln p))$ ($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0, p \in Z^{+}$ and p goes through all the prime numbers),

$(1 - a(p)p^{-(1-s)})$ =1- $a(p)p^{\sigma-1}(\cos(t\ln p) + i\sin(t\ln p)) = 1 - a(p)p^{\sigma-1}\cos(t\ln p) - a(p)p^{\sigma-1}i\sin(t\ln p)$
($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0, p \in Z^{+}$ and p goes through all the prime numbers,),
$\left(1 - a(p)p^{-\bar{s}}\right)$ =1- $a(p)(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p)) = 1 - a(p)p^{-\sigma}\cos(t\ln p) - ia(p)p^{-\sigma}\sin(t\ln p)$ ($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0, p \in Z^{+}$ and p goes through all the prime numbers),

When $\sigma=\frac{1}{2}$, then

$(1 - a(p)p^{-(1-s)}) = \left(1 - a(p)p^{-\bar{s}}\right)$ ($s \in C$ and $Re(s) > 1$),

$(1 - a(p)p^{-(1-s)})^{-1} = \left(1 - a(p)p^{-\bar{s}}\right)^{-1}$ ($s \in C$ and $Re(s) > 1$),

So
$\prod_{p}(1 - a(p)p^{-(1-s)})^{-1} = \prod_{p}(1 - a(p)p^{-\bar{s}})^{-1}$ ($s \in C$ and $Re(s) > 1$),becuse

$L(1-s, ✗(n)) = \prod_{p}(1 - a(p)p^{-(1-s)})^{-1}$ and $L(\bar{s}, ✗(n)) = \prod_{p}(1 - a(p)p^{-\bar{s}})^{-1}$, $n \in Z^{+}, p \in Z^{+}$, s∈ and $Re(s) > 1$, n goes through all the positive integers, p goes through all the prime numbers, ✗ (n)∈R and (✗(n) ≠ 0), $a(n) = a(p) =$ ✗(n), $P(p, s)=\frac{1}{1-a(p)p^{-s}}$),

So only $L\left(1-s, ✗(n)\right) = L\left(\bar{s}, ✗(n)\right)$

($s \in C$ and $Re(s) > 1, n \in Z^{+}$ and n goes through all positive integers),
and
Only $L(1-\bar{s}, ✗(n)) = L(s, ✗(n))$ ($s \in C$ and $Re(s) > 1$)
($s \in C$ and $s \neq 1, n \in Z^{+}$ and n goes through all positive integers),
Because $L(s, ✗(n)) =$ ✗(n)ζ(s) (s ∈C and $Re(s) > 1$, n∈ Z^{+} and n goes through all the positive integers), and $L(1-s, ✗(n)) =$ ✗(n) ζ(1-s) (s ∈C and $Re(s) > 1$,n∈ Z^{+} and n goes through all the positive integers),

so When only $\sigma=\frac{1}{2}$, it must be true that $L(s, ✗(n)) = \overline{L(\bar{s}, ✗(n))}$ (s ∈C and $Re(s) > 1$,n∈ Z^{+} and n goes through all the positive integers),and it must be true that
$L(1-s, ✗(n)) = L(\bar{s}, ✗(n))$ (s ∈ C and $Re(s) > 1$,n ∈ Z^{+} and n goes through all the positive integers), Suppose k∈ R,
$a(p)p^{k-s} = a(p)p^{(k-\sigma-ti)} = a(p)p^{k-\sigma}x^{-ti} = a(p)p^{k-\sigma}(\cos(\ln p) + i\sin(\ln p))-t = a(p)pk-\sigma\cos t\ln p - i\sin t\ln p$ (s∈C and Res>1,t∈C and t≠0,k∈R),
$a(p)p^{k-\bar{s}} = a(p)p^{(k-\sigma+ti)} = a(p)p^{k-\sigma}p^{ti} = a(p)p^{k-\sigma}(p^{ti}) = a(p)p^{k-\sigma}(\cos(\ln p) + i\sin(\ln p))^{t} = a(p)(p^{k-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$ ($s \in C$ and $Re(s) > 1, t \in C$ and $t \neq 0$, $k \in R$),

then

$a(p)p^{-(k-s)}$=$a(p)p^{\sigma-k}\frac{1}{(\cos(tlnp)-i\sin(tlnp))}$ =a(p)

$(p^{\sigma-k}(\cos(tlnp)+i\sin(tlnp))(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, k\in R)$,

$(1-a(p)p^{-(k-s)})$ =1- $(a(p)p^{\sigma-k}(\cos(tlnp)+i\sin(tlnp))$ = $1-a(p)p^{\sigma-k}\cos(tlnp)-ip^{\sigma-k}\sin(tlnp)(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, p\in Z^{+} \text{ and p is a prime numeber}, k\in R)$,

$(1-a(p)p^{-\bar{s}})$ =1- $(a(p)p^{-\sigma}(\cos(tlnp)+i\sin(tlnp))$ = $1-a(p)p^{-\sigma}\cos(tlnp)-ia(p)p^{-\sigma}\sin(tlnp)(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, \text{p is a prime numeber})$,

When $\sigma=\frac{k}{2}(k\in R)$,

then

$(1-a(p)p^{-(k-s)})$ =$(1-a(p)p^{-\bar{s}})(s\in C \text{and } Re(s)>1, \text{and p is a prime integer}, k\in R)$,

$(1-a(p)p^{-(k-s)})^{-1}$=$\left(1-a(p)p^{-\bar{s}}\right)^{-1}(s\in C \text{ and } Re(s)>1, \text{and p is a prime integer}, k\in R)$,

so

$\prod_p\left(1-a(p)p^{-(k-s)}\right)^{-1}$=$\prod_p\left(1-a(p)p^{-\bar{s}}\right)^{-1}$

$(s\in C \text{ and } Re(s)>1, k\in R, \text{and p goes through all the prime numbers}, k\in R)$,

becuse $L(k-s, \chi(n))$=$\prod_p\left(1-a(p)p^{-(k-s)}\right)^{-1}$), and $L(\bar{s}, \chi(n))$=0

$(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, k\in R, \text{and n goes through all positive integers})$, for the generalized Riemann function $L(s, \chi(n))(s\in C \text{ and } Re(s)>1$, and n goes through all the positive integers, $p\in Z^{+}$ and $\chi(n)\in R$ and $\chi(n)\neq 0$, $a(n)=a(p)=\chi(n)$),$P(p, s)=\frac{1}{1-a(p)p^{-s}}$).

So

Only $L\left(k-s, \chi(n)\right)$=$L\left(\bar{s}, \chi(n)\right)$

$(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, k\in R, n\in Z^{+} \text{ and n goes through all positive integers})$,

and

Only $L\left(k-\bar{s}, \chi(n)\right)$=$L\left(s, \chi(n)\right)$,

$(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, k\in R, \text{and n goes through all positive integers})$,

$k\in R)$,

And because Only $L\left(1-s, \chi(n)\right)$= $L\left(\bar{s}, \chi(n)\right)$

$(s\in C \text{ and } Re(s)>1, t\in C \text{ and } t\neq 0, k\in R, n\in Z^{+} \text{ and n goes through all positive integers})$,

,so only k=1 be true.

$$GRH\left(s, \chi(n)\right) = L\left(s, \chi(n)\right) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}}$$

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) ,

$$GRH(\overline{s}, \chi(n)) = L(\overline{s}, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\overline{s}}}$$

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) ,

$$GRH\left(1-s, \chi(n)\right) = L\left(1-s, \chi(n)\right) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{1-s}},$$

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) ,

Suppose

U=[$\chi(n)1^{-\sigma}$cos(tln1)$-\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-\chi(n)\,4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$Sin(tln1)$-$ $\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-\chi(n)4^{-\sigma}$sin(tln4)+...] ,

then

$L(s, \chi(n))=\overline{L(\overline{s}, \chi(n))}$

And n goes through all the positive numbers, so n=1,2,3,... ,let's just plug in, so

$L(s, \chi(n))= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{s}}$=[$\chi(n)1^{-\sigma}$cos(tln1)$-$ $\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-\chi(n)\,4^{-\sigma}$cos(tln4)+...]-i[$\chi(n)1^{-\sigma}$sin(tln1)$-$ $\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-$ $\chi(n)4^{-\sigma}$sin(tln4)+...]= U-Vi

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) ,

U=[$\chi(n)1^{-\sigma}$cos(tln1)$-\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-\chi(n)\,4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$sin(tln1)$-\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-\chi(n)4^{-\sigma}$sin(tln4)+...] ,

Then

$L(\overline{s}, \chi(n))= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{\overline{s}}}$=[$\chi(n)1^{-\sigma}$cos(tln1)$-\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-4^{-\sigma}$cos(tln4)+...]+i[$\chi(n)1^{-\sigma}$sin(tln1)$-\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-\chi(n)\,4^{-\sigma}$sin(tln4)+ ...]= U+Vi,

$(s \in C$and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) ,

U=[$\chi(n)1^{-\sigma}$cos(tln1)$-\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-\chi(n)\,4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$sin(tln1)$-\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-$ $\chi(n)4^{-\sigma}$sin(tln4)+...] ,

$L(s, \chi(n))$ and $L(\overline{s}, \chi(n))$ are complex conjugates of each other,that is

$L(s, \chi(n))=\overline{L(\overline{s}, \chi(n))}$

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^{+}$ and n goes through all positive integers) , When $\sigma=\frac{1}{2}$, then $L(s, \chi(n))=L(1-s, \chi(n))$= U-Vi

$(s \in C$ and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^+$ and n goes through all positive integers) ,

U=[$\chi(n)1^{-\sigma}$cos(tln1)$-\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)$-\chi(n)\ 4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$sin(tln1)$-\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)$-\ \chi(n)4^{-\sigma}$sin(tln4)+...] .

$GRH(k-s, \chi(n)) = L(k-s, \chi(n)) = \frac{\chi(n)\eta(k-s)}{(1-2^{1-k+s})} = \frac{\chi(n)}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-\rho-ti}} =$

$\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\chi(n)(\frac{1}{n^{k-\sigma}}\frac{1}{n^{-ti}}) = \frac{(-1)^{n-1}}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}(\chi(n)n^{\sigma-k}(\cos(tln(n)) + isin(tln(n))(s \in$ C and $Re(s) > 1$, $t \in C$ and $t \neq 0, k \in R, n \in Z^+$ and n goes through all positive integers) ,

W=[$\chi(n)1^{\sigma-k}$cos(tln1)$-\chi(n)2^{\sigma-k}$cos(tln2)+ $\chi(n)3^{\sigma-k}$cos(tln3)$-\chi(n)\ 4^{\sigma-k}$cos(tln4)+...]
U=[$\chi(n)1^{\sigma-k}$sin(tln1)$-\chi(n)2^{\sigma-k}$sin(tln2)+ $\chi(n)3^{\sigma-k}$sin(tln3)$-\ \chi(n)4^{\sigma-k}$sin(tln4)+...] .

When $\sigma=\frac{k}{2}$($k\in R$), then

Only $L(k-s, \chi(n))=L\left(\bar{s}, \chi(n)\right) = W - Ui.$

$(s \in C$ and $Re(s) > 1, k \in R, n \in Z^+$ and n goes through all positive integers) ,

which is $\zeta(k-s) = \zeta(1-s) = \zeta(\bar{s})$($s \in C$ and $Re(s) > 1$),so only k=1 be true.so only $Re(s)=\frac{k}{2}=\frac{1}{2}$ $(k \in R)$.So Only $L(1-s, \chi(n)) = L(\bar{s}, \chi(n))$($s\in C$ and $Re(s) > 1, n \in Z^+$) is true, so only k=1 is true.

Because when Re(s)>1 Euler ζ function is equivalent to the Riemann ζ function,so

ζ(s)= $\overline{\zeta(\bar{s})}$$(s = \sigma + ti, Re(s) > 1, \sigma \in R, t \in R$ and $s \neq 1)$ is true.According to the Euler product formula, when Re(s)>1, since every product factor in the Euler product formula is not equal to zero, so when Re(s)>1, Since each of the product factors in the Euler product formula is not equal to zero, when Re(s)>1, ζ(s) is not equal to zero, so the positive even number 2n($n \in Z^+$) can make sin($\frac{\pi s}{2}$) =0, but it is not the zero of Riemann ζ(s).

Because $L(s, \chi(n))=\chi(n)$ζ(s)(s∈C, Re(s)>1,$n \in Z^+$ and n traverse all positive integers, $\chi(n) \in R$, and $\chi(n) \neq$), because when Re(s)>1, ζ(s) has no zero, so when Re(s)>1, then $L(s, \chi(n))=\chi(n)$ζ(s)$\neq 0$(s∈C, Re(s)>1, , $n \in Z^+$ and n traverse all positive integers), so when Re(s)>1,L(s, ✗(n)) has no zero.

If $\mathrm{Re(s)} > 0$ and $\mathrm{s} \neq 1$,

when the Dirichlet eigen function ✗(n) is any real number that is not equal to zero, and η (s)=$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$ ($s \in C$ and $Rs(s) > 0$ and $s \neq 1$), $\zeta(s)$ is the Riemann $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$ ($s \in C$ and $Rs(s) > 0$ and $s \neq 1$, n∈ Z^+ and n goes through all the positive integers, $p \in Z^+$ and p goes through all the prime numbers),

$$L\left(s, ✗(n)\right)=\frac{✗(n)\eta(s)}{(1-2^{1-s})}=\frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma+ti}}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)\left(\frac{1}{n^{\sigma}}\frac{1}{n^{ti}}\right)=$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma})\frac{1}{(\cos\left(\ln((n)\right)+i\sin(\ln(n)))^{t}}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n))$$

$$L\left(\bar{s}, ✗(n)\right)=\frac{✗(n)\eta(s)}{\left(1-2^{1-\bar{s}}\right)}=\frac{✗(n)}{\left(1-2^{1-\bar{s}}\right)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}=\frac{✗(n)}{\left(1-2^{1-\bar{s}}\right)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma-ti}}$$

$$=\frac{(-1)^{n-1}}{\left(1-2^{1-\bar{s}}\right)}\sum_{n=1}^{\infty}✗(n)\left(\frac{1}{n^{\sigma}}\frac{1}{n^{-ti}}\right)=$$

$$\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma})\frac{1}{\left(\cos\left(\ln((n)\right)+i\sin(\ln(n))\right)^{-t}}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{t})$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)n^{-\sigma}(\cos(t\ln(n))+i\sin(t\ln(n))$$

When Rs(s)>0 and s≠1 ,because

$\frac{(-1)^{n-1}}{(1-2^{1-s})} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}}$,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \overline{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}}$,

$\prod_p(1-p^{-s})^{-1} = \overline{\prod_p(1-p^{-\bar{s}})^{-1}}$

$(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, p \in Z^+ \text{ and } p \text{ traves all prime numbers}),$
so

$\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \overline{\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}}$,

$\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}}$,

$\zeta(s) = \frac{1}{(1-2^{1-s})}\sum_1^{\infty}\frac{(-1)^{n-1}}{n^s} = \frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$,

$\zeta(\bar{s}) = \frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p\left(1-p^{-\bar{s}}\right)^{-1}$ $(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, n \in Z^+ \text{ and } n \text{ traves all positive integer}, p \in Z^+ \text{ and } p \text{ traves all prime numbers}),$
So

only $\zeta(s) = \overline{\zeta(\bar{s})}$ $(s \in C, Re(s) > 0 \text{ and } s \neq 1)$, so

$p^{1-s} = p^{(1-\sigma-ti)} = p^{1-\sigma}p^{-ti} = p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p))^{-t} = p^{1-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$,
$p^{1-\bar{s}} = p^{(1-\sigma+ti)} = p^{1-\sigma}p^{ti} = p^{1-\sigma}(p^{ti}) = p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p))^t = (p^{1-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,$(s\in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, t \in C \text{ and } t \neq 0, p \in Z+)$
then

$p^{-(1-s)} = p^{(-1+\sigma+ti)} = p^{\sigma-1}p^{ti} = p^{\sigma-1}\frac{1}{(\cos(t\ln p) - i\sin(t\ln p))} = (p^{\sigma-1}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(\bar{s})} = p^{-(\sigma-ti)} = p^{-\sigma}p^{ti} = (p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$

$(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, t \in C \text{ and } t \neq 0, p \in Z^+ \text{ and } p \text{ traves all prime numbers}),$
so
$(1-p^{-(1-s)}) = 1-(p^{\sigma-1}(\cos(t\ln p) + i\sin(t\ln p)) = 1 - p^{\sigma-1}\cos(t\ln p) - ip^{\sigma-1}\sin(t\ln p)$,

$(1-p^{-(\bar{s})}) = 1-(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p)) = 1 - p^{-\sigma}\cos(t\ln p) - ip^{-\sigma}\sin(t\ln p)$,
$(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, t \in C \text{ and } t \neq 0, p \in Z^+ \text{ and } p \text{ traves all prime numbers})$

when $\sigma = \frac{1}{2}$,then

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$ $(s \in C \text{ and } s \neq 1, \text{ and } n \text{ traves all positive integer}, k \in R)$,

$(1-p^{-(1-s)}) = (1-p^{-\bar{s}})$$(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, p \in Z^+)$,
and

$(1-p^{-(1-s)})^{-1} = \left(1-p^{-\bar{s}}\right)^{-1}$$(s \in C \text{ and } Rs(s) > 0 \text{ and } s \neq 1, p \text{ traves all prime numbers})$,

$\prod_p(1-p^{-(1-s)})^{-1} = \prod_p(1-p^{-\bar{s}})^{-1}$ $(s \in C \text{ and } s \neq 1, \text{ and } p \text{ traves all prime numbers}, k \in R)$,

$\prod_p \left(1-p^{-(1-s)}\right)^{-1}$=$\prod_p \left(1-p^{-\bar{s}}\right)^{-1}$

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1, p \in Z^+$ and p traves all prime numbers, $k \in R)$,

and

$\frac{1}{(1-2^s)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}} = \frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}}$,

$\frac{(-1)^{n-1}}{(1-2^s)} \prod_p (1-p^{-(1-s)})^{-1} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p (1-p^{-\bar{s}})^{-1}$

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1$, and n traverses all positive integers, $p \in Z^+$ and p traves all prime numbers),

And

$\zeta(1-s)$=$\frac{(-1)^{n-1}}{(1-2^s)} \prod_p (1-p^{-(1-s)})^{-1}$,

$\zeta(\bar{s})$=$\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p (1-p^{-\bar{s}})^{-1}$,

$\zeta(1-s)$=$\frac{1}{(1-2^s)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}}$,

$\zeta(\bar{s})$=$\frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}}$

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1, p \in Z^+$ and p traves all prime numbers, $n \in Z^+$ and n traves all positive integer),

so when σ=$\frac{1}{2}$, then only $\zeta(1-s)$=$\zeta(\bar{s})$($s \in C$ and $Rs(s) > 0$ and $s \neq 1$)must be true.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k-s}}$=[$1^{\sigma-k}$cos(tln1)$-$ $2^{\sigma-k}$cos(tln2)+ $3^{\sigma-k}$cos(tln3) $-4^{\sigma-k}$cos(tln4)-...]+i[$1^{\sigma-k}$sin(tln1) $-$ $2^{\sigma-k}$sin(tln2)+ $3^{\sigma-k}$sin(tln3) $-$ $4^{\sigma-k}$sin(tln4)+...],

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}}$=[$1^{-\sigma}$cos(tln1)$-2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)$-4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$Sin(tln1)$-$ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)$-4^{-\sigma}$sin(tln4)+...],

p^{k-s}=$p^{(k-\sigma-ti)}$=$p^{k-\sigma}p^{-ti}$=$p^{k-\sigma}(\cos(\ln p) + i\sin(\ln p))^{-t}$=$p^{k-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$,

$p^{1-\bar{s}} = p^{(1-\sigma+ti)} = p^{1-\sigma}p^{ti} = p^{1-\sigma}(p^{ti}) = p^{1-\sigma}(\cos(\ln p) + i\sin(\ln p))^t = (p^{1-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1, p \in Z^+$ and p traves all prime numbers, $n \in Z^+$ and n traves all positive integer, $k \in R)$,

Then

$p^{-(k-s)}$=$p^{(-k+\sigma+ti)}$=$p^{\sigma-k}p^{ti} = p^{\sigma-k}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$ =$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(\bar{s})}$=$p^{-(\sigma-ti)}$=$p^{-\sigma}p^{ti} = (p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(k-s)}$=$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1, p \in Z^+$ and and p is a prime number $k \in R)$,

so

$(1 - p^{-(k-s)})$=1-$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$ =$1 - p^{\sigma-k}\ \cos(t\ln p) - ip^{\sigma-k}\sin(t\ln p)$,

$(1 - p^{-\bar{s}})$=1-$(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$ =$1 - p^{-\sigma}\cos(t\ln p) - ip^{-\sigma}\sin(t\ln p)$,

$(s \in C$ and $Rs(s) > 0$ and $s \neq 1, p \in Z^+$ and and p is a prime number $k \in R)$, So when $\sigma=\frac{k}{2}$ (k ∈ R) ,then $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-k+s}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$ (s ∈ C and Rs(s) > 0 and s ≠ 1, k ∈ R, and n traves all positive integer,

$(1-p^{-(k-s)})=(1-p^{-\bar{s}})$ (s ∈ C and Rs(s) > 0 and s ≠ 1, k ∈ R, and p is a prime number),

and $(1-p^{-(k-s)})^{-1}=(1-p^{-\bar{s}})^{-1}$(s∈C, Rs(s) > 0 and s ≠ 1,k∈R, and p is a prime number),

$\prod_p\left(1-p^{-(k-s)}\right)^{-1}=\prod_p\left(1-p^{-\bar{s}}\right)^{-1}$

(s ∈ C and Rs(s) > 0 and s ≠ 1 and p traves all prime numbers, and n traves all positive integer, k ∈ R),

and

$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}=\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$

(s ∈ C and Rs(s) > 0 and s ≠ 1, p ∈ Z^+ and p traves all prime numbers, n ∈ Z^+ and n traves all positive integer, k ∈ R),

and

$\zeta(k-s)=\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\prod_p(1-p^{-(k-s)})^{-1}$,

$\zeta(\bar{s})=\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}$,

$\zeta(k-s)=\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$,

$\zeta(\bar{s})=\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$ (s∈Cand Rs(s) > 0),

(s ∈ C and Rs(s) > 0 and s ≠ 1, p ∈ Z^+ and p traves all prime numbers, n ∈ Z^+ and n traves all positive integer, k ∈ R),

so when $\sigma=\frac{k}{2}$(k∈R) then only $\zeta(k-s)=\zeta(\bar{s})$(s ∈ C and Rs(s) > 0 and s ≠ 1, k ∈ R).

According the equation $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s ∈ C and Rs(s) > 0 and s ≠ 1) obtained by Riemann,since Riemann has shown that the Riemann ζ(s) function has zero, that is, in

$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s ∈ C and Rs(s) > 0 and s ≠ 1)(Formula 6),

ζ(s)=0(s ∈ C and Rs(s) > 0 and s ≠ 1) is true.

When ζ(s)=0(s ∈ C and s ≠ 1), then only $\zeta(k-\bar{s})=\zeta(s)=0$(s ∈ C and s ≠ 1), and

When $\zeta(\bar{s})=0$(s ∈ C and s ≠ 1),then $\zeta(k-s)=\zeta(\bar{s})=0$(s ∈ C and s ≠ 1). And because when $\zeta(\bar{s})=0$(s ∈ C and s ≠ 1), then only $\zeta(1-s)=\zeta(\bar{s})=0$(s ∈ C and s ≠ 1), which is $\zeta(k-s)=\zeta(\bar{s})$(s ∈ C and s ≠ 1, k ∈ R),so only k=1 be true, so only $\zeta(s)=\overline{\zeta(\bar{s})}$ (s ∈ C Re(s) > 0, and s ≠ 1,is true.

$\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$ (s∈C and Rs(s) > 0 and s ≠ 1), when ζ(s)=0 and s≠ 2n(n ∈ Z^+) ,

then if $\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$ (s∈C and Rs(s) > 0 and s ≠ 1) is going to make sense, then the denominator $2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s) \neq 0$, Clearly indicates $2^s \neq 0$(s∈C and s≠ 1), $\pi^{s-1} \neq 0$(s∈C and s≠ 1), $\Gamma(1\text{-}s) \neq 0$(s∈C and $Rs(s) > 0$ and $s \neq 1$), so $\sin(\frac{\pi s}{2})$ can not equal to zero, so $\sin(\frac{\pi s}{2}) \neq 0$(s∈C and $Rs(s) > 0$ and $s \neq 1$), so So when ζ(s)=0(s∈C and $Rs(s) > 0$ and $s \neq 1$) and s≠ $2n(n \in Z^+)$, then $\zeta(1-s) = \zeta(s) = 0$ (s∈C(s∈C and $Rs(s) > 0$ and $s \neq 1$, and $s \neq -2n$, $n \in Z^+$).

Because $L(s, \chi(n))=\chi(n)\zeta(s)$
($s \in C$ ($s \in C$ and $Rs(s) > 0$ and $s \neq 1$, and n goes through all the positive integer) and
$L(1-s, \chi(n))=\chi(n)\zeta(1\text{-}s)$
(s∈C C and $Rs(s) > 0$ and $s \neq 1$, $n \in Z^+$ and n goes through all the positive integer) ,

and according to $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)$(s∈C C and $Rs(s) > 0$ and $s \neq 1$)(Formula 7), So only $L(s, \chi(n)) = 2^s\pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1\text{-}s) L(1-s, \chi(n))$($s \in C$ and $Rs(s) > 0$ and $s \neq 1$, $n \in Z^+$ and n goes through all the positive intege)(Formula 13).

According to the property that Gamma function Γ(s) and exponential function are nonzero, is also that $\Gamma(\frac{1-s}{2}) \neq 0$, and $\pi^{-\frac{1-s}{2}} \neq 0$, according to $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1\text{-}s)$(s∈C and $Rs(s) > 0$ and $s \neq 1$) (Formula 12), According the equation$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$

(s∈C and $Rs(s) > 0$ and $s \neq 1$)(Formula 6)obtained by Riemann,since Riemann has shown that the Riemann ζ(s) ($s \in C$ and $Rs(s) > 0$ and $s \neq 1$) function has zero, that is, in $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s∈C and s≠ 1) (Formula 6), so ζ(s)=0(s∈C and $Rs(s) > 0$ and $s \neq 1$) is true. According to the property that Gamma function Γ(s) and exponential function are nonzero, is also that $\Gamma(\frac{1-s}{2}) \neq 0$, and $\pi^{-\frac{1-s}{2}} \neq 0$,

So when ζ(s)=0(s∈C and $Rs(s) > 0$ and $s \neq 1$), then ζ(1-s)=0(s∈C and $Rs(s) > 0$ and $s \neq 1$), also must ζ(s)=ζ(1-s)=0(s∈C and $Rs(s) > 0$ and $s \neq 1$).

Because $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \sum_{n=0}^{\infty}\frac{1}{n!} \approx 2.7182818284...$,

and because $\sin(Z)=\frac{e^{iZ}-e^{-iZ}}{2i}$, Suppose Z=s= σ+ti (s∈C and $s \neq 1$, σ ∈R,t∈R), then

if s∈C and $s \neq 1$,because

$\sin(s)=\frac{e^{is}-e^{-is}}{2i} = \frac{e^{i(\sigma+ti)}-e^{-i(\sigma+ti)}}{2i}$,

$\sin(\overline{s})=\frac{e^{i\overline{s}}-e^{-i\overline{s}}}{2i} = \frac{e^{i(\sigma-ti)}-e^{-i(\sigma-ti)}}{2i}$,

according $x^s=x^{(\sigma+ti)}=x^\sigma x^{ti}=x^\sigma(\cos(\ln x) + i\sin(\ln x))^t=x^\sigma(\cos(t\ln x) + i\sin(t\ln x))(x > 0)$, then

$e^{s}=e^{(\sigma+ti)}=e^{\sigma}e^{ti}=e^{\sigma}(\cos(t)+i\sin(t)) = e^{\sigma}(\cos(t)+i\sin(t))$,

$e^{is}=e^{i(\sigma+ti)}=e^{\sigma i}(\cos(it)+i\sin(it)) = (\cos(\sigma)+i\sin(\sigma))(\cos(it)+i\sin(it))$,

$e^{i\bar{s}} = e^{i(\sigma-ti)}=e^{\sigma i}(\cos(-it)+i\sin(-it)) = (\cos(\sigma)+i\sin(\sigma))(\cos(it)-i\sin(it))$,

$e^{-is}=e^{-i(\sigma+ti)}=e^{-\sigma i}(\cos(-it)+i\sin(-it)) = (\cos(\sigma)-i\sin(\sigma))(\cos(it)-i\sin(it))$,

$e^{-i\bar{s}} = e^{-i(\sigma-ti)}=e^{-\sigma i}(\cos(it)+i\sin(it)) = (\cos(\sigma)-i\sin(\sigma))(\cos(it)+i\sin(it))$,

$2^{s}=2^{(\sigma+ti)}=2^{\sigma}2^{ti}=2^{\sigma}(\cos(\ln 2)+i\sin(\ln 2))^{t}=2^{\sigma}(\cos(t\ln 2)+i\sin(t\ln 2))$,

$2^{\bar{s}}=2^{(\rho-ti)}=2^{\sigma}2^{-ti}=2^{\sigma}(\cos(\ln 2)+i\sin(\ln 2))^{-t}=2^{\sigma}(\cos(t\ln 2)-i\sin(t\ln 2))$,

$\pi^{s-1}=\pi^{(\sigma-1+ti)}=\pi^{\sigma-1}\pi^{ti}=\pi^{\rho-1}(\cos(\ln\pi)+i\sin(\ln\pi))^{t}=\pi^{\sigma-1}(\cos(t\ln\pi)+i\sin(t\ln\pi))$,

$\pi^{\bar{s}-1}=\pi^{(\sigma-1-ti)}=\pi^{\sigma-1}\pi^{-ti}=\pi^{\sigma-1}(\cos(\ln\pi)+i\sin(\ln\pi))^{-t}=2^{\sigma-1}(\cos(t\ln\pi)-i\sin(t\ln\pi))$,

So

$2^{s}=\overline{2^{\bar{s}}}$, $\pi^{s-1}=\overline{\pi^{\bar{s}-1}}$,

and

$\frac{e^{is}-e^{-is}}{2i} = \overline{\frac{e^{i\bar{s}}-e^{-i\bar{s}}}{2i}}$,

So

$\sin(s)=\overline{\sin(\bar{s})}$,

and

$\sin(\frac{\pi s}{2})=\overline{\sin(\frac{\pi\bar{s}}{2})}$.

And the gamma function on the complex field is defined as:

$\Gamma(s)=\int_{0}^{+\infty} t^{s-1}\, e^{-t}dt$,

Among Re(s)>0, this definition can be extended by the analytical continuation principle to the entire field of complex numbers except for positive integers (zero and negative integers).

So

$\Gamma(s)=\overline{\Gamma(\bar{s})}$,

and

$\Gamma(1-s)=\overline{\Gamma(1-\bar{s})}$ .When $\zeta(1-\bar{s})=\overline{\zeta(1-\bar{s})}=0=\zeta(s)=\zeta(1-s)=0$ ($s\in C$ and $Rs(s) > 0$ and $s\neq 1$), and according $\zeta(s)=2^{s}\pi^{s-1}Sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ and $s\neq 1$ and $s \neq 0$)(Formula 7), then $\zeta(s)=\overline{\zeta(\bar{s})} = 0$($s\in C$,and $0 < Rs(s) < 1$ and $s\neq 1$),is also say $\zeta(s)=\zeta(\bar{s})=\zeta(1-\bar{s})=0$($s\in C$, and $0 < Rs(s) < 1$ and $s\neq 1$). so only $\zeta(\sigma+ti)=\zeta(\sigma-ti)=0$ is true.

If $\zeta(s)=0$($s = \sigma + ti, \sigma \in R, t \in R$ and $s \neq 1$), then $\zeta(s)= \zeta(\bar{s})=0$, it shows that the zeros of the Riemann ζ(s) function must be conjugate, then there must be $\zeta(s)=\zeta(\bar{s})=0$, indicating that the zeros of the Riemannian ζ(s) function must be conjugate , and in the critical band of Re(s)∈(0,1), there are no non-conjugate zeros.According $\zeta(s)= \zeta(\bar{s})=0$, if $s=\bar{s}$, then $s \in R$, because

$s=-2n$(n ∈ Z^+)make he function ζ(s)(s∈C and s≠1) has the value zero in 2sin(πs)Π(s − 1ζs=i0∞𝑥s−1d𝑥𝑥−1 and ζ(s)=2sπs−1sinπs2Γ1− sζ1− ss∈C and s≠1((Formula 7), so a negative even number can be the zero of Riemann ζ(s)(s∈C and s≠1). If $s \neq \bar{s}$, then s and $\bar{s}$ are not both real numbers but both imaginary numbers, t∈R and t≠0. And according to $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(s\in C\ and\ s\neq 1)$(Formula 7), if the ζ(s) = 0 (s ∈ C and s≠1) was established, then ζ(1-s)=ζ(s)=0(s ∈ C and s ≠ 1) must be true , because ζ(s)=$\overline{\zeta(\bar{s})}$,so when ζ(s)=0($s=\sigma+ti, \sigma\in R, t\in R, s\in C, Re(s)>0$ and $s\neq 1$ and $s\neq 1$),then ζ(s)= ζ($\bar{s}$)=0($s=\sigma+ti$, σ∈R,t∈R ,s∈C, *Res>0 and* s≠1, so the two zeros s and 1-s of Riemann ζ(s)(s∈C, *Res>0 and* s≠1) must also be conjugate. If either of s and 1-s are real numbers other than negative even numbers, since s and 1-s are conjugate, then s=1-s, then s=$\frac{1}{2}$. Since $\sin\left(\frac{\pi s}{2}\right)=\sin\left(\frac{\pi}{2}\times\frac{1}{2}\right)=\sin\left(\frac{\pi}{4}\right)\neq 0$, and because ζ($\frac{1}{2}$) diverge, then neither s nor 1-s are zeros of Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1), that is, Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1) has no real zeros other than negative even numbers. If Re(s)=1, then Re(1-s)=0, then s and 1-s are not conjugate, so Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1) has no zeros with real parts of 1 or 0. If Re(s)>1, then Re(1-s)<0, then s and 1-s are not conjugate, and because if Re(s)> 1, then Riemann ζ(s)(s∈C and s ≠1) has no zero, and according to ζ(s)=ζ(1-s)=0(s∈C and s≠1) was established, then when Re(s)<0, the ζ(s)(s ∈C and s≠1) is not equal to zero. Because when ζ(s)(s ∈C and s≠1), if the Re(s)=1, the Re(1-s)=0, then s and 1 s not conjugate, and according to ζ(s)=ζ(1-s)=0(s ∈C and s≠1) was established, so if Re(s)=0 or Re(s)=1, then ζ(s)(s∈C and s≠1) has no zero. Therefore, in addition to negative even numbers, Riemann ζ(s)+(s∈C and s≠1) has a zero if the value of Re(s) is in the interval (0,1).So in addition to negative even numbers, so the real part of Riemann ζ(s)(s∈C, $Re(s)>0$ and s≠1) zero s must be 0<Re(s)<1, that is, Re(s)∈(0,1), which shows that the prime number theorem holds. When 0<Re(s)<1,if s and 1-s are both real and imaginary, then s and 1-s are not conjugated, then s and 1-s cannot both be zeros of Riemann ζ(s)(s∈C, $0 < Re(s) < 1$,and s≠1), so 1-s and s can only be both imaginary and conjugate, and s cannot be pure imaginary, because if s is pure imaginary, then 1-s and s are not conjugated. So ζ(s)(s∈C, $0 < Re(s) < 1$ and s≠1) has no pure imaginary.And if Re(s)≠$\frac{1}{2}$, then Re(s)≠Re(1-s), then 1-s and s are not conjugate, so Re(s)≠$\frac{1}{2}$ cannot be true. So only 1-s=$\overline{s}$ is true, that is, only 1-σ-ti=σ-ti is true, so only σ=$\frac{1}{2}$, t∈R and t≠0, so the real part of the non-real zeros of Riemann ζ(s)(s∈C ,$0 < Re(s) < 1$) can only be , that is, only Re(s)=$\frac{1}{2}$ is true, Equivalent to $\xi(s) = 0\left(s = \frac{1}{2} + ti\right.$ or $s = 12 - ti, t$∈R and t≠0 , s∈C and s≠1or ξ12 +ti=0t∈R and t≠0 and ξ12 −ti=0t∈R and t≠0. Therefore, in the critical band of Re(s) ∈(0,1), Re(s)≠ $\frac{1}{2}$ is impossible, and there is no zero whose real part is not equal to $\frac{1}{2}$, so the Riemann conjecture holds. The symmetries of zeros s and zeros 1-s are not sufficient to prove that the nontrivial zeros of the Riemannn ζ(s)(s∈C, s≠1) function are on the critical line, and zeros s and zeros 1-s are symmetric only about the point ($\frac{1}{2}$,i) on the critical line. The conjugacy of s and 1-s is the fundamental reason why the nontrivial zeros of Riemann ζ(s)(s ∈ C, s ≠ 1) are all *located* on the critical line. According to ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C and s≠ 1) (Formula 6),so when ζ(s)=0, then ζ(s)=ζ(1-s)=0 is true. Because $\zeta(s) = \overline{\zeta(\overline{s})}(s \in C \text{ and } s \neq 1)$, then when ζ(s)=0 or ζ($\overline{s}$)=0, then it must be true that ζ(s) = ζ($\overline{s}$) =0. So when Riemann ζ(s)=0, then s and 1-s must also be conjugate. From this we get s=$\frac{1}{2}$+ti(t ∈ R and t ≠ 0), or s=$\frac{1}{2}$-ti(t ∈ R and t ≠ 0).According to the Euler product formula, when Re(s)>1, since every product factor in the Euler product formula is not equal to zero, so when Re(s)>1, Euler ζ function is equivalent to the Riemann ζ function,Since each of the product factors in the Euler product formula is not equal to zero, When Re(s)>1, ζ(s) is not equal to zero, so the positive even number 2n(n ∈ Z^+) can make sin($\frac{\pi s}{2}$) =0, but it is not the zero of

Riemann ζ(s). If s is any real number other than negative and positive even, and if it is the zero of the Riemannζ(s) function, then s and 1-s must be conjugate, for real numbers other than negative and positive even numbers, in addition to not making sin($\frac{\pi s}{2}$)=0, it must satisfy that s=1-s, then s=$\frac{1}{2}$, and function ζ($\frac{1}{2}$) diverge, so real numbers other than negative even numbers are not zeros of Riemann ζ(s). It holds that ζ(s)= ζ(1-s)=0($s \in \mathrm{C}$ and $s \neq 1$), and we know that the zero of ζ(s)(s∈C and $0 < Re(s) < 1$) is symmetric with respect to the point ($\frac{1}{2}$,0i). But is it possible to determine that the nontrivial zeros of the Riemann ζ(s) function are all on the critical boundary where the real part is equal to $\frac{1}{2}$, just because the zeros of ζ(s) are symmetric with respect to the point ($\frac{1}{2}$,0i)? Obviously not, when Re(s) ∈ (0,1), example s=0.54+ti($t \in \mathrm{R}$), Re(s)=0.54,then Re(1-s)=0.46,and 1-s are symmetric about the point ($\frac{1}{2}$,0i), but Riemann argued that such a complex number is not the zero of Riemann ζ(s).Riemann was right, and it is clear that when Re(s) is not equal to $\frac{1}{2}$, then s and 1-s must not be conjugate, and according to the zeros of the ζ(s) function must be conjugate, then if Re(s) is not equal to $\frac{1}{2}$, then it must not be the zero of the ζ(s) function. To sum up, the non-trivial zeros of the Riemann ζ(s) function must all lie on the critical boundary where the real part of the complex plane is equal to $\frac{1}{2}$, and the Riemann conjecture must be true.

According the equation ζ(1-s)=$2^{1-s}\pi^{-s}$cos($\frac{\pi s}{2}$)Γ(s)ζ(s)($s \in \mathrm{C}$ and $s \neq 1$)(Formula 6)obtained by Riemann,since Riemann has shown that the Riemann ζ(s)($s \in \mathrm{C}$ and $s \neq 1$) function has zero, that is, in ζ(1-s)= $2^{1-s}\pi^{-s}$ cos($\frac{\pi s}{2}$)Γ(s)ζ(s) ($s \in \mathrm{C}$, and $s \neq 1$)(Formula 6), ζ(s)=0 ($s \in \mathrm{C}$, and $s \neq 1$) is true, so when ζ(s)=0($s \in \mathrm{C}$, and $s \neq 1$), then only ζ(s)=ζ(1-s)=0($s \in \mathrm{C}$, and $s \neq 1$) is true.If ζ(s)=0($s = \sigma + ti, \sigma \in \mathrm{R}, t \in \mathrm{R}$ and $s \neq 1$), then ζ(s)= ζ($\overline{s}$)=0, it shows that the zeros of the Riemann ζ(s) function must be conjugate, then there must be ζ(s)=ζ($\overline{s}$)=0, indicating that the zeros of the Riemannian ζ(s) function must be conjugate , and in the critical band of Re(s)∈(0,1), there are no non-conjugate zeros.

So only when $\sigma = \frac{1}{2}$, it must be true that $L(s, ✗(n)) = \overline{L\left(\bar{s}, ✗(n)\right)}$ ($s \in C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers), and it must be true that $L(1-s, ✗(n))=L(\bar{s}, ✗(n))$($s∈C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers).
According $\zeta(1-s)=\zeta(s)=0$ ($s \in C$ and $s \neq 1$) and $\zeta(s)=\zeta(\bar{s})=\zeta(1-\bar{s})=0$ ($s \in C$ and $s \neq 1$), so $L(s, ✗(n))=L(1-s, ✗(n))=0$($s∈C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers) and $L(s, ✗(n))=L(\bar{s}, ✗(n))=L(1-\bar{s}, ✗(n))=0$($s∈C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers),
Because $L(s, ✗(n))= ✗(n)\zeta(s)$($s∈C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers) and $L(1-s, ✗(n))= ✗(n)\zeta(1-s)$($s∈C$ and $s \neq 1, n \in Z^{+}$ and n traverse all positive integers), and the Riemann conjecture must be correct.So $L\left(s, ✗(n)\right) = L\left(\bar{s}, ✗(n)\right) = 0$ and $L\Big(1-$

s, ✗ n=Ls, ✗ n=0,so s=s or s=1-s or s=1-s ,so s∈R and s=-2n(n∈Z+)),

or σ+ti=1-σ-ti, or σ-ti=1-σ-ti, so $s \in R$, and $s = -2n(n \in Z^{+})$, or $\sigma = \frac{1}{2}$ and t=0, or $\sigma = \frac{1}{2}$ and t ∈R and $t \neq 0$, so s∈ R, or s=$\frac{1}{2}$+0i ,or s=$\frac{1}{2}$+ti(t ∈R and $t \neq 0$),

because $\zeta\left(\frac{1}{2}\right) \to +\infty, \zeta(1) \to$

$+\infty, \zeta(1)$is divergent, $\zeta\left(\frac{1}{2}\right)$ is more divergent, then $L\left(1, ✗(n)\right) \to +\infty, L\left(\frac{1}{2}, ✗(n)\right) \to$

$+\infty, L\left(1, ✗(n)\right)$ is divergent, $L\left(\frac{1}{2}, ✗(n)\right)$ is more divergent, so drop $s = 1$ and drop $s =$

0. Only $s = \frac{1}{2} + ti$ ($t \in R$, and $t \neq 0$)and $s = \frac{1}{2} - ti$ ($t \in R$, and $t \neq 0$) are true, or say $s = \frac{1}{2} +$

ti ($t \in R$ and $t \neq 0$) and $s = \frac{1}{2} - ti$ ($t \in R$ and $t \neq 0$) are true. And beacause only when

σ=$\frac{1}{2}$,the next three equations,

$L\left(\sigma + ti, ✗(n)\right)$=0 (t∈R and $t \neq 0, n \in Z^{+}$ and n traverse all positive integers), $L\Big(1-\sigma-$

ti, ✗ n=0(t∈R and t≠0,n∈Z+ and n traverse all positive integers),and

$L\left(\sigma - ti, ✗(n)\right)$=0 (t∈R and $t \neq 0, n \in Z^{+}$ and traverse all positive integers) are all true. And

because $L(\frac{1}{2}, ✗(n))$>0, so only s=$\frac{1}{2}$+ti(t ∈R and $t \neq 0$) and s=$\frac{1}{2}$-ti(t ∈R and $t \neq 0$) are true.

The Generalized Riemann conjecture must satisfy the properties of the
$L(s, ✗(n)(s \in C$, and $s \neq 1$ and $s \neq -2n, n \in Z^{+}$ and n traverse all positive integers)
function, The properties of the

$L(s, \chi(n))(s \in C$, and $s1 \neq$ and $s \neq -2n, n \in Z^+$ and n traverse all positive integers)function are fundamental, the Generalized Riemann hypothesis and the Generalized Riemann conjecture must be correct to reflect the properties of the $L(s, \chi(n))(s \in C$, and $s \neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive integer) function , that is, the roots of $L(s, \chi(n))=0(s \in C$, and $s \neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive integers) can only be $s=\frac{1}{2}+ti(t \in R$ and $t \neq 0)$ and $s=\frac{1}{2}-ti(t \in R$ and $t \neq 0)$, that is, Re(s) must only be equal to $\frac{1}{2}$, and Im(s) must be real, and Im(s) is not equal to zero.So the Generalized Riemann conjecture must be correct.

According $L(1-s, \chi(n))=$ $L(s, \chi(n))=0(s \in C$, and $s \neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive integers),so the zeros Of $L(s, \chi(n))(s \in C$ and $s \neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive integers) function in the complex plane also correspond to the symmetric distribution of point $(\frac{1}{2}$,0i) on a line perpendicular to the real number line in the complexplane,

When $L(1-s, \chi(n)) = L(s, \chi(n)) = 0(s \in C$ and $s \neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive numbers), s and 1-s are pair of zeros of the function $L(s, \chi(n))(s \in C$ and $s \neq 1, n \in Z^+$ and n traverse all positive numbers) symmetrically distributed in the complex plane with respect to point $(\frac{1}{2}$,0i) on a line perpendicular to the real number line of the complex plane.We got $\overline{L(s, \chi(n))}$ $=L(\overline{s}, \chi(n))$(and $s \neq 1$and n traverse all positive numbers and n traverse all positive integers) before,When t in $s=\frac{1}{2}+ti(t \in C$ and $t \neq 0)$ defined by Riemann is a complex number, and then s in $\overline{L(s, \chi(n))} = L(\overline{s}, \chi(n))$ ($s \in C$ and $s \neq 1$ and $s \neq -2n, n \in Z^+$) is consistent with s in $s=\frac{1}{2}+ti(t \in C$ and $t \neq 0)$defined by Riemann, so only $\sigma = \frac{1}{2}$.When $L(s, \chi(n)) = L(\overline{s}, \chi(n)) = 0(s \in C$ and $s \neq 1$, $n \in Z^+$),since s and $\overline{s}$ are a pair of conjugate complex numbers, so s and $\overline{s}$ must be a pair of zeros of the Generalized function $L(s, \chi(n))(s \in C$ and $s \neq 1$, and $s \neq -2n, n \in Z^+$, and n traverse all positive numbers) in the complex plane with respect to point(ρ,0i) on a line perpendicular to the real number line.s is a symmetric zero of 1-s, and a symmetric zero of $\overline{s}$. By the definition of complex numbers, how can a symmetric zero of the same Generalized Riemann function $L(s, \chi(n))(s \in C$ and $s \neq 1$, and $s \neq -2n, n \in Z^+$, and n traverse all positivenumbers) of the same zero independent variable s on a line perpendicular to the real number axis of the complex plane be both a symmetric zero of 1-s on a line perpendicular to the real number axis of the complex plane with respect to point $(\frac{1}{2}$,0i) and a symmetric zero of $\overline{s}$ on a line perpendicular to the real number axis of the complex plane with respect to point (σ, 0i)? Unless σ and $\frac{1}{2}$ are the same value, is also that $\sigma = \frac{1}{2}$, and only 1-s=$\overline{s}$ is true, only s=$\frac{1}{2}$+ti($t \in R$ and $t \neq 0$) and s=$\frac{1}{2}$-ti($t \in R$ and $t \neq 0$) are true. Otherwise it's impossible,this is determined by the uniqueness of the zero of Generalized Riemann function

$L(s, \chi(n))(s \in C, \text{and n traverse all positive numbers})$ on the line passing through that point perpendicular to the real number axis of the complex plane with respect to the vertical foot symmetric distribution of the zero of the line and the real number axis of the complex plane,Only one line can be drawn perpendicular from the zero independent variable s of $L(s, \chi(n))(s \in C \text{ and } s \neq 1, \text{and } s \neq -2n, n \in Z^+, \text{and n traverse all positivenumbers})$on the real number line of the complex plane, the vertical line has only one point of intersection with the real number axis of the complex plane. In the same complex plane, the same zero point of

$L(s, \chi(n))(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, n \in Z^+ \text{ and n traverse all positive integers})$ on the line passing through that point perpendicular to the real number line of the complex plane there will be only one zero point about the vertical foot symmetric distribution of the line and the real number line of the complex plane,so I have proved the generalized Riemann conjecture when the Dirichlet eigen function χ(n)(n$\in Z^+$ and n traverse all positive numbers) is any real number that is not equal to zero,Since the nontrivial zeros of the Riemannian function $\zeta(s)$(s∈C and s$\neq 1$) and$L(s, \chi(n)(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, n \in Z^+ \text{ and n traverse all positive integers})$ are both on the critical line perpendicular to the real number line of Re(s)=$\frac{1}{2}$ and Im(s)$\neq 0$, these nontrivial zeros are general complex numbers of Re(s)=$\frac{1}{2}$ and Im(s)$\neq 0$,so I have proved the generalized Riemann conjecture when the Dirichlet eigen

function χ(n)($n \in Z^+$ and n traverse all positivel intergers) is any real number that is not equal to zero. The Generalized Riemann conjecture must satisfy the properties of the $L(s, \chi(n))$(s∈C and s$\neq 1, n \in Z^+$ and n traverse all positive intergers) function, The properties of the $L(s, \chi(n))$ (s∈C and s$\neq 1$ and $s \neq -2n, n \in Z^+$ and n traverse all positive numbers) function are fundamental, the Generalized Riemann conjecture must be correct to reflect the properties of the

$L\left(s, \chi(n)\right)(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, \text{ and n traverse all positive intergers})$ function, that is, the roots of the

$L(s, \chi(n))=0(s \in C \text{ and } s \neq 1 \text{ and } s \neq -2n, n \in Z^+ \text{ and n traverse all positive intergers})$ can only be s=$\frac{1}{2}$+ti(t∈Rand $t \neq 0$)or s=$\frac{1}{2}$-ti(t∈R and $t \neq 0$), that is, Re(s) can only be equal to $\frac{1}{2}$, and Im(s) must be real, and Im(s) is not equal to zero.When $L(s, \chi(n)) = 0$ (s∈C and s$\neq 1$, and $s \neq -2n, n \in Z^+$ and n traverse all positive numbers n goes through all the positive integers, $\chi(n) \in R$ and $\chi(n) \neq 0$), $a(n) = a(p)=\chi(n)$, P(p , s)=$\frac{1}{1-a(p)p^{-s}}$),then the

Generalized Riemann must be correct, and $s=\frac{1}{2}+ti(t\in R \text{ and } t\neq 0)$ or $s=\frac{1}{2}-ti(t\in R \text{ and } t\neq 0$.

because ζ(s)=$2^s\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)(s∈C and s$\neq$ 1)(Formula 7) .s=-2n(n∈ Z^+) is the trivial zero of the ζ(s)function, so s=-2n(n ∈ Z^+) is the trivial zero of the Landau-Siegel function L(β, ✗(n))(β∈R, ✗(n) ∈R and ✗(n)≠0,n∈ Z^+ and n traverses all positive integers). So when the

dirichlet characteristic function ✗(n)≡1, then s=-2n(n∈ Z^+) is the zero of landau - siegel function L(β, ✗(n)) (β∈R and ✗(n) = 1) . Soif s=β(β ∈R) and β=-2n(s∈C), then L(β, ✗(n))=0 and ζ(s)=0.

For any complex number s,when ✗(n) is the Dirichlet characteristic and satisfies the following properties:

1: There exists a positive integer q such that ✗(n+q)= ✗(n)(n ∈ Z^+);

2: when n and q are not mutual prime, ✗(n)=0(n ∈ Z^+);

3: ✗(a)✗(b)= ✗(ab) (a ∈ Z^+, b ∈ Z^+)for any integer a and b; Suppose q=2k(k ∈ Z^+),

if n and n+q are all prime number, and if ✗(Y) = 0 (Y traverses all positive odd numbers) and ✗(n + q) = ✗(n) =0(n and n + q traverses all positive odd numbers),because n(n traverses all prime numbers) and q=2k(k ∈ Z^+) are not mutual prime, then ✗(n)=0 (n ∈ Z^+ and n and n + q traverses all prime numbers) andfor any prime number a and b, ✗(a)✗(b) = ✗(ab) (a ∈ Z^+, b ∈ Z^+ , a traverses all prime numbers and b traverses all prime number, then the three properties described by the Dirichlet eigenfunction ✗(n)(n ∈ Z^+ and n traverses all prime numbers). above fit the definition of the Polignac conjecture, the Polignac conjecture states that for all natural numbers k, there are infinitely many pairs of prime numbers (p,p+2k)(k ∈ Z^+). In 1849, the French mathematician A. Polignac proposed the conjecture.When k=1, the Polygnac conjecture is equivalent to the twin prime conjecture.

$$GRH\left(s, ✗(n)\right)=L\left(s, ✗(n)\right)=\frac{✗(n)\eta(s)}{(1-2^{1-s})}=\frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma+ti}}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)\left(\frac{1}{n^{\sigma}}\frac{1}{n^{ti}}\right)=$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma})\frac{1}{(\cos\left(\ln((n)\right)+i\sin(\ln(n)))^t}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n))\ )$$

($t \in C$ and $t \neq 0, s \in C$ and $s \neq 1, n \in Z^{+}$ and n goes through all positive integers) ,because ζ(s)=$2^{s}\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)(s∈C and s≠ 1)(Formula 7) ,so if β∈R and $\beta = -2n(n \in Z^{+})$,then ζ(s)=0.

So L(β, ✗(n))=

$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{n=1}^{\infty}$ ✗$(n)(n^{-\beta}(\cos(0 \times \ln(n)) + i\sin(0 \times \ln(n)))$=$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{n=1}^{\infty}($✗$(n)n^{-\beta})$ =

$\frac{1}{\left(1-2^{1-\beta}\right)}($✗$(1)1^{-\beta} -$ ✗$(2)2^{-\beta} +$ ✗$(3)3^{-\beta} -$ ✗$(4)4^{-\beta} + \cdots)$, " × " is the symbol for multiplication, because the real exponential function of the real number has a function value greater than zero, because ✗(n) $\in R$ *and* ✗(1) = ✗(2) = ✗(3) = ✗(3), … , so $n^{-\beta} > 0$($n \in Z^{+}$ and n traverses all positive integers) and $1^{\beta} - 2^{\beta} < 0, 3^{\beta} - 4^{\beta} < 0, 5^{\beta} - 6^{\beta} < 0, \ldots, (n-1)^{\beta} - (n)^{\beta} < 0, \ldots,$ *or* $1^{\beta} - 2^{\beta} > 0, 3^{\beta} - 4^{\beta} > 0, 5^{\beta} - 6^{\beta} > 0, \ldots, (n-1)^{\beta} - (n)^{\beta} > 0$, and $\frac{1}{\left(1-2^{1-\beta}\right)} \neq 0$, it can be known that if ✗(n) ≠0(✗(n) $\in R$, $n \in Z^{+}$ and n traverses all positive integers), and β ∈R and $\beta \neq -2n(n \in Z^{+})$, then L(β, ✗(n))≠0(β∈R and $\beta \neq -2n$, $n \in Z^{+}$, ✗(n) ∈R and n traverses all positive integers) and L(β, 1)≠0(β∈R and $\beta \neq -2n$, $n \in Z^{+}$, and n traverses all positive integers), so for Riemann ζ(s)(s∈C and s≠ 1and $s \neq -2n$, $n \in Z^{+}$) functions, its corresponding landau-siegel function L(β, ✗(n))(β∈R and $\beta \neq -2n$, $n \in Z^{+}$, ✗(n) ∈R and n traverses all positive integers) of pure real zero does not exist, If s≠-2n($n \in Z^{+}$), the other Landau-Siegel functions L(β, ✗(n))(β∈R and β≠-2n, $n \in Z^{+}$, ✗(n) ∈R and n traverses all posite integers) also do not exist pure real zeros, this means that if s ≠-2n($n \in Z^{+}$),then the Riemann ζ(s)(s∈C and s≠ 1 and $s \neq -2n$, $n \in Z^{+}$) function does not have a zero of a pure real variable s, and this means that if s≠-2n($n \in Z^{+}$),then the generalized Riemannian L(s, ✗(n))=0(s∈C and s≠ 1, and $s \neq -2n$, $n \in Z^{+}$, ✗(n) ∈R and n traverses all positive integers) function also has no pure real zeros of the variable s,then the generalized Riemann conjecture L(s, ✗(n))=0(s∈C and s≠ 1, and $s \neq -2n$, $n \in Z^{+}$, ✗(n) ∈R and n traverses all positive integers) satisfies s=$\frac{1}{2}$+ti(t∈R,t≠0) and s=$\frac{1}{2}$-ti(t∈R,t≠0) is sufficient to prove that the twin primes, Polignac's conjecture and Goldbach's conjecture are almost true. And if ✗(n) = 0($n \in Z^{+}$ and n traverses all positive integers) or β ∈ R and $\beta = -2n(n \in Z^{+})$, then L(β, ✗(n))=0(β∈R and $\beta \neq -2n$, $n \in Z^{+}$, ✗(n) ∈R and n traverses all positive integers) and L(β, 1)=0(β∈R and $\beta \neq -2n$, $n \in Z^{+}$, and n traverses all positive integers), so for Riemann ζ(s)(s∈C and s≠ 1) functions, its corresponding landau-siegel function L(β, ✗(n))(β ∈ R, ✗(n) ∈ R , and $s \neq -2n$, $n \in Z^{+}$, and n traverses all positive integers) of pure real zero exist, this means that the Riemann ζ(s)(s∈C and s≠ 1) function have a zero of a pure real variable s, and the generalized Riemann conjecture L(s, ✗(n))=0(s∈C and s≠ 1,

χ(n) ∈R ,and $s \neq -2n$, $n \in Z^+$ and n traverses all positive integers) is sufficient to prove that the twin primes, Polignac's conjecture and Goldbach's conjecture are completely true.

when χ(n)≠1($n \in Z^+$ and n traverses all positive integers) and χ(n)≠0($n \in Z^+$ and n traverses all positive integers), because the real exponential function of the real number has a function value greater than zero, so

$n^{-\beta} > 0$($n \in Z^+$ and n traverses all positive integers) and $1^\beta - 2^\beta < 0, 3^\beta - 4^\beta < 0, 5^\beta - 6^\beta < 0, \ldots, (n-1)^\beta - (n)^\beta < 0, \ldots, or\ 1^\beta - 2^\beta > 0, 3^\beta - 4^\beta > 0, 5^\beta - 6^\beta > 0, \ldots, (n-1)^\beta - (n)^\beta > 0$ and $|\frac{1}{(1-2^{1-\beta})}| \neq 0$, it can be known that when χ(n)=1(n∈ Z^+ and n traverse all positivel numbers), then L(β,1)≠0(β∈R and β≠ $-2n$, $n \in Z^+$, χ(n) ∈R and χ(n)=1, n traverses all positive integers) so for Riemann ζ(s)(s∈C and s≠ 1) functions, its corresponding landau -siegel function L(β, χ(n))(β∈R, χ(n)∈R and χ(n) ≠0, $n \in Z^+$ and n traverses all positive integers) of pure real zero does not exist, this means that the generalized Riemann L(β, χ(n))(β∈R, χ(n)∈R and $n \in Z^+$ and n traverses all positive integers) function does not have a zero of a pure real variable s, and the generalized Riemann conjecture L(s, χ(n))=0(s∈C and s≠ 1, χ(n)∈R and χ(n)≡1 and $n \in Z^+$ and n traverses all positive integers) satisfies s=$\frac{1}{2}$+ti(t ∈ R,t≠0) and s=$\frac{1}{2}$-ti(t ∈ R,t≠0) is sufficient to prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are almost true.

When χ(n) ≠1 ($n \in Z^+$ and n traverses all positive integers) and χ(n) ≠0 ($n \in Z^+$ and n traverses all positive integers), because the real exponential function of the real number has a function value greater than zero, so

$n^{-\beta} > 0$($n \in Z^+$ and n traverses all positive integers) and $1^\beta - 2^\beta < 0, 3^\beta - 4^\beta < 0, 5^\beta - 6^\beta < 0, \ldots, (n-1)^\beta - (n)^\beta < 0, \ldots, or\ 1^\beta - 2^\beta > 0, 3^\beta - 4^\beta > 0, 5^\beta - 6^\beta > 0, \ldots, (n-1)^\beta - (n)^\beta > 0$ and $|\frac{1}{(1-2^{1-\beta})}| \neq 0$, it can be known that when χ(n) ≠1($n \in Z^+$ andn traverses all positive integers) and χ(n) ≠ 0($n \in Z^+$ and n traverses all positive integers), then L(β, χ(n))≠0(β∈R and β≠ $-2n$, $n \in Z^+$, χ(n)∈R and χ(n) ≠1 and χ(n) ≠0 and n traverses all positive integers) ,so for generalized Riemann L(s, χ(n)) (s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$ and n traverses all positive integers) functions, its corresponding landau-siegel function L(β, χ(n))(β∈R and β≠ $-2n$, $n \in Z^+$, χ(n)∈R and χ(n) ≠1 and χ(n) ≠0, $n \in Z^+$ and n traverses all positive integers) of pure real zero does not exist, this means that the generalized Riemann L(s, χ(n))(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$ and n traverses all positive integers) function does not have a zero of a pure real variable s. and the generalized Riemann conjecture L(s, χ(n))=0(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$, χ(n)∈R and χ(n)≠1 and χ(n)≠0 and n traverses all positive integers) satisfies s=$\frac{1}{2}$+ti(t∈R,t≠0) and s=$\frac{1}{2}$-ti(t∈R,t≠0)is sufficient to prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are all almost true.

When χ(n) ≡0 ($n \in Z^+$ and n traverses all positive integers), because the real exponential function of the real number has a function value greater than zero, so $n^{-\beta} > 0$($n \in Z^+$ and n traverses all positive integers) and $\chi(1)1^\beta = 0, \chi(2)2^\beta = 0, \chi(3)3^\beta = 0, \chi(4)4^\beta = 0, \chi(5)5^\beta = 0, \chi(6)6^\beta = 0, \ldots, \chi(n-1)(n-1)^\beta = 0,$ $\chi(n)n^\beta = 0, \ldots,$ and $|\frac{1}{(1-2^{1-\beta})}| \neq 0$, it can be known that when χ(n) ≡0 ($n \in Z^+$ and n traverses all positive integers), then L(β, ✗(n))≠0(β∈R and $\beta\neq-2n$, $n\in Z^{+}$, ✗(n)∈R and ✗(n) ≡1, $n\in Z^{+}$ and n traverses all positive integers) and L(β, 1)≠0(β∈R and $\beta\neq-2n$, $n\in Z^{+}$, and n traverses all positive integers), so for generalized Riemann L(s, ✗(n))(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$ and n traverses all positive integers) functions, its corresponding landau-siegel function L(β,0)(β∈R and $\beta\neq-2n$, $n\in Z^{+}$, ✗(n)∈R and ✗(n) ≡0 and n traverses all positive integers) of pure real zero exists, This means that the generalized Riemann L(s, ✗(n))(s∈C and s≠ 1and $s\neq-2n, n\in Z^{+}$ and n traverses all positive integers) function has a zero of a pure real variable s, that means the twin prime conjecture, Goldbach's conjecture, Polignac's conjecture are completely true.

When ✗(p)≡0(p ∈ Z^{+} and p traverses all odd primes, including 1), then L(s, ✗(p))=0(s∈C and s≠ 1and $s\neq-2n, n\in Z^{+}$, ✗(p) ∈R and ✗(p)≡0, p ∈ Z^{+} and p traverses all odd primes, including 1) was established. At the same time L(s, ✗(p))(s∈C and s≠ 1and $s\neq-2n, n\in Z^{+}$, ✗(p) ∈ R and ✗(p) ≡ 0, p ∈ Z^{+} and p traverses all odd primes, including 1) the corresponding landau-siegel function
L(β,0)(β∈R and $\beta\neq-2n$, $n\in Z^{+}$, ✗(p)∈R and ✗(p)≡0, p ∈ Z^{+} and p traverses all odd primes, including 1) expression as shown as follows: L(β, ✗(p))=$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{p=1}^{\infty}$ ✗$(p)p^{-\beta}(\cos(0\times \ln p)+i\sin(0\times\ln(p)))=$
$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{p=1}^{\infty}($✗$(p)p^{-\beta})$ = $\frac{(-1)^{n-1}}{(1-2^{1-\beta})}$[✗$(1)1^{-\beta}$ - ✗$(2)2^{-\beta}$ + ✗$(3)3^{-\beta}$ - ✗$(5)5^{-\beta}$ + ✗$(7)7^{-\beta}$ + … − ✗$(p)p^{-\beta}+\cdots$](β ∈R, p ∈ Z^{+} and p traverses all primes, including 1), " × " is the symbol for multiplication.

When ✗(p)≡0(p ∈ Z^{+} and p traverses all odd primes, including 1), then L(s, ✗(p))≡0(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$, ✗(n) ∈R and ✗(p)≡0, p trav erses all odd primes, including 1) was established. At the same time L(s, ✗(p))(s∈C and s ≠ 1 and $s\neq-2n, n\in Z^{+}$, ✗(p) ∈ R and ✗(p)≡0, p ∈ Z^{+} and p traverses all primes, including 1) the corresponding landau-siegel function L(β,0)=0(β∈R and $\beta\neq-2n$, $n\in Z^{+}$, ✗(p)∈R and ✗(p)≡0, p∈ Z^{+} and p traverses all primes, including 1), this means that the generalized Riemann L(s, ✗(n))(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$ and n traverses all positive integers) function has a zero of a pure real variable s, that means the twin prime conjecture, Goldbach's conjecture, Polignac's conjecture are all completely true.

Now I summarize the Dirichlet function L(s, ✗(n))(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$, ✗(n) ∈R, and n traverses all positive integers) as follows:

1: When ✗(n)≡1(s ∈ C and s ≠ 1 and $s\neq-2n, n\in Z^{+}$ and n traverses all positive integers), the generalized Riemannian hypothesis and the generalized Riemannian conjecture degenerate to the ordinary Riemannian hypothesis and the ordinary Riemannian conjecture, whose nontrivial zeros s satisfy s=$\frac{1}{2}$+ti(t∈R and t≠0), and ordinary Riemann ζ(s)=L(s, ✗(n))(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$, ✗(n)∈R and ✗(n) ≡ 1, $n\in Z^{+}$ and n traverses all positive integers) the corresponding Landau-siegel function L(β, ✗(n))≠0(β∈R,and $\beta\neq-2n$, $n\in Z^{+}$, ✗(n)∈R and ✗(n)≡1 and n traverses all positive integers), ordinary Riemann hypothesis and ordinary Riemann hypothesis all hold, and for Riemann ζ(s)(s∈C and s≠ 1 and $s\neq-2n, n\in Z^{+}$) function, its corresponding Landau-Siegel function L(β,1)(β ∈ R and $\beta \neq -2n$, $n \in Z^+$, ✗ (n)∈R and ✗(n)≡1, $n \in Z^+$ and n traverses all positive integers) does not exist pure real zero, which also shows that Riemann ζ(s)(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$) function does not exist zero when variable s is a pure real zero.

2: When ✗ (n)≡0($n \in Z^+$ and n traverses all positive odd numbers,including 1), then ✗ (p)≡0($p \in Z^+$ and p traverses all odd primes, including 1), a special Dirichlet function L(s, ✗ (p))(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$, ✗(p)∈R and ✗(p)≡0, $p \in Z^+$ and p traverses all odd primes, including 1) has zero, and when zero is obtained, the independent variable s is any complex number. This special dirichlet function L(s, ✗(p))(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$, ✗(p)∈R and ✗(p)≡0, $p \in Z^+$ and p traverses all odd prime, including 1) the corresponding Landau-siegel function L(β,0)=0(β∈R and β≠ $-2n$, $n \in Z^+$, ✗(p)∈R and ✗(p)≡0, $p \in Z^+$ and p traverses all odd prime, including 1) holds, so for this particular Dirichlet function L(s, ✗ (p))=0(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$, ✗ (p)∈R and ✗ (p) ≡ 0, $p \in Z^+$ and p traverses all odd primes, including 1) holds.The existence of a pure real zero of the corresponding Landau-Siegel function L(β,0)(β∈R and β≠ $-2n$, $n \in Z^+$, ✗(p)∈R and ✗(p)≡0, $p \in Z^+$ and p traverses all odd prime numbers, including 1) shows that the twin prime numbers, Polignac conjecture and Goldbach conjecture are all completely true.

3: When ✗(n)≠1 and ✗(n)≠0($n \in Z^+$ and n traverses all positive integers), Dirichlet function L(s, ✗ (n))(s∈C and s≠ 1 and $s \neq -2n, n \in Z^+$, ✗(n)∈ and ✗(n)≠0 and ✗(n)≠1, $n \in Z^+$ and n traverses all positive integers) has zero, it's nontrivial zero meet s=$\frac{1}{2}$+ti(t∈R and t≠0) and s=$\frac{1}{2}$-ti(t∈R and t≠0). For dirichlet function L(s, ✗ (n))(s∈C and s≠ 1and $s \neq -2n, n \in Z^+$, ✗(n)∈R and ✗(n)≠0, $n \in Z^+$ and n traverses all positive integes), it's corresponding Landau-siegel function L(β, ✗(n))(β∈R and β≠ $-2n$, $n \in Z^+$, ✗(n)∈R and ✗(n)≠0 and ✗(n)≠1, $n \in Z^+$ and n traverses all positive integers) of pure real zero does not exist, In other words, it shows that the Dirichlet function L(s, ✗ (n))(s∈C and s≠ 1and $s \neq -2n, n \in Z^+$,, ✗ (n)∈R and ✗(n)≠0 and ✗(n)≠1, $n \in Z^+$ and n traverses all positive integers) does not exist for the zero of a pure real variable s, so if ✗(n)≠0 and ✗(n)≠1 ($n \in Z^+$ and n traverses all positive integers), then both the generalized Riemannian hypothesis and the generalized Riemannian conjecture hold and the Generalized Riemann L(s, ✗ (n))(s∈C and s≠ 1,and $s \neq -2n, n \in Z^+$, ✗(n)∈R and ✗(n)≠0 and ✗(n)≠1, $n \in Z^+$ and n traverses all positive intege) function of nontrivial zero s also .0meet s=$\frac{1}{2}$+ti(t∈R and t≠0) and s=$\frac{1}{2}$-ti(t∈R,t≠0).Now we know that merely proving that the nontrivial zero s of the Riemann conjecture L(s,1)=0(s∈C and s≠ 1and $s \neq -2n, n \in Z^+$, ✗ (n)∈R and ✗ (n) ≡ 1, $n \in Z^+$ and n traverses all positive integers) and the generalized Riemann conjecture L(s, ✗(n))=0(s∈C and s≠ 1and $s \neq -2n, n \in Z^+$, ✗(n)∈R and ✗ (n)≠1 and ✗ (n)≠0 and n traverses all positive integers) satisfies s=$\frac{1}{2}$+ti(t∈R,t≠0) and s=$\frac{1}{2}$-ti(t∈R,t≠0) is sufficient to prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are all almost true.

Formula 2

Let's say I have any complex number Z= x+yi(x ∈R,y∈R), and I have any complex number s=σ+ui(σ ∈R,u∈R).We use r(r∈R,and r>0) to represent the module |Z| of complex Z= x+yi

(x ∈R ,y∈R), and ϕ to represent the argument Am(Z) of complex Z= x+yi(x∈R ,y∈R).That is |Z|=r,

then r= $(x^2+y^2)^{\frac{1}{2}}$, so Z=r(Cos(ϕ)+iSin(ϕ)) and ϕ=$|\arccos(\frac{x}{(x^2+y^2)^{\frac{1}{2}}})|$,and ϕ∈(-π, π], then ϕ=Am(Z).

Base on $x^s = x^{\sigma+ui} = x^{\sigma}x^{ui}$=$x^{\sigma}(\cos(\ln x) + i\sin(\ln x))^u$=$x^{\sigma}(\cos(u\ln x) + i\sin(u\ln x))$ can get $r^s = r^{(\sigma+ui)} = r^{\sigma}r^{ui} = r^{\sigma}(\cos(\ln x) + i\sin(\ln x))^u$ =$r^{\sigma}(\cos(u\ln x) + i\sin(u\ln x))$ (r>0), then $f(Z,s)=z^s=(r(\cos(\varphi) + i\sin(\varphi))^{\sigma+u\,i}=(r(\cos(\varphi) + i\sin(\varphi))^{\sigma}r(\cos(\varphi) + i\sin(\varphi))^{ui}$ = $r^{\sigma}(\cos(\sigma\varphi) + i\sin(\sigma\varphi))(r(\cos(\varphi) + i\sin(\varphi))^{ui} = r^{\sigma}(\cos(\sigma\varphi) + i\sin(\sigma\varphi))r^{ui}(\cos(\varphi) + i\sin(\varphi))ui$ = rσ(cos(σφ)+isin(σφ))(cos(ulnr)+isin(ulnr))(cos(uφ)+isin(uφ))i $=r^{\sigma}(\cos(\sigma\varphi + u\ln r) + i\sin(\sigma\varphi + u\ln r))(\cos(u\varphi) + i\sin(u\varphi))^i$.

Beacuse of

$Z = e^{\ln|Z|+iAm(Z)} = e^{\ln|Z|}e^{iAm(Z)} = e^{\ln|Z|}$ (cos(Am(Z))+isin(Am (Z)))=r(cos(Am(Z))+isin(Am (Z))),so lnZ=ln|Z|+iAm(Z)(−π<Am(Z)<= π).

Suppose a>0,then $a^x=e^{\ln(a^x)} = e^{x\ln a}$, then $z^s=e^{s\ln z}$.

Suppose any complex Number Q= cos(uφ) + isin(uφ), and Suppose

the complex ψ= i, then lnQ= ln|Q|+iAm(Q) (−π<Am(Q)<= π).

Because 0<=|sin(uφ)|<=1,

so

If −π<uφ<= π,then Am(Q)=uφ and − π<Am(Q)<= π；

If uφ>π, then Am(Q)=uφ-2kπ(k ∈ Z^+) and − π<Am(Q)<= π；

if uφ < $-\pi$,then Am(Q)=uφ+2kπ(k ∈ Z^+) and − π<Am(Q)<= π. Then

If Am(Q)=uφ, then

$(\cos(u\varphi) + i\sin(u\varphi))^i = Q^{\psi} = e^{\psi\ln Q} = e^{\psi(\ln|Q|+iAm(Q))}=e^{i(o+iAm(Q))}=e^{-u\varphi}$.then

$f(Z,s)=z^s=r^{\sigma}(\cos(\sigma\varphi + u\ln r) + i\sin(\sigma\varphi + u\ln r))(\cos(u\varphi) + i\sin(u\varphi))^i$

$=e^{-u\varphi}r^{\sigma}(\cos(\sigma\varphi + u\ln r)+ie^{-u\varphi}r^{\sigma}\sin(\sigma\varphi + u\ln r)$。Substituting

r= $(x^2+y^2)^{\frac{1}{2}}$ into the above equation gives:

$f(Z,s)=z^s=e^{-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\cos(\sigma\varphi + u\ln(x^2+y^2)^{\frac{1}{2}}))$

$+ie^{-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\sin(\sigma\varphi + u\ln(x^2+y^2)^{\frac{1}{2}}))$.

If Am(Q)= uφ-2kπ(k∈Z^+),then

$(\cos(u\varphi) + i\sin(u\varphi))^i = Q^{\psi} = e^{\psi\ln Q} = e^{\psi(\ln|Q|+iAm(Q))}=e^{i(o+i(u\varphi-2k\pi))}=e^{2k\pi-u\varphi}$, then

$f(Z,s)=z^s=r^{\sigma}(\cos(\sigma\varphi + u\ln r) + i\sin(\sigma\varphi + u\ln r))(\cos(u\varphi) + i\sin(u\varphi))^i$

$=e^{2k\pi-u\varphi}r^{\sigma}(\cos(\sigma\varphi + u\ln r)+ie^{2k\pi-u\varphi}r^{\sigma}\sin(\sigma\varphi + u\ln r)$.

Substituting r= $(x^2+y^2)^{\frac{1}{2}}$ into the above equation gives:

f(Z,s)=Z^s=$e^{2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\cos(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$

+$ie^{2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\sin(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$.

If Am(Q)=uφ+2kπ(k∈Z^+), then

$(\cos(u\varphi)+i\sin(u\varphi))^i=Q^\psi=e^{\psi\ln Q}=e^{\psi(\ln|Q|+iAm(Q))}$=$e^{i(o+i(u\varphi+2k\pi))}$=$e^{-2k\pi-u\varphi}$, then

f(Z,s)=Z^s=$r^\sigma(\cos(\sigma\varphi+u\ln r)+i\sin(\sigma\varphi+u\ln r))(\cos(u\varphi)+i\sin(u\varphi))^i$

=$e^{-2k\pi-u\varphi}r^\sigma(\cos(\sigma\varphi+u\ln r)$+$ie^{-2k\pi-u\varphi}r^\sigma\sin(\sigma\varphi+u\ln r)$.

Substituting r= $(x^2+y^2)^{\frac{1}{2}}$ into the above equation gives:

f(Z,s)=z^s=$e^{-2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\cos(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$

+$ie^{-2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\sin(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$.

III. Conclusion

After the Riemann hypothesis and the Riemann conjecture and the Generalized Riemann hypothesis and the Generalized Riemann conjecture are proved to be completely valid, the research on the distribution of prime numbers and other studies related to the Riemann hypothesis and the Riemann conjecture will play a driving role. Readers can do a lot in this respect.

IV.Thanks

Thank you for reading this paper.

V.Contribution

The sole author, poses the research question, demonstrates and proves the question.

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摘要:

为了从纯粹数学的角度严格证明 Riemann 1859 年的论文《论不大于x的素数的个数》中的猜想，并严格证明黎曼猜想的正确性，本文利用欧拉公式证明了如果ζ(s)函数的自变量共轭，则ζ(s)函数值也共轭，从而得到ζ(s)函数的零点自变量也共轭，并利用黎曼ζ(s)函数零点的共轭和ζ(s)=0 与ζ(1-s)=0 的零点自变量 s 和零点自变量 1-s 也必须共轭，得到了黎曼ζ(s)函数的非平凡零点必定满足 s=$\frac{1}{2}$+ti(t∈R 且 t ≠ 0)和 s=$\frac{1}{2}$-ti(t∈R 且 t ≠ 0)。而黎曼ζ(s)函数零点的对称性则是黎曼ζ(s)函数非平凡零点都位于临界线上的必要条件，根据黎曼ζ(s)函数零点 s 和黎曼ζ(s)函数零点 1-s 的对称性质，结合黎曼黎曼ζ(s)函数零点 s 和黎曼ζ(s)函数零点 1-s 的共轭性质，证明ζ(s)函数非平凡零点的实部必定只能等于$\frac{1}{2}$。又通过黎曼的设定的 s=$\frac{1}{2}+$ti(t∈C 且 t ≠ 0)和辅助函$\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)\left(s\in C\text{ 且 }s\neq 1\right)$，得到$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)=0$,结合黎曼ζ(s)函数的非平凡零点必定满足 s=$\frac{1}{2}$+ti(t∈R 且 t ≠ 0)和 s=$\frac{1}{2}$-ti(t∈R 且 t ≠ 0)，从而等价地证明了黎曼ξ(t)函数的零点必须都是不为零的实数，黎曼猜想是完全正确的。

关键词:

欧拉公式，黎曼ζ(s)函数，黎曼函数ξ(t)，黎曼猜想，共轭性。

I. 引论:

黎曼猜想是黎曼在其 1859 年的一篇论文《论不大于x的素数的个数》中留下的一个重要而著名的数学问题，对研究素数分布具有重要意义，被称为数学中最大的未解之谜。经过多年的努力，我解决了这个问题，并严格证明了黎曼猜想和广义黎曼猜想都是完全正确

的。波利尼亚克猜想、孪生素数猜想和哥德巴赫猜想也是完全正确的。如果你从黎曼的论文《论不大于x的素数》一开始就透彻地理解了黎曼的猜想并完全相信其背后的逻辑推理，那就太好了。你需要在阅读我的这篇论文之前做这些准备。下面介绍一下黎曼论文《论不大于x的素数的个数》的大约前半部分的内容，我对这部分内容作了讲解推导，这是你了解黎曼猜想的前提和基础。

1859 年，黎曼被柏林科学院接纳为通讯院士，为了表达自己被赐予这份殊荣的感谢之情，他想到最好方式是立即利用由此得到的许可向柏林科学院通报一项关于素数分布密度的研究，高斯和狄利克雷曾也长期对此问题抱有浓厚的兴趣，它似乎并不是完全配不上这样性质的一个报告。

黎曼以欧拉的发现，即下面这个等式作为本研究的起点：

$$\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

其中等式左边的 p 取遍所有质数，等式右边的 n 取遍所有自然数，用 ζ(s)表记由上面这两个级数（当它们收敛时）表示的复变量 s 的函数，也就是定义复变函数

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})。$$

上面这两个级数只有当 s 的实部大于 1 时才收敛，即当

Re(s)>1, $\sum_{n=1}^{\infty}\frac{1}{n^s}$和$\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})$才收敛。如果 s=1,那么$\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$,它叫调和级数，是发散的。如果 Re(s)<1, $\frac{1}{1^s}=\frac{1}{1},\frac{1}{2^s}>\frac{1}{2},\frac{1}{3^s}>\frac{1}{3},\frac{1}{4^s}>\frac{1}{4}$ … ，.它更是发散的。因为如果 Re(s)<1,那么$\frac{1}{1^s}=\frac{1}{1},\frac{1}{2^s}>\frac{1}{2},\frac{1}{3^s}>\frac{1}{3},\frac{1}{4^s}>\frac{1}{4},\ldots$。但是，如果 s 是一个负数，比如 s=-1,那么它就不满足 Re(s)>1 这个条件。因此需要找到一个对任意 s 总是有效的函数 ζ(s)的表达式。用现代数学语言讲，即要对复变函数ζ(s)进行解析延拓，而解析延拓的最好方法是寻找 一个该函数的更广泛有效的表示，如积分表示或适当的函数表示。因此我们要定义一个新的函数，这个新的函数也 ζ(s)来表示，这个新的函数的自变量s不但满足 Re(s)>1，还满足

$\mathrm{Re(s)}\leq 1(\mathrm{s}\neq 1)$，且函数图像是光滑的，函数图像上每一点都可求它的切线斜率，即函数处处都可求导数。不过它不再叫欧拉ζ函数，而是叫做黎曼ζ函数。黎曼用了积分来表示函数ζ(s)，在本论文中，我补充了另一种复变函数用来表示黎曼函数 ζ(s)。因为$\Pi(\mathrm{s}) = \Gamma(\mathrm{s}+1) = \mathrm{s}\Gamma(\mathrm{s})$，其中$\Pi(\mathrm{s})$为阶乘函数，$\Gamma(\mathrm{s})$为欧拉伽马函数，$\Gamma(\mathrm{s})=\int_0^{\infty} x^{\mathrm{s}-1}\,\mathrm{e}^{-x}\mathrm{d}x$，令积分符号中的变量

$x \to nx (n \in \mathrm{Z}^{+})$，那么

$\int_0^{\infty}(nx)^{\mathrm{s}-1}\,\mathrm{e}^{-\mathrm{n}x}\,\mathrm{d}(nx) = n\int_0^{\infty}\mathrm{e}^{-\mathrm{n}x}\;\; n^{\mathrm{s}-1}x^{\mathrm{s}-1} = n^{\mathrm{s}}\int_0^{\infty}\mathrm{e}^{-\mathrm{n}x}\;\; x^{\mathrm{s}-1} = \Gamma(\mathrm{s}) = \Pi(\mathrm{s}-1)$,所以

$$\int_0^{\infty}\mathrm{e}^{-\mathrm{n}x}\;\; x^{\mathrm{s}-1} = \frac{\Pi(\mathrm{s}-1)}{n^{\mathrm{s}}}$$

，这正是黎曼在它的论文中说的，他说他要利用

$$\int_0^{\infty}\mathrm{e}^{-\mathrm{n}x}\;\; x^{\mathrm{s}-1} = \frac{\Pi(\mathrm{s}-1)}{n^{\mathrm{s}}}$$

,因为 n 为全体正整数，所以需要对等式两边的$\mathrm{e}^{-\mathrm{n}x}$和$\frac{1}{\mathrm{n}^{\mathrm{s}}}$作$\sum$运算 ，所以

$\sum_{\mathrm{n}=1}^{\infty}\mathrm{e}^{-\mathrm{n}x} = 1+\sum_{\mathrm{n}=1}^{\infty}\mathrm{e}^{-\mathrm{n}x} - 1 = (1+\mathrm{e}^{-x}+\mathrm{e}^{-2x}+\mathrm{e}^{-3x}+\ldots) - 1 = \frac{1}{1-\mathrm{e}^{-x}} - 1 = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} = \frac{1}{\mathrm{e}^{x}-1}$ ，

公比 q 满足 $0<q=|\mathrm{e}^{-x}|<=1(0<=x\to+\infty)$，$\frac{\Pi(\mathrm{s}-1)}{\mathrm{n}^{\mathrm{s}}} = \frac{\Pi(\mathrm{s}-1)}{1^{\mathrm{s}}+2^{\mathrm{s}}+3^{\mathrm{s}}+4^{\mathrm{s}}+5^{\mathrm{s}}+\cdots}$ ，

并且$\sum_{\mathrm{n}=1}^{\infty}\frac{1}{\mathrm{n}^{\mathrm{s}}} = \frac{1}{1^{\mathrm{s}}+2^{\mathrm{s}}+3^{\mathrm{s}}+4^{\mathrm{s}}+5^{\mathrm{s}}+\cdots} = \zeta(\mathrm{s})$，所以由

$$\int_0^{\infty}\mathrm{e}^{-\mathrm{n}x}x^{\mathrm{s}-1} = \frac{\Pi(\mathrm{s}-1)}{n^{\mathrm{s}}}$$

,可以得到$\Pi(\mathrm{s}-1)\zeta(\mathrm{s})=\int_0^{\infty}\frac{x^{\mathrm{s}-1}\mathrm{d}x}{\mathrm{e}^{x}-1}$,这正是黎曼在他的论文中得到的。

现在考虑积分

$$\int\frac{(-x)^{\mathrm{s}-1}dx}{\mathrm{e}^{x}-1}$$

按现代数学记号，该积分应记成$\int_{\mathrm{C}}\frac{(-x)^{\mathrm{s}-1}\mathrm{d}x}{\mathrm{e}^{x}-1}$，或考虑到一般用 Z 表示复数，该积分应记成$\int_{\mathrm{C}}\frac{(-\mathrm{Z})^{\mathrm{s}-1}\mathrm{dZ}}{\mathrm{e}^{\mathrm{Z}}-1}$，它的积分路径从右向左从+∞到+∞，包含值 0 但不包含被积函数的任何其

他奇点的区域的正向边界进行，其中的积分路径 C 如下面的图 1 所示。

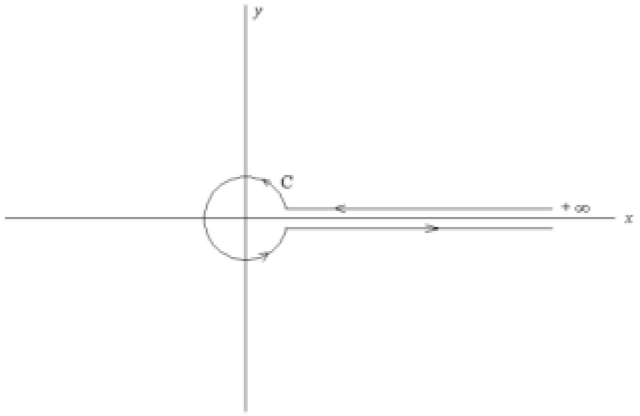

图 1

包含值 0 但不包含被积函数的任何其他奇点(比如 s=1)的区域的正向边界进行。容易该积分值为:

$(e^{-\pi si}-e^{\pi si})\int_0^{\infty}\frac{x^{s-1}dx}{e^{x}-1}$，

其中我们约定在多值函数$(-x)^{s-1}$中，$\ln(-x)$的取值对于负的x为实数，由此即得

$2\sin(\pi s)\,\Pi(s-1)\zeta(s)=i\int_{\infty}^{\infty}\frac{(-x)^{s-1}dx}{e^{x}-1}(x\ \in R)$。

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现在这一等式对于任意复变量s都给出了函数ζ(s)的值，并表明它是单值解析的，并且对于所有有限的s(除了 1 之外)都取有限值，当s等于一个负偶数时取零值。

上面等式的右边是一个整函数，故左边也是一个整函数，因为Π(s-1) = Γ(s)，而Γ(s)在s= 0,–1,–2,–3,...的一级极点和sin(πs)的零点抵消。

黎曼 ζ(s)函数是级数表达式$\sum_{n=1}^{\infty}\frac{1}{n^{s}}(n\in Z^{+})$(Re(s)>1)在复平面上的解析延拓。之所以要对上述级数表达式进行解析延拓，是因这一表达式只适用于复平面上 s 的实部 Re(s)>1 的区域（否则级数不收敛）。黎曼找到了这一表达式的解析延拓（当然黎曼没有使用“解析延拓”

这样的现代复变函数论术语）。运用围道积分,解析延拓后的黎曼ζ函数可以表示为:

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-z)^{s}}{e^{Z}-1}\frac{dz}{z}$$

上式中的积分实际上是一个环绕正实轴（即从+∞出发，沿实轴上方积分至原点附近，环绕原点积分至实轴下方，再沿实轴下方积分至+∞——离实轴的距离及环绕原点的半径均趋于0）进行的围道积分；式中的 Γ 函数Γ(s)是阶乘函数在复平面上的解析延拓，对于正整数 s ＞1:Γ(s)=(s-1)!。可以证明，上述ζ(s)的积分表达式除了在 s=1 处有一个简单极点外，在整个复平面上处处解析。这样的表达式是所谓的亚纯函数——即除了在一个孤立点集上存在极点外，在整个复平面上处处解析的函数——的一个例子。这就是黎曼ζ函数的完整定义。为了得到该积分的值分，我们假设有模任意小的一个复数 δ，且δ 的模｜δ｜→ 0,因为 $(-Z)^{s}=e^{s\ln(-Z)}$,并且 ln(-Z)=ln(Z)+πi 或 $\ln(-Z)=\ln(Z)-\pi i$, 那么

$\int_{C}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}=\int_{\infty}^{\delta}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}+\int_{\delta}^{\infty}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}+k\int_{|\delta|\to 0}^{\infty}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}=\int_{+\infty}^{\delta}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}+\int_{\delta}^{+\infty}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}$

$+k\int_{|\delta|\to 0}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}=(e^{\pi si}-e^{-\pi si})\int_{\delta}^{\infty}\frac{e^{s\ln(Z)}dZ}{(e^{Z}-1)Z}+k\int_{|\delta|\to 0}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}$,k为一常数。

复变量的三角函数的定义由欧拉公式 $\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}$给出，假设 z=πs，那么

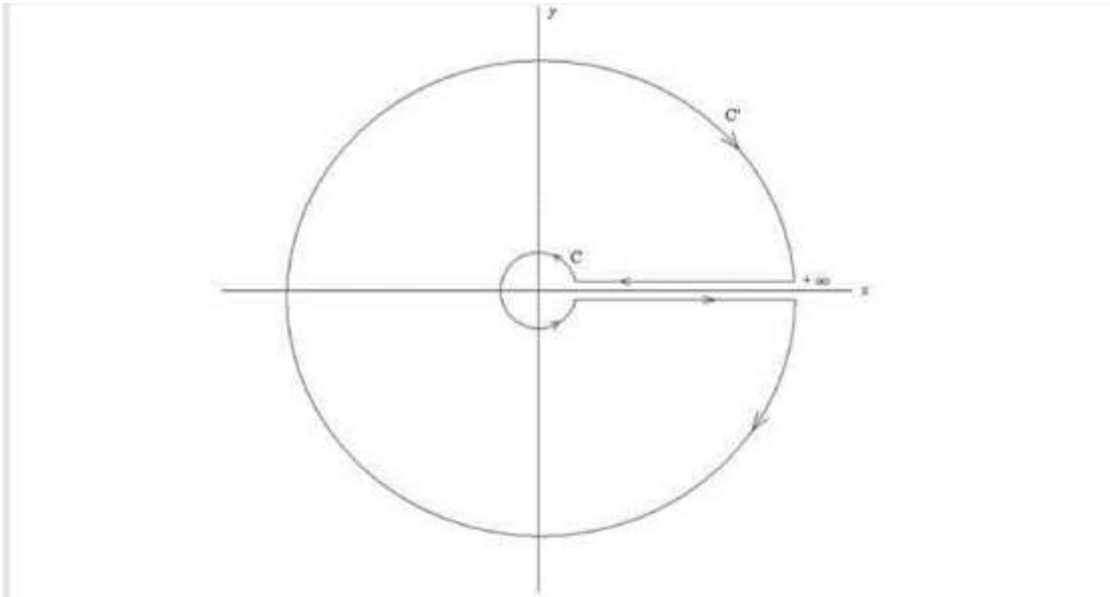
$\sin(\pi s)=\frac{e^{\pi si}-e^{-\pi si}}{2i}$.所以$e^{\pi si}-e^{-\pi si}=2i\sin(\pi s)$, $i=\frac{e^{\pi si}-e^{-\pi si}}{2\sin(\pi s)}$.所以

$\int_{C}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}=(e^{\pi si}-e^{-\pi si})\int_{\delta}^{\infty}\frac{e^{s\ln(Z)}dZ}{e^{Z}-1}+k\int_{|\delta|\to 0}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}$，如果δ 为一个实数，且 δ的绝对值｜δ｜→ 0，那么$\int_{|\delta|\to 0}\frac{(-Z)^{s}dZ}{(e^{Z}-1)Z}=0$,那么$\int_{C}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}=2i\sin(\pi s)\int_{0}^{\infty}\frac{x^{s-1}dx}{x-1}$ $(x\in R)$。那么 $\frac{1}{2i\sin(\pi s)}\int_{C}\frac{(-Z)^{s-1}dZ}{e^{Z}-1}=\int_{0}^{\infty}\frac{x^{s-1}dx}{e^{x}-1}$ $(x\in R)$。前面我们得到$\Pi(s-1)\zeta(s)=\int_{0}^{\infty}\frac{x^{s-1}dx}{e^{x}-1}(x\in R)$，

所以 $2\sin(\pi s)\,\Pi(s-1)\zeta(s)=i\int_{0}^{\infty}\frac{x^{s-1}dx}{e^{x}-1}$。

当 s 的实部为负时,上面的积分可以不沿正向围绕给定值的区域进行，而是沿负向包含所有剩下的复数值的区域进行，参见下面的图 2，其中的大圆 C′ 的半径趋向无穷大，从而包含被积函数的所有极点即分母$e^{x}-1$的所有零点 2nπi（n 为整数），接下来的计算应用柯西的

留数定理。



因为该积分的值对于模无限大的复数为无限小，而在该区域内部，被积函数只有当x等于 $2\pi i$ 的整数倍时才有奇点，于是该积分即等于负向围绕这些值的积分之和，但围绕值 $n2\pi i$ 的积分等于$(-n2\pi i)^{s-1}(-2\pi i)$，被积函数$\frac{(-x)^{s-1}}{(e^{x}-1)}$在 $n2\pi i(n\neq0)$的留数等于：

$[\frac{(-x)^{s-1}}{(e^{x}-1)}]_{x=n2\pi i}=[\frac{(-x)^{s-1}}{e^{x}}]_{x=n2\pi i}=(-n2\pi i)^{s-1}$,

于是我们得到

$2\sin(\pi s)\prod(s-1)\,\zeta(s)=(2\pi)^{s}\sum n^{s-1}((-i)^{s-1}+i^{s-1})$ [1] (等式 3),

它揭示了一个$\zeta(s)$ 和$\zeta(1-s)$之间的关系，利用函数$\Pi(s)$的已知性质，即利用$\Pi(s-1)=\Gamma(s)$和伽玛函数$\Gamma(s)$的余元公式和勒让德公式。也可以将它表述为：

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$在变换$s\to1-s$下不变。

根据欧拉的公式$e^{ix}=\cos(x)+i\sin(x)\ (x\in R)$可以得到

$e^{i(-\frac{\pi}{2})}=\cos(\frac{-\pi}{2})+i\sin(\frac{-\pi}{2})=0-i=-i$,

$e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})=0+i=i$,

那么

$(-i)^{s-1}+i^{s-1}=(-i)^{-1}(-i)^{s}+(i)^{-1}(i)^{s}=(-i)^{-1}e^{i\left(-\frac{\pi}{2}\right)s}+i^{(-1)}e^{i\left(\frac{\pi}{2}\right)s}=$

$ie^{i\left(-\frac{\pi}{2}\right)s}-ie^{i\left(\frac{\pi}{2}\right)s}=i(\cos\frac{-\pi s}{2}+i\sin\frac{-\pi s}{2})-i(\cos\frac{\pi s}{2}+i\sin\frac{\pi s}{2})=i\cos(\frac{\pi s}{2})-i\cos(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})$

$=2\sin(\frac{\pi s}{2})$ (等式 4).

根据伽马函数 Γ(s)的性质 Π(s-1)=Γ(s)，并且

$\sum_{n=1}^{\infty} n^{s-1}=\zeta(1-s)$($n\in Z^{+}$ 并且n遍取所有的正整数,$s\in C$，并且 $s\neq 1$),

把上面(等式 4)的结果代入上面(等式 3)右边，将得到:

$2\sin(\pi s)\Gamma(s)\,\zeta(s)=(2\pi)^{s}\zeta(1-s)2\sin\frac{\pi s}{2}$ (等式 5),

根据倍角公式 $\sin(\pi s)=2\mathrm{Sin}(\frac{\pi s}{2})\cos(\frac{\pi s}{2})$, 我们将得到:

$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\,\zeta(s)$($s\in C$ and $s\neq 1$) (等式 6),

作如下变换$s\to 1-s$,也就是把s当做 1-s代入等式 6, 将得到:

$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ 并且 $s\neq 1$) (等式 7)。

这就是ζ(s)的泛函方程$\zeta(s)$($s\in C$ 且 $s\neq 1$). 为了将它改写成一种对称的形式，用伽玛函数的

余元公式

$\Gamma(Z)\Gamma(1-Z)=\frac{\pi}{\sin(\pi Z)}$ (等式 8)

和勒让德公式

$\Gamma(\frac{Z}{2})\Gamma(\frac{Z}{2}+\frac{1}{2})=2^{1-Z}\pi^{\frac{1}{2}}\Gamma(Z)$ (等式 9)，

在等式 8 中，令 $z=\frac{s}{2}$ ，并把它代入等式 8，将得到:

$\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10)，

在 等式 9 中, 令 z=1-s 并把它代入等式 9，将得到:

$\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (等式 11)。

把 $\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10) 和 $\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (等式 11) 的结果代入

$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ 且 $s\neq 1$) (等式 7)将得到:

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\,\zeta(1-s)(s\in C$ 且 $s\neq 1)$ (等式 12),

也就是说

$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$在变换$s\to 1-s$下是不变的，这正是 Riemann 在他的论文中所说的。

即

$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1-s)(s\in C$ 并且 $s\neq 1)$ (等式 12)。

该函数的这一性质诱导我在级数$\sum_{n=1}^{\infty}\frac{1}{n^s}$的一般项中引入$\Pi(\frac{s}{2}-1)$而不是$\Pi(s-1)$，由此我们能得到函数$\zeta(s)$的一个很方便的表达式，事实上我们有

$$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx\ 。$$

为了推导上式，我们先来看$\Pi(\frac{s}{2}-1)=\Gamma(\frac{s}{2})=\int_0^{\infty}x^{\frac{s}{2}-1}e^{-x}dx$，在$\Pi(\frac{s}{2}-1)=\Gamma(s)=\int_0^{\infty}x^{\frac{s}{2}-1}e^{-x}dx$中，作如下替换$x\to n^2\pi x$，那么

$\Pi(\frac{s}{2}-1)=\Gamma(s)=\int_0^{\infty}(n^2\pi x)^{\frac{s}{2}-1}e^{-n^2\pi x}dx=n^s\times n^{-2}\times\pi^{\frac{s}{2}}\times\pi^{-1}\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}d(n^2\pi x)=$

$n^s\times n^{-2}\times\pi^{\frac{s}{2}}\times\pi^{-1}\times n^2\times\pi\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx=n^s.\pi^{\frac{s}{2}}\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx$，所以

$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx$。

因此，如果记$\sum_{n=1}^{\infty}e^{-n^2\pi x}=\psi(x)$, 即得

$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}=\int_0^{\infty}e^{-n^2\pi x}\ x^{-\frac{s}{2}}dx=\int_0^{\infty}(\sum_{n=1}^{\infty}e^{-n^2\pi x})x^{-\frac{s}{2}}dx=\int_0^{\infty}\psi(x)x^{-\frac{s}{2}}dx$。

根据雅可比 theta 函数

$\theta(x)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=e^{-0^2\pi x}+2\sum_{n=1}^{\infty}e^{-n^2\pi x}=1+2(e^{-\pi x}+e^{-4\pi x}+e^{-9\pi x}+e^{-16\pi x}+\cdots)$,

易见$\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}=\frac{\theta(x)-1}{2}$。

下面来推导 theta 函数的变换公式：$\theta(\frac{1}{x})=\sqrt{x}\,\theta(x)$:

设第一类完全椭圆积分

k, k'分别称为雅可比椭圆函数或椭圆积分的模（modulus）和补模。

$k=k(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$， k'= k(k') $=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-k'^2\sin^2\theta)}}$,

令$\tau=k'/k$ ，有

$\sqrt{\frac{2k}{\pi}}=\theta(\tau)=1+2(e^{-\pi\tau}+e^{-4\pi\tau}+e^{-9\pi\tau}+e^{-16\pi\tau}+\cdots)$,

将模 k 和补模 k'互换又有

$\sqrt{\frac{2k'}{\pi}}=\theta\left(\frac{1}{\tau}\right)=1+2(e^{-\pi/\tau}+e^{-4\pi/\tau}+e^{-9\pi/\tau}+e^{-16\pi/\tau}+\cdots)$,

上述两式相比即得$\theta(\frac{1}{\tau})=\sqrt{\tau}\,\theta(\tau)$。它先由柯西用傅立叶分析得到，后来雅可比又用椭圆函数给出了证明。

运用上面的积分表达式

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-z)^s}{e^z-1}\frac{dz}{z}$$

同样可以证明 ,黎曼 ζ 函数满足上面代数关系式——也叫函数方程$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ 且 $s\neq 1$) (等式 7)，从这个关系式中不难发现，黎曼ζ函数在 s=-2n（n 为正整数）处取值为零——因为 $\sin(\frac{\pi s}{2})$为零。复平面上的这种使黎曼ζ函数取值为零的点被称为黎曼ζ函数的零点。因此 s=-2n（n 为正整数）是黎曼ζ函数的零点。这些零点分布有序、性质简单，称为黎曼ζ函数的平凡零点。除了这些平凡零点外，黎曼ζ函数还有许多其他零点，它们的性质远比那些平凡零点来得复杂，被恰如其分地称为非平凡零点。

黎曼在他的论文中作了如下描述：

$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^{\infty}\psi(x)\,x^{\frac{s}{2}-1}dx+\int_1^{\infty}\psi(\frac{1}{x})\,x^{\frac{s-3}{2}}dx+\frac{1}{2}\int_0^1(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1})dx$

$=\frac{1}{s(s-1)}+\int_1^{\infty}\psi(x)\,(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx$,

我们来看上述等式的最后一个式子，如果 s→1-s,那么

$\frac{1}{s(s-1)}=\frac{1}{(1-s)(1-s-1)}=\frac{1}{(1-s)(-s)}\frac{1}{(s-1)s}$,

$x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}}=x^{\frac{1-s}{2}-1}+x^{-\frac{1+(1-s)}{2}}=x^{\frac{-1-s}{2}}+x^{-\frac{2-s}{2}}=x^{-\frac{1+s}{2}}+x^{\frac{s}{2}-1}$,所以

$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)$在变换$s \to 1-s$下不变。

这样黎曼就再次推导出了$\zeta(s)$的函数方程，这比前面用围道积分和留数定理的推导更简单。

若引入辅助函数函数方程$\Phi(s)=\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)$。

可以简洁地写为$\Phi(s)=\Phi(1-s)$，但更方便的做法是在$\Phi(s)$中添加因子$s(s-1)$，这正是黎曼接下来做的，即令(为了和黎曼的记号保持一致引入数字因子$\frac{1}{2}$)

$\xi(s)=\frac{1}{2}s(s-1)\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$。

因为因子$(s-1)$消去了$\zeta(s)$在$s=1$处的一阶极点，因子s消去了$\Gamma(\frac{s}{2})$在$s=0$处的极点，而$\zeta(s)$的平凡零点 -2，-4，-6,....和$\Gamma(\frac{s}{2})$的其余极点抵消，因此$\xi(s)$是一个整函数，且仅以$\zeta(s)$的 非平凡零点为零点。注意到因子$s(s-1)$显然在$s \to 1-s$下不变，所以有函数方程 $\xi(s)=\xi(1-s)$. $\zeta(s)$的零点除了平凡零点 $s=-2n$（n 为自然数），由于恰好是$\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$中的$\Gamma(\frac{s}{2}+1)$的极点，因而不是$\xi(s)$的零点外，其余全都是$\xi(s)$的零点，因此$\xi(s)$的零点与黎曼$\zeta$函数的非平凡零点相重合。换句话说，$\xi(s)$将黎曼$\zeta(s)$函数的非平凡零点从全体零点中分离了出来。

现在设 $s=\frac{1}{2}+ti(t \in C \text{ and } t \neq 0)$，$\prod\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)$,于是可得

$\xi(t)=\frac{1}{2}-(t^2+\frac{1}{4})\int_1^{\infty}\psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)dx$

或

$\xi(t)=4\int_1^{\infty}\frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx$。

黎曼定义的这个函数$\prod\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)$和现在通常使用的函数$\xi(s)=\frac{1}{2}s(s-1)\prod\left(\frac{s}{2}-\right.$ 1π−s2 Γ s2ζs 本质上完全相同,因为 s2=Γ s2+1= s2Γ s2,所以 s2(s–1)π−s2 ζs=s2Γ s2(s–1)π−s2 ζs=12s(s –1)π−s2 Γ s2ζs=$\xi(s)$仅有的差别是黎曼以 t 为自变量 ,而现在通常使用的$\xi(s)$仍以 s 为自变量，s 和 t 差一个线性变换：$s=\frac{1}{2}+ti$, 即一个 90°旋转加$\frac{1}{2}$的平移，也就是复数 t 逆时针方向旋转 90°，再往实数轴的正方向平移$\frac{1}{2}$， 即$t(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})+\frac{1}{2}$。

这样一来，s 复平面中的直线 Re(s)= $\frac{1}{2}$ 就对应于 t 平面中的实轴，黎曼$\zeta(s)$($s\in C$ 且 $s\neq 1$)函数在临界直线 Re(s)= $\frac{1}{2}$上的零点的实部就对应于函数$\xi(t)$的实根。注意在黎曼的记号中，函数方程$\xi(s)=\xi(1-s)$ 就变成了$\xi(t)=\xi(-t)$，即$\xi(t)$是偶函数，故而其幂级数展开只有偶次幂，且零点关于 t = 0 对称分布。 另外， 从上面的两个积分表示也可以明显看出$\xi(t)$是偶函数，因为$\cos(\frac{1}{2}t\ln x)$是 t 的偶函数。

对于所有有限的 t ，函数$\xi(t)=\frac{1}{2}-(t^2+\frac{1}{4})\int_1^\infty \psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)dx$

或

函数$\xi(t)=4\int_1^\infty \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx$

的值都是有限的，并可以按t^2的幂展开成一个快速收敛的级数，因为对于一个实部大于 1 的 s 值，$\ln\zeta(s)=-\sum\ln(1-p^{-s})$的值也是有限的，这对$\xi(t)$的其它因子的对数也同样成立，因此函数只$\xi(t)$有当 t 的虚部位于$\frac{1}{2}$i 和– $\frac{1}{2}$i 之间时才可能取零值。即 $\xi(s)$只有当 s 的实部位于 0 和 1 之间时才可能取零值。方程$\xi(t)$的实部在 0 和 T 之间的根的数目是 $N(T)=\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}+O(\ln T)$，约等于$\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$(黎曼对零点数目估计的这一结果直到 1859 年才由 Mangoldt 严格证明)。这是因为沿包含所有虚部位于$\frac{1}{2}$i 和– $\frac{1}{2}$i 之间、实部位 0 和 T 之间的 t 值的正向回路的积分$\int d\ln\xi(t)$ (略去和$\frac{1}{T}$同阶的小量后)的值约等于$(T\ln\frac{T}{2\pi}-T)i$，而该积分的值等于位于此区域内的方程的根的数目乘以 $2\pi i$(此即应用幅角原理)。事实上黎曼发现在该区域内的实根数目近似等于该数目，极有可能所有的根都是实数。对此黎曼自然希望能有一个严格的证明，然而在一些仓促的不成功的初步尝试之后，黎曼暂时把寻求证明搁在一边，因为对于黎曼接下来研究的目的来说它并不是必需的。黎曼轻描淡写写下的这几句话就是著名的黎曼猜想，数学中最著名的猜想！

根据黎曼在论文中的假设:$s=\frac{1}{2}+ti$($t\in C$ and $t\neq 0$)，那么黎曼猜想等价于，对于$\zeta(s)=0$($s\in C$ 且 $s\neq1$ 来说,它的复数根 s（负偶数除外）必定都是只满足 s=12+tit∈R 且 t≠0 和

$s=\frac{1}{2}-ti\left(t\in R且t\neq 0\right)$的复数，都位于满足 Re(s)=$\frac{1}{2}$的垂直实数轴的临界线上，这些复数根 s（负偶数除外） 叫做黎曼$\zeta(s)\left(n\in Z^{+}且s\neq 1且s\neq -2n, n\in Z^{+}\right)$函数的非平凡零点。

对黎曼ζ函数非平凡零点的研究构成了现代数学中最艰深的课题之一。我们所要讨论的黎曼猜想就是一个关于这些非平凡零点的猜想。在这里我们先把它的内容表述一下， 然后再叙述它的来龙去脉。黎曼猜想：黎曼 ζ 函数的所有非平凡零点都位于 Re(s)=$\frac{1}{2}$的直线上。在黎曼猜想的研究中，数学家们把复平面上 Re(s)=$\frac{1}{2}$的直线称为临界线（criticalline）。运用这一术语，黎曼猜想也可以表述为：黎曼 ζ 函数的所有非平凡零点都位于临界线上。这就是黎曼猜想的内容，它是黎曼在 1859 年在他的这篇《论不大于 x 的素数的个数》论文中提出的。从其表述上看，黎曼猜想似乎是一个纯粹的复变函数命题，但我们很快将会看到，它其实却是一曲有关素数分布的神秘乐章。

一个复数域上的函数——黎曼ζ函数——的非平凡零点平凡零点（在无歧义的情况下我们有时将简称其为零点）的分布怎么会与看似风马牛不相及的自然数（在本书中自然数指正整数）中的素数分布产生关联呢？这还得从所谓的欧拉乘积公式谈起。我们知道，早在古希腊时期，欧几里德就用精彩的反证法证明了素数有无穷多个。随着数论研究的深入，人们很自然地对素数在自然数集上的分布产生了越来越浓厚的兴趣。1737 年，数学家欧拉在俄国圣彼得堡科学院发表了一个极为重要的公式，为数学家们研究素数分布的规律奠定了基础。这个公式就是欧拉乘积公式，即

$\sum_{n} n^{-s}=\prod_{p}(1-p^{-s})^{-1}$ ，

这个公式左边的求和对所有的自然数进行，右边的连乘积则对所有的素数进行。可以证明这个公式对所有 Re(s)>1 的复数 s 都成立。这个公式的左边正是我们在上文中介绍过的黎曼 ζ 函数在 Re(s)>1 时的级数表达式，而它的右边则是一个纯粹有关素数（且包含所有素数）

的表达式，这样的形式正是黎曼ζ函数与素数分布之间存在关联的征兆。那么这个公式究竟蕴涵着有关素数分布的什么样的信息呢？黎曼ζ函数的零点又是如何出现在这种关联之中的呢？

欧拉本人率先对这个公式所蕴含的信息进行了研究。他注意到在 s=1 的时候，公式的左边

$$\sum_n n^{-1}$$

是一个发散级数（这是一个著名的发散级数，称为调和级数），这个级数以对数方式发散。这些对于欧拉来说都是不陌生的。为了处理公式右边的连乘积，他对公式两边同时取了对数，于是连乘积就变成了求和，由此他得到:

$\ln(\sum_n n^{-1})=\sum_p(p^{-1}+\frac{p^{-2}}{2}+\frac{p^{-3}}{3}+\cdots)$ ，

由于式中右端括号中除第一项外所有其他各项的求和都收敛，而且那些求和的结果累加在一起仍然收敛。因此右边只有第一项的求和是发散的。由此欧拉得到了这样一个有趣的渐近表达式：

$\sum_p p^{-1}\sim\ln(\sum_n n^{-1})\sim\ln\ln(\infty)$,

或者，更确切地说，

$\sum_{p<N} p^{-1}\sim\ln\ln(N)$,

这个结果, 以 lnln（N）的方式发散——是继欧几里得证明素数有无穷多个以来有关素数的又一个重要的研究结果。它同时也是对素数有无穷多个这一命题的一种崭新的证明（因为假如素数只有有限多个，则求和就只有有限多项，不可能发散）。但欧拉的这一新证明所包含的内容要远远多于欧几里得的证明，因为它表明素数不仅有无穷多个，而且其分布要比许多同样也包含无穷多个元素的序列——比如｛n｝序列——密集得多（因为后者的倒数之和收敛）。

不仅如此，如果我们进一步注意到 $\sum_{p<N} p^{-1}\sim\ln\ln(N)$,的右端可以改写为一个积分表达式：

$\ln\ln(N) \sim \int^{N} \frac{x^{-1}}{\ln(x)} dx$

而通过引进一个素数分布的密度函数 ρ(x)——它给出在x附近单位区间内发现素数的几率，

$\sum_{p<N} p^{-1} \sim \ln\ln(N)$左端也可以改写为一个积分表达式：

$\sum_{p<N} p^{-1} \sim \int^{N} x^{-1} \rho(x) dx$ ，

将这两个积分表达式进行比较，不难猜测到素数的分布密度为 $\rho(x) \sim 1/\ln x$，从而在x以内的素数个数——通常用 $\pi(x)$表示——为

$\pi(x) \sim \mathrm{Li}(x)$,

其中

$\mathrm{Li}(x) = \int \frac{1}{\ln x} dx$ ，

是对数积分函数。

这个结果正是著名的素数定理——当然这种粗略的推理并不构成对素数定理的证明。因此欧拉发现的这个结果可以说是一扇通向素数定理的暗门。可惜欧拉本人并没有沿着这样的思路走，从而错过了这扇暗门，数学家们提出素数定理的时间也因此而延后了几十年。

提出素数定理的荣誉最终落到了另外两位数学家的肩上：他们是德国数学家高斯（Friedrich Gauss ,1777—1855)和法国数学家勒让德(Adrien-Marie Legendre ,1752—1833)。

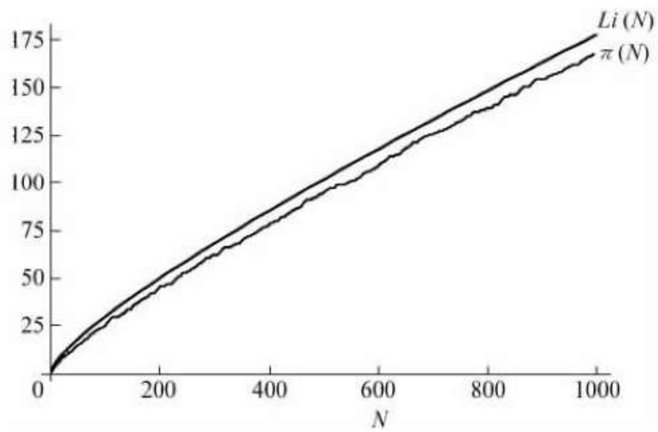
高斯对素数分布的研究始于 1792 到 1793 年间，那时他才十五岁。在那期间，每当“无所事事”的时候，这位早熟的天才数学家就会挑上几个长度为 1000 的自然数区间，计算这些区间中的素数个数，并进行比较。在做过了大量的计算和比较之后，高斯发现素数分布的密度可以近似地用对数函数的倒数来描述，即 $\rho(x) \sim 1/\ln x$，这正是上面提到的素数定理的主要内容。但是高斯并没有发表这一结果。高斯是一位追求完美的数学家，他很少发表自己认为还不够完美的结果，而他的数学思想与灵感犹如浩瀚奔腾的江水，汹涌激荡，常常让他还没来得及将一个研究结果完美化就又展开了新课题的研究。因此高斯一生所做的数学研究远

远多过他正式发表的。但另一方面，高斯常常会通过其他的方式——比如书信——透露自己的某些未发表的研究成果，他的这一做法给一些与他同时代的数学家带来了不小的尴尬。其中“受灾”较重的一位便是勒让德。这位法国数学家在 1806 年率先发表了线性拟合中的最小平方法，不料高斯在 1809 年出版的一部著作中提到自己曾在 1794 年（即比勒让德早了 12 年）就发现了同样的方法，使勒让德极为不快。

有道是：不是冤家不聚首。在素数定理的提出上，可怜的勒让德又一次不幸地与数学巨匠高斯撞到了一起。勒让德在 1798 年发表了自己关于素数分布的研究，这是数学史上有关素数定理最早的文献。由于高斯没有发表自己的研究结果，勒让德便理所当然地成为素数定理的提出者。勒让德的这个优先权一共维持了 51 年。但是到了 1849 年，高斯在给德国天文学家恩克（Johann Encke，1791—1865）的一封信中提到了自己在 1792—1793 年间对素数分布的研究，从而把尘封了半个世纪的优先权从勒让德的口袋中勾了出来，挂到了自己那已经鼓鼓囊囊的腰包之上。幸运的是，高斯给恩克写信的时候勒让德已经去世 16 年了，他用最无奈的方式避免了再次遭受残酷打击。无论高斯还是勒让德，他们对于素数分布规律的研究都是以猜测的形式提出的（勒让德的研究带有一定的推理成分，但离证明仍相距甚远）。因此确切地说，素数定理在那时还只是一个猜想，即素数猜想，我们所说的提出素数定理指的也只是提出素数猜想。素数定理的数学证明直到一个世纪之后的 1896 年，才由法国数学家阿达马（Jacques Hadamard，1865—1963）与比利时数学家普森（Charlesde la Vallée-Poussin，1866—1962）彼此独立地给出。他们的证明与黎曼猜想有着很深的渊源，其中阿达马的证明所出现的时机和场合还有着很大的戏剧性，这些我们将在后文中加以叙述。

素数定理是简洁而优美的，但它对于素数分布的描述仍然是比较粗略的，它给出的只是素数分布的一个渐近形式——小于 N 的素数个数在 N 趋于无穷时的分布形式。从有关素数分布与素数定理我们也可以看到，$\pi(x)$与 $\mathrm{Li}(x)$之间是有偏差的，而且这种偏差的绝对值随着x的

增加似有持续增加的趋势（所幸的是，这种偏差的增加与 π(x)及 Li(x)本身的增加相比仍是微不足道的——否则素数定理也就不成立了）。那么有没有一个公式可以比素数定理更精确地描述素数的分布呢？这便是黎曼在 1859 年想要回答的问题。那一年是高斯去世后的第五年，32 岁的黎曼继德国数学家狄利克雷（Johann Dirichlet，1805—1859）之后成为高斯在哥廷根大学的继任者。同年的 8 月 11 日，他被选为柏林科学院的通信院士。作为对这一崇高荣誉的回报，黎曼向柏林科学院提交了一篇论文——一篇只有短短八页的论文，标题是：论小于给定数值的素数个数。正是这篇论文将欧拉乘积公式所蕴含的信息破译得淋漓尽致，也正是这篇论文将黎曼ζ函数的零点分布与素数的分布联系在了一起



（上图为素数分布与素数定理的示意图）

这篇论文把人们对素数分布的研究推向壮丽的巅峰，并为后世的数学家们留下一个魅力无穷的伟大谜团。

根据欧拉公式：$\zeta(s) \equiv \sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})$，这是研究素数分布规律的基础。黎曼的研究

也是以这一公式作为起点的。为了消除这一公式右边的连乘积，欧拉曾经对公式的两边取对数，黎曼也如法炮制（连乘积真是一个人人欲除之而后快的东西），从而得到

$\ln\zeta(s) \equiv \sum_p \ln(1-p^{-s})$=$\sum_p \sum_n \frac{p^{-s}}{n}$，但过了这一步，黎曼就和欧拉就分道扬镳了：欧拉证明了素数有无穷多个后就鸣金收兵了；而黎曼则沿着一条布满荆棘的道路继续走了下去，走出了素数研究的一片崭新的天地。

可以证明，上面给出的 $\ln\zeta(s) \equiv \sum_p \ln(1-p^{-s})$=$\sum_p \sum_n \frac{p^{-s}}{n}$右边的双重求和在复平面上 Re(s)＞1 的区域内是绝对收敛的，并且可以改写成斯蒂尔切斯积分：

$\ln\zeta(s)=\int_0^{\infty} x^s \mathrm{d}\,J(x)$,

其中$J(x)$是一个特殊的阶梯函数，它在x=0 处取值为零，以后每越过一个素数就增加 1，每越过一个素数的平方就增加 1/2，……，每越过一个素数的 n 次方就增加 1/n，……而在$J(x)$不连续的点（即x等于素数、素数的平方、……、素数的 n 次方、……的点）上，其函数值则用$J(x)=\frac{1}{2}[J(x^-)+J(x^+)]$来定义。显然，这样的一个阶梯函数可以用素数分布函数 $\pi(x)$ 表示为:

$J(x)=\sum_n \frac{\pi(x^{\frac{1}{n}})}{n}$。

对上述斯蒂尔切斯积分进行一次分部积分便可得到:

$\ln\zeta(s) = s\int_0^{\infty} J(x)\, x^{-s-1}\mathrm{d}x$。

这个公式的左边是黎曼ζ函数的自然对数，右边则是对$J(x)$——一个与素数分布函数 $\pi(x)$有直接关系的函数的积分，它可以被视为欧拉乘积公式的积分形式。这一结果的方法与黎曼有所不同，黎曼发表论文时还没有斯蒂尔切斯积分——那时候荷兰数学家斯蒂尔切斯（Thomas Stieltjes，1856—1894）才三岁。

如果说传统形式下的欧拉乘积公式只是黎曼ζ函数与素数分布之间存在关联的朦胧征兆，那么在上述积分形式的欧拉乘积公式下，这两者间的关联就已是确凿无疑并且完全定量的了。

接下来首先要做的显然是从上述积分中解出$J(x)$来，黎曼解出的$J(x)$是：

$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(Z)}{Z}x^{Z}dz$,

其中 a 为大于 1 的实数。上面这个积分是一个条件收敛的积分，它的确切定义是从 a-ib 积分到 a+ib(b 为正实数),然后取 b→∞的极限。黎曼说这个结果是完全普遍的。实际上与实际上黎曼所说的普遍性结果相匹配的完整结果直到 40 年后才由芬兰数学家梅林（Robert Mellin，1854—1933）所发表，现在被称为梅林变换（Mellin transform）。

像这样一种被黎曼随手写下、却让数学界花费几十甚至上百年的时间才能证明的命题在黎曼那篇论文中还有好几处。这是黎曼那篇论文的一个极为突出的特点：它有一种高屋建瓴的宏伟视野，远远超越了同时代的其他数学文献。它那高度浓缩的文句背后包含着的极为丰富的数学结果，让后世的数学家们陷入漫长的深思之中。人们在对黎曼的部分手稿进行研究时发现，黎曼对自己论文中的许多语焉不详的命题是做过扎实的演算和证明的，只不过他和高斯一样追求完美，发表的东西远远少于自己研究过的。更令人钦佩的是，黎曼手稿中的一些演算和证明哪怕是时隔了几十年之后才被整理出来，也往往还是大大超越当时数学界的水平。我们有较强的理由相信，黎曼在论文中以陈述而不是猜测的语气所表述的内容，无论有没有给出证都是有着深入的演算和证明背景的。

好了，现在回到 $J(x)$的表达式来，这个表达式给出了 $J(x)$与黎曼 ζ 函数之间的确切关联。换句话说，只要知道了ζ(s)，通过这个表达式原则上就可以计算出 $J(x)$。知道了 $J(x)$，下一步显然就是计算 $\pi(x)$。这并不困难，因为上面提到的 $J(x)$与 $\pi(x)$之间的关系式可以通过所谓的默比乌斯反演（Möbius inversion）来反解出 $\pi(x)$与 $J(x)$的关系式，其结果为:

$\pi(x)=\sum_{n}\frac{\mu(n)}{n}J(x^{\frac{1}{n}})$,

这里 μ(n)称为默比乌斯函数（Möbius function），它的取值如下：

●μ(1)=1;

●μ(n)=0（如果 n 可以被任一素数的平方整除;

●μ(n)=-1（如果 n 是奇数个不同素数的乘积);

●μ(n)=1（如果 n 是偶数个不同素数的乘积);

因此知道 J(x)就可以计算出 π(x)　,即素数的分布函数。把这些步骤连接在一起 ,我们看到，从ζ(s)到 J(x)，再从 J(x)到 π(x)，素数分布的秘密完全定量地蕴涵在了黎曼ζ函数之中。这就是黎曼研究素数分布的基本思路。

素数的分布与黎曼 ζ 函数之间存在着深刻关联。这一关联的核心就是 J(x)的积分表达式：$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(s)}{s}x^{s}ds$,由于黎曼 ζ 函数具有极为复杂的性质，这一积分同样也是极为复杂的。为了对这一积分做进一步的研究，黎曼引进了一个辅助函数 ξ(s)：

$\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$。

但更好的做法是将ξ(s)定义为

$\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$。

因为因子(s–1)消去了ζ(s)在s=1 处的一阶极点，因子s 消去了$\Gamma(\frac{s}{2})$在s=0 处的极点，而ζ(s)的平凡零点 -2，-4，-6,....和$\Gamma(\frac{s}{2})$的其余极点抵消，因此ξ(s)是一个整函数，且仅以ζ(s)的 非平 凡 零 点 为 零 点 。

引进这样一个辅助函数有什么好处呢？首先，由式 $\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$定义的辅助函数 ξ(s)可以被证明是整函数，即在复平面上所有 s≠∞的点上都解析的函数。这样的函数在性质上要比黎曼ζ函数简单得多，处理起来也容易得多。事实上，在所有非平庸的复变函数中，整函数是解析区域最为宽广的（解析区域比它更大，即包括 s=∞的函数只有一种，那就是常数函数）。这是引进 ξ(s)的好处之一。

其次 ,利用这一辅助函数 ,我们在前面得到过的黎曼ζ函数所满足的代数关系式$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$(s∈C 且 $s\neq 1$)(等式 7)可以表述为一个对于 s 与 1-s 对称的简单形式：ξ(s)= ξ(1-s)。这是引进 ξ(s)的好处之二。

此外，从 ξ(s)的定义中不难看到，ξ(s)的零点必定是 ζ(s)的零点。另一方面，ζ(s)的零点除了平凡零点 s=-2n(n 为自然数)由于恰好是黎曼引进的辅助函数 ξ(s)= $\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$中的 $\Gamma(\frac{s}{2}+1)$ 的极点，因而不是 ξ(s)的零点外，其余全都是 ξ(s)的零点，因此 ξ(s)的零点与黎曼ζ函数的非平凡零点相重合。换句话说，ξ(s)将黎曼 ζ函数的非平凡零点从全体零点中分离了出来。这是引进 ξ(s)的好处之三。

这里我们有必要先提一下黎曼 ζ 函数的一个简单性质，即 ζ(s)在 Re(s)>1 的区域内没有零点。没有零点当然就更没有非平凡零点，而后者跟 ξ(s)的零点是重合的，因此上述性质表明 ξ(s)在 Re(s)>1 的区域内也没有零点；又由于 ξ(s)=ξ(1-s)，因此 ξ(s)在 Re(s)<0 的区域内也没有零点。这表明 ξ(s)的所有零点——从而也就是黎曼 ζ函数的所有非平凡零点都位于 0≤Re(s)≤1 的区域内。由此我们得到了一个有关黎曼ζ函数零点分布的重要结果，那就是：黎曼ζ函数的所有非平凡零点都位于复平面上 0≤Re(s)≤1 的区域内。

好了，现在回到黎曼的论文中来。引进了 ξ(s)之后，黎曼便用 ξ(s)的零点对 lnξ(s)进行了分解：

$\ln\zeta(s)= \ln\xi(0)+\sum_{\rho}\ln(1-\frac{s}{\rho})-\ln\Gamma(s/2+1)+\frac{s}{2}\ln\pi-\ln(s-1)$，

其中 ρ 为 ξ(s)的零点(也就是黎曼 ζ 函数的非平凡零点)。分解式中的求和对所有的 ρ 进行，并且是以先将 ρ 与 1-ρ 配对的方式进行的。由于 ξ(s)=ξ(1-s)，因此零点总是以 ρ 与 1-ρ 成对的方式出现的。这一点很重要，因为上述级数是条件收敛的，但是在将 ρ 与 1-ρ 配对之后则是绝对收敛的。这一分解式也可以写成等价的连乘积关系式：

$\xi(s)=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})$。

这样的连乘积关系式对于有限多项式来说是显而易见的(只要满足 ξ(0)≠0 这一条件即可)，但对于无穷乘积来说却绝非一目了然，它有赖于 ξ(s)是整函数这一事实。其完整证明直到 1893 年才由阿达马在对整函数的无穷乘积表达式进行系统研究时给出。阿达马对这一关系式

的证明是黎曼的论文发表之后这一领域内第一个重要进展。

很明显，上述级数分解式的收敛与否与 ξ(s)的零点分布有着密切的关系。为此黎曼研究了 ξ(s)零点分布，并由此而提出了三个重要命题：

命题一　在 $0<\mathrm{Im}(s)<T$ 的区域内，ξ(s)的零点数目约为$(T/2\pi)\ln(T/2\pi)-(T/2\pi)$。

命题二　在 $0<\mathrm{Im}(s)<T$ 的区域内， ξ(s)的位于 $\mathrm{Re}(s)=1/2$ 的直线上的零点数目也约为$(T/2\pi)\ln(T/2\pi)-(T/2\pi)$。

命题三　ξ(s)的所有零点都位于 $\mathrm{Re}(s)=1/2$ 的直线上。（注：后面我会严格证明这个命题）

在这三个命题之中，第一个命题是证明级数分解式的收敛性所需要用到的（不过黎曼建立在这一命题基础上的说明因过于简略而不足以构成证明）。对于这个命题黎曼的证明是指出在 $0<\mathrm{Im}(s)<T$ 的区域内 ξ(s)的零点数目可以由 $d\xi(s)/2\pi i\xi(s)$沿矩形区域$\{0<\mathrm{Re}(s)<1，0<\mathrm{Im}(s)<T\}$的边界作围道积分得到。在黎曼看来，这点小小的积分算不上什么，因此他直接写下了结果（即命题一）。黎曼并且给出了该结果的相对误差为 1/T。但黎曼显然大大高估了他的读者的水平，因为直到 46 年后的 1905 年，他所写下的这一结果才由德国数学家曼戈尔特（Hansvon Mangoldt，1854—1925）所证明（这一结果因此而被称为了黎曼-曼戈尔特公式，它除了补全黎曼论文中的一个小小证明外，也确立了黎曼ζ函数的非平凡零点有无穷多个）。

将黎曼的第二个命题与前一个命题相比较可以看出，这第二个命题实际上是表明 ξ(s)的几乎所有零点——从而也就是黎曼ζ函数的几乎所有非平凡零点——都位于 $\mathrm{Re}(s)=1/2$ 的直线上。这是一个令人吃惊的命题，因为它比迄今为止——也就是黎曼的论文发表一个半世纪以来——人们在研究黎曼猜想上取得的所有结果都要强得多！而且黎曼在叙述这一命题时所用的语气是完全确定的，这似乎表明，当他写下这一命题时，他认为自己对此已经有了证明。可惜的是，他完全没有提及证明的细节，因此他究竟是怎么证明这一命题的？他的证明

究竟是正确的还是错误的？我们全都无从知晓。除了 1859 年的论文外，黎曼还曾在一封信件中提到过这一命题，他说这一命题可以从对 ξ 函数的一种新的表达式中得到，但他还没有将之简化到可以发表的程度。这就是后人从黎曼留下的片言只语中得到的有关这一命题的全部信息。

黎曼的这三个命题就像是三座渐次升高的山峰，一座比一座巍峨，攀登起来一座比一座困难。他的第一个命题让数学界等待了 46 年；他的第二个命题已经让数学界等待了超过一个半世纪；而他的第三个命题想必大家都看出来了，正是大名鼎鼎的黎曼猜想！今天，黎曼猜想已经被我攻克，它确实完全成立，后面我将会严格证明这个黎曼猜想。

惯常在谈笑间让定理灰飞烟灭的黎曼到了表述这第三个命题——也就是黎曼猜想——的时候，也终于一改举重若轻的风格，用起了像“非常可能”这样的不确定语气。黎曼并且写道：“我们当然希望对此能有一个严格的证明，但是在经过了一些快速而徒劳的尝试之后，我已经把对这种证明的寻找放在了一边，因为它对于我所研究的直接目标不是必需的。”黎曼把证明放在了一边，整个数学界的心弦却被提了起来。黎曼猜想的成立与否对于黎曼的“直接目标”——证明 lnξ(s)的级数分解式的收敛性——的确不是必需的（因为那只要上述第一个命题就足够了），但对于今天的数学界来说却是至关重要的。粗略的统计表明，在当今的数学文献中已经有超过一千条数学命题或“定理”以黎曼猜想（或其推广形式）的成立作为前提。黎曼猜想的命运与提出这些命题或“定理”的所有数学家们的“直接目标”息息相关，并通过那些命题或“定理”而与数学的许多分支有着千丝万缕的联系。另一方面，黎曼对于黎曼猜想的表述方式也从一个侧面表明黎曼对于自己写下的命题是属于猜测性的还是肯定性的是加以区分的。现在让我们回到对 J(x)的计算上来。利用 ξ(s)的定义及其分解式，可以将 lnζ(s)表示为:$\ln\zeta(s)=\ \ln\xi(0)+\sum_{\rho}\ln(1-\frac{s}{\rho})-\ln\Gamma(s/2+1)+\frac{s}{2}\ln\pi-\ln(s-1)$;对 lnζ(s)作这样的分解，目的是为了计算 J(x)。但是将这一分解式直接代入 J(x)的积分表达式所得到的

各个单项积分却并不都收敛，因此黎曼在代入之前先对 J(x)作了一次分部积分，由此得到:

$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\zeta(Z)}{Z}x^{Z}dz$,将 lnζ(s)的分解式代入上式 ,各单项便可分别积出。下表是 lnζ(s)分解式中的项及其对应的积分结果：

lnζ(s)分解式中的项	对应的积分结果
$-\ln(s-1)$	$Li(x)$
$\sum_{\rho}\ln\left(1-\frac{s}{\rho}\right)$	$-\sum_{Im(\rho)>0}[Li(x^{\rho})+Li(x^{1-\rho})]$
$-\ln\Gamma\left(\frac{s}{2}+1\right)$	$\int_{x}^{\infty}\frac{dt}{t(t^2-1)\ln t}$
$\ln\xi(0)$	$\ln\xi(0)=-\ln 2$
$\frac{s}{2}\ln\pi$	0

在上述结果中，对级数

$$\sum_{p}\ln(1-\frac{s}{\rho})$$

的积分最为复杂，其结果是对级数逐项积分的结果。这一结果$-\sum_{Im(\rho)>0}[Li(x^{\rho})+Li(x^{1-\rho})]$是条件收敛的，不仅要如 lnξ(s)的级数表达式中一样将 ρ 与 1-ρ 进行配对，而且还必须依照 Im(ρ)从小到大的顺序求和。黎曼在给出这一结果时承认逐项积分的有效性有赖于对 ξ 函数的“更严格”的讨论，但他表示这是容易证明的。这一“容易证明”的结果在 36 年后的 1895 年被曼戈尔特所证明。另外值得指出的一点是，在黎曼对这一级数的各个项进行积分时隐含了一个要求，那就是对所有的零点 ρ，0<Re(ρ)<1， 这比我们在前面提到过的 0≤Re(ρ)≤1 要强。这一加强看似细微(只不过是将等号排除掉而已)，其实却是数论中一个非同小可的结果,后面我会对它加以证明。黎曼在文章中不仅没有对这一结果加以证明，连暗示性的说明也没有，应该被视为他论文的一个漏洞。这一漏洞在曼戈尔特的证明中也同样存在。不过这一漏洞只是论证方法上的漏洞，是可以弥补的，论证的结果本身并不依

赖于 0<Re(ρ)<1 这样的条件。

由上面这些结果黎曼得到了 J(x)的显形式：

$J(x) = \mathrm{Li}(x) - \sum_{\mathrm{Im}(\rho)>0}[\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})] + \int_{x}^{+\infty} \frac{dt}{t(t^2-1)\ln t}$ -ln2,

$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln t} (x \in Z^+)$，

这一结果，连同 $\pi(x)$与 $J(x)$的关系式：

$\pi(x) = \sum_n \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$,

便是黎曼所得到的素数分布的完整表达式，也是他 1859 年论文的主要结果。黎曼的这一结果给出的是素数分布的精确表达式，它的第一项(由 $J(x)$及 $\pi(x)$的第一项共同给出)正是当时尚未得到证明的素数定理所预言的结果 $\mathrm{Li}(x)$。黎曼既然已经给出了素数分布的精确表达式，却没能直接证明远比该结果粗糙的素数定理，这是为什么呢？这其中的奥秘就在于黎曼ζ函数的非平凡零点，在于 J(x)的表达式中那些与零点有关的项，即- $\sum_{\mathrm{Im}(\rho)>0}[\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x\ 1-\rho)]$。在 $J(x)$的表达式中，所有其他的项都十分简单，也比较光滑，因此素数分布的细致规律——那些细致的疏密涨落——主要就蕴涵在了这个与黎曼ζ函数的非平凡零点有关的级数之中。如上所述，这个级数是条件收敛的，也就是说它的收敛有赖于参与求和的各项——即来自不同零点的贡献——之间的相互抵消。这些来自不同零点的贡献就像一首盘旋起伏的舞曲，引导着素数的细致分布。而这首舞曲的奔放程度——也就是这些贡献相互抵消的方式和程度——则决定了素数的实际分布与素数定理给出的渐近分布之间的接近程度。所有这一切都定量地取决于黎曼ζ函数非平凡零点的分布。黎曼给出的素数分布的精确表达式之所以没能立即使得对素数定理的直接证明成为可能，原因正是因为当时人们对黎曼ζ函数非平凡零点的分布还知道得太少（事实上当时人们所知道的也就是我们在上面已经提到过的 0≤Re(ρ)≤1,无法有效地估计那些来自零点的贡献，从而也就无法有效地估计素数定理与素数实际分布——即黎曼给出的精确表达式——之间的偏差。

那么黎曼 ζ 函数非平凡零点的分布对素数定理与素数实际分布之间的偏差究竟有什么样的影响呢？在这个问题上数学家们已经取得了一系列结果。素数定理的证明本身就是其中一个。在素数定理被证明之后，1901 年，瑞典数学家科赫（von Koch，1870—1924）进一步证明了（这正是我们前面提到过的以黎曼猜想的成立为前提的数学命题的一个例子），假如黎曼猜想成立，那么素数定理与素数实际分布之间的绝对偏差为 $O(x^{\frac{1}{2}}\ln x)$。$Li(x^{\rho})$的模随的增加以$x^{Re(\rho)}/\ln x$的方式增加，因此任何一对非平凡零点 ρ 与 1-ρ 所给出的渐近贡献 $Li(x^{\rho})+Li(x^{1-\rho})$起码是 $Li(x^{\frac{1}{2}})\sim x^{\frac{1}{2}}/\ln x$ 。这一结果暗示素数定理与素数实际分布之间的偏差不可能小于 $Li(x^{\frac{1}{2}})$。事实上，英国数学家李特尔伍德（John Littlewood，1885—1977）曾经证明，素数定理与素数实际分布之间的偏差起码有 $Li(x^{\frac{1}{2}})\ln\ln\ln x$ 。这与科赫的结果已经非常接近（其主项都是$x^{\frac{1}{2}}$）。因此黎曼猜想的成立意味着素数的分布相对有序；而反过来，假如黎曼猜想不成立，假如黎曼 ζ 函数的某一对非平凡零点 ρ 与 1-ρ 偏离了临界线（即 Re(ρ)>1/2 或 Re(1-ρ)>1/2），那么它们所对应的渐近贡献的主项就会大于$x^{\frac{1}{2}}$，从而素数定理与素数实际分布之间的偏差就会变大。因此，对黎曼猜想的研究使数学家们看到了貌似随机的素数分布背后奇异的规律和秩序。这种规律和秩序就体现在黎曼ζ函数非平凡零点的分布之中。

1885 年，一位叫做斯蒂尔切斯（Thomas Stieltjes，1856—1894）的年轻的荷兰数学家，在巴黎科学院发表了一份简报，声称自己证明了以下结果：

$M(N)\equiv\sum_{n<N}\mu(n)=O(N^{\frac{1}{2}})$,

这里的 μ(n)是我们前面提到过的默比乌斯函数，由它的求和所给出的函数 M(N)称为梅尔滕斯函数（Mertens function）。这个命题看上去倒是“面善”得很：默比乌斯函数 μ(n)是一个整数函数，其定义虽有些琐碎，却也并不复杂，而梅尔滕斯函数 M(N)不过是对 μ(n)的求和，证明它按照 $O(N^{\frac{1}{2}})$增长似乎不像是一件太困难的事情。但这个其貌不扬的命题事实

上却是一个比黎曼猜想更强的结果！换句话说，证明了上述命题就等于证明了黎曼猜想（但反过来则不然，否证了上述命题并不等于否证了黎曼猜想）。因此斯蒂尔切斯的简报意味着声称自己证明了黎曼猜想。虽然当时黎曼猜想还远没有像今天这么热门，消息传得也远没有像今天这么飞快，但有人证明了黎曼猜想仍是一个非同小可的消息。别的不说，证明了黎曼猜想就意味着证明了素数定理，而后者自高斯等人提出以来折磨数学家们已近一个世纪之久，却仍未得到证明。与在巴黎科学院发表简报几乎同时，斯蒂尔切斯给当时法国数学界的一位重量级人物埃尔米特（Charles Hermite，1822—1901）发去了一封信件，重复了这一声明。但无论在简报还是在信件中斯蒂尔切斯都没有给出证明，他说自己的证明太复杂，需要简化。换作是在今天，一位年轻数学家开出这样一张空头支票，是很难引起数学界的任何反响的。但是 19 世纪的情况有所不同，因为当时学术界常有科学家做出成果却不公布（或只公布一个结果）的事，高斯和黎曼都是此道中人。因此像斯蒂尔切斯那样声称自己证明了黎曼猜想，却不给出具体证明，在当时并不算离奇。学术界对之的反应多少有点像现代西方法庭所奉行的无罪推定原则，即在出现相反证据之前倾向于相信声明成立。

但相信归相信，数学当然是离不开证明的，而一个证明要想得到最终的承认，就必须公布细节、接受检验。因此大家就期待着斯蒂尔切斯发表具体的证明，其中期待得最诚心实意的当属接到斯蒂尔切斯来信的埃尔米特。埃尔米特自 1882 年起就与斯蒂尔切斯保持着通信关系，直至 12 年后斯蒂尔切斯过早地去世为止。在这期间两人共交换过 432 封信件。埃尔米特是当时复变函数论的大家之一，他与斯蒂尔切斯的关系堪称数学史上一个比较奇特的现象。斯蒂尔切斯刚与埃尔米特通信时还只是莱顿天文台（Leiden Observatory）的一名助理，而且就连这个助理的职位还是靠了他父亲（斯蒂尔切斯的父亲是荷兰著名的工程师兼国会成员）的关照才获得的。在此之前他在大学里曾三度考试失败。好不容易“拉关系、走后门”进了天文台，斯蒂尔切斯却“身在曹营心在汉”，手上干着天文观测的活，心里惦记的却是数学，

并且给埃尔米特写了信。照说当时一无学位、二无名声的斯蒂尔切斯要引起像埃尔米特那样的数学元老的重视是不容易，甚至不太可能的。但埃尔米特是一位虔诚的天主教徒，他恰巧对数学怀有一种奇特的信仰，他相信数学存在是一种超自然的东西，寻常的数学家只是偶尔才有机会了解数学的奥秘。那么，什么样的人能比“寻常的数学家”更有机会了解数学的奥秘呢？埃尔米特凭着自己的神秘主义眼光找到了一位，那就是默默无闻的观星之人斯蒂尔切斯。埃尔米特认为斯蒂尔切斯具有上帝所赐予的窥视数学奥秘的眼光，他对之充满了信任。在他与斯蒂尔切斯的通信中甚至出现过“你总是对的，我总是错的”那样极端的赞许。在这种奇特信仰与 19 世纪数学氛围的共同影响下，埃尔米特对斯蒂尔切斯的声明深信不疑。但无论埃尔米特如何催促，斯蒂尔切斯始终没有公布他的完整证明。一转眼 5 年过去了，埃尔米特对斯蒂尔切斯依然“痴心不改”，他决定向对方“诱之以利”。在埃尔米特的提议下，法国科学院将 1890 年数学大奖的主题设为“确定小于给定数值的素数个数”。这个主题大家想必有似曾相识的感觉，是的，它跟我们前面刚刚介绍过的黎曼那篇论文的题目十分相似。事实上，该次大奖的目的就是征集对黎曼那篇论文中提及过却未予证明的某些命题的证明（这一点明确写入了征稿要求之中）。至于那命题本身，则既可以是黎曼猜想，也可以是其他命题，只要其证明有助于“确定小于给定数值的素数个数”即可。在如此灵活的要求下，不仅证明黎曼猜想可以获奖，就是证明比黎曼猜想弱得多的结果——比如素数定理——也可以获奖。在埃尔米特看来，这个数学大奖将毫无悬念地落到斯蒂尔切斯的腰包里，因为即便斯蒂尔切斯对黎曼猜想的证明仍然“太复杂，需要简化”，他依然能通过发表部分结果或较弱的结果而领取大奖。可惜直至大奖截止日期终了，斯蒂尔切斯依然毫无动静。

但埃尔米特也并未完全失望，因为他的学生阿达马提交了一篇论文，领走了大奖——肥水总算没有流入外人田。阿达马获奖论文的主要内容正是前面提到过的对黎曼论文中辅助函数 ξ(s)的连乘积表达式的证明。这一证明虽然不仅不能证明黎曼猜想，甚至离素数定理的证明

也还有一段距离，却仍是一个足可获得大奖的进展。几年之后，阿达马再接再厉，终于一举证明了素数定理。埃尔米特放出去的这根长线虽未能如愿钓到斯蒂尔切斯和黎曼猜想，却错钓上了阿达马和素数定理，斩获亦是颇为丰厚(素数定理的证明在当时其实比黎曼猜想的证明更令数学界期待）。

那么斯蒂尔切斯呢？没听过这个名字的读者可能会觉得他是一个浮夸无为的家伙，事实却不然。斯蒂尔切斯在分析与数论的许多方面都做出过重要贡献。他在连分数方面的研究为他赢得了“连分数分析之父”的美誉；挂着他名字的黎曼-斯蒂尔切斯积分（Riemann-Stieltjes integral）更是将他与黎曼的大名联系在了一起（不过两人之间并无实际联系——黎曼去世时斯蒂尔切斯才 10 岁）。但他那份有关黎曼猜想的声明却终究没能为他赢得永久的悬念。现在数学家们普遍认为斯蒂尔切斯所宣称的关于 $M(N)=O(N^{\frac{1}{2}})$的证明即便有也是错误的。不仅如此，就连命题 $M(N)=O(N^{\frac{1}{2}})$本身的成立也已受到了越来越多的怀疑。

素数定理自高斯与勒让德以经验公式的形式提出以来，许多数学家对此做过研究。其中一个比较重要的结果是由俄国数学家切比雪夫（Pafnuty Chebyshev，1821—1894）做出的。早在 1850 年，切比雪夫就证明了对于足够大的x，素数分布 π(x)与素数定理给出的分布 Li(x)之间的相对误差不会超过 1%。但在黎曼 1859 年的研究以前，数学家们对素数分布的研究主要局限在实分析手段上。从这个意义上讲，即使撇开具体的结果不论，黎曼建立在复变函数之上的研究仅就其方法而言，也是对素数分布研究的重大突破。这一方法上的突破为素数定理的最终证明铺平了道路。

前面曾经提到，黎曼对素数分布的研究之所以没能直接导致素数定理的证明，是因为人们对黎曼ζ函数非平凡零点的分布还知道得太少。那么，为了证明素数定理，我们起码要知道多少有关黎曼 ζ 函数非平凡零点分布的信息呢？这一问题的答案到了 1895 年随着曼戈尔特对黎曼论文的深入研究而变得明朗起来。曼戈尔特的研究我们在前面已经提到过，正是他证黎

曼关于 J(x)的公式。但曼戈尔特那项研究的价值比仅仅证明黎曼关于 J(x)的公式要深远得多。

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曼戈尔特在研究中使用了一个比黎曼的 J(x)更简单有效的辅助函数 Ψ(x)，它的定义为:

$\Psi(x)=\sum_{n<x}\Lambda(n)$，其中 Λ(n)被称为曼戈尔特函数（von Mangoldt function），它对于 n =pk（p 为素数，k 为自然数）取值为 ln(p)；对于其他 n 取值为 0。应用 Ψ(x)，曼戈尔特证明了一个本质上与黎曼关于 J(x)的公式相等价的公式：

$\Psi(x)=x-\sum_\rho \frac{x^\rho}{\rho}-\frac{1}{2}\ln(1-x^{-2})-\ln(2\pi)$,

其中有关 ρ 的求和与黎曼的 J(x)中的求和一样，也是先将 ρ 与 1-ρ 配对，再依 Im(ρ)从小到大的顺序进行。

很明显，曼戈尔特的 Ψ(x)表达式比黎曼的 J(x)简单多了。时至今日，Ψ(x)在解析数论的研究中差不多已完全取代了黎曼的 J(x)。引进 Ψ(x)的另一个重大好处是早在几年前，前面提到的切比雪夫就已经证明了素数定理 π(x)～Li(x)等价于 Ψ(x)～x。为了纪念切比雪夫的贡献，曼戈尔特函数也被称为第二切比雪夫函数（second Chebyshevfunction）。

将这一点与曼戈尔特有关 Ψ(x)的那个本质上与黎曼关于 J(x)的公式相等价的公式联系在一起，不难看到素数定理成立的条件是:

$\lim_{x\to\infty}\sum_\rho (x^{\rho-1}/\rho)=0$,

这一条件启示我们考虑$x^{\rho-1}$在x→∞时趋于零的情形。而要让$x^{\rho-1}$在x→∞时趋于零，Re(ρ)

必须小于 1。换句话说黎曼 ζ 函数在直线 Re(s)=1 上必须没有非平凡零点。这就是我们为证明素数定理而必须知道的有关黎曼ζ函数非平凡零点分布的信息。由于黎曼 ζ 函数的非平凡零点是以 ρ 与 1-ρ 成对的方式出现的，因此这一信息等价于 0<Re(s)<1。

前面曾经提到过，黎曼ζ函数的所有非平凡零点都位于 0≤Re(s)≤1 的区域内。因此为了证明素数定理，我们所需知道的有关黎曼ζ函数非平凡零点分布的信息要比我们已知的（也是当时数学家们已知的）略多一些（但仍大大少于黎曼猜想所要求的）。这样，在经过了切比雪夫、黎曼、阿达马和曼戈尔特等人的卓越努力之后，我们离素数定理的证明终于只剩下了最后一小步：即把已知的零点分布规律中那个小小的等号去掉。这一小步虽也绝非轻而易举，却已难不住在黎曼峰上攀登了三十几个年头 ,为素数定理完整证明的到来等待了一个世纪的数学家们。(注;在后面我的论文中我也证明了这一点)曼戈尔特的结果发表后的第二年(即 1896 年），阿达马与普森就几乎同时独立地给出了对这最后一小步的证明，从而完成了自高斯以来数学界的一个重大心愿。那时斯蒂尔切斯已经去世两年了。

经过素数定理的证明，人们对于黎曼 ζ 函数非平凡零点分布的了解又推进了一步，那就是证明了黎曼 ζ 函数的所有非平凡零点都位于复平面上 0<Re(s)<1 的区域内。在黎曼猜想的研究中数学家们把这个区域称为临界带（critical strip）。

素数定理的证明——尤其是以一种与黎曼的论文如此密切相关的方式所实现的证明——让数学界把更多的注意力放到了黎曼猜想上来。四年后（即 1900 年）的一个夏日，两百多位当时最杰出的数学家会聚到了巴黎，一位 38 岁的德国数学家走上了讲台，作了一次永载数学史册的伟大演讲。演讲的题目叫做《数学问题》，演讲者的名字叫做希尔伯特(David Hilbert，1862—1943），他恰好来自高斯与黎曼的学术故乡——群星璀璨的哥廷根大学。他是哥廷根数学精神的伟大继承者，一位与高斯及黎曼齐名的数学巨匠。希尔伯特在演讲稿中列出了 23 个对后世产生深远影响的数学问题 ,黎曼猜想被列为其中第八个问题的一部分，

从此成为整个数学界瞩目的难题之一。

20 世纪的数学大幕在希尔伯特的演讲声中徐徐拉开，黎曼猜想也迎来了一段新的百年征程。

我们把质数计数函数记作 $\pi(x)(x \in R^{+})$,这个函数的名称与圆周率毫无关系。根据素数定理，$\pi(x) \approx \frac{x}{\ln x}(x \in \mathrm{R}^{+})$。小于等于 1 的质数是 1,除外 1 以外的质数的个数是 0，所以 $\pi(1)=0$。小于等于 2 的质数是 1 和 2，除 1 以外的质数的个数是 1，所以$\pi(2)=1$。小于等于 3 的质数是 1、2、3，除外 1 以外的质数的个数是 2，所以$\pi(3)=2$。小于等于 4 的质数是 1、2、3，除外 1 以外的质数的个数是 2，所以$\pi(4)=2$。小于等于 5 的质数是 1、2、3，5，除 1 以外的质数的个数是 3，所以$\pi(5)=3$。所以$\pi(6)=3$，$\pi(7)=4$，$\pi(11)=5$，$\pi(13)=6$， ... ,如此等等。如果我们得到了质数计数函数得到了一个简便的计算表达式，这将会造成惊人的结果，对于关于数学分布的理论和应用及对数学学科的发展有重大的意义。

黎曼改进了质数计数函数，黎曼得到的质数计数函数叫做$J(x)(x \in Z^{+})$。J(x)(x ∈ Z+和 πx≈xlnxx∈Z+的关系如下:

$\pi(x)=\sum_{\mathrm{n}=1}^{\infty} \frac{\mu(\mathrm{n})}{\mathrm{n}} J\left(x^{\frac{1}{\mathrm{n}}}\right)=J(x)-\frac{1}{2} J\left(x^{\frac{1}{2}}\right)-\frac{1}{3} J\left(x^{\frac{1}{3}}\right)-\frac{1}{5} J\left(x^{\frac{1}{5}}\right)+\frac{1}{6} J(x^{\frac{1}{6}})$-...

$(x \in R^{+}, n \in Z^{+})$，且

$\frac{1}{s}\ln\zeta(s)=\int_0^{\infty} \mathrm{J}(x) x^{-s-1} \mathrm{d}x$,

$\mu(n)$叫做莫比乌斯函数。莫比乌斯函数$\mu(n)$的取值只有三种，就是 0 和正负 1,如果n可以被任何质数的平方整除，也就是说在n的质因数分解中除 1 以外有一个或多个质因数的指数的次方出现了二次或更高次方，那么$\mu(n)=0$。如果n不可以被任何质数的平方整除，也就是说 n 的质因数分解中除 1 以外任何质因数的指数的次方都是一次，那么我们来数质因数的个数。如果质因数的个数是偶数个，那么$\mu(n)=1$。而假如质因数的个数是奇数个，那么$\mu(\mathrm{n})=-1$。这里还包括了 n=1 的情况，因为 1 没有除了 1 以外的质因数，那么 1 的除 1 以外的质因数的个数就是 0，0 算作偶数，所以

μ(1)=1。 在上面的展开式中，随着 n($n \in Z^{+}$)的增加，$\frac{1}{n}$就变得越来越小，$x^{\frac{1}{n}}$($n \in Z^{+}$)也变得越来越小，相应的第 n($n \in R^{+}$and $n \to +\infty$)项也就变得越来越小。表明对于π(x)的值来说，贡献最大的就是第一项$J(x)$。下面再来看黎曼得到的下面这个公式:

$J(x) = \mathrm{Li}(x)$-$\sum_{\rho} \mathrm{Li}(x^{\rho})$ $+\int_{x}^{+\infty} \frac{\mathrm{dt}}{\mathrm{t}^{2}(\mathrm{t}^{2}-1)\ln\mathrm{t}} - \ln 2$ ($x \in \mathrm{R}^{+}$),

其中 Li(x)=$\int_{0}^{x} \frac{\mathrm{dt}}{\ln\mathrm{t}}$($x \in \mathrm{R}^{+}$), $J(x)$又可以描述为

$J(x)$叫做阶梯函数，它在x等于零的地方等于零，即$J(0)$=0,然后随着x的值不断增大，每经过一个质数(比如 2,3,5，...)，$J(x)$的值就增加 1。每经过一个质数的平方(比 4，9，25)，，$J(x)$的值就增加$\frac{1}{2}$。每经过一个质数的 3 次 方(比如 8,9,25，...)，J(x)的值就增加$\frac{1}{3}$。每经过一个质数的 4 次 方(比如 16, 81,256，625, ...)，J(x)的值就增加$\frac{1}{4}$。以此类推，每经过一个质数的x^{n}($n \in Z^{+}$，$n \to +\infty$)次方，J(x)的值就增加$\frac{1}{\mathrm{n}}$($n \in Z^{+}$and $n \to +\infty$)。你可以把它理解为，每经过过一个x^{n}($n \in R^{+}$ and $n \to +\infty, n$是一个质数),$J(x)$的值就被增加了$\frac{1}{\mathrm{n}}$($n \in Z^{+}$and $n \to +\infty$)。显然，这个函数跟质数的分布密切相关。来看等式的右边，第一项叫做对数积分函数 $\mathrm{Li}(x) = \int_{0}^{x} \frac{\mathrm{dt}}{\ln\mathrm{t}}$($x \in \mathrm{Z}^{+}$)，在 x 充分大的时候，$\mathrm{Li}(x) \approx \frac{x}{\ln x}$ ($x \in \mathrm{Z}^{+}$)。所以$\pi(\mathrm{x}) \approx \mathrm{Li}(\mathrm{x}) \approx \frac{\mathrm{x}}{\ln\mathrm{x}}$(x $\in \mathrm{Z}^{+}$,并且 x 充分大)。再看第二项$\mathrm{Li}(x^{\rho})$ ($x \in$ Z+,并且 ρ∈C,ρ 是除负偶数之外的复数,ρ 被黎曼称为黎曼 ζs n∈R+ and s≠1 and s≠−2n 函数的非平凡零点。ρ 被记作：$\rho = \sigma + it$($\sigma \in R$，$t \in \mathrm{R}$), 在实数轴上，除了负偶数，黎曼ζ(s)($s \in C$,且 $s \neq 1$ 且 $s \neq -2n$, $n \in R^{+}$)函 数 就 再 也 没 有 零 点 了 ，所 以 ρ(ρ ∈ C, $n \in$ R+且 ρ≠1 且 ρ≠−2n 肯定不是实数，那么$x$$\rho$ρ∈$C$,$x$∈$R$+, andρ≠1 andρ≠−2n,n∈R+ 也肯定不是实数，那么$\mathrm{Li}(x^{\rho})$ ($x \in Z^{+}$，$\rho \in C$,且 $\rho \neq 1$ 且 $\rho \neq -2n, n \in R^{+}$)如何计算? 只要将 Li($x$)=$\int_{0}^{x} \frac{\mathrm{dt}}{\ln\mathrm{t}}$($x \in \mathrm{R}^{+}$)的定义域解析延拓到除以 1 以外的全体复数。黎曼证明了黎曼ζ(ρ)(ρ ∈ C 且 $s \neq 1$ 且 $\rho \neq -2n, n \in Z^{+}$)函数的非平凡零点ρ必定满足 0≤Re(ρ)≤1，人们把这复平面上这宽度为 1 的竖直条带内称为临界带，并把满足

Re(s)= $\frac{1}{2}\left(s\in C\text{ 且 }s\neq 1\text{ 且 }s\neq -2n, n\in Z^{+}\right)$的垂直于实数轴的直线叫做临界线，也就是临界带的中心线。黎曼猜测黎曼 $\zeta(s)\left(s\in C\text{ 且 }s\neq 1\text{ 且 }s\neq -2n, n\in Z^{+}\right)$函数的非平凡零点都位于临界线上，这是一个非常令人惊讶的结论。假如黎曼 $\zeta(s)\left(s\in R^{+}\text{ 且 }s\neq 1\text{ 且 }s\neq -2n, n\in R+\right.$ 函数的非平凡零点的实部是在 0 到 1 之间随机取值的 ,那么它刚好取到 12 的概率应该等于 0,而黎曼却认为这个概率是 100%。如果黎曼猜想严格成立，那么素数的出现或素数的分布就一点都不随机，而是以确定的方式出现的，在这背后必定有有深刻的原因。质数定理的证明是研究黎曼猜想的过程中的中间产物。1896 年，阿达玛和德拉瓦布桑，证明了黎曼$\zeta(\rho)\left(\rho\in C\text{ 且 }\rho\neq 1\text{ 且 }\rho\neq -2n\right)$函数的非平凡零点ρ在满足 Re(ρ)=0 和 Re(ρ)=1 时无零点，从而轻松地证明了素数定理$\pi(x)\approx\frac{x}{\ln x}(x\in Z^{+})$。素数定理$\pi(x)\approx\frac{x}{\ln x}(x\in Z+$成立，表明对于质数计数函数 πx来说，它的值最大部分来自于对数积分函数 Li(x)= $\int_0^x\frac{dt}{\ln t}(x\in Z^{+})$，而它的值次要部分来自于$\mathrm{Li}(x^{\rho})\left(x\in Z^{+}, \rho\in C\text{ 且 }s\neq 1\text{ 且 }\rho\neq -2n, n\in Z+\right.$，由于 $x\ln x$ $x\in Z+$的计算很简单，所以但对于质数计数函数 πx的精确计算来说，黎曼$\zeta(\rho)\left(\rho\in C\text{ 且 }s\neq 1\text{ 且 }\rho\neq -2n, n\in Z^{+}\right)$函数的非平凡零点ρ的计算就至关重要，黎曼猜想的严格证明就至关重要。1921 年英国数学家哈代证明了黎曼 $\zeta(s)(s\in C\ and\ s\neq 1\ and\ s\neq -2n, n\in Z+$ 函数有无穷多个非平凡零点位于临界线上。但这个结论实际上跟黎曼猜想是完全两码事，因为有无穷多个非平凡零点位于临界线上，并不代表所有零点都位于临界线上。就像一条线段上有无穷多个点，但一条直线有无穷多条线段一样，哈代证明的结果跟所有非平凡零点的数目比起来，所占的百分比几乎是零。而数学家们将这个百分比推进到明显大于零的数还是到了 1942 年。那一年，挪威数学家塞尔伯格证明出百分比大于零，但没有给出具体数值是多少。1974 年美国数学家列森证明了至少有 34%的非平凡零点位于临界线上。1980 年中国数学家楼世拓和姚琦证明有 35%的非平凡零点位于临界线上。1989

年美国数学家康瑞证明有 40%的非平凡零点位于临界线上。黎曼 ζ(s) $\left(s \in R^{+}\right.$ 且 $s \neq$ 1 且 $s\neq-2n,n\in Z+$ 函数的非平凡零点的计算比较复杂，格兰姆算出了黎曼 ζs 函数的前 15 个非平凡零点，如下图所示(列出了其中的十个)：

零点序号	格拉姆的零点数值	现代数值
1	1/2+14.134 725i	1/2+14.134 725 1i
2	1/2+21.022 040i	1/2+21.022 039 6i
3	1/2+25.010 856i	1/2+25.010 857 5i
4	1/2+30.424 878i	1/2+30.424 876 1i
5	1/2+32.935 057i	1/2+32.935 061 5i
6	1/2+37.586 176i	1/2+37.586 178 1i
7	1/2+40.918 720i	1/2+40.918 719 0i
8	1/2+43.327 073i	1/2+43.327 073 2i
9	1/2+48.005 150i	1/2+48.005 150 8i
10	1/2+49.773 832i	1/2+49.773 832 4i

过了 25 年，又有 138 个非平凡零点被计算出。打那以后，黎曼 ζ(s)函数的非平凡零点的计算就陷入了停滞，因为当时的计算方法十分笨拙，也没有计算机做辅助。在计算被停顿了 7 年后，僵局被打破了，德国的数学家西格尔在黎曼的手稿里发现了黎曼那遥遥领先那个时代 70 年的的高明算法，让非平凡零点的计算一下子柳暗花明。为了表彰西格尔，这个算法公式也被称为黎曼-西格尔公式，西格尔本人也因此获得了菲尔兹奖。数学家的手稿比古董要值钱得多。自那以后，黎曼 ζ(s)函数的非平凡零点的计算要快得多了。哈代的学生将黎曼 ζ(s)函数的非平凡零点的计算推进到了 1041 个，人工智能之父图灵将黎曼 ζ(s)函数的非平凡零点的计算推进到了 11041 个，后来随着计算机的应用，对黎曼 ζ(s)函数的非平凡零点的计算从 350 万个推进到 3 亿个，15 亿个，8500 亿个，乃至目前的十万亿个，这些非平凡零点都位于黎曼所说的临界线上。但十万亿零点位于临界线上相对于无限个零点都位于临界线上这一个猜测来说根本不算什么，无论计算出的位于临界线上的零点数目有多大，都还不足

以证明黎曼猜想就是正确。黎曼猜想的正确性需要严格的理论证明。人们依据十万亿零点位于临界线上猜测黎曼 $\zeta(s)\left(s\in C 且 s\neq 1 且 s\neq -2n, n\in R^{+}\right)$函数的非平凡零点对于实数轴是对称的，但猜测终归是猜测，需要对这个猜测进行严格的证明，否则这样的猜测没有意义。在后面我的这篇论文中对这个猜测进行了严格的证明，并对黎曼猜想作了严格的证明，黎曼猜想确实成立。

素数定理$\pi(x)\approx\frac{x}{\ln x}(x\in Z^{+})$已由阿达马（Hadamard）与德·拉·瓦·布桑（dela Valee Poussin）于 1896 年独立地证明了。但人们期望有一个具有精密误差项的素数定理。在 RH 之下，可以证明$\pi(x)=\mathrm{Li}(x)+O(\sqrt{x}\ln x)$。反之，由这个公式也可以推出 RH。所以，这个公式可以看作 RH 的算术等价形式。由此足见 RH 的极端重要性了。黎曼的文章中还包括了几个未经严格证明的命题。除了 RH 之外，都被阿达马与曼戈尔特（Mangoldt）证明了，只剩下现在所谓的 RH。命 N（T）表示 ζ(s)在矩形 $0\leq\sigma\leq 1$，$0<t<T$ 中的零点个数，黎曼作了猜想：

$N(T)\sim\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$，这个结果已由曼戈尔特证明。命 N（T）

表示在线段 $\sigma=\frac{1}{2}$，$0<t<T$ 上，$\zeta(x)$的零点个数，塞尔伯格（Selberg）证明了，存在正常数 c 与 T,则

$$N_0(T)>c\,N(T)$$

这个结果是非常惊人的。它说明了ζ(s)在线段 $\sigma=\frac{1}{2}$,$0<t<T$ 上的零点个数与它在矩形 $0\leq\sigma\leq 1$，$0<t<T$ 上的零点个数相比，占有一个正密度，而线段的二维测度为零。黎曼ζ函数ζ(s）与 RH 都是“原型”，有不少ζ(s)与 RH 的类似及推广。这些类似及推广都有强烈的数学背景,有许多 RH 的某种推广，它们的数学背景都是极其重要的。比如有限域 F 上的平面代数曲线对应的 RH，即每一条满足一定条件的代数曲线都对应于一个 L 函数，它们的零点都位于直线 $\sigma=\frac{1}{2}$上。这一命题已由韦伊（Weil）证明，而且韦伊对于高维代数簇的 RH 也作了

猜想。这个猜想已由德利涅（Deligne）证明。这些无疑都是 20 世纪最伟大的数学成就之一。据我所知韦伊与德利涅的结果对解析数论就有极大的推动。例如，由韦伊证明的 RH 可以推出模素数 p 的克卢斯特曼（Kloosterman）和与完整三角和的最佳阶估计。

下面来介绍黎曼ζ(s)函数等式，对于欧拉ζ(s)函数等式

$\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in R 且 s\neq 1)$和

$\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}$ ($s\in C$ ，$Re(s)>1$ 且 $s\neq 1$）演变为黎曼 ζ(s) 函数等式 $\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in C$ 且 $s\neq 1)$，所以要利用欧拉公式

$e^{ix}=\cos(x)+i\sin(x)(x\in R)$和$e^{iZ}=\cos(Z)+i\sin(Z)(Z\in C)$，并将复数的三角函数表达式中的乘方运算中的指数从正整数推广到指数为一般实数,从而将欧拉级数 $\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in R 且 s\neq 1)$和$\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\sum_{n=1}^{\infty}\frac{1}{n^s}(s\in C$ ，$Re(s)>1$ 且 $s\neq 1)$的定义域解析延拓到整个复数平面，这样，除了 s=1 以外，它处处解析，这样得到的 ζ 函数就和黎曼的 ζ 函数等价。

黎曼猜想等价于$\zeta(s)=\zeta(\overline{s})=0(s\in C 且 s\neq 1)$和$\zeta(1-s)=\zeta(s)=0(s\in C 且 s\neq 1)$都成立。

$\zeta(1-s)=\zeta(s)=0$ 可以由 $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ 且 $s\neq 1)$得到，$\zeta(s)=\zeta(\overline{s})=0$ 可以当 $\zeta(s)=0$, 由 $\zeta(s)=\overline{\zeta(\overline{s})}$ 得到，为了得到 $\zeta(s)=\overline{\zeta(\overline{s})}$ ，必须利用欧拉公式 $e^{ix}=\cos(x)+i\sin(x)(x\in R)$和$e^{iZ}=\cos(Z)+i\sin(Z)(Z\in C)$，并将复数的三角函数表达式中的乘方运算中的指数从正整数推广到指数为一般实数后，进行严格的证明。如果想解决黎曼猜想，它的证明就必须遵循这样的原则和方法，否则可能就不正确。

下面我们来看看$\sum\frac{1}{n^s}=\prod(\frac{1}{1-p^{-s}})$是怎么得到的。这是欧拉的一个公式，公式中 n 为自然数，p 为质数，s 为大于的正整数。欧拉早已将之证明，下面我将重复这个证明。

欧拉刚提出这个公式的时候，显然这个公式的两边都是级数，欧拉发现有这么一个级数：

$\sum\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots$ 记为 (1 式)

上面等式两边同乘$\frac{1}{2^s}$，左边多了一乘积项$\frac{1}{2^s}$，右边作幂运算，得：

$\frac{1}{2^s}\sum\frac{1}{n^s}=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\frac{1}{12^s}+$…(2 式)

将(1 式)和(2 式)这两个等式左右两边相减，得：

$(1-\frac{1}{2^s})\sum\frac{1}{n^s}=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{15^s}+$…(3 式)

可以观察到，相对于(1 式)，(3 式)左边的乘积项增加了 1-$\frac{1}{2^s}$，(3 式)右边为(1 式)右边去除了所有分母为偶数项后留下的各项。

将(3 式)左右两边同乘以$\frac{1}{3^s}$得:

$\frac{1}{3^s}(1-\frac{1}{2^s})\sum\frac{1}{n^s}=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\frac{1}{33^s}+\frac{1}{39^s}+\frac{1}{45^s}$ … (4 式)

将(3 式)和(4 式)这两个等式左右两边相减，得：

$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{25^s}+\frac{1}{29^s}+\frac{1}{31^s}+$ … (5 式)

同理，将(5 式)左右两边同乘$\frac{1}{5^s}$得:

$(\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{35^s}+\frac{1}{55^s}+\frac{1}{65^s}+\frac{1}{85^s}+\frac{1}{95^s}+\frac{1}{115^s}+\frac{1}{145^s}+$ … (6 式)

将(5 式)和(6 式)这两个等式左右两边相减，得：

$(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum\frac{1}{n^s}=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{29^s}+\frac{1}{31^s}+\frac{1}{37^s}+$ … (7 式)

参照如此方法继续下去，在(2k-1 式)左右两边同乘$\frac{1}{{p_{(i)}}^s}$ (i 为正整数) ，$p_{(i)}$为(2k-1 式)左边第一项中$(1-\frac{1}{{p_{(i-1)}}^s})$的素数$p_{(i-1)}$的上一个素数，这里的“上一个素数”指的是在数值大小上最接近$p_{(i-1)}$的，两者之间没有别的素数的素数，且$p_{(i)}>p_{(i-1)}$。(2k-1 式)左边乘积项就增加一项$\frac{1}{{p_{(i)}}^s}$,(2k-1 式)右边就变为：第 1 项为$\frac{1}{{p_{(i)}}^s}\frac{1}{{p_{(i)}}^s}$，第 2 项为$\frac{1}{{p_{(i)}}^s}*\frac{1}{{p_{(i+1)}}^s}$，第 3 项为 $\frac{1}{{p_{(i)}}^s}*\frac{1}{{p_{(i+2)}}^s}$，第 4 项为$\frac{1}{{p_{(i)}}^s}*\frac{1}{{p_{(i+3)}}^s}$，第 5 项为$\frac{1}{{p_{(i)}}^s}*\frac{1}{{p_{(i+4)}}^s}$，… ，$\frac{1}{{p_{(i)}}^s}*\frac{1}{{p_{(i+k)}}^s}$，…，k. 为正整数。如此下去，并把所有这些项都加起来，其中$p_{(1)}$、$p_{(2)}$、$p_{(3)}$ ，… ，$p_{(i)}$ ，$p_{(i+1)}$ ，$p_{(i+2)}$ ，$p_{(i+3)}$、$p_{(i+4)}$、… ，$p_{(i+k)}$ ，…，是由所有质数按照数值大小从小到大的顺序排列起来的无穷数列，且$p_{(3)}=5$，

$p_{(2)}=3$,$p_{(1)}$=2 ,这样就得到(2k-1 式)右边的表达式 ,把这整个等式记为(2k-1 式)后的(2k 式)。

参照如此方法并往复做下去，最终将得到这么一个等式：

它的左边$\sum\frac{1}{n^s}$的系数是一些形如 $(1-\frac{1}{p^s})$的连乘积，n 为自然数，p 遍取所有质数，为了书写方便，引入连乘符号Π，将左边记为:

$\prod(1-\frac{1}{p^s})\sum\frac{1}{n^s}$

右边为 1 加上一个分数$\frac{1}{{p_{(i)}}^s * {p_{(i+k)}}^s}$ ，${p_{(i)}}^s$与${p_{(i+k)}}^s$是两个无穷的大素数，所以 $\frac{1}{{p_{(i)}}^s * {p_{(i+k)}}^s}$的值为零，该分数可以略去。所以，右边只剩下 1。那么，$\sum\frac{1}{n^s}=\frac{1}{\prod(1-\frac{1}{p^s})}=\prod\frac{1}{(1-\frac{1}{p^s})}=\prod\frac{1}{1-p^{-s}}$，即：

$\sum_{n=1}^{\infty} n^{-s}=\prod_p(1-p^{-s})^{-1}(s\in Z^+$ 并且 $s\neq 1$, $n\in Z^+$ ，n 便取所有正整数, $p\in Z^+$ ，p 便取所质数)。这个乘积公式是由瑞士数学家欧拉于 1737 年在《无穷级数的若干观察》一文中提出并证明的，欧拉乘积公式将自然数的求和表达式与素数的连续乘积表达式联系起来，包含了关于素数分布的重要信息。这一信息在 122 年之后终于被黎曼破译，这导致了黎曼著名的论文“关于小于给定值的质数的个数”的发表。为了纪念黎曼，欧拉乘积公式的左端以黎曼命名，并采用黎曼使用的符号ζ(s)(s∈C, 并且 s≠1) ,称为黎曼ζ函数。前面我介绍了黎曼用积分的形式对欧拉复变函数ζ(s)进行了解析延拓，将欧拉定义的正整数 s 解析延拓到复数 ,定义变量 s 为复数 ,得到了黎曼ζ(s)函数。下面我应用欧拉公式$e^{ix}=\cos(x)+i\sin(x)(x\in R)$和$e^{iZ}=\cos(Z)+i\sin(Z)(Z\in C)$及指数为实数的幅角原理，得到了一个适当的函数，同样对欧拉复变函数 ζ(s)进行了解析延拓，将 Re(s)(s≠ 1)推广到全体实数条件下 ζ(s)函数处处解析且都能收敛，得到了 s 为除了 1 以外的全体复数下的解析函数，即黎曼复变 ζ(s)(s≠ 1)函数。我在解决黎曼猜想的过程中, 必须用到欧拉公式$e^{ix}=\cos(x)+i\sin(x)$(x为实数，表示角的弧度数)或$e^{iZ}=\cos(Z)+i\sin(Z)$(Z 为复数)。这个公式欧拉已经将之证明了，可直接使用。下面我用自己的方法再来证明一遍：

设有函数$f_1(x)=e^x$,   我们对$f_1(x)=e^x(x\in R)$求导数，“′”号表示求导，那么$(e^x)'=e^x$，e^x的导

数就是它自身。那么如果我们令函数$f_1(x)$=e^x的自变量为 cx （c 为常数）将得到函数$f_1(cx)$=e^{cx}，对$f_1(cx)$=e^{cx}求导数，那么[$f_1(x)$] ‘=(e^{cx})’= ce^{cx}，如果令函数$f_1(cx)$=e^{cx}中的 c=i(i 也为常数)，那么$f_1(ix)$= e^{ix}，那么[$f_1(ix)$]‘=[e^{ix}]’= ie^{ix}。又假设$f_2(x)$=cos(x)+isin(x)=S，则 S 是一个复数。现在对函数$f_2(x)$求导，有[$f_2(x)$]’=[cos(x)+isin(x)]’=[cos(x)]’+[isin(x)]’=-sin$x$+icos$x$ （1 式），如果$f_1(ix)$= e^{ix}=Cos$x$+iSin$x$成立，那么根据上面得到的[$f_1(x)$]‘=[e^{ix}]’=ie^{ix}，假设e^{ix}=Cos(x)+iSin(x)成立，把e^{ix}=Cos$x$+iSin$x$代入等式[$f_1(ix)$]‘=[e^{ix}]’= ie^{ix}的右边得[$f_1(ix)$]‘=[e^{ix}]’=ie^{ix}=i(cos$x$+isinx)= -sin(x)+icos(x) （2 式），对比(1 式)和(2 式)，可以发现$f_1(ix)$与$f_2(x)$的导数相等，又由于$f_1(ix)$与 f_2 (x) 均 没 有 常 数 项 ， 那 么 f_1 (i x) 与 f_2 (x) 的 表 达 式 应 该 一 致 。 我 们 发 现 f(ix)=e^{ix} =Cos$x$+iSin$x$ =$f_2(x)$，$f_1(ix)$与$f_2(x)$的表达式确实是一致，说明假设 e^{ix}=cos(x)+isin(x)是成立的，也就是说欧拉的公式$e^{ix} = \cos(x)$+isin(x)是对的。要证明e^{ix} =cos$x$+isin$x$(x ∈R)，更好的方法是下面一个方法，不过比较复杂。

大家都可能曾经碰到过这么一个等式：$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$+…($x$ ∈R)，它是如何来的?首先观察函数 y=e^x，如果对这个函数求导数，将得到 y’=(e^x)’=e^x，也就是说 y=e^x的导数是它本身，这是个非常特殊的指数函数，令 y’=$\frac{dy}{dx}$，当$\frac{dy}{dx} = 0$时，y=$e^x = 1$，当$\frac{dy}{dx} = 1$时，y=$e^x = 1+ x$，当$\frac{dy}{dx} = 1 + x$时，y=$e^x = 1+ x+\frac{1}{2}x^2$，当$\frac{dy}{dx} = 1 + x + \frac{1}{2}x^2$时，y=$e^x = 1+ x +\frac{1}{2}x^2+\frac{1}{6}x^3$，当$\frac{dy}{dx} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$时，y=$e^x = 1+ x +\frac{1}{2}x^2+\frac{1}{6}x^3 + \frac{1}{24}x^4$，当$\frac{dy}{dx} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3+\frac{1}{24}x^4$时，y=$e^x = 1+ x +\frac{1}{2}x^2+\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$，依此类推，这就初步证明了 y=$e^x = 1+ x +\frac{1}{2}x^2+\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$+…，但是，一般地 y=$x^n$的级数是怎么样的？y=$e^x$的级数又是怎样的？当把$x$当作 e，n 当作$x$时，就得到 $y = e^x$，这就需要引入幂级数的概念。

设有幂级数：1+ x +x^2+x^3+x^4+x^5+…(x ∈R)，各项都是形如x^n的幂，令函数 f(x)= 1+ x +x^2+x^3+x^4+x^5+…(x ∈R)，等于各项的和，如果用一些数字作为各项的系数，假如这些数字

是a_0，a_1，a_2，a_3，a_4，a_5，a_6…，a_{i-1}，a_i，它们分别是函数 f(x)=x^n的 0 阶导数$f^{(0)}(x)$，f(x)=x^n的 1 阶导数$f^{(1)}(x)$=nx^{n-1}，f(x)=x^n的 2 阶导数$f^{(2)}(x)$=n(n-1)x^{n-2}，f(x)=x^n的 3 阶导数$f^{(3)}(x)$=n(n-1)(n-2)x^{n-3}，… ，f(x)=x^n 的 n 阶导数$f^{(n)}(x)$=n(n-1)(n-2)(n-3)…2∗1∗ x^0在x =0 时的值，分别记作$a_0 = f^{(0)}(0)$，$a_1 = f^{(1)}(0)$，$a_2 = f^{(2)}(0)$，$a_3 = f^{(3)}(0)$，…，$a_{i-1} = f^{(n-1)}(0)$，a_i=$f^{(n)}(0)$。假如把 f(x)=x^n作 n 次求导，将会得到：$f^{(n)}(x)$=n(n-1)(n-2)(n-3)…2×1× x^0，那么$f^{(n)}(0)$=n! ,对于特定的函数 f=e^x 来说 ,所有这些各阶导数在x=0 时的值$f^{(0)}(0)$,$f^{(1)}(0)$,$f^{(2)}(0)$，$f^{(3)}(0)$，…，$f^{(n-1)}(0)$，$f^{(n)}(0)$，…，都必须为 1，因为e^x 的任何阶次的导数都是它本身。但x^n的各阶导数在$x = 0$ 时的值$f^{(n)}$ (0)=n(n-1)(n-2)(n-3)… × 2 × 1 × 0^0 =n! ，因此这些 a_0，a_1，a_2，a_3，a_4，a_5，…，a_{i-1}，a_i必须用 1 除以 n!， 才能让$f^{(0)}(0)$=1，$f^{(1)}(0)$=1，$f^{(2)}(0)$=1，$f^{(3)}(0)$=1，…，$f^{(n-1)}(0)$=1，$f^{(n)}(0)$=1，才能满足正确地写出函数 f(x)=e^x的级数表达式的各项的系数：a_0，a_1，a_2，a_3，a_4，a_5，…，a_{i-1}，a_i，…，即：$a_0=\frac{1}{0!}=1$ ，$a_1=\frac{1}{1!}$，$a_2=\frac{1}{2!}$，$a_3=\frac{1}{3!}$，$a_4=\frac{1}{4!}$ ，$a_5=\frac{1}{5!}$ ，…，$a_n=\frac{1}{n!}$，…，对于特定的函数 f=e^x来说，所有这些各阶导数在x=0 时的值$f^{(0)}(0)$，$f^{(1)}(0)$，$f^{(2)}(0)$，$f^{(3)}(0)$，…，$f^{(n-1)}(0)$都必须为 1，因为e^x的任何阶次的导数都是它本身 ,但x^n的各阶导数在x=0 时的值$f^{(n)}(0)$=n(n-1)(n-2)(n-3)…2×1× 0^0=n!，因此这些 a_0，a_1，a_2，a_3，a_4，a_5，…，a_{i-1}，a_i等于用 1 除以 n! ,才能让$f^{(0)}(0)$=1 ,$f^{(1)}(0)$=1，$f^{(2)}(0)$=1，$f^{(3)}(0)$=1，…，$f^{(n-1)}(0)$=1，$f^{(n)}(0)$=1，才能满足正确地写出函数 f(x)=e^x的级数表达式的各项的系数a_0，a_1，a_2，a_3，a_4，a_5，…，a_{i-1}，a_i ,即：$a_0=\frac{1}{0!}=1$,$a_1=\frac{1}{1!}$,$a_2=\frac{1}{2!}$，$a_3=\frac{1}{3!}$,$a_4=\frac{1}{4!}$ ，$a_5=\frac{1}{5!}$ ，…，$a_n=\frac{1}{n!}$,… ,对于特定的函数 f(x)=e^x来说 ,这里的方法是通过x的各次幂x^0、x^1、x^2、x^3、 …. 、x匹配乘上x^n的各阶导数函数在自变量x=0 处的值 n!的倒数，n 为函数 y=e^x的各阶次导数函数的阶次数，也是x的 n 次幂。所以对于特定的函数 f(x)=e^x来说，$a_0=\frac{1}{0!}=1$ ，$a_1=\frac{1}{1!}=1$，$a_2=\frac{1}{2!}=\frac{1}{2}$，$a_3=\frac{1}{3!}=\frac{1}{6}$，$a_4=\frac{1}{4!}=\frac{1}{24}$，$a_5=\frac{1}{5!}=\frac{1}{120}$，…，所以可以再次写出函数 f($x$)=$e^x$的级数：$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\ldots+\frac{1}{n!}x^n+\cdots$。

下面令 f(x)=cos(x)，来求 cos(x)的幂级数。函数 f(x)=cos(x)的 0 阶导数是$f^{(0)}(x)$=cos(x)(函数的 0 阶导数是它自己本身)，f(x) =cos(x)的 1 阶导数是 $f^{(1)}(x)=-\sin(x)$，f(x)=cos(x)的 2 阶导数是 $f^{(2)}(x)$= - $\cos(x)$，f(x)=cos(x)的 3 阶导数是 $f^{(3)}(x)=\sin(x)$，f(x) =cos(x)的 4 阶导数是 $f^{(4)}(x)$= $\cos(x)$，...，f(x)=cos(x) 的 n 阶导数是: $f^{(n)}(x)$=... ,如果代入x=0，将得各阶次导数函数在x=0 处的值。因为级数是函数的各阶次导数函数在自变量x=0 处的值除以 n!，并乘上x^n展开得到的。因此将x=0 代入各阶次导数函数的自变量x求解，就很容易得到：$f^{(0)}(0)=\cos(0)$=1，$f^{(1)}(0)=-\sin(0)$=0，$f^{(2)}(0)$= - $\cos(0)$=-1，$f^{(3)}(0)=\sin(0)$=0，$f^{(4)}(0)=\cos(0)=1$，f(5)(0)=$-\sin0=0$，f60=$-\cos0=-1$，f(7)(0)=$\sin(0)$=0，...，按照 1，0，-1，0，1，0，-1，0，...，的形式，以 1,0,-1,0 为循环节无限循环下去。函数 f=cos(x)的各阶次导数函数在它的自变量为x=0 处的函数值可用来构建 cos(x)的幂级数需要的系数，它们是构建 cos(x)的幂级数需要匹配的函数 f(x)=cos(x)的各阶次导数函数在它的自变量为x=0 处的函数值，它们除以 n 的阶乘，就是x的各次幂的系数，n 为函数 f(x)=cos(x)的各阶次

导数函数的阶次数，也是x的 n 次幂。现在可以参照上面构建e^x的幂级数的方式来构建 cos(x)的幂级数了。cos(x)展开的幂级数是：它从$\frac{f^{(0)}(0)}{0\,!}x^0=\frac{\cos(0)}{0\,!}x^0=\frac{1}{0\,!}*1$=1开始，作为第 0 项 ，即常数项。

接着是$\frac{f^{(1)}(0)}{1\,!}x^1=\frac{-\sin(0)}{1\,!}x^1=\frac{0}{1\,!}*x$=0 ，作为第 1 项，结果是 0 ，等于是没有第 1 项，也等于是没有x的 1 次方项。

接着是：$\frac{f^{(2)}(0)}{2\,!}x^2=\frac{-\cos(0)}{2\,!}x^2=\frac{-1}{2!}\times x^2$=-$\frac{1}{2}x^2$ ，作为第 2 项。

接着是：$\frac{f^{(3)}(0)}{3\,!}x^3=\frac{\sin(0)}{3\,!}x^3=\frac{0}{3!}\times x^3$=0，作为第 3 项，结果是 0 ，等于是没有第 3 项，也等于是没有x的 3 次方项。

再接着是：$\frac{f^{(4)}(0)}{4\,!}x^4=\frac{\cos(0)}{4\,!}x^4=\frac{1}{4!}x^4$，作为第 4 项。

...，如此往复做下去，将会发现，对于 f(x)= $\cos(x)$的 n 阶导数$f^{(n)}(x)$，若 n 为非负整数，

从 0 开始，如果 n 是偶数，则$f^{(n)}(0)$的值不是+1 就是-1，按照 1，-1，1，-1，1，-1，1，-1，…，的规律排列，所以对于 cos(x)的幂级数展开式来说其x的偶次方项前面的系数的值的符号是按+，-，+，-，+，-，+，-，…规律排列的，其系数的值$\frac{f^{(n)}(0)}{n!}=\frac{1}{n!}$或$\frac{f^{(n)}(0)}{n!}=-\frac{1}{n!}$，若 n 是奇数，则其系数的值$\frac{f^{(n)}(0)}{n!}=0$，所以对于 cos($x$)的幂级数的展开式来说没x 的奇次方项。

所以函数 f(x)=cos(x)的幂级数是：cos(x)= $\frac{1}{0!}x^0$ -$\frac{1}{2!}x^2$+ $\frac{1}{4\,!}x^4$ - $\frac{1}{6!}x^6$+ $\frac{1}{8!}x^8$- $\frac{1}{10!}x^{10}$ +…= 1 -$\frac{1}{2!}x^2$+ $\frac{1}{4\,!}x^4$ -$\frac{1}{6!}x^6$+$\frac{1}{8!}x^8-\frac{1}{10!}x^{10}$…。

下面令 f(x)=sin(x)，来求 sin(x)的幂级数。函数 f(x)=sin(x)的 0 阶导数是$f^{(0)}(x)$=sin(x) (函数的 0 阶导数是它自己本身)，f(x)的 1 阶导数是$f^{(1)}(x)=\cos(x)$，f(x)的 2 阶导数是 $f^{(2)}(x)= -\sin(x)$，f(x) 的 3 阶导数是 $f^{(3)}(x)$=- $\cos(x)$，f(x) 的 4 阶导数是$f^{(4)}(x)$=sin(x)，…，f(x) 的 n 阶导数是 $f^{(n)}(x)$=…，如果代入x =0，将得到各阶导数函数在x=0 处的值。因为级数是各阶导数函数在自变量x=0 处的值除以 n!并乘上x^n展开得到的。因此x=0 处，代入各阶导数函数求解，就容易得到：$f^{(0)}(0)=\sin(0)$=0，$f^{(1)}(0)=\cos(0)$=1，$f^{(2)}(0)$= - $\sin(0)$ =0 ，$f^{(3)}(0)=-\cos(0)$=- 1 ，$f^{(4)}(0)=\sin(0)=0$，$f^{(5)}(0)=\cos(0)=1$，$f^{(6)}(0)=-\sin(0)$=0，f(7)(0)=cos(0) = -1，…，按照 0，1，0，-1，0，1，0，-1，…，的形式，以 0，1，0，-1 为循环节无限循环下去。这些就是构建 sin(x)的幂级数需要匹配的导数值，现在可以参照上面构建e^x的幂级数的方式来构建 sin(x)的幂级数了，各阶导数函数在x=0 处的值除以 n!就是x的各次幂的系数。函数 f=sin(x)的各阶次导数函数在它的自变量x=0 处的函数值可用来构建 sin(x)的幂级数需要的系数，它们是构建 sin(x)的幂级数需要匹配的函数 f=sin(x)的各阶次导数函数在它的自变量x=0 处的函数值，它们除以 n 的阶乘，就是x的各次幂的系数，n 为函数 f(x)=sin(x)的各阶次导数函数的阶次数，也是x的 n 次幂。现在可以参照上面构建e^x的幂级数的方式来构建 sin(x)的幂级数了。

所以 sin(x)展开的幂级数是：它从$\frac{f^{(0)}(0)}{0\,!}x^0=\frac{\sin(0)}{0\,!}x^0=\frac{0}{0\,!}*1$=0开始，作为第 0 项，即常数

项，由于是 0，等于是没有常数项。

接着是$\frac{f^{(1)}(0)}{1!}x^1=\frac{\cos(0)}{1!}x^1=\frac{1}{0!}\times x= x$ ，作为第 1 项，结果是x。

又接着是：$\frac{f^{(2)}(0)}{2!}x^2=\frac{-\sin(0)}{2!}x^2=\frac{0}{2!}\times x^2=0$ ，作为第 2 项，由于是 0，等于是没有第 2 项。

接着是：$\frac{f^{(3)}(0)}{3!}x^3=\frac{-\cos(0)}{3!}x^3=\frac{-1}{3!}\times x^3= -\frac{1}{3!}x^3$，作为第 3 项。

再接着是：$\frac{f^{(4)}(0)}{4n!}x^4=\frac{\sin(0)}{4!}x^4=0$ 作为第 4 项，由于是 0，等于是没有第 4 项。…，如此往复做下去，将会发现对于 f (x) =sin(x)的 n 阶导数$f^{(n)}(x)$，n 为非负正数，从 0 开始，若 n 是偶数，则$f^{(n)}$(0)的值均为 0，所以对于 cos(x)的幂级数来说没有常数项和 x 的偶次方项。若 n 是奇数，则$f^{(n)}$(0)的值不是+1 就是-1，按照 1，-1，1，-1，1，-1，1，-1，… 的规律排列，其系数的值为$\frac{f^{(n)}(0)}{n!}=\frac{1}{n!}$或$\frac{f^{(n)}(0)}{n!}=-\frac{1}{n!}$，所以对于 sin($x$)的幂级数来说其$x$的奇次方项前面的系数的值符号是按+，-，+，-，+，-，+，-，…规律排列的，所以函数 f(x)=sin(x)的幂级数展开式是 sin(x)=$\frac{1}{1!}x^1-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\ldots= x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\ldots$前面得到

$e^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\ldots+\frac{1}{n!}x^n=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\ldots+\frac{1}{n!}x^n+\ldots$($x$ ∈R)，如果将x换为 ix，得到：$e^{ix}=1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\frac{1}{4!}(ix)^4+\frac{1}{5}(ix)^5+\ldots+\frac{1}{n!}(ix)^n=(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\frac{1}{10!}x^{10}+\ldots)+i(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\ldots)$ (x ∈R)，因为 cos(x)=$1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\frac{1}{10!}x^{10}+\ldots$，而 sin($x$)=$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\ldots$，所以$e^{ix}$=cos($x$)+i sin($x$) ($x$ ∈R)。这是另一个欧拉公式。

上式中如果令x=π，将得到：

$e^{i\pi}$=cos(π)+isin(π)=-1+0=-1 ,所以 $e^{i\pi}$+1=0。它也叫欧拉公式 ,把数学中最重要的一些东西 0，1，e，i，π全放到一个公式里去了，它是欧拉公式e^{ix}= cos(x)+isin(x) (x ∈R)的特例。当 Z∈C,那么 e^{iZ}= cos(Z)+isin(Z)(Z∈C)。

下面我来对上面的内容做一个总结: 黎曼猜想：所有非平凡零点的实部都是 1/2。

首先，黎曼 ζ 函数是ζ(s)是一个复变量函数，定义为 s=σ+ti(σ ∈R,t∈R)，当 Re(s)>1 时，ζ

函数可以由级数$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$来表示，不过这个级数在 Re(s)≤ 1 时发散，所以需要通过解析延拓来扩展到整个复平面，除了在 s=1 处有一个简单极点。解析延拓的 ζ 函数在复平面上除了 s=1 外都是解析的。

黎曼 ζ 函数的零点就是那些使得$\zeta(s)=0$ 的s的值。黎曼 ζ 函数的零点分为两种:平凡零点和非平凡零点。平凡零点是负偶数，比如 s=-2,-4,-6, ... ,等等，这些被证明是黎曼 ζ 函数的零点。而非平凡零点则位于所谓的“临界带”内，也就是实部在 0 到 1 之间的区域，也就是是 0<Re(s)<1。黎曼猜想说所有这些非平凡零点的实部都是 1/2,也就是说它们都位于临界线 Re(s)=1/2 上。

黎曼在 1859 年的论文中提出了这个猜想，但没有给出证明，只是通过计算和观察支持了这一猜想，后来很多数学家通过数值计算验证了数十亿个非平凡零点都位于临界线上，但当然，这还不能算是证明，因为数值计算只能覆盖有限的情况。

我们应该理解为什么这个猜想如此重要，因为黎曼猜想与数论中的素数分布有密切关系。例如，素数定理告诉我们小于x的素数个数大约是$x/\ln x$,而这个定理的证明就用到了黎曼ζ函数在 Re(s)=1 附近没有零点的性质当黎曼猜想成立，那么我们可以得到素数分布的更精确估计，比如误差项会更小。此外，黎曼猜想在密码学和其它数学领域也有应用，所以它的解决可能会带来很多突破。

为什么非平凡零点的位置会影响素数的分布？这需要更深入的学习，比如了解 ζ 函数与素数之间的联系，比如欧拉乘积公式，或者素数的显式公式，比如黎曼提出的那个涉及ζ 函数零点的公式。

非平凡零点的实部都是 1/2,那它们的位置在复平面上排成一条直线，中间在 1/2 的位置，这样的话，ζ函数的对称性也与此有关，因为ζ函数满足函数方程

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$且$s\neq 1$),这个方程可能使得零点关于点(1/2,0i)对称,

或者说，如果 s 是一个零点，那么 1-s 也是一个零点。所以如果某个零点不在临界线上，那么它关于点(1/2,0i)对称的点也不在临界线上。可能这样会出现对称的零点对，但根据黎曼猜想，这样的零点是不存在的，所有的非平凡零点都必须位于 Re(s)=1/2 这条直线上。不过之前已经证明的是有无限多个非平凡零点位于 Re(s)=1/2 的临界线上，但当然无限多个并非意味着全部都是。可以有某些方法能够排除掉有存在于临界线外的零点的可能性。另外，有些结果是在某些比例上零点满足条件，比如，有百分之多少的零点在临界线上，但100 %的结果还没有出来，而我在我的这篇论文中将要证明的是 100 %的零点位于Re(s)=1/2 的临界线上。

历史上有很多尝试，比如哈代证明了有无限多个零点位于临界线上，后来塞尔伯格改进了这个结果。证明了有正比例的零点在临界线上。但这些都是部分结果。最近的进展比如德利涅证明的韦伊猜想相关的结果，可能间接有关连，但具体有什么联系，需要进一步研究，不过我在我的这篇论文中，对黎曼猜想的证明，可以不必考虑这些。

另外，黎曼猜想有很多等价的描述，比如涉及数论函数的误差项，或者关于其它数学结构的性质，比如随机矩阵中的某些性质，可能与ζ函数的零点分布有关联，不过我的这个证明中不必从这个视角来解决黎曼猜想，当然也可以尝试从多个视角来攻击这个问题。

我们需要深入研究这些内容，并理解黎曼猜想的基本陈述和重要性。我在这里回顾一下黎曼ζ的基本性质，比如它的欧拉乘积公式，这显示了ζ函数与素数的直接联系。因为欧拉乘积是对于每个素数 p 的乘积形式，即 $\zeta(s)=\prod(\frac{1}{1-p^{-s}})$，这在 Re(s)>1 时成立。这个乘积形式意味着黎曼ζ函数的零点可能与素数的某种分布规律有关联，特别是当s的实部小于等于 1 时，乘积不再收敛，但解析延拓后的ζ函数零点可能携带了素数分布的信息。

再想想，当黎曼猜想成立，那么我们可以利用零点都在临界线上这一点，来更精确地估计素

数计数函数$\pi(x)$和 $\mathrm{Li}(x)$之间的误差，也就是 $\pi(x) - \mathrm{Li}(x) < C(\sqrt{x}\ln x)$，这对于密码学中的素数应用可能有实际意义。比如 RSA 加密算法依赖大素数的生成。

为什么零点在临界线上就能改进误差估计？这是因为在显式公式中，零点的位置会影响余项的大小，如果所有零点的实部都不超过 1/2,那么每个非平凡零点对误差项的贡献就会被控制在某个范围之内，进而整体误差项就可以被精确地估计。

黎曼猜想是希尔伯特 23 个问题中的第 8 个问题的一部分， 解决这个问题需要新的数学方法，对黎曼ζ 函数的零点的对称和共轭性质有更深入的了解。在我的这篇论文中，将会展示这我的这种方法和理解。

另外有人尝试过物理中的量子力学或者统计力学方法来研究黎曼ζ 函数的零点，因为它们似乎与某些量子系统的能级分布有相似之处，比如随机矩阵理论预测的能级间距，与黎曼ζ 函数的非平凡零点的间距相似，这可能暗示某种深层的数学结构。不过，这似乎更像一种类比，而不是直接的数学证明路径。

回到问题本身，黎曼猜想问的是黎曼ζ 函数的非平凡零点的实部是否都是 1/2。目前经过大量的数值计算验证了这个猜想对于非常大的范围内的零点都是成立的，比如 ZetaGrid 项目验证了超过十亿个零点都在临界线上。但数学证明显然不能依赖数值计算。

另外，需要排除黎曼ζ 函数在临界带内的Re(s)=1/2 的临界线上之外不存在零点，还需要排除黎曼ζ 函数在复平面临界带之外的其它地方没有零点。比如在 Re(s)>1 的区域，由于欧拉乘积的存在，$\zeta(s) \neq 0$，每个因子都是 $1/(1-1/p^s)$,而每个这样的因子都不为零。所以黎曼 ζ(s)在 Re(s)>1 时没有零点。在 Re(s)=1 这条直线上，黎曼 ζ(s)有没有零点呢？根据素数定理的证明，我们知道在 Re(s)=1 这条直线上，黎曼 ζ(s)没有零点，这帮助证明了素数定理。而通过函数方程$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$ ($s\in C$且$s\neq 1$),因为如果ζ(s)=0，那么$\zeta(s)=\zeta(1-s)=0$($s\in C$且$s\neq 1$,且$s \neq -2n$ 且 $s \neq 2n, n \in Z^+$),很容易知道 Re(s)=0 的地方也没有

非平凡零点。所有综合起来，黎曼ζ 函数的非平凡零点都在 $0<$Re(s)<1 的临界带内部，而黎曼猜想断言它们都在中间的临界线 Re(s)=1/2 上。需要深入了解黎曼 ζ 函数的函数方程，即 $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C且s\neq 1)$,这个方程表明，如果 s 是零点，那么 1-s 也是零点，同时因为 Γ 函数的性质，还涉及到对称性。例如，如果 s=1/2+it 是一个零点，那么 1-s=1/2-it 也是一个零点。这样，零点关于实数轴和点(1/2,oi)对称，并且在临界线上成对出现。不过对于不在临界线上的零点，比如 s=σ+ti(σ∈R,t∈R),其中σ≠1/2,那么 1-s=1-σ-ti,如果σ 在 0 到 1 之间，那么1-σ 也在 0 到 1 之间，所有这样一对零点也是存在的，但黎曼认为这样的零点不存在。黎曼的这个观点是正确的。在我的论文中，我证明了在 $0<$ Re(s)<1 的临界带内部和复平面的其它区域，黎曼ζ 函数的零点必定是共轭对称的。在 $0<$ Re(s)<1 的临界带内部，根据 $\zeta(s)=\zeta(1-s)=0(s\in C, 0<\mathrm{Re}(s)<1$ 且$s\neq 1$,且$s\neq -2n$ 且 $s\neq 2n, n\in Z^+)$,s 和 1-s 也必定是共轭的，而当 Re(s)$\neq 1/2$，s 和 1-s 则是不共轭的，这就相互矛盾，所以在$0<$Re(s)<1 的临界带内部，不存在实部不等于$\frac{1}{2}$的零点，自然而然地，也就不存在实部不等于$\frac{1}{2}$的非平凡零点。

另外，之前我还听说过一些接近的结果，比如所谓的弱黎曼猜想，即存在一个常数 c<1,使得所有的非平凡零点的实部都小于 c,但这没有被证明。实际上之前已经知道非平凡零点的实部趋向于 1/2,在某种平均意义上，具体的每个非平凡零点都满足实部都为 1/2，在后面我的这篇论文中给出了严格证明。

黎曼猜想：黎曼 ζ 函数的所有非平凡零点的实部均为 1/2，即它们位于复平面上的临界线 Re(s)=$\frac{1}{2}$上。黎曼猜想涉及黎曼 ζ 函数的非平凡零点的分布。具体陈述如下：

关键点解析：

1. 黎曼 ζ 函数：

-定义：对于复变量 s=σ+ti(σ ∈R,t∈R),当 Re(s)>1 时，ζ 函数由级数 $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$定义。

- 解析延拓：ζ 函数可解析延拓至整个复平面（除 s=1 处的简单极点外）。

2. 零点分类：

-平凡零点：位于负偶数点(s=-2,-4,-6, … ,)，由 ζ 函数的函数方程 $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$(s∈C且s≠ 1)中的 $\sin(\frac{\pi s}{2})$直接得出。

- 非平凡零点：位于临界带(0<Re(s)<1)内，其存在性与素数分布密切相关。

3. 重要性：

-素数定理的强化：当黎曼猜想成立，素数计数函数$\pi(x)\sim \mathrm{Li}(x)$的误差项可被大幅优化，如$\pi(x)-\mathrm{Li}(x)<C(\sqrt{x}\ln x)$。

-数学与物理的交叉：ζ 函数零点分布与量子混沌、随机矩阵理论等领域的现象存在深刻联系。

4. 研究进展：

-数值验证：已计算超过(10^{13})个非平凡零点，均位于临界线上。

-部分结果：哈代（1914）证明无限多零点在临界线上；塞尔伯格(1942)证明正比例零点位于临界线上。

-等价命题：存在多种与数论、代数几何等相关的等价表述，如涉及 Mertens 函数、Dirichlet L 函数的推广形式等。

5.挑战与现状：

- 尽管有大量证据支持，但更需要严格的数学证明。

- 解决猜想可能需要发展新的数学方法或揭示 ζ 函数更深层的对称性。

结论：

黎曼猜想的核心断言是 ζ 函数所有非平凡零点的实部均为 1/2。其证明将对数论、密码学及

物理学产生深远影响。

答案：黎曼猜想断言所有非平凡零点的实部都是 1/2，下面我将会给出严格的数学证明。数学界普遍认为其正确，并基于此发展了许多理论。

黎曼猜想的核心是黎曼ζ 函数的非平凡零点的实部都是 1/2。这个猜想在后面我的这篇论文中被我证明成立。将极大地推动数论和其它数学领域的发展，阅读我的这篇论文需要一些复分析和数论方面的知识。才能更深入地理解这个问题背后的数学结构和证明方法。

II .推理

下面开始我的对黎曼猜想的证明:

公式 1:

因为 $e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\sum_{n=0}^{\infty}\frac{1}{n!}\approx 2.7182818284...$,e 是自然常数, 我使用 "×" 来表示相乘，那么根据欧拉公式 $e^{ix}=\cos(x)+i\sin(x)(x\in R)$，可以得到 $(e^{3i})^{2}=(\cos(3)+i\sin(3))2=\cos(2\times3)+i\sin(2\times3)=\cos(6)+i\sin(6)$。一般地, 如果将复数三角函数的乘方扩展到指数为实数的情形，那么 $(e^{bi})^{c}=e^{b\times ci}(b\in R, c\in R)$ 成立。当 $x>0(x\in R)$,假设 $e^{y}=x$ ($e=2.718281828459045...$，e 是自然常数, $x\in R$ 并且 $x>0$, $y\in R$),那么 $y=\ln(x)(x>0)$, 根据欧拉公式 $e^{ix}=\cos(x)+i\sin(x)$ $(x\in R)$, 可以得到 $e^{yi}=e^{\ln(x)i}=\cos(\ln x)+i\sin(\ln x)(x\in R$ 并且 $x>0)$。

假设 $t\in R$ 并且 $t\neq 0$,我们可以求出 $x^{ti}(x\in R$ and $x>0$, $t\in R$ and $t\neq 0)$ 的表达式是 $x^{ti}=(e^{y})^{ti}=(e^{yi})^{t}=(\cos(\ln x)+i\sin(\ln x))^{t}(x>0)$。

假设 s 是任意一个复数，并且假设 $s=\sigma+ti(\sigma\in R,t\in R)$,那么让我们来找出 $x^{s}(x\in R$ and $x>0$, $s\in C)$的表达式 ,你可以将 $s=\sigma+ti(\sigma\in R,t\in R)$ 和

$x^{ti}=(e^{y})^{ti}=(e^{yi})^{t}=(\cos(\ln x)+i\sin(\ln x))^{t}(x>0)$ 代入 $x^{s}(x>0)$ 中，你将得到：$x^{s}=x^{(\sigma+ti)}=x^{\sigma}x^{ti}=x^{\sigma}(\cos(\ln x)+i\sin(\ln x))^{t}=x^{\sigma}(\cos(t\ln x)+i\sin(t\ln x))(x>0)$,

如果你将 s=σ-ti(σ ∈R, t∈R）和 $x^{ti}=(e^{y})^{ti}=(e^{yi})^{t}=(\cos(\ln x)+i\sin(\ln x))^{t}(x>0)$ 代入x^{s},

你将得到：

$x^{\bar{s}}=x^{(\sigma-ti)}=x^{\sigma}(x^{ti})^{-1}=x^{\sigma}(\cos(\ln x)+i\sin(\ln x))^{-t}=x^{\sigma}(\cos(t\ln x)-i\sin(t\ln x))(x>0)$.

那么

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+ti}}=\sum_{n=1}^{\infty}(\frac{1}{n^{\sigma}}\times\frac{1}{n^{ti}})=\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^{t}}$$

$$=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})$$

(s ∈ C , n∈ Z⁺ ，并且 n 便取所有正整数), 或者

$\zeta(s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\prod_{p=1}^{\infty}(1-p^{-s})^{-1}=\prod_{p=1}^{\infty}(1-p^{-\rho-ti})^{-1}=\prod_{p=1}^{\infty}(1-\frac{1}{p^{\sigma+ti}})^{-1}=\prod_{p=1}^{\infty}[1-$

p−σ1(coslnp+isin(lnp))t]−1=p=1∞[1−p−σ(costlnp−isin(tlnp))]−1

(s ∈ C 并且 s≠ 1, p ∈ Z⁺，并且 p 便取所有质数)，并且

$\zeta(\bar{s})=\sum_{n=1}^{\infty}\frac{1}{n^{\bar{s}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\bar{s}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma-ti}}=\sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\times\frac{1}{n^{-ti}}\right)=$

$\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{\left(\cos(\ln(n))+i\sin(\ln(n))\right)^{-t}}=\sum_{n=1}^{\infty}(n^{-\sigma}(\cos\left(\ln(n)\right)+\ i\sin(\ln(n)))^{t})\ =$

$\sum_{n=1}^{\infty}(n^{-\sigma}(\cos\left(t\ln(n)\right)+i\sin(t\ln(n)))$ (s ∈ C，并且 s≠ 1,n∈ Z⁺ 且 n 便取所有正整数),或

者

$\zeta(\bar{s})=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-\bar{s}}})=\prod_{p=1}^{\infty}(1-p^{-\bar{s}})^{-1}=\prod_{p=1}^{\infty}(1-p^{-\sigma+ti})^{-1}=\prod_{p=1}^{\infty}(1-\frac{1}{p^{\sigma-ti}})^{-1}=$

$$\prod_{p=1}^{\infty}[1-(p^{-\sigma})\frac{1}{(\cos(\ln p)-i\sin(\ln p))^{t}}]^{-1}=\prod_{p=1}^{\infty}[1-(p^{-\sigma})(\cos(t\ln p)+i\sin(t\ln p))\ ]^{-1}$$

(s ∈ C, p ∈ Z⁺ ，并且p便取所有质数). 并且

$\zeta(1-s)=\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}=\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma-ti}}=\sum_{n=1}^{\infty}(n^{\sigma-1})\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^{-t}}=$

$\sum_{n=1}^{\infty}(n^{\sigma-1})(\cos(\ln(n))+i\sin(\ln(n)))^{t})=\sum_{n=1}^{\infty}(n^{\sigma-1})(\cos(t\ln(n))+i\sin(t\ln(n)))$

(s ∈ C, n∈ Z⁺ ，并且 n 便取所有正整数),

或者如果有任意实数 k(k∈ R), 那么

$\zeta(k-s)=\sum_{n=1}^{\infty}\frac{1}{n^{k-s}}=\sum_{n=1}^{\infty}\frac{1}{n^{k-\sigma-ti}}=\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)\frac{1}{(\cos(\ln(n))+i\sin(\ln(n)))^{-t}}=$

$\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)(\cos(\ln(n))+i\sin(\ln(n)))^{t})=\sum_{n=1}^{\infty}\left(n^{\sigma-k}\right)(\cos(t\ln(n))+i\sin(t\ln(n)))$

($s\in C$, k并且 $\in R$, n$\in Z^{+}$ $n\in Z^{+}$ ，并且 n 便取所有正整数),并且

$\zeta(k-s)=\prod_{p=1}^{\infty}(\frac{1}{1-p^{(-k+s)}}$)= $\prod_{p=1}^{\infty}(1-p^{s-k})^{-1}$ = $\prod_{p=1}^{\infty}(1-p^{\sigma-k+ti})^{-1}$ = $\prod_{p=1}^{\infty}[1-$

pσ−k(costlnp+isin(tlnp))]−1(s∈ C,k∈R,p∈Z+ ，并且 p 便取所有质数).

因此

X=$n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n)))$,

Y=$n^{-\sigma}(\cos(t\ln(n))+i\sin(t\ln(n)))$,

G=$[1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))\,]^{-1}$,

H=$[1-(p^{-\sigma})(\cos(t\ln p)+i\sin(t\ln p))\,]^{-1}$,

也就是说，X 和 Y 是彼此的复共轭，即 X=$\overline{Y}$, 并且

G 和 H 是彼此的复共轭，也就是

G=$\overline{H}$,

因此 $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\sum_{n=1}^{\infty}X=\prod_{p=1}^{\infty}G$ ($s\in C$ 并且 $s\neq 1$), 并且 $\zeta(\overline{s})=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\sum_{n=1}^{\infty}Y=$

p=1∞H(s∈C 并且 s≠1),

因此

ζ(s)=$\overline{\zeta(\overline{s})}$($s\in C$),并且仅当 $\sigma=\frac{1}{2}$ ，$\zeta(1-s)$=$\zeta(\overline{s})$ $\left(s\in C \text{ 并且 } s\neq 1\right)$,

仅当 $\sigma=\frac{k}{2}(k\in R)$, $\zeta(k-s)=\zeta(\overline{s})\left(s\in C \text{ 并且 } s\neq 1, k\in R\right)$, 那么， $\zeta(1-s)=\zeta(\overline{s})=$

$\zeta(k-s)\left(s\in C \text{ 并且 } s\neq 1, k\in R\right)$, 因此仅 k=1($k\in R$)成立，并且当 ζ(s)=0, 那么

$\zeta(1-s)$=$\zeta(k-s)=\zeta(\overline{s})$=$\zeta(s)$=0$\left(s\in C \text{ 并且 } s\neq 1, k\in R\right)$。

因为

$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+ti}}=\sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\times\frac{1}{n^{ti}}\right)=\sum_{n=1}^{\infty}(n^{-\sigma})\frac{1}{\left(\cos(\ln(n))+i\sin(\ln(n))\right)^t}=$

$\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})\ =$

$\sum_{n=1}^{\infty}(n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n)))\ =\ \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}\ )=\ \prod_{p=1}^{\infty}(1-p^{-s})^{-1}\ =\ \prod_{p=1}^{\infty}(1-$

$p^{-\sigma-ti})^{-1}\ =\ \prod_{p=1}^{\infty}\left(1-\frac{1}{p^{\sigma+ti}}\right)^{-1}\ =\ \prod_{p=1}^{\infty}[1-(p^{-\sigma})\frac{1}{(\cos(\ln p)+i\sin(\ln p))^t}]^{-1}=$

$\prod_{p=1}^{\infty}[1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))\ ]^{-1}$ ($s\in C$ 并且 $s\neq 1, t\in C$ 并且 $t\neq$

0 p 为质数，并且 $p\neq 1$)。

所以当σ=1,如果 $1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\ \neq 0$，那么

$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})\neq 0$. 如果 $1-\frac{1}{p}\cos(t\ln p)\neq 0$ 并且 $\frac{1}{p}\sin(t\ln p)\neq$

0 ,那么 $\sin(t\ln p)\neq 0$ 并且 $\frac{1}{p}\cos(t\ln p)\neq 1$, 那么 $t\neq\frac{k\pi}{\ln p}$ (k∈ Z，p 为质数，并且 $p\neq 1$)，

并且 cos(tlnp) $\neq p$ ($t\in R$ 并且 $t\neq 1$, p 为质数，并且 $p\neq 1$)。因此如果 $p>$

1 (p 为质数)，那么 t≠ $\frac{k\pi}{\ln p}$(k∈ Z , p 为质数) 并且cos(tlnp) ≠ p(p 为质数,并且p>1) 。或者

$p=1$, 那么|t| ≠ | $\frac{k\pi}{\ln 1}$ | ≠ +∞(k∈ Z , p 为质数) ,并且cos(tln1)= 1 ($t\in R$ 并且 $t\neq 1$). 因此，

并且cos(tln1)= 1 ($t\in R$ 并且 $t\neq 1$). 因此，如果 σ =Re(s)=1并且t≠ $\frac{k\pi}{\ln p}$(k∈ Z , p 为质数,

且 $p\neq 1, t\in R$ 且 $t\neq 0$), 那么 $\zeta(1+ti)=\prod_{p=1}^{\infty}[1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\ ]^{-1}\neq$

0 ($s\in C$ 并且 $s\neq 1$)。

所以当 Re(s)=1 并且 p ≠ 1 那么 $\zeta(1+ti)=\prod_{p=1}^{\infty}[1-\frac{1}{p}\cos(t\ln p)+i\frac{1}{p}\sin(t\ln p)\ ]^{-1}\neq$

0 ($t\in C$ 并且 $t\neq 0$). 并且当 Re(s)=1 并且 p=1(p 为质数), 那么 $\zeta(1+ti)=$

$=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})=\prod_{p=1}^{\infty}[1-\cos(t\ln p)+i\sin(t\ln p)\ ]^{-1}=\prod_{p=1}^{\infty}\frac{1}{1-(p^{-1})(\cos(t\ln p)-i\sin(t\ln p))}$

$=\ \prod_{p=1}^{\infty}\frac{1}{1-(1^{-1})(\cos(t\ln 1)-i\sin(t\ln 1))}\ =\ \frac{1}{0}\rightarrow+\infty$ ($t\in C$ 并且 $t\neq 0$) ,then $\zeta(1+ti)\neq 0\rightarrow$

$+\infty$ ($t\in C$ 并且 $t\neq 0$), 发散，没有零点。

当 σ =0, 那么如果 $1-\cos(t\ln p)+i\sin(t\ln p)\ \neq 0$ 那么 $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}})\neq$

0， $1-\cos(t\ln p)\neq 0$ 并且 $\sin(t\ln p)\neq 0$, p 为质数，那么 $t\ln p\neq k\pi$ ($k\in Z$，p 为质数)

并且 $p \neq 1$ (p 为质数),那么 $t \neq \frac{k\pi}{\ln p}$(k∈ Z 并且 p 为质数， $p \neq 1$) and $\cos(t\ln p) \neq 1$($t \in C$ 并且 $t \neq 0$，p 为质数，并且 $p \neq 1$)， 所 以 如 果 $p > 1$ (p为质数),那么 $t \neq \frac{k\pi}{\ln p}$ (k ∈ Z，p 为质数，并且 $p \neq 1$) ，并且 cos(tlnp) $\neq 1$(p 为质数,并且 $p \neq 1$) 。 或 者 如 果 $p = 1$, 那么$|t| \neq \left|\frac{k\pi}{\ln 1}\right| \neq +\infty$(k∈ Z，p 为质数，并且 $p = 1$) 并且 $|t| \neq +\infty$ ，$t \in R$ 并且 $t \neq 0$,那么 $\zeta(0 + ti) = \prod_{p=1}^{\infty}[1 - \cos(t\ln p) + i\sin(t\ln p)]^{-1} \neq 0$ ($t \in C$ 并且 $t \neq 0$).如果

Re(s)=0，并且 p≠1(p 为质数), 那么 $\zeta(0 + ti) = \prod_{p=1}^{\infty}[1 - \cos(t\ln p) + i\sin(t\ln p)]^{-1} \neq 0$. 另外当 Re(s)=0 并且 p=1 (p 为质数)，那么$\prod_{p=1}^{\infty}[1 - (p^{-\sigma})(\cos(t\ln p) - i\sin(t\ln p))]^{-1} = \prod_{p=1}^{\infty}\frac{1}{1-(p^{-\sigma})(\cos(t\ln p)-i\sin(t\ln p))} = \prod_{p=1}^{\infty}\frac{1}{1-(1^{-1})(\cos(t\ln 1)-i\sin(t\ln 1))} = \frac{1}{0} \to +\infty$,那么 $\zeta(0 + ti) \neq 0 \to +\infty$($t \in R$ 并且 $t \neq 0$), 发散，没有零点， 所以 $\zeta(0 + ti) \neq 0$($t \in R$ 并且 $t \neq 0$)。 事实上黎曼ζ(s)函数的非平凡零（也就是除了负偶数以外的零）是存在的，黎曼证明了黎曼ζ(s)(s∈C, s≠1)函数的非平凡零点 s 的实部 Re(s)(s∈C,s≠1)必须满足 Re(s) ∈[0,1]。手工计算ζ(s)(s∈C, s≠1)函数的非平凡零点并不容易，黎曼计算了最初的十几个，它们的实部 Re(s) 都等于$\frac{1}{2}$。所以黎曼ζ(s)(s∈C, s≠1)函数的非平凡零点（即 s 不是负偶数）是存在的,并且黎曼ζ(s)(s∈C, s≠1)函数的非平凡零点 s 的实部 Re(s)(s∈C, s≠1)必须满足 Re(s)∈(0,1)。当 s=1+ti($t \in R$ 且 $t \neq 0$), Rs(s)= σ =1, 那么 $\zeta(s) = \zeta(1 + ti) = \prod_{p=1}^{\infty}(\frac{1}{1-p^{-s}}) = \prod_{p=1}^{\infty}(1 -$ p−s)−1

$= \prod_{p=1}^{\infty}(1 - p^{-1-ti})^{-1} = \prod_{p=1}^{\infty}[1 - (p^{-1})\frac{1}{(\cos(\ln p)+i\sin(\ln p))^{t}}]^{-1} = \prod_{p=1}^{\infty}[1 - (p^{-1})(\cos(t\ln p) - i\sin(t\ln p))]^{-1} = \prod_{p=1}^{\infty}\frac{1}{[1-\frac{1}{p}\cos(t\ln p)]+i\frac{1}{p}\sin(t\ln p)]} \neq 0$ ($s \in C$ 且 $s \neq 1$，$t \in C$ 且 $t \neq 0$，$p \in Z^{+}$ 且 p 取所有质数)。当自变量 s 由正整数推广为一般复数时，在欧拉积公式中，每个乘积因子的分子为 1，每个乘积因子的分母为与自然对数 函 数 有 关 的 多 项 式 。 当 $p \in Z^{+}$且 p 遍历所有质数时，$\zeta(1 + ti) \neq 0$($t \in R$ 且 $t \neq$ 于x的质数是有限的。由解析推广到一般复数后的欧拉积公式可知，对于不大于x的正整数，0)，说明不大

每增加一个质数 p，欧拉乘积公式中与 ln(p)相关的分数因子就会增加一个，表明在x附近(即

x=p）存在质数 p 的概率约为$\frac{1}{\ln(p)}$,就是$\frac{1}{\ln(x)}$（即x=p）。如果我们用$\pi(x)$表示不大于x的质数的个数，那么对于一个不大于x的正整数 p（p 为质数）,那么它是质数的概率近似为$\frac{\pi(x)}{x}$，那么$\frac{\pi(x)}{x}\approx\frac{1}{\ln(x)}$。$\frac{\pi(x)}{x}\approx\frac{1}{\ln(x)}$是质数定理的表达式。正如 Riemann 在他的论文中所说，n 取所有的正整数，所以 n=1,2,3...,代入$\sum\frac{1}{n^s}$，显然：

$\zeta(s)=\zeta(\sigma+ti)=\sum\frac{1}{n^s}=\sum X=[1^{-\sigma}\cos(t\ln 1)+2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)+4^{-\sigma}\cos(t\ln 4)+...]-i[1^{-\sigma}\sin(t\ln 1)+2^{-\sigma}\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)+4^{-\sigma}\sin(t\ln 4)+...]=U-Vi(s=\sigma+ti,\sigma\in R,t\in R,$且$s\neq 1)$,

$U=[1^{-\sigma}\cos(t\ln 1)+2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)+4^{-\sigma}\cos(t\ln 4)+...]$,

$V=[1^{-\sigma}\sin(t\ln 1)+2^{-\sigma}\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)+4^{-\sigma}\sin(t\ln 4)+...]$,那么

$\zeta(\bar{s})=\zeta(\sigma-ti)=\sum\frac{1}{n^{\bar{s}}}=\sum Y=[1^{-\sigma}\cos(t\ln 1)+2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)+4^{-\sigma}\cos(t\ln 4)+...]+i[1^{-\sigma}\sin()t\ln 1)+2^{-\sigma}\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)+4^{-\sigma}\sin(t\ln 4)+...]=U+Vi(s\in C$ and $s\neq 1,t\in C$ 且 $t\neq 0)$

$U=[1^{-\sigma}\cos(t\ln 1)+2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)+4^{-\sigma}\cos(t\ln 4)+...]$,

$V=[1^{-\sigma}\sin(t\ln 1)+2^{-\sigma}\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)+4^{-\sigma}\sin(t\ln 4)+...]$,

$\zeta(1-s)=\sum(x^{\sigma-1})(\cos(t\ln x)+i\sin(t\ln x))=[1^{\sigma-1}\cos(t\ln 1)+2^{\sigma-1}\cos(t\ln 2)+3^{\sigma-1}\cos(t\ln 3)+4^{\sigma-1}\cos(t\ln 4)+...]+i[1^{\sigma-1}\sin(t\ln 1)+2^{\sigma-1}\sin(t\ln 2)+3^{\sigma-1}\sin(t\ln 3)+4^{\sigma-1}\sin(t\ln 4)+...](s=\sigma+ti,\sigma\in R,t\in R,$且$s\neq 1)$,

根据$\sum_n n^{-s}=\prod_p(1-p^{-s})^{-1}$，公式的左边是与所有自然数有关的和，公式的右边是与所有素数的乘积。这个公式适用于所有满足 Re(s)>1 复数 s。这个公式的左边是 Riemann ζ函数对于 Re(s)>1 的级数表达式，我们在上面已经描述过，右边是一个纯粹关于质数的表达式(并且包含所有质数)，这是黎曼 zeta 函数与质数分布之间关系的标志,所以我先假设 Re(s)>1。因为当 Re(s)>1 欧拉ζ函数等价黎曼ζ函数时， 所以$\zeta(s)=\overline{(s)}(s=\sigma+ti,Re(s)>1,\sigma\in R,t\in R$ 成立。根据欧拉乘积公式 p(1−p−s)−1,当 Re(s)>1,因为每个欧拉乘积因子(1−p−s)−1 都不等于零,所以当 Re(s)>1，ζ(s)不存在零点。根据 ζ(s)=2sπs−1Sin(πs2)Γ(1-s)ζ(1-s)(s∈C 且 s≠ 1) (公式 7),所以尽管正偶数 2n ($n\in Z^+$)可以使得$\sin(\frac{\pi s}{2})=0$,但它不是黎曼ζ(s)函数的零点。

推论 1:

对于任意复数 s, 当 $Rs(s)>0$ 且 $s\neq 1$,如果 $s=\sigma+ti(\sigma\in R,t\in R)$，那么

那么黎曼 $\zeta(s)\left(s \in C 且 Rs(s) > 0 \text{ and } s \neq 1\right)$函数和狄利克雷 $\eta(s)\left(s \in C 且 Rs(s) >\right.$ 0 且 $s\neq1$ 函数的关系就是：

$\eta(s)=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\ldots\left(s \in C 且 Rs(s) > 0 且 s \neq 1\right)$，

$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots\left(s \in C 且 Rs(s) > 0 且 s \neq 1\right)$，

$\eta(s)-\zeta(s)=-(\frac{2}{2^s}+\frac{2}{4^s}+\frac{2}{6^s}+\cdots)=-\frac{2}{2^s}(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots)=-\frac{2}{2^s}\zeta(s)\left(s \in\right.$ C 且 $Rss>0$ 且 $s\neq1$，

$\eta(s)=1-\frac{2}{2^s}\zeta(s)=(1-2^{1-s})\zeta(s)\left(s \in C 且 Rs(s) > 0 且 s \neq 1\right)$,那么

$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\left(s \in C 且 Rs(s) > 0 且 s \neq 1\right)$且 $\eta(s)=(1-2^{1-s})\zeta(s)\left(s \in C 且 Rs(s) > 0 且 s \neq 1\right)$，$\zeta(s)$是黎曼 Zeta 函数,$\eta(s)$是狄利克雷$\eta(s)$函数,

因此黎曼 $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$ ($s \in C$ 且 $Rs(s) > 0$ 且 $s \neq 1$, $n\in Z^+, p \in Z^+, s \in C$，n 遍取所有正整数, p 遍取所有质数)。现在让我们来证明 $\zeta(s)$和$\zeta((\bar{s})$彼此复共轭。

因为：

$\frac{(-1)^{n-1}}{(1-2^{1-s})}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}}$，

$\prod_p(1-p^{-s})^{-1}=\overline{\prod_p(1-p^{-\bar{s}})^{-1}}$，

($s \in C$，$Rs(s) > 0$ 且 $s \neq 1, p \in Z^+$ 且 p 遍取所有的质数),

因此

$\frac{(-1)^{n-1}}{(1-2^{1-s})}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}}$，

因此

$\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\overline{\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}}$，

$\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}}$， ζ(s)=$\frac{1}{(1-2^{1-s})}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{n^{s}}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_{p}(1-p^{-s})^{-1}$,

ζ($\overline{s}$)=$\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}=\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_{p}\left(1-p^{-\overline{s}}\right)^{-1}$ ($s \in C$ ， $Rs(s) > 0$ 且 $s \neq 1, n \in Z^{+}$ 且 n 遍取所有的正整数, $p \in Z^{+}$ p 遍取所有的质数),

因此

仅 ζ(s)=$\overline{\zeta(\overline{s})}$ ($s \in C$，$Rs(s) > 0$ 且 $s \neq 1$), [2]

因此

p^{1-s}=$p^{(1-\sigma-ti)}$=$p^{1-\sigma}p^{-ti}$=$p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}$=$p^{1-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$,
$p^{1-\overline{s}}=p^{(1-\sigma+ti)}=p^{1-\sigma}p^{ti}=p^{1-\sigma}(p^{ti})\ =p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t}$ = $(p^{1-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ (s∈C，Rss>0 且 s≠1, t∈C 且 s≠0,p∈Z+),

因此

$p^{-(1-s)}$=$p^{(-1+\sigma+ti)}$=$p^{\sigma-1}p^{ti}=p^{\sigma-1}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$=$(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$,

$p^{-(\overline{s})}$=$p^{-(\sigma-ti)}$=$p^{-\sigma}p^{ti}=(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$

($s \in C$ ，$Rs(s) > 0$ 且 $s \neq 1, t \in C$ 且 $s \neq 0, p \in Z^{+}$),

因此

$(1-p^{-(1-s)})$=1-$(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$ =$1-p^{\sigma-1}\ \cos(t\ln p)-ip^{\sigma-1}\sin(t\ln p)$,
$(1-p^{-(\overline{s})})$=1-$(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ =$1-p^{-\sigma}\ \cos(t\ln p)-ip^{-\sigma}\sin(t\ln p)$,

($s \in C$ ，$Rs(s) > 0$ 且 $s \neq 1, t \in C$ 且 $t \neq 0, p \in Z^{+}$),

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$ =[$1^{\sigma-1}$cos(tln1)− $2^{\sigma-1}$cos(tln2)+ $3^{\sigma-1}$cos(tln3)−$4^{\sigma-1}$cos(tln4)-...]+i[$1^{\sigma-1}$sin(tln1) − $2^{\sigma-1}$sin(tln2)+ $3^{\sigma-1}$sin(tln3) − $4^{\sigma-1}$sin(tln4)+...] ,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$=[$1^{-\sigma}$cos(tln1)−$2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)−$4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$sin(tln1)− $2^{-\sigma}$ sin(tln2)+ $3^{-\sigma}$sin(tln3)−$4^{-\sigma}$sin(tln4)+...]

($s \in C$，$Rs(s) > 0$ 且 $s \neq 1, t \in C$ 且 $s \neq 0, n \in Z^{+}$ 且 n 遍取所有的正整数),

当 σ=$\frac{1}{2}$,

那么

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$=$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ ($s \in C$，$Rs(s) > 0$ 且 $s \neq 1, n \in Z^{+}$ 且 n 遍取所有的正整数),

$(1-p^{-(1-s)})$ =$(1-p^{-\bar{s}})$$(s \in C$，$Rs(s)>0$ 且 $s \neq 1, p \in Z^{+})$,

且

$(1-p^{-(1-s)})^{-1}$=$\left(1-p^{-\bar{s}}\right)^{-1}$$(s \in C$，$Rs(s)>0$ 且 $s \neq 1, p \in Z^{+})$,

$\prod_p(1-p^{-(1-s)})^{-1}$=$\prod_p\left(1-p^{-\bar{s}}\right)^{-1}$, $(s \in C$，$Rs(s)>0$ 且 $s \neq 1, p \in Z^{+}$ and p 遍取所有的质数),

并且

$$\frac{1}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}=\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}} \ ,$$

$$\frac{(-1)^{n-1}}{(1-2^{s})}\prod_p(1-p^{-(1-s)})^{-1}=\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}$$

$(s \in C$，$Rs(s)>0$ 且 $s \neq 1, t \in C$ 且 $t \neq 0, n \in Z^{+}$ 且 n 遍取所有的正整数, $p \in$

Z^{+} 且 p 遍取所有的质数),

而且

$\zeta(1-s)$=$\frac{(-1)^{n-1}}{(1-2^{s})}\prod_p(1-p^{-(1-s)})^{-1}$,

$\zeta(\bar{s})$=$\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}\prod_p(1-p^{-\bar{s}})^{-1}$,

$\zeta(1-s)$=$\frac{1}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$,

$\zeta(\bar{s})$=$\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$

$(s \in C$ ，$Rs(s)>0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, $n \in Z^{+}$且 n 遍取所有的正整数),

因此当 σ=$\frac{1}{2}$, 那么

仅 $\zeta(1-s)$=$\zeta(\bar{s})$ $\left(s \in C\ \ Rs(s)>0 \text{ 且 } s \neq 1\right)$成立。

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$=[$1^{\sigma-k}$cos(tln1)$-$ $2^{\sigma-k}$cos(tln2)+ $3^{\sigma-k}$cos(tln3) $-4^{\sigma-k}$cos(tln4)-...]+i[$1^{\sigma-k}$sin(tln1) $-$ $2^{\sigma-k}$sin(tln2)+ $3^{\sigma-k}$sin(tln3) $-$ $4^{\sigma-k}$sin(tln4)+...],

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$=[$1^{-\sigma}$cos(tln1)$-2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)$-4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$Sin(tln1)$-$ $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)$-4^{-\sigma}$sin(tln4)+...],

p^{k-s}=$p^{(k-\sigma-ti)}$=$p^{k-\sigma}p^{-ti}$=$p^{k-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}$=$p^{k-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$,

$p^{1-\bar{s}} = p^{(1-\sigma+ti)} = p^{1-\sigma}p^{ti} = p^{1-\sigma}(p^{ti}) = p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t} = (p^{1-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$,

$(s \in C$, $Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, n 遍取所有的正整数, $k \in R$),

那么

$p^{-(k-s)}$=$p^{(-k+\sigma+ti)}$=$p^{\sigma-k}p^{ti} = p^{\sigma-k}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$=$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(\overline{s})}$=$p^{-(\sigma-ti)}$=$p^{-\sigma}p^{ti} = (p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$,

$p^{-(k-s)}$=$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$,

$(s \in C$, $Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, $k \in R$),

因此

$(1 - p^{-(k-s)})$=1-$(p^{\sigma-k}(\cos(t\ln p) + i\sin(t\ln p))$ =$1 - p^{\sigma-k}\cos(t\ln p) - ip^{\sigma-k}\sin(t\ln p)$,

$(1 - p^{-\overline{s}})$ =1- $(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$ = $1 - p^{-\sigma}\cos(t\ln p) - ip^{-\sigma}\sin(t\ln p)$ $(s \in C$, $Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, $k \in R$) , 因 此 当 $\sigma = \frac{k}{2}$ $(k \in R)$, 那 么

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-k+s}}$=$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ $\left(s \in C\right.$, $Rs(s) > 0$ 且 $s \neq 1, k \in R$, n 遍取所有的正整数$\left.\right)$,

$(1 - p^{-(k-s)})$ = $(1 - p^{-\overline{s}})$ ($s \in C$, $Rs(s) > 0$ 且 $s \neq 1$, n 遍取所有的正整数 , $k \in R$,p∈+ 且 p 是质数),

and $(1 - p^{-(k-s)})^{-1}$=$(1 - p^{-\overline{s}})^{-1}$(s∈C , $Rs(s) > 0$ 且 s≠1,k∈R, p∈Z^{+} 且 p 是质数),

$\prod_p(1 - p^{-(k-s)})^{-1}$=$\prod_p\left(1 - p^{-\overline{s}}\right)^{-1}$ $(s \in C$, $Rs(s) > 0$ 且 $s \neq 1$, p 遍取所有的质数, $k \in R$),

且

$$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}} = \frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$$

$(s \in C$, $Rs(s) > 0$ 且 $s \neq 1$, p 遍取所有的质数, n 遍取所有的正整数, $k \in R$),且

$\zeta(k - s)$=$\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\prod_p(1 - p^{-(k-s)})^{-1}$,

$\zeta(\overline{s})$=$\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p(1 - p^{-\overline{s}})^{-1}$,

$\zeta(k - s)$=$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$ (s∈C 且 s≠ 1, k ∈R),

$\zeta(\overline{s})$=$\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ (s∈C 且 s≠ 1),

$(s \in C$, $Rs(s) > 0$ 且 $s \neq 1$, p 遍取所有的质数, n 遍取所有的正整数, $k \in R$),

因此当 $\sigma=\frac{k}{2}(k\in R)$,那么仅 $\zeta(k-s)=\zeta(\overline{s})(s\in C$， $Rs(s)>0$ 且 $s\neq 1, k\in R)$ 。黎曼已经知道黎曼ζ(s)函数有零点,通过黎曼得到的ζ(1-s)得到的等式 $\zeta(1-s)2^{1-s}\pi^{-s}Cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)\left(s\in C\right.$ 且 $s\neq 1$(等式 6), 我们就知道 ζ(1-s)=21−sπ −scos(πs2)Γ(s)ζ(s) s∈C 且 s≠1(等式 6)中,ζ(s)=0(s∈C 且 $s\neq 1$)成立。当ζ(s)=0($s\in C$, $Rs(s)>0$ 且 $s\neq 1$)，那么 $\zeta(\overline{s})=0(s\in C, Rs(s)>0$ 且 $s\neq 1)$，那么 $\zeta(k-\overline{s})$ = $\zeta(s)$ =0 ($s\in C, Rs(s)>0$ 且 $s\neq 1$)， 当 $\zeta(s)$ =0 ($s\in C, Rs(s)>0$ 且 $s\neq 1$),那么 $\zeta(\overline{s})=0(s\in C$ 且 $s\neq 1)$，那么 $\zeta(k-s)=\zeta(\overline{s})=0(s\in C$ 且 $s\neq 1)$．并且因为当 ζ(s)=0 $\left(s\in C\right.$ 且 $\left.s\neq 1\right)$成立，那么 $\zeta(1-s)=0(s\in C$ 且 $s\neq 1)$， 那 么 仅 $\zeta(1-s)=\zeta(\overline{s})$ =0 $\left(s\in C\right.$,Rss>0 且 s≠1 成立，即 ζk−s=ζ(s)(s∈C ,Rss>0 and s≠1,k∈R)，因此仅 k=1 成立.

Reasoning 2:

下面是黎曼在他的论文中得到的一个结果：

$2\sin(\pi s)\prod(s-1)\zeta(s)=(2\pi)^{s}\sum n^{s-1}((-i)^{s-1}+i^{s-1})$ [1] (等式 3),

根据欧拉公式 $e^{ix}=\cos(x)+i\sin(x)$ $(x\in R)$可以得到：

$e^{i(-\frac{\pi}{2})}=\cos(\frac{-\pi}{2})+i\sin(\frac{-\pi}{2})=0-i=-i$,

$e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})=0+i=i$,

那么

$(-i)^{s-1}+i^{s-1}=(-i)^{-1}(-i)^{s}+(i)^{-1}(i)^{s}=(-i)^{-1}e^{i\left(-\frac{\pi}{2}\right)s}+i^{(-1)}e^{i\left(\frac{\pi}{2}\right)s}=$

$ie^{i\left(-\frac{\pi}{2}\right)s}-ie^{i\left(\frac{\pi}{2}\right)s}=i(\cos\frac{-\pi s}{2}+i\sin\frac{-\pi s}{2})-i(\cos\frac{\pi s}{2}+i\sin\frac{\pi s}{2})=i\cos(\frac{\pi s}{2})-i\cos(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})+\sin(\frac{\pi s}{2})$

$=2\sin(\frac{\pi s}{2})$ (等式 4).

根据伽马函数 Γ(s)的性质 Π(s-1)=Γ(s)，并且

$\sum_{n=1}^{\infty}n^{s-1}$=ζ(1-s)($n\in Z^{+}$ 并且n遍取所有的正整数,$s\in C$，并且 $s\neq 1$),

把上面(等式 4)的结果代入上面(等式 3)右边，将得到:

$2\sin(\pi s)\Gamma(s)\zeta(s)=(2\pi)^{s}\zeta(1-s)2\sin\frac{\pi s}{2}$(等式 5)。根据倍角公式 $\sin(\pi s)=2\sin(\frac{\pi s}{2})\cos(\frac{\pi s}{2})$ ，

如果把它代入 $2\sin(\pi s)\Gamma(s)\zeta(s)=(2\pi)^s\zeta(1-s)2\sin\frac{\pi s}{2}$(等式 5)，

我们将得到: $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s\in C$ and $s\neq 1$)(等式 6),

因此当 $\zeta(s)=0$ ($s\in C$ 且 $s\neq 1$)，那么 $\zeta(1-s)=0$ ($s\in C$ 且 $s\neq 1$)，也就是 $\zeta(s)=\zeta(1-s)=0$ ($s\in$ C 且 $s\neq 1$。

作如下变换$s\to 1-s$,也就是把s当做 1-s代入等式 6, 将得到:

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ 并且 $s\neq 1$) (等式 7)。

这就是$\zeta(s)$的泛函方程$\zeta(s)$($s\in C$ 且 $s\neq 1$). 为了将它改写成一种对称的形式，用伽玛函数的余元公式

$\Gamma(Z)\Gamma(1-Z)=\frac{\pi}{\sin(\pi Z)}$ (等式 8)

和勒让德公式

$\Gamma(\frac{Z}{2})\Gamma(\frac{Z}{2}+\frac{1}{2})=2^{1-Z}\pi^{\frac{1}{2}}\Gamma(Z)$ (等式 9)，

在 $\Gamma(Z)\Gamma(1-Z)=\frac{\pi}{\sin(\pi Z)}$ (等式 8) 中，令 $z=\frac{s}{2}$ ，将得到:

$\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10)，

在 $\Gamma(\frac{Z}{2})\Gamma(\frac{Z}{2}+\frac{1}{2})=2^{1-Z}\pi^{\frac{1}{2}}\Gamma(Z)$中, 令 $z=1-s$，将得到:

$\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (等式 11)，

把 $\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10) 和 $\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$ (等式 11) 的结果代入

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s\in C$ 并且 $s\neq 1$) (等式 7) 将得到:

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$($s\in C$ 且 $s\neq 1$) (等式 12),

也就是说

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$在变换$s\to 1-s$下是不变的，这正是 Riemann 在他的论文中所说的。

也就是

$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1-s)$($s\in C$ 且 $s\neq 1$)。

同时 $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)$(s∈C 且 s≠ 1) (等式 7)在变换s→1- s下是不变的 将得到：

$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C 且 s≠ 1)(等式 6)。

那么 $\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$ (s∈C 且 s≠ 1), 当 ζ(s)=0，如果

$\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}$ (s∈C 且 s≠ 1) 在单调函数中有意义 ,则分母$2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s)$ ≠0。 显然 $2^s\neq 0$(s∈C 且 s≠ 1), $\pi^{s-1}\neq 0$(s∈C 且 s≠ 1), $\Gamma(1\text{-}s)\neq 0$(s∈C 且 s≠ 1), 所以 $\sin(\frac{\pi s}{2})\neq 0$(s∈C 且 s≠ 1),因此当 ζ(s)=0，且 $s\neq 2n(n\in Z^+)$,且s≠ 1，那么$\zeta(1-s)=\zeta(s)=0$(s∈C 且s≠ 1 且 $s\neq -2n$，$n\in Z^+$)。

因为

$L(s,\chi(n))=\chi(n)\zeta(s)$($s\in C$ 且 $s\neq 1, n\in Z_+$ ，且n遍取得所有的正整数) 且

$L(1-s,\chi(n))=\chi(n)\zeta(1\text{-}s)$($s\in C$ 且 $s\neq 1, n\in Z_+$ ，且n遍取得所有的正整数)，

根据 $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)$(s∈C and s≠ 1)(等式 7),

因此仅 $L(s,\chi(n))=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)L(1-s,\chi(n))$($s\in C$ 且 $s\neq 1, n\in Z^+$)成立。根据函数Γ(s) 和 指 数 函 数 非 零 的 性 质 ， 也 就 是 $\Gamma(\frac{1-s}{2})\neq 0$,且$\pi^{-\frac{1-s}{2}}\neq 0$， 根 据 $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1\text{-}s)$(s∈C 且 s≠ 1) (Formula 12),数学家已经证明黎曼ζ(s)(s∈C 且 s≠ 1)函数的复自变量 s 的实部只有在满足 0<Re(s)<1 且 Im(s)≠0 时，ζ(s)(s∈C 且 s≠ 1,且$s\neq -2n, n\in Z^+$)的函数值才为零，所以我们应用 $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=$ −1n−11−21−sp1−p−s−1(s∈C 且 0<Rs(s)<1 且 s≠1 且 Im(s)≠0，n∈Z+,p∈Z+,s∈C，n遍取得所有的正整数, p 遍取得所有的质数)。根据

$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C 且 s ≠ 1)(等式 6)和 $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)$(s∈C 且 s≠ 1)(等式 7)，既然黎曼已经知道且计算出黎曼 ζ(s)(s∈C and s≠ 1)函数有非平凡零点,那我们就应用 $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$ ，且 $Im(s)\neq 0$, n∈ Z^+, n遍取得所有的正整数) 和 ζ(s)=$2^s\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)(s ∈ C 且 0 < Rs(s) < 1 且 s ≠ 1，且$s \neq -2n$, 且 Im(s) ≠ 0 ,n∈ Z^+, n遍取得所有的正整数)，根据函数Γ(s)和指数函数非零的性质，也就是 $\Gamma(\frac{1-s}{2}) \neq 0$,且 $\pi^{-\frac{1-s}{2}} \neq 0$,所以当 ζ(s)=0 (s ∈ C 且 0 < Rs(s) < 1 且 s ≠ 1，$s \neq -2n$,且 Im(s) ≠ 0 ,n ∈ Z^+, n遍取得所有的正整数)，那么 ζ(1-s)=0 (s ∈ C 且 0 < Rs(s) < 1 且 s ≠ 1，且$s \neq -2n$,且 Im(s) ≠ 0 ,n∈ Z^+, n遍取得所有的正整数)，所以

ζ(s)=ζ(1-s)=0(s ∈ C 且 0 < Rs(s) < 1 且 s ≠ 1 , $s \neq -2n$,且 Im(s) ≠ 0, ,n∈ Z^+, n遍取得所有的正整数)。

Because e = $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \sum_{n=0}^{\infty}\frac{1}{n!} \approx 2.7182818284...$ ，e 是自然常数， 且因为 $\sin(Z)=\frac{e^{iZ}-e^{-iZ}}{2i}$, 假设 Z=s= σ+ti (σ ∈R,t∈R 且 t ≠ 0)，那么

$\sin(s)=\frac{e^{is}-e^{-is}}{2i} = \frac{e^{i(\sigma+ti)}-e^{-i(\sigma+ti)}}{2i}$,

$\sin(\overline{s})=\frac{e^{i\overline{s}}-e^{-i\overline{s}}}{2i} = \frac{e^{i(\sigma-ti)}-e^{-i(\sigma-ti)}}{2i}$,

根据 $x^s=x^{(\sigma+ti)}=x^\sigma x^{ti}=x^\sigma(\cos(\ln x)+i\sin(\ln x))^t=x^\sigma(\cos(t\ln x)+i\sin(t\ln x))(x>0)$，那么

$e^s=e^{(\sigma+ti)}=e^\sigma e^{ti}=e^\sigma(\cos(t)+i\sin(t)) = e^\sigma(\cos(t)+i\sin(t))$，
$e^{is}=e^{i(\sigma+ti)}=e^{\sigma i}(\cos(it)+i\sin(it)) = (\cos(\sigma)+i\sin(\sigma))(\cos(it)+i\sin(it))$，
$e^{i\overline{s}} = e^{i(\sigma-ti)}=e^{\sigma i}(\cos(-it)+i\sin(-it)) = (\cos(\sigma)+i\sin(\sigma))(\cos(it)-i\sin(it))$，
$e^{-is}=e^{-i(\sigma+ti)}=e^{-\sigma i}(\cos(-it)+i\sin(-it)) = (\cos(\sigma)-i\sin(\sigma))(\cos(it)-i\sin(it))$，
$e^{-i\overline{s}} = e^{-i(\sigma-ti)}=e^{-\sigma i}(\cos(it)+i\sin(it)) = (\cos(\sigma)-i\sin(\sigma))(\cos(it)+i\sin(it))$，
$2^s=2^{(\sigma+ti)}=2^\sigma 2^{ti}=2^\sigma(\cos(\ln 2)+i\sin(\ln 2))^t=2^\sigma(\cos(t\ln 2)+i\sin(t\ln 2))$，
$2^{\overline{s}}=2^{(\rho-ti)}=2^\sigma 2^{-ti}=2^\sigma(\cos(\ln 2)+i\sin(\ln 2))^{-t}=2^\sigma(\cos(t\ln 2)-i\sin(t\ln 2))$，
$\pi^{s-1}=\pi^{(\sigma-1+ti)}=\pi^{\sigma-1}\pi^{ti}=\pi^{\rho-1}(\cos(\ln\pi)+i\sin(\ln\pi))^t=\pi^{\sigma-1}(\cos(t\ln\pi)+i\sin(t\ln\pi))$，
$\pi^{\overline{s}-1}=\pi^{(\sigma-1-ti)}=\pi^{\sigma-1}\pi^{-ti}=\pi^{\sigma-1}(\cos(\ln\pi)+i\sin(\ln\pi))^{-t}=2^{\sigma-1}(\cos(t\ln\pi)-i\sin(t\ln\pi))$，

因此

$2^s=\overline{2^{\overline{s}}}$，$\pi^{s-1}=\overline{\pi^{\overline{s}-1}}$ ，

且

$\frac{e^{is}-e^{-is}}{2i} = \overline{\frac{e^{i\overline{s}}-e^{-i\overline{s}}}{2i}}$ ，

因此

$\sin(s)=\overline{\sin(\bar{s})}$ ，

且

$\sin(\frac{\pi s}{2})=\overline{\sin(\frac{\pi \bar{s}}{2})}$ 。

根据复数域上 Γ(s)函数的定义 ：

$\Gamma(s)=\int_0^{+\infty} t^{s-1} e^{-t}dt(s \in C)$,

在 Re(s)>0 中，该定义可由解析延拓原理推广到除非正整数外的整个复数域，

因此

$\Gamma(s)=\overline{\Gamma(\bar{s})}$ ，

且

$\Gamma(1-s)=\overline{\Gamma(1-\bar{s})}$.当

$\zeta(1-\bar{s})=\overline{\zeta(1-\bar{s})}=0$ 且

$\zeta(1-s)=\zeta(\bar{s})=0(s \in C, Rs(s)>0$ 且 $s \neq 1$ ，且 $Im(s) \neq 0$,n∈ Z^+, n遍取得所有的正整数), 并根据 $\zeta(s)=2^s\pi^{s-1}Sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s \in C$,且 $s \neq 1$ ，且 $Im(s) \neq 0$,n∈ Z^+, n遍取得所有的正整数),

那么 $\zeta(s)=\overline{\zeta(\bar{s})}(s \in C$ 且 $0 < Rs(s) < 1$ 且 $s \neq 1$ ，且 $Im(s) \neq 0$,n∈ Z^+, n遍取得所有的正整数)

成立,当

$\zeta(s)= \overline{\zeta(\bar{s})} = 0(s \in C, Rs(s) > 0$ 且 $s \neq 1$ ，且 $Im(s) \neq 0$,n ∈ Z^+, n遍取得所有的正整数）则

$\zeta(1-s)=\zeta(\bar{s})$ =0($s \in C$ 且 $0 < Rs(s) < 1$ 且 $s \neq 1$ ，且 $Im(s) \neq 0$,n∈ Z^+, n遍取得所有的正整数)

也成立. 因此仅有 $\zeta(\sigma+ti)=\zeta(\sigma-ti)=0$ 成立.

因　为　$\zeta(s) = \overline{\zeta(\bar{s})}\left(s = \sigma + ti, \sigma \in R, Re(s) > 0\right.$ 且 $\left.s \neq 1, t \in R\right)$成立，所以当$s \neq -2n$(n ∈ Z+,如果 ζ(s)=0s=σ+ti, σ∈R,t∈R, *Res>0* 且*s≠1*，则必有 ζ(s)= ζ(s)=0s=σ+ti, σ∈R,*Res>0* 且 *s≠1*,t∈R，表明除了负偶数，黎曼 ζ(s)函数的零点必定是共轭的，

黎曼 ζ(s)函数在 Re(s) ∈ （ 0,1 ）的临界带内 ,不存在不共轭的零点。根据 $\zeta(s)=\zeta(\overline{s})=0\left(s=\sigma+ti, \sigma\in R, Res>0\right.$ 且 $s\neq 1$, $t\in R$，如果 s=s， 那么 $s\in R$，由于 $s=-2m$ m∈Z+ 使得

$2\sin(\pi s)\Pi(s-1)\zeta(s)=i\int_0^{\infty}\frac{x^{s-1}dx}{e^x-1}$ 和 $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\left(s\in\right.$ C 并且 s≠1 等式 7 中的函数ζss∈C 且 s≠1 的值为零,所以负偶数可以是黎曼 ζ(s)s∈C 且 s≠1 的零点。如果 s≠s， 那么 s和 s 不是同为实数,而是同为虚数，t∈R 且 t≠0，而且根据

$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\left(s\in C \text{ 并且 } s\neq 1\right)$（等式 7）， 如 果 ζ(s)=0 （s ∈ C 且 s≠1 成立,那么 ζ(1-s)=ζ(s)=0s∈C 且 s≠1 必定成立。

如果 $\zeta(s)=0\left(s\in C \text{ 且 } Re(s)>0 \text{ 且 } s\neq 1\right)$成立,那么

$\zeta(1-s)=\zeta(s)=0\left(s\in C \text{ 且 } Re(s)>0 \text{ 且 } s\neq 1\right)$必定成立。

因为 $\zeta(s)=\overline{\zeta(\overline{s})}\ (s\in C, Re(s)>0 \text{ and } s\neq 1)$ ， 所以如果 ζ(s)=0 （$s\in C, Re(s)>0$ 且 $s\neq 1$ and s≠1, 那么

$\zeta(s)=\zeta(\overline{s})=0\left(s\in C, Re(s)>0 \text{ 且 } s\neq 1\right)$,又根据 $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(s\in$ C *and* s≠1(等式 7), 如果*Res>0* 且 ζ(s)=0s∈C, *Res>0* 且 s≠1,那么 ζ(s)=ζ(1-s)=0s∈C, *Res>0* 且 s≠1, 所以当 *Res>0*，黎曼 ζss∈C 且 *Res>0*且*s≠1* 函数的两个零点s和 $1-s$也必须是共轭的。

如果s和 $1-s$是除负偶数以外的实数，由于s和 $1-s$共轭，则$s=1-s$，那么$s=\frac{1}{2}$,由于 $\sin\left(\frac{\pi s}{2}\right)=\sin\left(\frac{\pi}{2}\times\frac{1}{2}\right)=\sin\left(\frac{\pi}{4}\right)\neq 0$，且因为 $\zeta\left(\frac{1}{2}\right)$发散，所以如果$s$和 $1-s$是除负偶数以外的实数， 那么s和 $1-s$都不是黎曼 $\zeta(s)\left(s\in C \text{ 且 } Re(s)>0 \text{ 且 } s\neq 1\right)$的零点，也就是说黎曼 $\zeta(s)\left(s\in C \text{ 且} Re(s)>0 \text{ 且} s\neq 1\right)$没有除负偶数以外的实数零点。当$Re(s)>0$，如果$Re(s)=1$,则$Re(1-s)=0$,那么$s$和 $1-s$不共轭，所以黎曼 $\zeta(s)\left(s\in C \text{ 且 } s\neq 1\right)$没有实部为 1 或 0 的零点。 当 $Re(s)>1, \zeta(s)\left(s\in\right.$

C 并且 $s \neq 1\big)$ 没有零点,又根据 $\zeta(\mathrm{s}) = \zeta(1-\mathrm{s}) = 0\left(\mathrm{s} \in \mathrm{C}\,,\text{且}\,\mathrm{s} \neq 1\right)$ 成立，那么当 $\mathrm{Re(s)} <$

0，则 $\zeta(s)$ $\left(\mathrm{s} \in \mathrm{C} \text{ 并且 } s \neq 1\right)$ 也不等于零。又因为当 $\zeta(\mathrm{s}) = 0\left(\mathrm{s} \in \mathrm{C}\,, Re(s) > 0 \text{ 且 } s \neq\right.$

$\left.1\right)$，如果 $\mathrm{Re(s)} = 1$，则 $\mathrm{Re}(1-\mathrm{s}) = 0$,那么 s 和 $1-\mathrm{s}$ 不共轭，又根据 $\zeta(\mathrm{s}) = \zeta(1-\mathrm{s}) =$

$0\left(\mathrm{s} \in \mathrm{C}\,,\text{且}\,\mathrm{s} \neq 1\right)$ 成立,所以如果 $\mathrm{Re(s)} = 0$ 或 $\mathrm{Re(s)} = 1$，则 $\zeta(\mathrm{s})\left(\mathrm{s} \in \mathrm{C} \text{ 并且 } s \neq\right.$

$\left.1\right)$ 无零点。所以除了负偶数，黎曼 $\zeta(\mathrm{s})\left(\mathrm{s} \in \mathrm{C} \text{ 并且 } s \neq\right.$

$\left.1\right)$ 有零点的前提条件是 $\mathrm{Re(s)}$的值位于区间$(0,1)$。黎曼 $\zeta(\mathrm{s})\left(\mathrm{s} \in \mathrm{C} \text{ 且 } \mathrm{s} \neq\right.$

$\left.1\right)$ 零点s的实部必须 满足 $0 < Re(s) < 1$ ，也就是$Re(s) \in$

$(0,1)$，这 表明素数定理 成立。 当 $0 < Re(s) <$

1 ，如果s和 $1-s$一个为实数，一个为虚数，则s和 $1-s$不共轭，那么s和 $1-$

s不可能同为黎曼 $\zeta(\mathrm{s})\left(\mathrm{s} \in \mathrm{C}\,, Re(s) > 0 \text{ 且 } \mathrm{s} \neq 1\right)$ 的零点，所以 $1-$

s和s只能同为虚数且共轭，s不能为纯虚数，因为如果s为纯虚数，则 $1-$

s和s不共轭，所以 $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}\,, 0 < Re(s) < 1)$无纯虚数零点。而且如果

$Re(s) \neq \frac{1}{2}$， 那么$Re(1-s) \neq 0$， 而且必定有$Re(s) \neq$

$Re(1-s)$，那么 $1-s$和s不共轭，所以$Re(s) \neq \frac{1}{2}$不可能成立。所以仅有 $1-\mathrm{s} =$

$\overline{\mathrm{s}}$ 成立,即仅有 1-σ-ti= σ-ti 成立，所以仅有 $\sigma = \frac{1}{2}$ ，$\mathrm{t} \in \mathrm{R}$ 且 $\mathrm{t} \neq 0$，因此黎曼 $\zeta(\mathrm{s})(\mathrm{s} \in \mathrm{C}\,, 0 <$

$Re(s) < 1)$的非实数零点的实部只能是$\frac{1}{2}$，即仅有 $\mathrm{Re(s)} = \frac{1}{2}$成立，等价于 $\xi(\mathrm{s}) =$

$0\left(\mathrm{s} = \frac{1}{2} + \mathrm{ti} \text{ 或 } \mathrm{s} = \frac{1}{2} - \mathrm{ti}\,, \mathrm{t} \in \mathrm{R} \text{ 且 } \mathrm{t} \neq 0\text{，}\mathrm{s} \in \mathrm{C},\text{且}\,\mathrm{s} \neq 1\right)$ 或 $\xi\left(\frac{1}{2} + \mathrm{ti}\right) = 0\left(\mathrm{t} \in \mathrm{R} \text{ 且 } \mathrm{t} \neq\right.$

$\left.0\right)$ 和 $\xi\left(\frac{1}{2} - \mathrm{ti}\right) = 0\left(\mathrm{t} \in \mathrm{R} \text{ 且 } \mathrm{t} \neq 0\right)$ 成立，所以在 $\mathrm{Re(s)} \in (0,1)$的临界带内，$\mathrm{Re(s)} \neq$

$\frac{1}{2}$不可能,不存在实部不等于$\frac{1}{2}$的零点，所以黎曼猜想成立。零点 s 和零点 $1-$

s 的对称性不足以说明黎曼 $\zeta(\mathrm{s})\left(\mathrm{s} \in \mathrm{C} \text{ 且 } \mathrm{s} \neq\right.$

$\left.1\right)$ 的非平凡零点都位于临界线上，零点 s 和零点 $1-\mathrm{s}$ 的对称性表示它们关于 $\mathrm{Re(s)} =$

$\frac{1}{2}$的临界线上的一个点$\left(\frac{1}{2}, \mathrm{oi}\right)$对称，而不是零点 s 和零点 $1-\mathrm{s}$ 关于 $\mathrm{Re(s)} =$

$\frac{1}{2}$的临界线对称，s 和 $1-$

s 的共轭性才是黎曼 $\zeta(s)\left(s \in C \text{ 且 } s \neq 1\right)$ 的非平凡零点都位于临界线上的根本原因。

根据$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C and $s \neq 1$) (等式 6),那么当 ζ(s)=0 ，就会有 ζ(1-s)= ζ(s)=0(s∈C and $s \neq 1$) 成立。由于$\zeta(s) = \overline{\zeta(\overline{s})}\left(s \in C \text{ 并且 } Re(s) > 0 \text{ 且 } s \neq 1\right)$，那么当ζ(s)=0$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$或 $\zeta(\overline{s})$=0$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$，那么就必定会有ζ(s)=$\zeta(\overline{s})$=0$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$ 成立。当ζ(s)=0$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$，根据$\zeta(1-s)=\zeta(s)$=0$\left(s \in C \text{ 并且 } Re(s) > 0 \text{ 且 } s \neq 1\right)$成立，那么 s 和 1-s 也必定共轭。由此我们就得到 s=$\frac{1}{2}$+ti(t∈ R 且≠ 1)，或 s=$\frac{1}{2}$-ti(t∈ R 且≠ 1)，当 Re(s)>1,欧拉ζ与黎曼ζ函数等价，同时一个乘积因子都不等于零，所以当 Re(s)>1, ζ(s)就不等于零，又根据 $\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$(s∈C 且 $s \neq 1$)(等式 7) ,所以正偶数 2n(n 为正整数)虽然能够使 $\sin(\frac{\pi s}{2})$为零，但它不是黎曼ζ(s)的零点。如果 s 为除负偶数和正偶数以外的其它实数，如果它是黎曼ζ(s)$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$函数的零点，那么$s$和 $1-s$就必定共轭，所以它除了不能使得 $\sin(\frac{\pi s}{2})$为零，而且必须满足 s=1-s，那么 s=$\frac{1}{2}$, 而$\zeta(\frac{1}{2})$发散， 所以负偶数以外的实数不是黎曼ζ(s)$\left(s \in C, Re(s) > 0 \text{ 且 } s \neq 1\right)$的零点。根据 ζ(s)= ζ(1-s)=0 (s∈C 且 $s \neq 1$)成立，我们知道除了负偶数，ζ(s)(s∈C 且$0 < Re(s) < 1$)的零点关于点($\frac{1}{2}$,0i)对称。但是仅仅根据ζ(s)(s∈C 且$0 < Re(s) < 1$)的零点关于点($\frac{1}{2}$,0i)对称，就判定黎曼ζ(s)(s∈C 且 $0 < Re(s) < 1$)函数的非平凡零点就都位于实部等于$\frac{1}{2}$的临界线上，这可以吗？显然不可以, 当 $Re(s) \in (0,1)$, 假如 s=0.54+ti($t \in R$),Re(s)=0.54，那么 Re(1-s)=0.46, s 和 1-s 关于点($\frac{1}{2}$,0i)对称，但是黎曼认为这样的复数不是黎曼ζ(s)的零点。黎曼的看法是正确的，很显然，当 Re(s)≠ $\frac{1}{2}$, 那么 s 和 1-s 就必定不共轭 ,根据 黎曼 ζ(s)(s∈C 且$0 < Re(s) < 1$)函数的零点必定共轭，那么, 如果 Re(s)≠ $\frac{1}{2}$, 那么它必定不是ζ(s) (s∈C 且$0 < Re(s) < 1$)函数的零点。综上所述，黎曼ζ(s)$\left(s \in C \text{ 并且 } s \neq 1\right)$函数的非平凡零点必定都位于复平面 Re(s)= $\frac{1}{2}$的临界线上，黎曼猜想必定成立。

根据黎曼的论文《论不大于 x 的质数的个数》，我们可以得到关于黎 ζ(s)(s ∈ C 且 s ≠ 1)函数的表达式$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s)\left(s \in C \text{ 并且 } s \neq 1\right)\left(\text{等式 } 7\right)$，这是现代数学家早就知道的，而我呢，也得到了得到了这个结果。既然黎曼计算出了 ζ(s)(s ∈ C 且 s ≠ 1)函数有零点，也就是说在

$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s)\left(s \in C \text{ 并且 } s \neq 1\right)\left(\text{等式 } 7\right)$中, ζ(s)=0(s ∈ C 且 s ≠ 1)成立,那么根据$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s)\left(s \in C \text{ 并且 } s \neq 1\right)\left(\text{等式 } 7\right)$和 ζ(s)=ζ($\bar{s}$)=0(s ∈ C 且 s ≠ 1)， 当 ζ(s)=0，下面三个等式： ζ(σ+ti)=0, ζ(1-σ-ti)=0, and ζ(σ-ti)=0 才都成立。并且当 ζ(s)=0(s ∈ C 且 s ≠ 1)，根据 ζ(s)=ζ(1-s)=0(s ∈ C ，$Re(s) > 0$ 且 s ≠ 1)和 ζ(s)=ζ($\bar{s}$)=0(s ∈ C ，$Re(s) > 0$ 且 s ≠ 1),因此当 ζ(s)=0，s 和 1-s 也是共轭的。所以仅有 $1 - s = \bar{s}$ 成立,即仅有 1-σ-ti= σ-ti 成立。在我的分析中， ζ(s)(s ∈ C 且 s ≠ 1)、ζ(1-s)(s ∈ C 且 s ≠ 1)和 ζ($\bar{s}$)(s ∈ C 且$Re(s) > 0$)这些函数表达式都来自$\sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$(s ∈ C 且s≠ 1, n∈ Z^+ 且 n 遍取所有正整数, p ∈ Z^+ 且 p 遍取所有质数)。ζ(s)($s = \sigma + ti, \sigma \in R, t \in R, t \neq 0$,且 s ≠ 1是自变量为 s 的函数 ,$s$与 $\bar{s}$与 1-s 的关系只有C_3^2=3 种，即 s=$\bar{s}$ 或 s=1-s 或 $\bar{s}$=1-s，且 sin($\frac{\pi s}{2}$ = 0), 所以 s∈R 且 s=-2n(n ∈ Z^+),舍去 s=2n(n ∈ Z^+)，或 σ+ti=1-σ-ti ,或 σ-ti=1-σ-ti, 即 s ∈ R, drop s = 2n(n ∈ Z^+)或 σ=$\frac{1}{2}$ 且 t=0,或 $\sigma = \frac{1}{2}$， t ∈R 且 t≠ 0,也就是 s ∈ R, 或 s=$\frac{1}{2}$+0i ,或 s=$\frac{1}{2}$+ti(t ∈R 且 t ≠ 0)和 s=$\frac{1}{2}$-ti(t ∈R 且 t ≠ 0)。因为 $\zeta\left(\frac{1}{2}\right) \to +\infty, \zeta(1) \to +\infty$, ζ(1) 是发散的, ζ($\frac{1}{2}$)更是发散，舍弃 s=1 和 s=$\frac{1}{2}$ ，只取 s=$\frac{1}{2}$+ti(t ∈R 且 t ≠ 0)和 s=$\frac{1}{2}$-ti(t ∈R 且 t ≠ 0)。根据黎曼得到的等式$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$(s ∈ C 且 s ≠ 1) ,所以ξ(s)=ξ(1 − s)(s ∈ C 且 s ≠ 1), 因为 Γ($\frac{s}{2}$)=$\overline{\Gamma(\frac{\bar{s}}{2})}$ 、$\pi^{-\frac{s}{2}}$==$\overline{\pi^{-\frac{\bar{s}}{2}}}$。通过 ζ(s)=ζ(1-s) (s ∈ C 且 s ≠ 1)=0 和 ζ(s)=ζ($\bar{s}$)=ζ(1-$\bar{s}$)=0(s ∈ C 且 s≠1 成立,那么 s=s 或者 s=1-s 或者 s=1-s ，且 sin(πs2=0), 所以 s∈R 且 s=-2n(n ∈ Z^+),舍去 s=2n(n ∈ Z^+)，

或者 σ+ti=1-σ-ti , 或者σ-ti=1-σ-ti,因此 s ∈ R,或者 σ=$\frac{1}{2}$ and t=0, 或者$\sigma = \frac{1}{2}$或者 t ∈R 或

者 $t\neq 0$，因此 $t\in R$，且 $s=-2n(n\in Z^{+})$，或者 $s=\frac{1}{2}+0i$ ，或者 $s=\frac{1}{2}+ti\,(t\in R$ 且 $t\neq 0)$ 和 $s=\frac{1}{2}-ti(t\in R$ 且 $t\neq 0)$。由于 $\zeta\left(\frac{1}{2}\right)\to+\infty,\zeta(1)\to+\infty$，$\zeta(1)$是发散的，$\zeta(\frac{1}{2})$ 更是发散的，因此舍去 s=1 和 $s=\frac{1}{2}$。因为仅当 $\sigma=\frac{1}{2}$,下面三个等式: $\zeta(\sigma+ti)=0$，$\zeta(1-\sigma-ti)=0$，和 $\zeta(\sigma-ti)=0$ 才都成立。 因此当 $\zeta(s)=0\left(t\in C \text{ 且 } t\neq 0, s\in C \text{ 且 } s\neq 1 \text{ 且 } s\neq 0, s\neq -2n, n\in Z^{+}\right)$，那么仅 $s=\frac{1}{2}+ti(t\in R$ 且 $t\neq 0)$ 和 $s=\frac{1}{2}-ti(t\in R$ 且 $t\neq 0)$ 成立. 通过黎曼定义的等式

$\xi(s)=\frac{1}{2}s(s-1)\,\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)\left(s\in C \text{ 且 } s\neq 1\right)$，因此 $\xi(s)=\xi(1-s)(s\in C$ 且 $s\neq 1)$，因为 $\Gamma(\frac{s}{2})=\overline{\Gamma(\frac{\bar{s}}{2})}$，且 $\pi^{-\frac{s}{2}}=\overline{\pi^{-\frac{\bar{s}}{2}}}$，并且因为 $\zeta(s)=\overline{\zeta(\bar{s})}(s\in C$ 且 $s\neq 1)$，因此 $\xi(s)=\overline{\xi(\bar{s})}(s\in C$ 且 $s\neq 1)$。所以当 $\zeta(s)=0\ (s\in C$ 且 $s\neq 1)$ ，那么 $\zeta(s)=\zeta(1-s)=\zeta(\bar{s})=0(s\in C$，$s\neq 1)$ 且 $\xi(s)=\xi(1-s)=\xi(\bar{s})=0(s\in C$ 且 $s\neq 1)$ 必定成立，$\zeta(s)$的零点除了平凡零点 s=-2n（n为自然数），由于恰好是$\xi(s)=\Gamma(\frac{s}{2}+1)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$中的$\Gamma(\frac{s}{2}+1)$的极点，因而不是 $\xi(s)$的零点外，其余全都是 $\xi(s)$的零点，因此 $\xi(s)$的零点与黎曼ζ函数的非平凡零点相重合，因此黎曼函数 $\zeta(s)$的非平凡零点和黎曼$\xi(s)(s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z^{+})$的函数的零点是相同的。换句话说，$\xi(s)$将黎曼 $\zeta(s)$函数的非平凡零点从全体零点中分离了出来。因为黎曼 $\zeta(s)$函数的非平凡零点满足 $s=\frac{1}{2}+ti(t\in R$ 且 $t\neq 0)$ 和 $s=\frac{1}{2}-ti(t\in R$ 且 $t\neq 0)$，因此黎曼引进的函数 $\xi(s)=0(s\in C$ 且 $s\neq 1$ 且 $s\neq 0, s\neq -2n, n\in Z^{+})$的复数根必定满足 $s=\frac{1}{2}+ti(t\in R$ 且 $t\neq 0)$ 和 $s=\frac{1}{2}-ti(t\in R$ 且 $t\neq 0)$，和黎曼黎曼 $\zeta(s)$函数的非平凡零点一致，$\xi(t)=0$ 根也和黎曼 $\zeta(s)$函数的非平凡零点的虚部一致，是一个实数。通过黎曼定义的函数

$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)(s\in C$ 且 $s\neq 1, t\in C$ 且 $t\neq 0)$ 和黎曼定义的 $s=\frac{1}{2}+ti(t\in C$ 且 $t\neq 0)$，因为 $s\neq 1$，且 $\prod\frac{s}{2}\neq o$，$\pi^{-\frac{s}{2}}\neq 0$，因此 $\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\neq 0(s\in C$ 且 $s\neq 1)$，并且当 $\xi(t)=0(t\in C$ 且 and $t\neq 0)$，那么

$\prod\frac{s}{2}(s\text{-}1)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0(s\in C$ 且$s\neq 1, t\in C$ 且 $t\neq 0)$, 且

$\zeta(\frac{1}{2}+ti)=\frac{\xi(t)}{\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}}=\frac{0}{\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}}=0(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1$，$s\neq -2n, n\in Z^{+})$,

so $t\in R$ 且 $t\neq 0$. 因此等式

$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z^{+})$ 和等式

$4\int_{1}^{\infty}\frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)dx=\xi(t)=0(t\in C$ 且 $t\neq 0)$和等式

$\xi(t)=\frac{1}{2}\text{-}(t^{2}+\frac{1}{4}\int_{1}^{\infty}\Psi(x)\,x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)=0(t\in C$ 且 $t\neq 0)$ 的根 t 必定是实数且 $t\neq 0$。如果 $Re(s)=\frac{k}{2}(k\in R)$,那么 $\zeta(k\text{-}s)=2^{k-s}\pi^{-s}Cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$且$s\neq 1)$ 和

$\xi(k-s)=\frac{1}{2}s(s\text{-}k)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)\left(t\in C\right.$ 且 $t\neq 0, s\in C$ 且 $\left.s\neq 1, s\neq -2n, n\in Z^{+}\right)$都成立。因此当 $\zeta(s)=0(s\in C$ 且 $s\neq 1)$,那么 $\zeta(s)=\zeta(k-s)=\zeta(\overline{s})=0(s\in C$ 且 $s\neq 1, s\in C)$且$\xi(s)=\xi(k-s)=\xi(\overline{s})=0(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1$ 且 $s\neq 0, s\neq -2n, n\in Z^{+})$ 必定都成立，且 $s=\frac{k}{2}+ti(k\in R$，$t\in R$ 且 $t\neq 0)$ 必定成立，因此$\prod\frac{s}{2}(s\text{-}k)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z^{+}, k\in R)$, 并且 $\zeta(\frac{k}{2}+ti)=$

$\frac{\xi(t)}{\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=\frac{0}{\prod\frac{(\frac{k}{2}+ti)}{2}(-k+\frac{1}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=0(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z^{+}, k\in R)$,因此 $t\in R$ 且 $t\neq 0$。 因此等式$\prod\frac{s}{2}(s\text{-}k)\pi^{-\frac{s}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0(s\in C$ and $s\neq 1, t\in C$ and $t\neq 0, k\in R)$ 的根 t 必定是实数， 且 $t\neq 0$。但是黎曼 $\zeta(s)(s\in C$ and $s\neq 1)$函数仅满足 $\zeta(1\text{-}s)=2^{1-s}\pi^{-s}Cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$且$s\neq 1)$ 和 $\xi(s)=\frac{1}{2}s(s\text{-}1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ $(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1$ 且 $s\neq -2n, n\in Z^{+}, k\in R$，$t\in R$) 和 $\xi(s)=\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=(s-1\pi-s2\Gamma(s2+1)\zeta s(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z+)$，也就是说仅 ζ(1-s)=21−sπ −sCos(πs2)Γ(s)ζ(s)(s∈C and s≠1) 和 ξ(s)=12s(s-1)Γ(s2)π− s2ζs (s∈C 且 s≠1) 和

$\xi(s)=\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}+1)\zeta(s)(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1$，$s\neq -2n, n\in Z^{+})$成立，所以仅 Re(s)=$\frac{k}{2}$=$\frac{1}{2}$ 成立，所以仅 k=1 成立。黎曼猜想必须满足黎曼ζ(s)(s∈C and s≠ 1)函数和黎曼 $\xi(s)(s\in C$ 且 $s\neq 1$，$s\neq -2n, n\in Z^{+})$函数的性质。黎曼ζ(s)（s∈C, s≠1）函数和黎曼ξ(s)（$s\in C$ 且 $s\neq 1$，$s\neq -2n, n\in Z^{+}$）函数的性质是基本的，黎曼猜想必须是正确的，这样才能正确反映黎曼ζ(s)（s∈C, s≠1）函数和黎曼ξ(s)（$s\in C$ 且 $s\neq 1, s\neq -2n, n\in Z^{+}$）函数的性质，即黎曼ξ(t)（t∈C, t≠0）函数的根只能是实数，也就是说，Re(s)只能等于$\frac{1}{2}$, Im(s)必须是实数，且 Im(s)不等于 0。黎曼猜想必定成立，是完全正确的。

因此仅有 ζ(σ+ti)=ζ(σ-ti)=0 成立.根据 ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)，既然黎曼已经知道且计算出黎曼ζ(s)(s∈C and s≠ 1)函数有非平凡零点，也就是说在 ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$，$n\in Z^{+}, n$遍取得所有的正整数)中， ζ(s)=0 ($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)成立，所以当ζ(s)=0($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)，那么ζ(s)=ζ(1-s)=0 ($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)成立. 在证明

ζ(s)=ζ(1-s)=0 ($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)和 ζ(s)=ζ($\overline{s}$)=0($s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq 1$，且 $Im(s)\neq 0$,n∈ Z^{+}, n遍取得所有的正整数)的过程中，ζ(s)是一个泛函数。在我的分析中， ζ(s)=0, ζ(1-s)=0 和 ζ($\overline{s}$)=0 的函数表达式可以是相同的或等价的 ,都符合$\sum_{n=1}^{\infty}n^{-s}(s\in C,$且 $s\neq 1$，$n\in Z^{+}$，且 n遍取得所有的正整数)。通过 ζ(s)= $0\left(s\in C,\text{且 } s\neq 1\right)$，可知函数 ζ(s)的自变量 s 与$\overline{s}$和 1-s 的关系只有$C_3^2$=3 种等值关系，也就是 s=$\overline{s}$ 或 s=1-s 或 $\overline{s}$=1-s.再加上 s 与 $\overline{s}$不相等但共轭，且当 $\sin(\frac{\pi s}{2})=0$,所以 $s\in R$ 且 $s=-2n(n\in Z^{+})$,舍去 $s=2n(n\in Z^{+})$后，一共四种关系。正如下:通过

ζ(s)=ζ(1-s)=0($s \in C$ 且 $s \neq 1$)和 ζ(s)=ζ($\overline{s}$)=ζ(1-s)=0($s \in C$ 且 $s \neq 1$)，当 ζ(s)=0 ，可知

ζ(s)= ζ(1-s)=0($s \in C$ 且 $s \neq 1$) 表明 s 和 1-s 要对称，既可以关于点($\frac{1}{2}$, 0i)对称，也可以在垂直于实数轴的直线上关于垂足($\frac{1}{2}$, 0i)对称，而 ζ(s)= ζ($\overline{s}$)=0 表明要 s 和$\overline{s}$要共轭，而 s 和$\overline{s}$显然是复平面上位于垂直于实数轴的直线上的关于垂足$\left(\frac{1}{2}, 0i\right)$对称的两个复数，所以既要和$1-s$关于垂足($\frac{1}{2}$, 0i)共轭对称，s 又要和 $\overline{s}$在垂直于实数轴的直线上关于垂足($\frac{1}{2}$, 0i)共轭对称，而 s 和$\overline{s}$ 共轭本身包含了对称，那么只能有 1-s=$\overline{s}$成立。那么仅有 s=$\overline{s}$ 或 s=1-s 或 $\overline{s}$=1-s, 且 sin($\frac{\pi s}{2}$) = 0, 所以 s∈R 且 s=-2n($n \in Z^+$),舍去 s=2n($n \in Z^+$)，因此 s∈R 且 $s = -2n(n \in Z^+)$， 或 σ +ti=1- σ -ti ， 或 σ -ti=1- σ − ti ≠ 0， 即 $s \in R$ 且 $s = -2n(n \in Z^+)$，或 σ=$\frac{1}{2}$ ，$t \in R$且$t = 0$,或 σ=$\frac{1}{2}$ ，$t \in R$且$t \neq 0$。因为

$\zeta\left(\frac{1}{2}\right) \to +\infty, \zeta(1) \to$

$+\infty$, ζ(1)是调和级数，它发散，而 $\zeta\left(\frac{1}{2}\right)$更是发散,因此它们都应该被舍去.所以仅当 ρ=$\frac{1}{2}$，下面三个等式ζ(σ + ti)= 0, ζ(1 − σ − ti)= 0, 和 ζ(σ-ti)=0 才都成立。因此如果不考虑

$s = -2n(n \in Z^+)$，那仅有 s=$\frac{1}{2}$+ti(t∈R 且 $t \neq 0$) 和 s=$\frac{1}{2}$-ti(t∈R且 $t \neq 0$)成立。由于黎曼已经证明黎曼 ζ(s)($s \in C$,且 $s \neq 1$)函数有零点，也就是说，在 ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s \in C$ and $s \neq 1$)(Formula 7)中，ζ(s)=0($s \in C$,且 $s \neq 1$) 为真。根据黎曼得到得的这个等式 $\xi(s) = \frac{1}{2}s(s-1)\,\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$ ($s \in C$ 且$s \neq 1$，且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数), 因 此 $\xi(s) = \xi(1-s)$($s \in C$ 且$s \neq 1$，且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数)。 因为 $\Gamma(\frac{s}{2}) = \overline{\Gamma(\frac{\overline{s}}{2})}$，且$\pi^{-\frac{s}{2}} = \overline{\pi^{-\frac{\overline{s}}{2}}}$,且因为ζ(s)=$\overline{\zeta(\overline{s})}$($s \in C$,且 $s \neq 1$)，因此ξ(s)=$\overline{\xi(\overline{s})}$($s \in C$,且 $s \neq 1$,且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数)。

根据, ξ(s)=ξ(1 − s)和 ξ(s)=$\overline{\xi(\overline{s})}$($s \in C$ 且$s \neq 1$,且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数)，通过 ξ(s)= 0 ($s \in C$ 且$s \neq 1$ 且$s \neq 0$ 且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数),可知函数 ξ(s) ($s \in C$ 且$s \neq 1$，且 $s \neq -2n, n \in Z^+$，n 遍取所有正整数)的自变量 s 与 $\overline{s}$和 1-s 的关系只有

C_3^2=3 种等值关系，也就是 s=$\bar{s}$ 或 s=1-s 或 $\bar{s}$=1-s.再加上 s 与

$\bar{s}$不相等但共轭，且 $\sin(\frac{\pi s}{2}=0)$,所以 $s\in R$ 且 $s=-2n(n\in Z^+)$,舍去 $s=2n(n\in Z^+)$，一共四种关系。正如通过 ζ(s)=ζ(1-s)=0 $(s\in C$,且 $s\neq 1)$ 和 ζ(s)=ζ($\bar{s}$)=ζ(1-s)=0 $(s\in C$ 且 $s\neq 1)$,那么仅有 s=$\bar{s}$ 或 s=1-s 或 $\bar{s}$=1-s, 且 $\sin(\frac{\pi s}{2}=0)$, 所以 s∈R 且 s=-2n$(n\in Z^+)$,舍去 s=2n$(n\in Z^+)$，因此 s∈R 且 $s=-2n(n\in Z^+)$，或 σ+ti=1-σ-ti，或 σ-ti=1-σ − ti ≠ 0，所以当 ζ(s)=0 $(s\in C$,且 $s\neq 1)$ ，那么 $\zeta(s)=\zeta(1-s)=\zeta(\bar{s})=0(s\in C$ 且 $s\neq 1)$ 和 $\xi(s)=\xi(1-s)=\xi(\bar{s})=0\left(s\in C\right.$ 且 $s\neq 1$, 且 $s\neq -2n, n\in Z^+$，n 遍取所有正整数$\left.\right)$)必定都正确。又由于黎曼 ζ(s) $(s\in C$,且 $s\neq 1)$ 函数的平凡零点与ξ(s)中的 $\Gamma\left(\frac{s}{2}+1\right)$ 中的极点相抵消，所以黎曼 ζ(s) $(s\in C$,且 $s\neq 1)$函数的非平凡零点和黎曼

ξ(s) $\left(s\in C\right.$ 且 $s\neq 1$，且 $s\neq -2n, n\in Z^+$，n 遍取所有正整数$\left.\right)$ 函数的零点是相同的。ζ(s)$(s\in C$ 且 $s\neq 1)$和黎曼ξ(s) $\left(s\in C\right.$ 且 $s\neq 1$ 且 $s\neq 0$ 且 $s\neq -2n, n\in Z^+\left.\right)$函数中的复数根满足 s=$\frac{1}{2}$+ti(t∈R 且 t≠0)和 s=$\frac{1}{2}$-ti(t∈R 且 t≠0)。通过黎曼定义的函数$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)(t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1)$ 和黎曼定义的 s=$\frac{1}{2}$+ti(t∈C 且 $t\neq 0$), 因为 s≠1, 且 $\prod\frac{s}{2}\neq 0$, $\pi^{-\frac{s}{2}}\neq 0$, 因此 $\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\neq 0$。当 ξ(t)= 0, 那么

$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0$, 且

$\zeta(\frac{1}{2}+ti)=\frac{\xi(t)}{\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}}=\frac{0}{\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}}=0$ ，所以 $t\in R$ 且 $t\neq 0$。 所以方程

$\prod\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{(\frac{1}{2}+ti)}{2}(-\frac{1}{2}+ti)\pi^{-\frac{\frac{1}{2}+ti}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0$ 与方程 $4\int_1^{\infty}\frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx}x^{-\frac{1}{4}}\cos(\frac{1}{2}t\ln x)$ dx=ξ(t)=0 $(t\in C$ 且 $t\neq 0)$和方程 $\xi(t)=\frac{1}{2}-(t^2+\frac{1}{4})\int_1^{\infty}\Psi(x)\,x^{-\frac{3}{4}}\cos(\frac{1}{2}t\ln x)=0(t\in C$ 且 $t\neq 0)$ 的根必定是实数且 $t\neq 0$。如果 Re(s)=$\frac{k}{2}(k\in R)$,

那么 $\zeta(k-s)=2^{k-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ 且 $s\neq 1, k\in R)$ 和

$\xi(k-s)=\frac{1}{2}$ s(s-k) $\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)\left(\left(s\in C\text{ 且 }s\neq 1\text{，且 }s\neq -2n\text{，n 遍取所有正整数}\right), k\in\right.$ R 都成立。因此当 ζ(s)=0(s∈C,且 s≠1,那么 ζs=ζk−s=ζs=0s∈C,且 s≠1,k∈R

和$\xi(s)=\xi(k-s)=\xi(\overline{s})=0\left(s\in C\text{ 且 }s\neq 1\text{，且 }s\neq -2n, n\in Z^{+}\text{，n 遍取所有正整数}, k\in R\right)$ 必定成立，且 $s=\frac{k}{2}+ti$(k∈R,t∈R and $t\neq 1$) 必定

成立，那么

$\prod\frac{s}{2}(s-k)\pi^{-\frac{s}{2}}\zeta(s)=\prod\frac{s}{2}(-\frac{k}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0$(k∈R,t∈R 且 $t\neq 0, k\in R$)，且

$\zeta(\frac{k}{2}+ti)=\frac{\xi(t)}{\prod\frac{s}{2}(-\frac{k}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=\frac{0}{\prod\frac{s}{2}(-\frac{k}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}}=0$(k∈R,t∈R 且 $t\neq 0, s\in C$ 且 $s\neq 1$),因此 t∈R 且 $t\neq$ 0. 所以方程

$\prod\frac{s}{2}(s-k)\pi^{-\frac{s}{2}}\zeta(\frac{k}{2}+ti)=\prod\frac{s}{2}(-\frac{k}{2}+ti)\pi^{-\frac{\frac{k}{2}+ti}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0$(k∈R,t∈R 且 $t\neq 0, s\in C$ 且 $s\neq 1$，且 $s\neq -2n, n\in Z^{+}$，n 遍取所有正整数)的根必定是实数且$t\neq 0$。碟但是黎曼 ζ(s)函数仅满足 $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s\in C$ 且 $s\neq 1$) 和 $\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$($s\in C$ 且$s\neq 1$，且 $s\neq -2n, n\in Z^{+}$，n 遍取所有正整数)，也就是说仅 $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s ∈ C and $s\neq 1$) 成立，因此仅 Re(s)=$\frac{k}{2}=\frac{1}{2}$ (k ∈ R) 成立，所以仅 k=1 成立。黎曼猜想必须满足黎曼ζ(s)（s∈C, s≠1）函数和黎曼ξ(s)（$s\in C$ 且$s\neq 1$ 且 $s\neq -2n, n\in Z^{+}$）函数的性质，黎曼 ζ(s)（s∈C 且 s≠1）函数和黎曼ξ(s)（$s\in C$ 且$s\neq 1$ 且 $s\neq -2n, n\in Z^{+}$）函数的性质是基本的，黎曼猜想必须是正确的，才能正确反映黎曼ζ(s)（s∈C 且 s≠1）函数和黎曼ξ(s)（$s\in C$ 且$s\neq 1$ 且 $s\neq -2n, n\in Z^{+}$，n 遍取所有正整数）函数的性质，即黎曼ξ(t)（t∈C 且 t≠0）函数的根只能是实数，即 Re(s)只能等于$\frac{1}{2}$，且 Im(s)必须是实数，并且 Im(s)不等于零。所以黎曼猜想必定是正确的。

黎曼在他的论文中得到

$\prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^{\infty}\psi(x)\ x^{\frac{s}{2}-1}dx+\int_1^{\infty}\psi(\frac{1}{x})\ x^{\frac{s-3}{2}}dx+\frac{1}{2}\int_0^1(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1})dx$

$=\frac{1}{s(s-1)}+\int_1^{\infty}\psi(x)\ (x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx$(s∈C and s≠ 1)，因为$\frac{1}{s(s-1)}$ 和

$\int_1^{\infty}\psi(x)(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}})dx$ 在变换 s→1-s 下都是不变的。如果我引入辅助函数ψ(s)=$\prod\left(\frac{s}{2}-\right.$

1π−s2 ζs(s∈C and s≠1)，那么我可以把它写成ψ(s)=ψ(1-s)。但是把因子 s(s-1)加到ψ(s)

中引入系数$\frac{1}{2}$会更方便，这正是黎曼所做的，就是要取$\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$($s\in C$ 且$s\neq$ 1，且$s\neq-2n,n\in Z^{+}$，n 遍取所有正整数)。因为因子（s-1）消掉了 s=1 时ζ(s)的第一个极点，因子 s 消掉了$\Gamma\left(\frac{s}{2}\right)$在 s=0 时的极点，且 s 等于-2，-4,-6，…，和$\Gamma\left(\frac{s}{2}\right)$的其余极点也抵消了，所以ξ(s)是一个整函数。而因子 s（s-1）在 s→1-s 变换下显然没有变化，所以我们仍有有函数方程ξ(s)=ξ(1-s)=0($s\in C$ 且$s\neq1$ 且$s\neq-2n,n\in Z^{+}$，n 遍取所有正整数)，同时根据$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s ∈ C and ≠ 1)（等式 6），如果 ζ(s)=0（s∈C 且 s≠ 1),那么必ζ(1-s)=0（s∈C 且 s≠ 1)成立，也就是说 ζ(s)=ζ(1-s)=0(s∈C 且 s≠ 1)。根据黎曼的假设 s=$\frac{1}{2}$+ti(t∈C且 t ≠ 0), S 和 t 差一个线性变换，它是 90 度旋转加上$\frac{1}{2}$的平移。所以 s 平面上的直线 Re(s)=$\frac{1}{2}$对应于 t 平面上的实数轴，黎曼 ζ(s)（s∈C,且 s≠1）函数和黎曼 ξ(s)（$s\in C$ 且$s\neq$ 1，且$s\neq-2n,n\in Z^{+}$）函数在临界线上的的零点的虚部对应于函数ξ(t)（t∈C 且 t≠0）的实根，所以$\xi(t)=0$中的t与$\zeta\left(\frac{1}{2}+ti\right)$=0 中的 t 是一致的。在黎曼的函数ξ(t)(t ∈ C 且 t ≠ 0)中，函数方程 ξ(s)=$\xi(1-s)$($s\in C$ 且$s\neq1$ 且$s\neq-2n,n\in Z^{+}$，n 遍取所有正整数)变为方程

$\xi(t)=$
$\xi(-t)$（t ∈ C 且 t ≠ 0）。ξ(t)是一个偶函数，故而其幂级数展开只有偶次幂，它的零点在$t=0$时关于零点(0,0)对称分布。$\xi(t)=0$中的t，黎曼设计的函数ξ(t)（t∈C, 且 t≠0）和黎曼定义的 s=$\frac{1}{2}$+t(t∈C 且 t≠0)和ξ(s)=ξ(1-s)($s\in C$ 且$s\neq1$ 且$s\neq-2n,n\in Z^{+}$，n 遍取所有正整数）中的 t 等价于ξ(t)=ξ(-t)（t∈C 且 t≠0）中的 t。所以 ξ(s)（$s\in C$ 且$s\neq1$ 且$s\neq-2n,n\in$ Z+，n 遍取所有正整数 是个偶函数，

ξ(s) $\left(s \in \mathrm{C}\ 且 s \neq 1\ 且\ \mathrm{s} \neq -2n, n \in \mathrm{Z}^{+}\right.$，n 遍取所有正整数$\left.\right)$在复平面上的零点对称地分布在与复平面的实数轴相垂直的直线上。当 ξ(t)=0 $\left(\mathrm{t} \in \mathrm{C}\ 且\ \mathrm{t} \neq 0\right)$，也就是ξ(t)=ξ(−t)=0 $\left(\mathrm{t} \in\right.$ C 且 t≠0 的零点是关于 t = 0 对称分布的,当 ξs=0s∈C 且s≠1 且 s≠$-2n$,n∈Z+,

也就是 ξ(s)=ξ(1 − s)=0 $\left(s \in \mathrm{C}\ 且 s \neq 1\right.$，且 $\mathrm{s} \neq -2n, n \in \mathrm{Z}^{+}$，n 遍取所有正整数$\left.\right)$, ξ(s) $\left(s \in\right.$ C 且s≠1 且 s≠$-2n$,n∈Z+，n 遍取所有正整数 的零点位于垂直于复平面实数轴的直线上，并关于点（$\frac{1}{2}$,0i）对称分布。所以当 ξ(s) = ξ(1 − s) = 0 $\left(s \in \mathrm{C}\ 且 s \neq 1\ 且\ \mathrm{s} \neq -2n, n \in\right.$ Z+，n 遍取所有正整数, 那么 s 和 1-s 是函数ξ(s)（s∈C 且s≠1 且 s≠$-2n$,n∈Z+，n 遍取所有正整数）与复平面的实数轴垂直的直线上的关于点（$\frac{1}{2}$,0i）对称分布的一对零点。当 ζ(s)=0(s∈C 且 s≠ 1), 那么 ζ(1-s)=0(s∈C 且 s≠ 1) ,也就是说 ζ(s)=ζ(1-s)=0$\left(\mathrm{s} \in \mathrm{C}\ 且\ \mathrm{s} \neq 1\right)$.

ζ(s)=ζ(1-s)=0 $\left(\mathrm{s} \in \mathrm{C}\ 且\ \mathrm{s} \neq 1\right)$ 和 ξ(s)=ξ(1 − s)=0 $\left(s \in \mathrm{C}\ 且 s \neq 1\ 且\ \mathrm{s} \neq -2n, n \in \mathrm{Z}^{+}\right.$，n 遍取所有正整数$\left.\right)$只是函数的名字不一样的，这意味着函数 ζ(s)(s ∈ C,s ≠ 1 ）和函数 ξ(s)($s \in \mathrm{C}$ 且$s \neq 1$ 且 $\mathrm{s} \neq -2n, n \in \mathrm{Z}^{+}$，n 遍取所有正整数）的零点的自变量 s 的值，除了负偶数与其余零点数的值都是完全相同的，并且关于点($\frac{1}{2}$,0i)对称分布，因此当 ξ(s)=ξ(1−s)= 0 $\left(s \in \mathrm{C}\ 且 s \neq 1\ 且\ \mathrm{s} \neq -2n, n \in \mathrm{Z}^{+}\right.$，n 遍取所有正整数$\left.\right)$，则 s 和 1 − s 是函数 ξ(s) $\left(s \in\right.$ C 且s≠1 且 s≠$-2n$,n∈Z+，n 遍取所有正整数 1 是与复平面内实数轴相垂直的直线上关于点$\left(\frac{1}{2}, 0\mathrm{i}\right)$对称的一对零

点。因为$\overline{\zeta(\mathrm{s})}$=ζ($\overline{\mathrm{s}}$)(s=σ+ti,s ∈ C 且 s ≠ 1),在黎曼定义的 s=$\frac{1}{2}$+ti(t∈C 且 t ≠ 0) 中,s 是一个复数，而 $\overline{\zeta(\mathrm{s})}$=ζ($\overline{\mathrm{s}}$)(s=σ+ti, σ ∈ R, t ∈R and t ≠ 0)中的s 与黎曼定义的s=$\frac{1}{2}$+ti(t ∈ C and t ≠ 0))中的s 是一致的。当ζ(s)=ζ($\overline{\mathrm{s}}$)=0(s=σ+ti, σ ∈R, t∈R, t≠0)，因为 s 和 $\overline{\mathrm{s}}$是一对共轭复数，所以 s 和 $\overline{\mathrm{s}}$必定是同一个复平面内经过点（σ,0i）与该复平面的实数轴相垂直的直线上关于点（σ,0i）共轭且对称的零点。同时 s 又是 1-s 的共轭且对称零点。黎曼 ζ(s)函数的同一个零点 s 在垂

直于复平面的实数轴的直线上的共轭且对称零点怎么会是既关于点($\frac{1}{2}$,0i)共轭且对称分布的一个零点 $1-s$，又是垂直于复平面实数轴的直线上关于点(σ,0i)共轭且对称分布的一个零点$\overline{s}$？除非σ和$\frac{1}{2}$是同一个值，即σ=$\frac{1}{2}$，而且仅有 1-s=$\overline{s}$成立，也就是 1-s 和 s 也共轭，否则绝无可能。这是由经过($\frac{1}{2}$,0i)垂直于复平面实数轴的直线上的黎曼 ζ(s)函数的零点关于该直线与复平面实数轴的垂足($\frac{1}{2}$,0i)对称分布的零点的唯一性决定的。在同一个复平面内，从零点 s 向复平面实数轴作垂线，仅能作出一条,垂足也只有一个。在同一个复平面，经过($\frac{1}{2}$,0i)垂直于复平面实数轴的直线上的 ζ(s)函数的同一个零点关于该直线与复平面实数轴的垂足($\frac{1}{2}$,0i)共轭且对称分布的零点只会有一个。 从方程的角度来考虑，因为$\overline{\zeta(s)}$=ζ($\overline{s}$) (s=σ+ti, σ∈R,t∈R 且 $t\neq0$)，那么当 ζ(σ+ti)=0,则 ζ(σ-ti)=0，又因为 ζ(s)=ζ(1−s)=0，那么 ζ(1-σ-ti)=0。因为下面三个方程 ζ(σ+ti)=0， ζ(σ-ti)=0，ζ(1-σ-ti)=0 都成立，那么只能 1-σ=σ，σ=$\frac{1}{2}$。调和级数 ζ(1)是发散的，已由中世纪晚期法国学者奥雷姆(1323-1382)证明。 因为级数 ζ($\frac{1}{2}$)的各项的分母比级数ζ(1)的各对应项的分母更小，所以 ζ($\frac{1}{2}$)更是发散的。所以 s=$\frac{1}{2}$不是级数 ζ(s)

的零点，舍去。当 ζ(s)=0，求解 1-s=s，也得到仅有 s=$\frac{1}{2}$+ti(t∈R 且 t≠0)和 s=$\frac{1}{2}$-ti(t∈R 且 t≠0)成立。所以当α=$\frac{1}{2}$并且 ζ(s)=0(s∈C 且 s≠1)。黎曼假设的 s= $\frac{1}{2}$+ti(t∈C 且 $t\neq0$) 必须等价于 s=$\frac{1}{2}$+ti(t∈R 且 t≠0)。黎曼的定义 s= $\frac{1}{2}$+ti(t∈C 且 $t\neq0$)和黎曼猜想必须满足黎曼ζ(s)=0 (s∈C, s≠1) 函数和黎曼ξ(s)=0 ($s\in$ C 且$s\neq1$ 且 $s\neq-2n, n\in Z^+$，n 遍取所有正整数) 函数的性质，黎曼ζ(s)(s∈C, s≠1)函数和黎曼ξ(s)($s\in$ C 且$s\neq1$,且 $s\neq-2n, n\in Z^+$，n 遍取所有正整数) 函数的性质是基本的，黎曼猜想必须是正确的，才能反映黎曼ζ(s) (s∈C 且 s≠1) 函数和黎曼ξ(s)($s\in$ C 且$s\neq1$ 且 $s\neq-2n, n\in Z^+$，n 遍取所有正整数)函数的性质 ,即黎曼ξ(t)(t∈C 且 t≠0) 函数的根只能是实数，即：Re(s)只能等于$\frac{1}{2}$，而 Im(s)必须是实数，而且 Im(s)不等于 0。所以黎曼猜想必定是正确的。黎曼在他的论文中得到了$\prod\frac{s}{2}$(s-1)$\pi^{-\frac{s}{2}}$ζ(s)=ξ(t) ($s\in$ C 且$s\neq1$，且 $s\neq-2n, n\in Z^+$，n 遍取所有正整数t∈R 且 $t\neq0$,)和

ξ(t)=$\frac{1}{2}$ -$(t^2+\frac{1}{4})\int_1^{\infty}\Psi(x)\,x^{-\frac{3}{4}}\cos\left(\frac{1}{2}\mathrm{tln}x\right)\mathrm{d}x$(t∈R 且 $t\neq 0$) ，或

$\prod\frac{s}{2}$ (s-1) $\pi^{-\frac{s}{2}}$ ζ(s)= $\prod\frac{(\frac{1}{2}+\mathrm{ti})}{2}\left(-\frac{1}{2}+\mathrm{ti}\right)\pi^{-\frac{\frac{1}{2}+\mathrm{ti}}{2}}\zeta(\frac{1}{2}+\mathrm{ti})$ =ξ(t)($t\in R$ and $t\neq 0, s\in C$ 且$s\neq 1$，且 $s\neq -2n, n\in Z^{+}$) 和 ξ(t)=4 $\int_1^{\infty}\frac{\mathrm{d}(x^{\frac{3}{2}}\Psi'(x))}{\mathrm{d}x}x^{-\frac{1}{4}}\cos(\frac{1}{2}\mathrm{tln}x)\ \mathrm{d}x$($t\in R$ 且 $t\neq 0$) . 因 为 ζ($\frac{1}{2}$+ti)=0(t∈R 且 $t\neq 0$)，所以方程

$\prod\frac{s}{2}$(s-1) $\pi^{-\frac{s}{2}}$ ζ(s) = $\prod\frac{(\frac{1}{2}+\mathrm{ti})}{2}\left(-\frac{1}{2}+\mathrm{ti}\right)\pi^{-\frac{\frac{1}{2}+\mathrm{ti}}{2}}\zeta(\frac{1}{2}+\mathrm{ti})$ =ξ(t)= 0 ($t\in R$ and $t\neq 0, s\in C$ 且$s\neq 1$ 且 $s\neq -2n, n\in Z^{+}$) 和 $4\int_1^{\infty}\frac{\mathrm{d}(x^{\frac{3}{2}}\Psi'(x))}{\mathrm{d}x}x^{-\frac{1}{4}}\cos(\frac{1}{2}\mathrm{tln}x)\mathrm{d}x$=ξ(t)= 0(t∈R and $t\neq 0$) 和 ξ(t)=$\frac{1}{2}$ -$(t^2+\frac{1}{4})\int_1^{\infty}\Psi(x)\ x^{-\frac{3}{4}}\cos(\frac{1}{2}\mathrm{tln}x)\ \mathrm{d}x$=0(t∈R and $t\neq 0$) 的根必定实数。当ζ(s)= 0($s\in C$ 且 $s\neq 1$) 且ξ(t)=0($t\in C$ and $t\neq 0$)，如果方程 ξ(t)=0(t∈C,and $t\neq 0$)的根的实数部分在 0和 T 之间，那么方程ξ(t)=0 的实部在 0 到 T 之间的复数根的个数近似等于$\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$ [1]，黎曼对零数估计的这个结果在 1895 年被曼戈尔特严格地证明了。那么，当ζ(s)=0（s∈C 且 s≠1）且ξ(t)=0（t∈C 且 t≠0）时，方程ξ(t)=0（t∈C 且 t≠0）的实部在 0 和 T 之间的实根个数年必须近似等于$\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi}$ [1],因此当黎曼 ζ(s)（s∈C 且 s≠1）函数有非平凡零点时，黎曼猜想必定是完全正确的。N=$\lim_{T\to+\infty}(\frac{T}{2\pi}\ln\frac{T}{2\pi}-\frac{T}{2\pi})\to\infty$,所以黎曼ζ(s)（s∈C 且 s≠1）函数在 Re(s)= $\frac{1}{2}$的临界直线上的非平凡零点就有无穷多个，1921 年，英国数学家哈代已经证明黎曼ζ(s)（s∈C 且 s≠1）函数在 Re(s)= $\frac{1}{2}$的临界直线上的非平凡零点就有无穷多个，但他没有证明黎曼 ζ(s)（s∈C 且 s≠1）函数的非平凡零点全部都位 Re(s)= $\frac{1}{2}$的临界直线上。

根据黎曼论文中得到的 $2\sin(\pi s)\prod(s-1)\zeta(s)=\int_{\infty}^{\infty}\frac{x^{s-1}\mathrm{d}x}{e^{x}-1}$函数，我们知道黎曼 ζ(s)（s∈C, 且 s ≠1）函数是由二对一映射，甚至是由多对一映射的确定的泛函数，或者是一对二映射如果我们把黎曼ζ(s)s∈C 且 s≠1）函数看作是定义域包含实数的一般复数，那么 s=-2n（n 是正整数的是黎曼ζ(s)（s∈C 且 s≠1）函数的唯一的一类实数零点根，也是朗道-西格尔函数 L(β,1)(β∈R）唯一的一类实数零点根。如果我们把黎曼ζ(s)（s∈C 且 s≠1）函数看作是定义域不包含实数的一般复数，那么 s=$\frac{1}{2}$+ti（t∈R 且 t≠0）和 s=$\frac{1}{2}$-ti（t∈R 且 t≠0）是黎曼ζ(s)（s∈C, s≠1）

函数唯一的一类复数零点根 ,除了负偶数， 朗道-西格尔函数 L(β, X(n))=0(β∈R 且$\beta\neq -2n$, n

是正整数, X(n)= 1) 的实数根不存在。

定义:

假设 a(n)是一个单位积函数，则狄利克雷级数

$\sum_{n=1}^{\infty} a(n)n^{-s}$ $\left(s \in C\text{ 且 } s \neq 1, n \in Z^{+}\text{ 且}n\text{遍取得所有的正整数}\right)$等于欧拉积$\prod_{p} P(p，s)$ $\left(s \in\right.$ C 且 s≠1,p∈Z+ 且 p 遍取得所有的质数。将乘积作用于所有素数 p，可表示为：

1+$a(p)p^{-s}$+$a(p^2)p^{-2s}$+...，这可以看作是一个形式生成函数，其中形式欧拉积展开的存在和 a(n)为积函数是相互充要条件。当 a(n)是完全积分函数时，得到一个重要的特例: 其中 $P(p，s)$ $\left(s \in C\text{ 且 } s \neq 1, p \in Z^{+}\text{ 且 } p\text{ 遍取得所有的质数}\right)$是一个几何级数，且$P(p,s)$=$\frac{1}{1-a(p)p^{-s}}$ $\left(s \in C\text{ 且 } s \neq 1, p \in Z^{+}\text{且 } p\text{ 遍取得所有的质数}\right)$。且 $P(p,s)$=$\frac{1}{1-a(p)p^{-s}}$ $\left(s \in C\text{ 且 } s \neq 1, p \in\right.$ Z+且 p 遍取得所有的质数。当 a(n)=1(n∈Z+且 n 遍取得所有的正整数)时，它是黎曼ζ(s)(s∈C 且 s≠1）函数，更一般地说是狄利克雷特征。

欧拉积公式：对于任意复数 s，当

$Rs(s) > 1$ 且 $s \neq 1$,那么 $\sum_{n=1}^{\infty} n^{-s} =$

$\prod_{p}(1-p^{-s})^{-1}$ $\left(s \in C\text{ 且 } s \neq 1, p \in Z^{+}\text{ 且 } p\text{ 遍取得所有的质数}, n \in\right.$

Z+ 且n遍取得所有的正整数，且当 Rss>1 黎曼 ζ(s)=n=1∞n−s=p(1−p−s)−1(s∈C 且

$Rs(s) > 0$ ， $s \neq 1$,n∈ $Z^{+}, p \in Z^{+}$, s ∈C，且n遍取得所有的正整数，p 遍取得所有的质数)。

黎曼ζ函数表达式：ζ(s)=1/1^s+1/2^s+1/3^s+...+1/m^s （m 趋于无穷，且 m 总是偶数）(1 式)，

(1 式)表达式两边同时乘以（1/2^s ）那么，(1/2^s)ζ(s)=1/1^s(1/2^s)+1/2^s(1/2^s) +1/3^s(1/2^s) +...+1/m^s(1/2^s)=1/2^s+1/4^s+1/6^s+...+1/$(2m)^s$ (2 式)，

(1 式) - (2 式)得:

ζ(s)-(1/2^s)ζ(s)=1/1^s+1/2^s+1/3^s+...+1/m^s-[1/2^s+1/4^s+1/6^s+...+1/$(2m)^s$]，

欧拉积公式的推导如下：

$\zeta(s)-(1/2^s)\zeta(s)=1/1^s+1/3^s+1/5^s+...+1/(m-1)^s$。广义欧拉积公式：

假设 f(n)是一个满足 $f(n_1)f(n_2)=f(n_1n_2)$ 且 $\sum_n |f(n)|<+\infty$ (n_1 和 n_2 都是正整数)，那么

$\sum_n f(n)=\prod_p[1+f(p)+f(p^2)+f(p^3)+\ldots]$。

欧拉积公式的证明很简单，唯一要注意的是处理无穷级数和无穷乘积时，不能随意使用有限级数和有限乘积的性质。下面我证明的是一个更一般的结果，欧拉积公式将作为这个结果的一个特例出现。

由于$\sum_{n=1}^{\infty}|f(n)|<+\infty$，so $1+f(p)+f(p^2)+f(p^3)+\ldots$ 绝对收敛。考虑连积（有限积）中 p<N 的部分，由于级数是绝对收敛的，乘积只有有限项，所以同样的结合律和分配律可以作为普通的有限求和和有限乘积。利用 f(n)的乘积性质，可得：

$\prod_{p<N}[1+f(p)+f(p^2)+f(p^3)+\ldots]=\sum f(n)$，对所有质因数小于 N 的自然数执行求和的右端（每个这样的自然数在求和中只出现一次，因为自然数的质因数分解是唯一的）。因为所有小于 N 的自然数显然只包含小于 N 的质数因子，因此 $\Sigma' f(n) = \sum_{n<N} f(n) + R(N)$,式中 R(N)是所有大于或等于 N 但只包含小于 N 的质数的自然数的和，由此得到：$\prod_{p<N}[1+f(p)+f(p^2)+f(p3)+\ldots]=n<N\quad f(n)+R(N)$。要使广义欧拉积公式成立，只需要证明 $\lim n\to\infty R(N)=0$，这很明显，因为$|R(N)| \le \sum_{n\ge N}|f(n)|$，且$\sum_n |f(n)|<+\infty$ 导出

$\lim_{n\to\infty}\sum_{n\ge N}|F(n)|=0$,因此 $\lim_{n\to\infty}R(N)=0$.因为 $1+f(p)+f(p^2)+f(p^3)+\ldots=1+f(p)+f(p)^2+f(p)^3+\ldots=[1-f(p)]^{-1}$，所以广义欧拉积公式也可以写成：$\sum_n f(n)=\prod_p[1-f(p)]-1$。 在广义欧拉积公式中，取 $f(n)=n-s$，则显然 $n\quad fn<+\infty$，对应欧拉积公式中的条件 Rs(s)>1，将广义欧拉积公式化简为欧拉积公式。由以上证明可知，欧拉积公式的关键在于每一个自然数都有一个唯一的质因数分解的基本性质，即所谓算术基本定理。

对于任何复数 s，✗(n)是 Dirichlet 特征并且满足以下性质：

1: 存在一个正整数 q，使得✗(n+q)= ✗(n);

2: 当 n 和 q 不互素数时，✗(n)=0;

3: 对任意整数 a 和 b 来说，✗(a).✗(b)=✗(ab);

推论 3:

接下来我们证明广义黎曼猜想，当 Dirichlet 特征函数(n)是任何不等于零的实数时，

如果 $Re(s) > 1$,那么

L(s,✗(n))=$\sum_{n=1}^{\infty}\frac{✗(n)}{n^s}$(n∈ Z_+, p ∈ Z_+, s ∈C 且 Re(s)>1，n 遍取所有正整数, ✗(n)∈R 且 ✗(n) ≠ 0,a(n) = a(p)=✗(n)),P(p，s)=$\frac{1}{1-a(p)p^{-s}}$)。

GRH(s,✗(n))=L(s,✗(n))=$\sum_{n=1}^{\infty}\frac{✗(n)}{n^s} = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p P(p,s) = \prod_p(\frac{1}{1-a(p)p^{-s}})$ (n∈ Z^+, p ∈ Z^+, s∈C 且 Re(s)>1，n 遍取所有正整数, p ∈ Z^+，p 遍遍取所有质数，✗(n)∈R 且(✗(n) ≠ 0),a(n) = a(p)=✗(n)),P(p，s)=$\frac{1}{1-a(p)p^{-s}}$).

$a(p)p^{-s}$=$a(p)p^{-\sigma}\frac{1}{(\cos(t\ln p)+i\sin(t\ln p))}$ =$a(p)(p^{-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$($s \in C$ 且 $Re(s) > 1, t \in R$ 且 $t \neq 0$),

$(1 - a(p)p^{-s})$ =1- $a(p)(p^{-\sigma}(\cos(t\ln p) - i\sin(t\ln p))$ = $1 - a(p)p^{-\sigma}\cos(t\ln p) + ia(p)p^{-\sigma}\sin(t\ln p)$($s \in C$ ，$Re(s) > 1, t \in R$ 且 $t \neq 0$),

$a(p)p^{-\bar{s}}$=$a(p)p^{-\sigma}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$ =$a(p)(p^{-\sigma}(\cos(t\ln p) + i\sin(t\ln p))$($s \in C$ ，$Re(s) > 1, t \in R$ 且 $t \neq 0$),

$(1 - a(p)p^{-\bar{s}})$=1-$a(p)p^{-\sigma}\cos(t\ln p) - ia(p)p^{-\sigma}\sin(t\ln p)$($s \in C$ ，$Re(s) > 1, t \in R$ 且 $t \neq 0$) ,因为

$(1- a(p)p^{-s})$=$\overline{1 - a(p)p^{-\bar{s}}}$ ($s \in C$ ，$Re(s) > 1, p \in Z^+$ 且 p 遍遍取所有质数),

因此

$(1 - a(p)p^{-s})^{-1}$ = $\overline{(1 - a(p)p^{-\bar{s}})^{-1}}$ ($s \in C$ ，$Re(s) > 1, p \in Z^+$ 且 p 遍遍取所有质数 ，✗(n) ∈ R 且✗(n) ≠ 0),

因此

$\prod_p(1-a(p)p^{-s})^{-1}$=$\overline{\prod_p(1-a(p)p^{-\bar{s}})^{-1}}$

($s\in C$ 且 $Re(s)>1, p\in Z^{+}$且 p 遍遍取所有质数）.

becuse L(s, ✗(n))=$\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_p(1-a(p)p^{-s})^{-1}$和

L($\bar{s}$, ✗(n))=$\sum_{n=1}^{\infty}a(n)n^{-\bar{s}}=\prod_p\left(1-a(p)p^{-\bar{s}}\right)^{-1}$

($s\in C$，$Re(s)>1$, 且 $s\neq-2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，p ∈

Z^{+}，且 p 遍取所有质数)。

对于广义黎曼 L(s, ✗(n))=$\sum_{n=1}^{\infty}\frac{✗(n)}{n^{s}}=\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_p\frac{1}{1-a(p)p^{-s}}$

($s\in C$ 且 $s\neq1$ 且 $s\neq-2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，p ∈

Z^{+}，且 p 遍取所有质数,a(n) = a(p)= ✗(n)),

P(p，s) = $\frac{1}{1-a(p)p^{-s}}$($s\in C$，$Re(s)>1$ 且 $s\neq-2n, n\in Z^{+}$，n 遍取所有正整数, ✗(n) ∈ R 且✗n≠0，p∈Z+，且 p 遍取所有质数,

因此 $L\left(s, ✗(n)\right)$=$\overline{L\left(\bar{s}, ✗(n)\right)}$($s\in C$ 且 $s\neq1$ 且 $s\neq-2n, n\in Z^{+}$，n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，$p\in Z^{+}$，且 p 遍取所有质数).

$a(p)p^{1-s}=a(p)p^{(1-\sigma-ti)}=a(p)p^{1-\sigma}x^{-ti}=a(p)p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}=a(p)p^{1-\sigma}(\cos(t\ln p)-$ isintlnp s∈C，Re(s)>1 且 s≠−2n,n∈Z+，n 遍取所有正整数, ✗n∈ R 且✗n≠0，p∈Z+，且 p 遍取所有质数,

$a(p)p^{1-\bar{s}}=a(p)p^{(1-\sigma+ti)}=a(p)p^{1-\sigma}p^{ti}=a(p)p^{1-\sigma}(p^{ti})=a(p)p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t}=a(p)p^{1-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$ ($s\in C$，$Re(s)>1$ 且 $s\neq$ −2n,n∈Z+，n 遍取所有正整数, ✗n∈ R 且✗n≠0，p∈Z+，且 p 遍取所有质数,

那么 $a(p)p^{-(1-s)}=a(p)p^{\sigma-1}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}=a(p)(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$ ($s\in$ C，Re(s)>1 且 s≠−2n,n∈Z+，n 遍取所有正整数, ✗n∈ R 且✗n≠0，p∈Z+，且 p 遍取所有质数,

$(1-a(p)p^{-(1-s)})$ =1- $a(p)p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$ = $1-a(p)p^{\sigma-1}\cos(t\ln p)-a(p)p^{\sigma-1}i\sin(t\ln p)$ ，

$(s\in C$，$Re(s)>1$ 且 $s\neq-2n$, n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$，$p\in Z^{+}$，且 p 遍取所有质数),

$\left(1-a(p)p^{-\bar{s}}\right)$=1-a(p)$(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ =$1-a(p)p^{-\sigma}\cos(t\ln p)-ia(p)p^{-\sigma}\sin(t\ln p)$,

当 $\sigma=\frac{1}{2}$, 那么

$(1-a(p)p^{-(1-s)})$ =$\left(1-a(p)p^{-\bar{s}}\right)\left(s\in C \text{ 且 } Re(s)>1\right)$,

$(1-a(p)p^{-(1-s)})^{-1}$=$\left(1-a(p)p^{-\bar{s}}\right)^{-1}(s\in C$ 且 $Re(s)>1)$,

因此 $\prod_p(1-a(p)p^{-(1-s)})^{-1}$=$\prod_p\left(1-a(p)p^{-\bar{s}}\right)^{-1}(s\in C$ 且 $s\neq 1)$,

因为 $L\left(1-s,\chi(n)\right)$=$\prod_p\left(1-a(p)p^{-(1-s)}\right)^{-1}$ and $L\left(\bar{s},\chi(n)\right)$=$\prod_p\left(1-a(p)p^{-\bar{s}}\right)^{-1}$,

$s\in C$，$Re(s)>1$ 且 $s\neq-2n, n\in Z^{+}$，n 遍取所有正整数，$p\in Z^{+}$，且 p 遍取所有质数，χ(n)∈R 且 $\chi(n)\neq 0, a(n)=a(p)$=$\chi(n)$,P(p，s)=$\frac{1}{1-a(p)p^{-s}}$)。因此仅

$L(1-s,\chi(n))$ = $L(\bar{s},\chi(n))(s\in C$ 且 $s\neq 1$ 且 $s\neq-2n, n\in Z^{+}$，n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0)$,且

仅 $L(1-\bar{s},\chi(n))$ = $L(s,\chi(n))(s\in C$ ，$Re(s)>1$, 且 $s\neq-2n$, n 遍取所有正整数，$\chi(n)\in R$ 且 $\chi(n)\neq 0)$成立,

因此 $L(s,\chi(n))$ = $\chi(n)$ ζ(s) $(s\in C$ ，$Re(s)>1$ 且 $s\neq-2n$, n 遍取所有正整数，$\chi(n)\in R$ 且 $\chi(n)\neq 0)$,

$L(1-s,\chi(n))$ = $\chi(n)$ ζ(1-s) $(s\in C$，$Re(s)>1$ 且 $s\neq-2n$, n 遍取所有正整数，$\chi(n)\in R$ 且 $\chi(n)\neq 0)$,

且 $L(s,\chi(n))$ = $\overline{L\left(\bar{s},\chi(n)\right)}(s\in$ ，$C\ Re(s)>1$, 且 $s\neq-2n$, n 遍取所有正整数，$\chi(n)\in R$ 且 $\chi(n)\neq 0)$,

且

$L(1-s,\chi(n)) = L\left(\bar{s},\chi(n)\right)\left(s\in C\right.$ ，$Re(s)>1$ 且 $s\neq -2n, n\in Z^{+}$， n 遍取所有正整数，$\chi(n)\in$ R 且 $\chi(n)\neq 0$ 必定成立,

Suppose k∈ R,

$a(p)p^{k-s}=a(p)p^{(k-\sigma-ti)}=a(p)p^{k-\sigma}x^{-ti}=a(p)p^{k-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}=a(p)p^{k-\sigma}(\cos(t\ln p)-i\sin t\ln p$ (s∈C 且 s≠1,t∈C 且 t≠0,k∈R),

$a(p)p^{k-\bar{s}}=a(p)p^{(k-\sigma+ti)}=a(p)p^{k-\sigma}p^{ti}=a(p)p^{k-\sigma}(p^{ti}) = a(p)p^{k-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t}=$

$a(p)(p^{k-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ ($s\in C$ ，$Re(s)>1$, $p\in Z^{+}$ 且 p 是质数,且 $s\neq -2n, n\in Z^{+}$， n 遍取所有正整数，, $k\in R$),

那么

$a(p)p^{-(k-s)}=a(p)p^{\sigma-k}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}=a(p)$

$(p^{\sigma-k}(\cos(t\ln p)+i\sin(t\ln p))(s\in C$ ，$Re(s)>1, t\in R$ 且 $t\neq 0, k\in R)$,

$(1-a(p)p^{-(k-s)})$ =1- $(a(p)p^{\sigma-k}(\cos(t\ln p)+i\sin(t\ln p))$ = $1-a(p)p^{\sigma-k}\cos(t\ln p)-ip^{\sigma-k}\sin(t\ln p)(s\in C$， $Re(s)>1$,且 $s\neq -2n$, n 遍取所有正整数, $p\in$ Z+ 且 p 是质数,k∈R) ,

$(1-a(p)p^{-\bar{s}})$ =1- $(a(p)p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ = $1-a(p)p^{-\sigma}\cos(t\ln p)-ia(p)p^{-\sigma}\sin(t\ln p)(s\in C$ 且 $s\neq 1$ 且 $s\neq -2n$, n 遍取所有正整数，$p\in$ Z+ 且 p 是质数,) ,

当 $\sigma=\frac{k}{2}$(k∈ R) ,那么

$\left(1-a(p)p^{-(k-s)}\right)=\left(1-a(p)p^{-\bar{s}}\right)$

($s\in C$ ，$Re(s)>1$ ，且 $s\neq -2n$, $n\in Z^{+}$，n 遍取所有正整数 $p\in Z^{+}$ 且 p 是质数, $k\in R$),

$\left(1-a(p)p^{-(k-s)}\right)^{-1}=\left(1-a(p)p^{-\bar{s}}\right)^{-1}$

($s\in C$， $Re(s)>1$, 且 $s\neq -2n$, $n\in Z^{+}$，n 遍取所有正整数 $p\in Z^{+}$ 且 p 是质数，$k\in R$),

因此

$\prod_p(1-a(p)p^{-(k-s)})^{-1}$=$\prod_p\left(1-a(p)p^{-\bar{s}}\right)^{-1}$

$(s\in C$ ，$Re(s)>1$,且 $s\neq -2n$, $n\in Z^+$ ，n 遍取所有正整数 $p\in Z^+$ 且 p 是质数，$k\in R)$,

becuse $L\left(k-s, ✗(n)\right)$=$\prod_p\left(1-a(p)p^{-(k-s)}\right)^{-1}$

$(s\in C$， $Re(s)>1$,且 $s\neq -2n, n\in Z^+$， n 遍取所有正整数 $p\in Z^+$ 且 p 是质数，$k\in R)$,

$L(\bar{s}, ✗(n))$=$\prod_p\left(1-a(p)p^{-\bar{s}}\right)$ $(s\in$ C ，$Re(s)>1$ 且 $s\neq -2n$, n 遍取所有正整数, $p\in$

Z+ 且 p 遍取所有质数 ,✗(n)∈R 且 ✗(n)≠0, a(n)=a(p)=✗(n)),P(p ,s)= 1 1−a(p)p−s),

且

$L\left(s, ✗(n)\right)=$

$\prod_p\left(1-a(p)p^{-\bar{s}}\right)$ $(s\in$ C $Re(s)>1$ 且 $s\neq -2n$,

n∈Z+ ,n 遍取所有正整数,p∈Z+ 且 p 遍取所有质数 ,✗(n)∈R 且 ✗(n)≠0, a(n)=a(p)=

✗(n)),P(p，s)= 1 1−a(p)p−s)

因此

仅 $L\left(k-s, ✗(n)\right)$=$L\left(\bar{s}, ✗(n)\right)$

(s∈C 且 Re(s)>1, $n\in Z^+$,且n 遍取所有正整数, $✗(n)\in R$ 且 $✗(n)\neq 0$，$k\in R)$,

且 $L\left(k-\bar{s}, ✗(n)\right)$=$L\left(s, ✗(n)\right)$,

(s∈C 且 Re(s)>1, $s\neq -2n, n\in Z^+$，且n 遍取所有正整数, $✗(n)\in R$ 且 $✗(n)\neq 0$，$k\in R)$,

因为$L\left(1-s, ✗(n)\right)$= $L\left(\bar{s}, ✗(n)\right)$

(s∈C 且 $s\neq 1, s\neq -2n$，$n\in Z^+$，且n 遍取所有正整数, $✗(n)\in R$ 且 $✗(n)\neq 0$，$k\in R)$,

,因此仅 k=1 成立.

因为当 Re(s)>1 欧拉 ζ 函数等价于黎曼 ζ 函数，根据欧拉积公式，当 Re(s)>时，由于欧拉乘积公式中的每个乘积因子不等于零，所以当 Re(s)>1 时，由于欧拉积公式中的每个乘积因子乘积公式不等于零，所以当 Re(s)>1, ζ(s)不等于零，所以尽管正偶数 2n($n\in Z^+$)) 可

以使 $\sin(\frac{\pi s}{2})$ =0=0，但它不是黎曼ζ(s)的零。因为$L(s, ✗(n))=✗(n)\zeta(s)$(s∈C, Re(s)>1,$n \in Z^{+}$,且 n 遍取所有正整数，，$✗(n) \in R$,且 $✗(n) \neq 0$), 因为当 Re(s)>1 时，ζ(s)没有零点，所以当 Re(s)>1，那么$L(s, ✗(n))=✗(n)\zeta(s)\neq 0$(且 n 遍取所有正整数，，$✗(n) \in R$,且 $✗(n) \neq$), 所以当 Re(s)>1,L(s, ✗(n)) (s∈C, Re(s)>1,$n \in Z^{+}$,且 n 遍取所有正整数, $✗(n) \in R$,且 $✗(n) \neq 0$)没有零点。

如果 $Re(s) > 0$ 且 $s \neq 1$,

当狄利克雷特征函数✗(n)值是任意不等于零的实数，且

$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$ ($s \in C$, $Rs(s) > 0$ 且 $s \neq 1$), 黎曼 ζ 函数是 $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$ ($s \in C$，$Rs(s) > 0$ 且 $s \neq 1$，

$n\in Z^{+}$ 且 n 遍取所有正整数,$p \in Z^{+}$且 p 遍取所有素数)。

$$GRH\left(s, ✗(n)\right) = L\left(s, ✗(n)\right) = \frac{✗(n)\eta(s)}{(1-2^{1-s})} = \frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

$$= \frac{✗(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma+ti}} = \frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)\left(\frac{1}{n^{\sigma}}\frac{1}{n^{ti}}\right) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma})\frac{1}{(\cos\left(\ln((n)\right)+i\sin(\ln(n)))^{t}}$$

$$= \frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})$$

$$= \frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}✗(n)n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n))\ )$$

(s∈C，$Rs(s) > 0$ 且 $s\neq 1$, $s \neq -2n$, n 遍取所有正整数, $✗(n) \in R$ 且$✗(n) \neq 0$，$k \in R$),

$$GRH(\overline{s}, ✗(n)) = L(\overline{s}, ✗(n)) = \frac{✗(n)\eta(\overline{s})}{(1-2^{1-\overline{s}})} = \frac{✗(n)}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}} = \frac{✗(n)}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma-ti}}$$

$$= \frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty} ✗(n)(\frac{1}{n^{\sigma}}\frac{1}{n^{-ti}})$$

$$= \frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}(✗(n)\frac{1}{n^{\sigma}}\frac{1}{(\cos(\ln(n)) + i\sin(\ln(n)))^{-t}})$$

$$= \frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}(✗(n)n^{-\sigma}(\cos(\ln(n)) + i\sin(\ln(n)))^{t}) =$$

$$\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}(✗(n)n^{-\sigma}(\cos(t\ln(n)) + i\sin(t\ln(n)))$$

($s\in C$，$Rs(s) > 0$ 且 $s\neq 1, s \neq -2n, n$ 遍取所有正整数, $✗(n) \in R$ 且 $✗(n) \neq 0$，$k \in R$),

$$GRH\left(1-s, ✗(n)\right) = L\left(1-s, ✗(n)\right) = \frac{✗(n)\eta(1-s)}{(1-2^{s})} = \frac{✗(n)}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-\sigma-ti}}$$

$$= \frac{(-1)^{n-1}}{(1-2^{s})}\sum_{n=1}^{\infty} ✗(n)\left(\frac{1}{n^{1-\sigma}}\frac{1}{n^{-ti}}\right)$$

$$= \frac{(-1)^{n-1}}{(1-2^{s})}\sum_{n=1}^{\infty}(✗(n)n^{\sigma-1}(\cos(t\ln(n)) + i\sin(t\ln(n))),$$

($s\in C$ ，$Rs(s) > 0$且 $s\neq 1, s \neq -2n, n$ 遍取所有正整数, $✗(n) \in R$ 且 $✗(n) \neq 0$，$k \in R$),

因为:

$\frac{(-1)^{n-1}}{(1-2^{1-s})} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}}$,

$\prod_p(1-p^{-s})^{-1} = \overline{\prod_p(1-p^{-\overline{s}})^{-1}}$,

($s \in C$，$Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数),

因此

$\frac{(-1)^{n-1}}{(1-2^{1-s})} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}}$,

因此

$\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\overline{\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}}$,

$\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}=\overline{\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p(1-p^{-\overline{s}})^{-1}}$,

$\zeta(s)=\frac{1}{(1-2^{1-s})}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\prod_p(1-p^{-s})^{-1}$,

$\zeta(\overline{s})=\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}=\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_p\left(1-p^{-\overline{s}}\right)^{-1}$ $(s\in$ ，$Rs(s)>0$ 且 $s\neq 1, n\in$

Z^+ 且 n 遍取所有的正整数, $p\in Z^+$ p 遍取所有的质数),

因此

仅 $\zeta(s)=\overline{\zeta(\overline{s})}$ ($s\in C$，$Rs(s)>0$ 且 $s\neq 1$), [2]

因此

$p^{1-s}=p^{(1-\sigma-ti)}=p^{1-\sigma}p^{-ti}=p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}=p^{1-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$,
$p^{1-\overline{s}}=p^{(1-\sigma+ti)}=p^{1-\sigma}p^{ti}=p^{1-\sigma}(p^{ti})=p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t}=(p^{1-\sigma}(\cos(t\ln p)+$

$i\sin(t\ln p))$ (s∈C，Rss>0 且 s≠1, t∈C ,p∈Z+),

因此

$p^{-(1-s)}=p^{(-1+\sigma+ti)}=p^{\sigma-1}p^{ti}=p^{\sigma-1}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}=(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))$,

$p^{-(\overline{s})}=p^{-(\sigma-ti)}=p^{-\sigma}p^{ti}=(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$

($s\in C$，$Rs(s)>0$ 且 $s\neq 1, t\in C$ 且 $s\neq 0, p\in Z^+$),

因此

$(1-p^{-(1-s)})$=1-$(p^{\sigma-1}(\cos(t\ln p)+i\sin(t\ln p))=1-p^{\sigma-1}\cos(t\ln p)-ip^{\sigma-1}\sin(t\ln p)$,
$(1-p^{-(\overline{s})})$=1-$(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))=1-p^{-\sigma}\cos(t\ln p)-ip^{-\sigma}\sin(t\ln p)$,

($s\in C$，$Rs(s)>0$ 且 $s\neq 1, t\in C$ 且 $t\neq 0, p\in Z^+$),

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$=[$1^{\sigma-1}\cos(t\ln 1)-2^{\sigma-1}\cos(t\ln 2)+3^{\sigma-1}\cos(t\ln 3)-4^{\sigma-1}\cos(t\ln 4)$-...]+i[$1^{\sigma-1}\sin(t\ln 1)$ $-2^{\sigma-1}\sin(t\ln 2)+3^{\sigma-1}\sin(t\ln 3)-4^{\sigma-1}\sin(t\ln 4)$+...] ,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$=[$1^{-\sigma}\cos(t\ln 1)-2^{-\sigma}\cos(t\ln 2)+3^{-\sigma}\cos(t\ln 3)-4^{-\sigma}\cos(t\ln 4)$-...]+i[$1^{-\sigma}\sin(t\ln 1)-2^{-\sigma}$ $\sin(t\ln 2)+3^{-\sigma}\sin(t\ln 3)-4^{-\sigma}\sin(t\ln 4)$+...]

$(s \in C\ ，Rs(s) > 0$ 且 $s \neq 1, t \in C$ 且 $s \neq 0, n \in Z^{+}$ 且 n 遍取所有的正整数),

当 $\sigma=\frac{1}{2}$,

那么

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ $(s \in C\ ，Rs(s) > 0$ 且 $s \neq 1, n \in Z^{+}$ 且 n 遍取所有的正整数),

$(1-p^{-(1-s)})=(1-p^{-\overline{s}})(s \in C，Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+})$,

且

$(1-p^{-(1-s)})^{-1}=\left(1-p^{-\overline{s}}\right)^{-1}(s \in C, Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+})$,

$\prod_{p}(1-p^{-(1-s)})^{-1}=\prod_{p}\left(1-p^{-\overline{s}}\right)^{-1}, (s \in C, Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ p 遍取所有的质数),

并且

$\frac{1}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}=\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$,

$\frac{(-1)^{n-1}}{(1-2^{s})}\prod_{p}(1-p^{-(1-s)})^{-1}=\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_{p}(1-p^{-\overline{s}})^{-1}$

$(s \in C, Rs(s) > 1$ 且 $s \neq 1, t \in C$ 且 $t \neq 0, n \in Z^{+}$ 且 n 遍取所有的正整数, $p \in$

Z^{+} 且 p 遍取所有的质数),

而且

$\zeta(1-s)=\frac{(-1)^{n-1}}{(1-2^{s})}\prod_{p}(1-p^{-(1-s)})^{-1}$,

$\zeta(\overline{s})=\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_{p}(1-p^{-\overline{s}})^{-1}$,

$\zeta(1-s)=\frac{1}{(1-2^{s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-s}}$,

$\zeta(\overline{s})=\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$

$(s \in C\ ，Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, $n \in Z^{+}$且 n 遍取所有的正整数),

因此当 $\sigma=\frac{1}{2}$, 那么

仅 $\zeta(1-s)=\zeta(\overline{s})\left(s \in C\ ，Rs(s) > 0 \text{ 且 } s \neq 1\right)$成立。

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$=[$1^{\sigma-k}$cos(tln1)− $2^{\sigma-k}$cos(tln2)+ $3^{\sigma-k}$cos(tln3) −$4^{\sigma-k}$cos(tln4)-...]+i[$1^{\sigma-k}$sin(tln1) − $2^{\sigma-k}$sin(tln2)+ $3^{\sigma-k}$sin(tln3) − $4^{\sigma-k}$sin(tln4)+...],

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$=[$1^{-\sigma}$cos(tln1)−$2^{-\sigma}$cos(tln2)+ $3^{-\sigma}$cos(tln3)−$4^{-\sigma}$cos(tln4)-...]+i[$1^{-\sigma}$Sin(tln1)− $2^{-\sigma}$sin(tln2)+ $3^{-\sigma}$sin(tln3)−$4^{-\sigma}$sin(tln4)+...],

p^{k-s}=$p^{(k-\sigma-ti)}$=$p^{k-\sigma}p^{-ti}$=$p^{k-\sigma}(\cos(\ln p)+i\sin(\ln p))^{-t}$=$p^{k-\sigma}(\cos(t\ln p)-i\sin(t\ln p))$,

$p^{1-\bar{s}}$ = $p^{(1-\sigma+ti)}$ = $p^{1-\sigma}p^{ti}$ = $p^{1-\sigma}(p^{ti})$ = $p^{1-\sigma}(\cos(\ln p)+i\sin(\ln p))^{t}$ = $(p^{1-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$,

($s\in C$ ，$Rs(s)>0$ 且 $s\neq 1, p\in Z^{+}$ 且 p 遍取所有的质数, $n\in Z^{+}$ 且 n 遍取所有的正整数, $k\in R$),

那么

$p^{-(k-s)}$=$p^{(-k+\sigma+ti)}$=$p^{\sigma-k}p^{ti}=p^{\sigma-k}\frac{1}{(\cos(t\ln p)-i\sin(t\ln p))}$ =$(p^{\sigma-k}(\cos(t\ln p)+i\sin(t\ln p))$,

$p^{-(\bar{s})}$=$p^{-(\sigma-ti)}$=$p^{-\sigma}p^{ti}=(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$,

$p^{-(k-s)}$=$(p^{\sigma-k}(\cos(t\ln p)+i\sin(t\ln p))$,

($s\in C$，$Rs(s)>0$ 且 $s\neq 1, p\in Z^{+}$ 且 p 遍取所有的质数, $k\in R$),

因此

$(1-p^{-(k-s)})$=1-$(p^{\sigma-k}(\cos(t\ln p)+i\sin(t\ln p))$ =$1-p^{\sigma-k}\cos(t\ln p)-ip^{\sigma-k}\sin(t\ln p)$,

$(1-p^{-\bar{s}})$=1-$(p^{-\sigma}(\cos(t\ln p)+i\sin(t\ln p))$ =$1-p^{-\sigma}\cos(t\ln p)-ip^{-\sigma}\sin(t\ln p)$,

($s\in C$，$Rs(s)>0$ 且 $s\neq 1, p\in Z^{+}$ 且 p 遍取所有的质数, $k\in R$) , 因 此 当 $\sigma=\frac{k}{2}$ (k ∈ R) ， 那 么

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-k+s}}$=$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$ ($s\in C$，$Rs(s)>0$ 且 $s\neq 1, k\in R, n\in Z^{+}$且 n 遍取所有的正整数),

$(1-p^{-(k-s)})$ = $(1-p^{-\bar{s}})$ ($s\in C$ ， n 遍取所有的正整数，$Rs(s)>1$ 且 $s\neq 1, k\in R$,p∈+ 且 p 遍取所有的质数),

and $(1-p^{-(k-s)})^{-1}$=$(1-p^{-\bar{s}})^{-1}$(s∈C，$Rs(s)>1$ 且 s≠1,k∈R, p∈Z^{+} 且 p 遍取所有的质数),

$\prod_{p}(1-p^{-(k-s)})^{-1}$=$\prod_{p}(1-p^{-\bar{s}})^{-1}$ ($s\in C$ 且 $s\neq 1, p\in Z^{+}$且 p 遍取所有的质数, $k\in R$),

且

$$\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}=\frac{1}{(1-2^{1-\bar{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$$

$(s \in C$，$Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$且 p 遍取所有的质数, $n \in Z^{+}$ 且 n 遍取所有的正整数, $k \in R$),且

$\zeta(k-s)=\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\prod_{p}(1-p^{-(k-s)})^{-1}$,

$\zeta(\overline{s})=\frac{(-1)^{n-1}}{(1-2^{1-\overline{s}})}\prod_{p}(1-p^{-\overline{s}})^{-1}$,

$\zeta(k-s)=\frac{1}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-s}}$ (s∈C 且 s≠ 1, k ∈R),

$\zeta(\overline{s})=\frac{1}{(1-2^{1-\overline{s}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{s}}}$ (s∈C 且 s≠ 1),

$(s \in C$，$Rs(s) > 0$ 且 $s \neq 1, p \in Z^{+}$ 且 p 遍取所有的质数, $n \in Z^{+}$ 且 n 遍取所有的正整数, $k \in R$),因此当 $\sigma=\frac{k}{2}$(k∈R),那么仅 $\zeta(k-s)=\zeta(\overline{s})$(s∈C且s≠ 1))$(s \in C$，$Rs(s) > 0$ 且 $s \neq 1, k \in R)$ 。黎曼已经知道黎曼 ζ(s) 函数有零点，通过黎曼得 ζ(1-s) 得到的等式ζ(1 − s)21−sπ −sCos(πs2)Γ(s)ζ(s)s∈C 且 s≠1(等式 *6*), 我们就知道

ζ(1-s)= $2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ $(s \in C$ 且 $s \neq 1)$ (等式 6)中,ζ(s)=0 $(s \in C$ 且 $s \neq 1)$成立。那么 ζ(1-s)= ζ(s)=0 $(s \in C$ 且 $s \neq 1)$ 成立。当ζ(s) =0 $(s \in C, Rs(s) > 0$ 且 $s \neq 1)$，那么 $\zeta(\overline{s})$ =0 $(s \in C, Rs(s) > 0$ 且 $s \neq 1)$，仅 $\zeta(1-s) = \zeta(\overline{s})$ =0 $(s \in C, Rs(s) > 0$ 且 $s \neq 1)$成立，那么 ζ(k − s=ζs=0(s∈C,Rss>*0*且 s≠1)成立，ζk−s=ζ(s)(s∈C，Rss>*0* and s≠1,k∈R)，因此仅 k=1 成立.

Reasoning 2:

下面是黎曼在他的论文中得到的一个结果：

$2\sin(\pi s)\prod(s-1)\zeta(s)=(2\pi)^{s}\sum n^{s-1}((-i)^{s-1}+i^{s-1})$ [1] (等式 3),

根据欧拉公式 $e^{ix}=\cos(x)+i\sin(x)$ $(x \in R)$可以得到：

$e^{i(-\frac{\pi}{2})}=\cos(\frac{-\pi}{2})+i\sin(\frac{-\pi}{2})$ =0-i= -i ,

$e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})$=0+i=i ,

那么

$(-i)^{s-1}+i^{s-1}=(-i)^{-1}(-i)^{s}$+$(i)^{-1}(i)^{s}$=$(-i)^{-1}e^{i\left(-\frac{\pi}{2}\right)s}+i^{(-1)}e^{i\left(\frac{\pi}{2}\right)s}$=

$ie^{i\left(-\frac{\pi}{2}\right)s}$-$ie^{i\left(\frac{\pi}{2}\right)s}$ =i(cos$\frac{-\pi s}{2}$+isin$\frac{-\pi s}{2}$)-i(cos$\frac{\pi s}{2}$+isin$\frac{\pi s}{2}$)=icos($\frac{\pi s}{2}$)-icos($\frac{\pi s}{2}$)+sin($\frac{\pi s}{2}$)+sin($\frac{\pi s}{2}$)

=2sin($\frac{\pi s}{2}$) (等式 4).

根据伽马函数 Γ(s)的性质 Π(s-1)=Γ(s)，并且

$\sum_{n=1}^{\infty} n^{s-1}$=ζ(1-s)($n \in Z^{+}$ 并且n遍取所有的正整数,$s \in C$，并且 $s \neq 1$),

把上面(等式 4)的结果代入上面(等式 3)右边，将得到:

2sin(πs)Γ(s)ζ(s)=$(2\pi)^{s}\zeta(1-s)2\sin\frac{\pi s}{2}$(等式 5)。根据倍角公式 sin(πs)=2sin($\frac{\pi s}{2}$)cos($\frac{\pi s}{2}$) ，

如果把它代入 2sin(πs)Γ(s)ζ(s)=$(2\pi)^{s}\zeta(1-s)2\sin\frac{\pi s}{2}$(等式 5)，

我们将得到: ζ(1- s)=$2^{1-s}\pi^{-s}$cos($\frac{\pi s}{2}$)Γ(s)ζ(s)(s∈C and $s \neq 1$)(等式 6),

因此当 ζ(s)=0$(s \in C$ 且 $s \neq 1)$，那么 ζ(1-s)=0$\left(s \in C\right.$ 且 $\left.s \neq 1\right)$,也就是 ζ(s)=ζ(1-s)=0$\Big(s \in$

C 且 s≠1。

作如下变换s→1- s,也就是把s当做 1-s代入等式 6, 将得到:

ζ(s)=$2^{s}\pi^{s-1}$sin($\frac{\pi s}{2}$)Γ(1- s)ζ(1- s)(s∈C 并且 $s \neq 1$) (等式 7)。

这就是ζ(s)的泛函方程ζ$(s)(s \in C$ 且 $s \neq 1)$. 为了将它改写成一种对称的形式，用伽玛函数的

余元公式

Γ(Z)Γ(1-Z)= $\frac{\pi}{\sin(\pi Z)}$ (等式 8)

和勒让德公式

Γ($\frac{Z}{2}$)Γ($\frac{Z}{2}$+$\frac{1}{2}$)=$2^{1-Z}\pi^{\frac{1}{2}}$Γ(Z) (等式 9)，

在 Γ(Z)Γ(1-Z)= $\frac{\pi}{\sin(\pi Z)}$ (等式 8) 中，令 z= $\frac{s}{2}$ ，将得到:

sin($\frac{\pi s}{2}$)= $\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10)，

在 Γ($\frac{Z}{2}$)Γ($\frac{Z}{2}$+$\frac{1}{2}$)=$2^{1-Z}\pi^{\frac{1}{2}}$Γ(Z)中, 令 z=1-s，将得到:

$\Gamma(1\text{-}s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1\text{-}\frac{s}{2})$ (等式 11)，

把 $\sin(\frac{\pi s}{2})=\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$ (等式 10) 和 $\Gamma(1\text{-}s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1\text{-}\frac{s}{2})$ (等式 11) 的结果代入

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)(s\in C$ 并且 $s\neq 1)$ (等式 7) 将得到:

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1\text{-}s)(s\in C$ 且 $s\neq 1)$ (等式 12),

也就是说

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$在变换$s\rightarrow 1\text{-}s$下是不变的，这正是 Riemann 在他的论文中所说的。

也就是

$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1\text{-}s)(s\in C$ 且 $s\neq 1)$。

同时 $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)(s\in C$ 且 $s\neq 1)$ (等式 7)在变换$s\rightarrow 1\text{-}s$下是不变的 将得到：

$\zeta(1\text{-}s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ 且 $s\neq 1)$(等式 6)。

那么 $\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}(s\in C$ 且 $s\neq 1)$, 当 $\zeta(s)=0$，如果

$\zeta(1\text{-}s)=\frac{\zeta(s)}{2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)}(s\in C$ 且 $s\neq 1)$ 在单调函数中有意义 ,则分母$2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1\text{-}s)$

$\neq 0$。 显然 $2^s\neq 0(s\in C$ 且 $s\neq 1)$, $\pi^{s-1}\neq 0(s\in C$ 且 $s\neq 1)$, $\Gamma(1\text{-}s)\neq 0(s\in C$ 且 $s\neq 1)$, 所以

$\sin(\frac{\pi s}{2})\neq 0(s\in C$ 且 $s\neq 1)$,因此当 $\zeta(s)=0$，且 $s\neq 2n(n\in Z^+)$,且$s\neq 1$，那么$\zeta(1-s)=\zeta(s)=0(s\in C$ 且 $s\neq 1$ 且 $s\neq -2n$，$n\in Z^+)$。

因为

$L(s,\chi(n))=\chi(n)\zeta(s)(s\in C$ 且 $s\neq 1, n\in Z_+$ ，且n遍取得所有的正整数) 且

$L(1-s,\chi(n))=\chi(n)\zeta(1\text{-}s)(s\in C$ 且 $s\neq 1, n\in Z_+$ ，且n遍取得所有的正整数)，

根据 $\zeta(s)=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)\zeta(1\text{-}s)(s\in C$ and $s\neq 1)$(等式 7),

因此仅 $L(s,\chi(n))=2^s\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1\text{-}s)L(1-s,\chi(n))(s\in C$ 且 $s\neq 1, n\in Z^+$)成立。根据函数

$\Gamma(s)$和指数函数非零的性质，也就是 $\Gamma(\frac{1-s}{2})\neq 0$,且 $\pi^{-\frac{1-s}{2}}\neq 0$, 根据

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$(s∈C 且 s≠ 1) (等式 12),数学家已经证明黎曼ζ(s)(s∈C 且 s≠ 1)函数的复自变量 s 的实部只有在满足 0<Re(s)<1 且 Im(s)≠0 时，ζ(s)(s∈C 且 s≠ 1,且$s \neq -2n, n \in Z^{+}$)的函数值才为零，所以我们应用 $\zeta(s)=\frac{\eta(s)}{(1-2^{1-s})}=\frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{s}}=$ −1n−11−21−sp1−p−s−1(s∈C 且 0<Rs(s)<1 且 Im(s)≠0，s∈C，n遍取得所有的正整数, p 遍取得所有的质数)。根据

$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$(s∈C 且 s ≠ 1)(等式 6)和 $\zeta(s)=2^{s}\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$(s∈C 且 s≠ 1)(等式 7)，既然黎曼已经知道且计算出黎曼 ζ(s)(s∈C and s≠ 1)函数有非平凡零点,那我们就应用 $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$($s \in C$ 且 $0 < Rs(s) < 1$ 且 $s \neq 1$ ，且 $Im(s) \neq 0$,n∈ Z^{+},n遍取得所有的正整数) 和 $\zeta(s)=2^{s}\pi^{s-1}\mathrm{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$($s \in C$ 且 $0 < Rs(s) < 1$，且$s \neq -2n$, 且 $Im(s) \neq 0$,n∈ Z^{+},n遍取得所有的正整数)，根据函数Γ(s)和指数函数非零的性质，也就是 $\Gamma(\frac{1-s}{2}) \neq 0$,且$\pi^{-\frac{1-s}{2}} \neq 0$，所以当 ζ(s)=0 ($s \in C$ 且 $0 < Rs(s) < 1$ 且 $s \neq 1$ ，$s \neq -2n$,且 $Im(s) \neq 0$,n $\in Z^{+}$,n遍取得所有的正整数)，那么 ζ(1-s)=0 ($s \in C$ 且 $0 < Rs(s) < 1$ 且 $s \neq 1$，且$s \neq -2n$,且 $Im(s) \neq 0$,n∈ Z^{+},n遍取得所有的正整数)，所以

ζ(s)=ζ(1-s)=0($s \in C$ 且 $0 < Rs(s) < 1, s \neq -2n$,且 $Im(s) \neq 0$, n遍取得所有的正整数)。

Because $e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\sum_{n=0}^{\infty}\frac{1}{n!}\approx 2.7182818284\ldots$ ，e 是自然常数， 且因为 $\sin(Z)=\frac{e^{iZ}-e^{-iZ}}{2i}$, 假设 Z=s= σ+ti (σ ∈R,t∈R 且 t ≠ 0)，那么

$\sin(s)=\frac{e^{is}-e^{-is}}{2i}=\frac{e^{i(\sigma+ti)}-e^{-i(\sigma+ti)}}{2i}$,

$\sin(\bar{s})=\frac{e^{i\bar{s}}-e^{-i\bar{s}}}{2i}=\frac{e^{i(\sigma-ti)}-e^{-i(\sigma-ti)}}{2i}$,

根据 $x^{s}=x^{(\sigma+ti)}=x^{\sigma}x^{ti}=x^{\sigma}(\cos(\ln x)+i\sin(\ln x))^{t}=x^{\sigma}(\cos(t\ln x)+i\sin(t\ln x))(x>0)$，那么

$e^{s}=e^{(\sigma+ti)}=e^{\sigma}e^{ti}=e^{\sigma}(\cos(t)+i\sin(t))=e^{\sigma}(\cos(t)+i\sin(t))$,
$e^{is}=e^{i(\sigma+ti)}=e^{\sigma i}(\cos(it)+i\sin(it))=(\cos(\sigma)+i\sin(\sigma))(\cos(it)+i\sin(it))$,
$e^{i\bar{s}}=e^{i(\sigma-ti)}=e^{\sigma i}(\cos(-it)+i\sin(-it))=(\cos(\sigma)+i\sin(\sigma))(\cos(it)-i\sin(it))$,
$e^{-is}=e^{-i(\sigma+ti)}=e^{-\sigma i}(\cos(-it)+i\sin(-it))=(\cos(\sigma)-i\sin(\sigma))(\cos(it)-i\sin(it))$,
$e^{-i\bar{s}}=e^{-i(\sigma-ti)}=e^{-\sigma i}(\cos(it)+i\sin(it))=(\cos(\sigma)-i\sin(\sigma))(\cos(it)+i\sin(it))$,
$2^{s}=2^{(\sigma+ti)}=2^{\sigma}2^{ti}=2^{\sigma}(\cos(\ln 2)+i\sin(\ln 2))^{t}=2^{\sigma}(\cos(t\ln 2)+i\sin(t\ln 2))$,

$2^{\overline{s}}$=$2^{(\rho-ti)}$=$2^{\sigma}2^{-ti}$=$2^{\sigma}(\cos(\ln2)+i\sin(\ln2))^{-t}$ =$2^{\sigma}(\cos(t\ln2)-i\sin(t\ln2))$,
π^{s-1}=$\pi^{(\sigma-1+ti)}$=$\pi^{\sigma-1}\pi^{ti}$=$\pi^{\rho-1}(\cos(\ln\pi)+i\sin(\ln\pi))^{t}$=$\pi^{\sigma-1}(\cos(t\ln\pi)+i\sin(t\ln\pi))$,
$\pi^{\overline{s}-1}$=$\pi^{(\sigma-1-ti)}$=$\pi^{\sigma-1}\pi^{-ti}$=$\pi^{\sigma-1}(\cos(\ln\pi)+i\sin(\ln\pi))^{-t}$=$2^{\sigma-1}(\cos(t\ln\pi)-i\sin(t\ln\pi))$,

因此

2^{s}=$\overline{2^{\overline{s}}}$, π^{s-1}=$\overline{\pi^{\overline{s}-1}}$,

且

$\frac{e^{is}-e^{-is}}{2i}$ =$\overline{\frac{e^{i\overline{s}}-e^{-i\overline{s}}}{2i}}$,

因此

sin(s)=$\overline{\sin(\overline{s})}$,

且

sin($\frac{\pi s}{2}$)=$\overline{\sin(\frac{\pi\overline{s}}{2})}$ 。

根据复数域上 Γ(s)函数的定义 ：

Γ(s)=$\int_0^{+\infty} t^{s-1}\, e^{-t}dt(s\in C)$,

在 Re(s)>0 中，该定义可由解析延拓原理推广到除非正整数外的整个复数域，

因此

Γ(s)=$\overline{\Gamma(\overline{s})}$,

且

Γ(1-s)=$\overline{\Gamma(1-\overline{s})}$.当

ζ(1-$\overline{s}$)=$\overline{\zeta(1-\overline{s})}$=0 且

ζ(1-s)=ζ($\overline{s}$)=0$(s\in C, Rs(s)>0$ 且 $s\neq1$ ，且 $Im(s)\neq0$,n∈ Z^{+}, n遍取得所有的正整数)，并根

据 ζ(s)=$2^{s}\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)$(s\in C$,且 $s\neq1$ ，且 $Im(s)\neq0$,n∈ Z^{+}, n遍取得所有的正整数),

那么 ζ(s)=$\overline{\zeta(\overline{s})}$$(s\in C$ 且 $0<Rs(s)<1$ 且 $s\neq1$ ，且 $Im(s)\neq0$,n∈ Z^{+}, n遍取得所有的正整数)

成立,当

ζ(s)= $\overline{\zeta(\overline{s})}=0$(s ∈ C , Rs(s) > 0 且 s ≠ 1 ，且 Im(s) ≠ 0 ,n ∈ Z^+, n遍取得所有的正整数 ）则

ζ(1-s)=ζ($\overline{s}$) =0(s ∈ C 且 0 < Rs(s) < 1 且 s ≠ 1 ，且 Im(s) ≠ 0 ,n∈ Z^+, n遍取得所有的正整数)

也成立. 因此仅有 ζ(σ+ti)=ζ(σ-ti)=0 成立.

假设

U=[$\chi(n)1^{-\sigma}$cos(tln1)− $\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)− $\chi(n)\ 4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$Sin(tln1)− $\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)− $\chi(n)4^{-\sigma}$sin(tln4)+...] ,

那么

$$L\left(s, \chi(n)\right)=\overline{L\left(\overline{s}, \chi(n)\right)}$$

(s∈C, Rs(s) > 0 且 s≠ 1 ， s ≠ −2n, n 遍取所有正整数, χ(n) ∈ R 且 χ(n) ≠ 0 , k ∈ R),

或

$$L\left(s, \chi(n)\right)=\overline{L\left(\overline{s}, \chi(n)\right)}$$

(s ∈ C, 0 < Rs(s) < 1 ， s ≠ −2n, n ∈ Z^+ ，且n 遍取所有正整数, χ(n) ∈ R 且 χ(n) ≠ 0 , k ∈

R, χ(n) ∈ R 且 χ(n) ≠ 0),

当 $\sigma=\frac{1}{2}$,

那么

L(s, χ(n))=L(1 − s, χ(n))= U-Vi,

(s∈C , $Re(s) > 0$且 s≠ 1 , s ≠ −2n, n 遍取所有正整数, χ(n) ∈ R 且 χ(n) ≠ 0 , k ∈ R),

U=[$\chi(n)1^{-\sigma}$cos(tln1)− $\chi(n)2^{-\sigma}$cos(tln2)+ $\chi(n)3^{-\sigma}$cos(tln3)− $\chi(n)\ 4^{-\sigma}$cos(tln4)+...],

V=[$\chi(n)1^{-\sigma}$sin(tln1)− $\chi(n)2^{-\sigma}$sin(tln2)+ $\chi(n)3^{-\sigma}$sin(tln3)− $\chi(n)4^{-\sigma}$sin(tln4)+...] .

且当 $\sigma=\frac{1}{2}$, 那么仅 $L\left(1-s, \chi(n)\right)=L\left(\overline{s}, \chi(n)\right)$

(s∈C,$0<\mathrm{Rs}(s)<1$，$s\neq-2n$,且n遍取所有正整数, $\chi(n)\in R$ 且$\chi(n)\neq0$，$k\in R$),

$\mathrm{GRH}(k-s,\chi(n))=L(k-s,\chi(n))=\frac{\chi(n)\eta(k-s)}{(1-2^{1-k+s})}=\frac{\chi(n)}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k-\rho-ti}}=$

$\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}\chi(n)(\frac{1}{n^{k-\sigma}}\frac{1}{n^{-ti}})=\frac{(-1)^{n-1}}{(1-2^{1-k+s})}\sum_{n=1}^{\infty}(\chi(n)n^{\sigma-k}(\cos(t\ln(n))+i\sin(t\ln(n))(s\in$

C ,*Res>0*且 s≠1 ,s≠*−2n,n* 遍取所有正整数， k∈R,χ n∈ R 且χ(n)≠0)，

W=[$\chi(n)1^{\sigma-k}$cos(tln1)− $\chi(n)2^{\sigma-k}$cos(tln2)+ $\chi(n)3^{\sigma-k}$cos(tln3)− $\chi(n)\,4^{\sigma-k}$cos(tln4)+...]

U=[$\chi(n)1^{\sigma-k}$sin(tln1)− $\chi(n)2^{\sigma-k}$sin(tln2)+ $\chi(n)3^{\sigma-k}$sin(tln3)− $\chi(n)4^{\sigma-k}$sin(tln4)+...] .

当 $\sigma=\frac{k}{2}$(k∈ R),

那么

仅 $L(k-s,\chi(n))=L\left(\bar{s},\chi(n)\right)=W-Ui$

(s∈C 且$0<\mathrm{Rs}(s)<1$, $s\neq-2n$,n遍取所有正整数, $\chi(n)\in R$ 且$\chi(n)\neq0$，$k\in R$),

$\chi(n)\in R$ 且$\chi(n)\neq0$),

但是黎曼 ζ(s) 函数仅满足 ζ(1-s)=$2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ (s∈C and s≠ 1), 因此当 ζ(s)=0(s∈C 且 s≠ 1), 那么仅$\zeta(1-s)=\zeta(s)=0$(s∈C 且 s≠ 1),且当 $\zeta(\bar{s})=0$, 那么仅 $\zeta(1-s)=\zeta(\bar{s})=0$(s∈C 且$0<\mathrm{Rs}(s)<1$), 且 $\zeta(k-s)=\zeta(1-s)=\zeta(\bar{s})$(s∈C 且$0<\mathrm{Rs}(s)<1$),因此仅 k=1 成立，仅有 Re(s)=$\frac{k}{2}=\frac{1}{2}$$(k\in R)$。

因此仅 $L(1-s,\chi(n))=L(\bar{s},\chi(n))$(s∈C 且 $0<\mathrm{Rs}(s)<1$,n遍取所有正整数, $\chi(n)\in R$ 且$\chi(n)\neq0$) 成立, 仅 k=1 成立。

因 为 $\zeta(s)=\overline{\zeta(\bar{s})}\left(s=\sigma+ti,\sigma\in R,Re(s)>0\text{ 且 }s\neq1,t\in R\right)$成立，所以当$s\neq-2n(n\in$ Z+,如果 ζ(s)=0s=σ+ti, σ∈R,t∈R, *Res>0*且*s≠1*，则必有 ζ(s)= ζ(s)=0s=σ+ti, σ∈R,*Res>0*且*s≠1*,t∈R ,表明除了负偶数 黎曼 ζ(s)函数的零点必定是共轭的 黎曼 ζ(s)函数在 Re(s) ∈(0,1) 的临界带内 ,不存在不共轭的零点。根据 ζ(s)= $\zeta(\bar{s})=0\left(s=\sigma+ti,\sigma\in R,Re(s)>0\text{ 且 }s\neq1,t\in\right.$ R，如果 s=s， 那么*s*∈*R*，由于*s*=*−2n*n∈Z+ 使得

$2\sin(\pi s)\Pi(s-1)\zeta(s)=i\int_0^{\infty}\frac{x^{s-1}dx}{e^x-1}$ 和 $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\Big(s\in$

C 并且 s≠1 等式 7 中的函数ζss∈C 且 s≠1 的值为零,所以负偶数可以是黎曼 ζ(s)s∈C 且

s≠1 的零点。如果 s≠s， 那么s和 s 不是同为实数,而是同为虚数，t∈R 且 t≠0，而且根据

$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\left(s\in C\text{ 并且 }s\neq 1\right)$(等式 7)， 如 果 ζ(s)=0 (s ∈

C 且 s≠1 成立,那么 ζ(1-s)=ζ(s)=0s∈C 且 s≠1 必定成立。

如果 ζ(s)=0$\left(s\in C\text{ 且 }Re(s)>0\text{ 且}s\neq 1\right)$成立,那么

ζ(1-s)=ζ(s)=0$\left(s\in C\text{ 且 }Re(s)>0\text{ 且}s\neq 1\right)$必定成立。

因 为 $\zeta(s)=\overline{\zeta(\overline{s})}$ $\left(s=\sigma+ti,\sigma\in R,Re(s)>0\text{ 且}s\neq 1,t\in R\right)$成立，所以当$s\neq -2n$(n ∈

Z+,如果 ζ(s)=0s=σ+ti, σ∈R,t∈R, *Res>0* 且*s≠1*，则必有 ζ(s)= ζ(s)=0s=σ+ti, σ∈R,*Res>0* 且

s≠1,t∈R 表明除了负偶数 黎曼 ζ(s)函数的零点必定是共轭的 黎曼 ζ(s)函数在 Re(s) ∈(0,1)

的临界带内 ,不存在不共轭的零点。根据 ζ(s)= ζ($\overline{s}$)=0$\big(s=\sigma+ti,\sigma\in R,Re(s)>0\text{ 且}s\neq 1,t\in$

R，如果 s=s， 那么$s\in R$，由于$s=-2n$n∈Z+ 使得

$2\sin(\pi s)\Pi(s-1)\zeta(s)=i\int_0^{\infty}\frac{x^{s-1}dx}{e^x-1}$ 和 $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\Big(s\in$

C 并且 s≠1 等式 7 中的函数ζss∈C 且 s≠1 的值为零,所以负偶数可以是黎曼 ζ(s)s∈C 且

s≠1 的零点。如果 s≠s， 那么s和 s 不是同为实数,而是同为虚数，t∈R 且 t≠0，而且根据

$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\left(s\in C\text{ 并且 }s\neq 1\right)$(等式 7)， 如 果 ζ(s)=0 (s ∈

C 且 s≠1 成立,那么 ζ(1-s)=ζ(s)=0s∈C 且 s≠1 必定成立。

如果 ζ(s)=0$\left(s\in C\text{ 且 }Re(s)>0\text{ 且}s\neq 1\right)$成立,那么

ζ(1-s)=ζ(s)=0$\left(s\in C\text{ 且 }Re(s)>0\text{ 且}s\neq 1\right)$必定成立。

因为 ζ(s)= $\overline{\zeta(\overline{s})}$ $(s \in C, Re(s) > 0 \text{ and } s \neq 1)$ ， 所以如果 ζ(s)=0 $\big(s \in C, Re(s) > 0$ 且 $s \neq$ 1 and s≠1，那么

ζ(s)= ζ($\overline{s}$)=0$\big(s \in C, Re(s) > 0$ 且 $s \neq 1\big)$,又根据 $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(s \in$ C *and* s≠1(等式 7)，如果*Res>0* 且 ζ(s)=0s∈C，*Res>0* 且 s≠1,那么 ζ(s)=ζ(1-s)=0s∈C，*Res>0* 且 s≠1，所以当 *Res>0*，黎曼 ζss∈C 且 *Res>0* 且*s≠1* 函数的两个零点*s*和 1−*s*也必须是共轭的。

如果s和 $1-s$是除负偶数以外的实数，由于s和 $1-s$共轭，则$s = 1-s$，那么$s = \frac{1}{2}$,由于 $\sin\left(\frac{\pi s}{2}\right) = \sin\left(\frac{\pi}{2} \times \frac{1}{2}\right) = \sin\left(\frac{\pi}{4}\right) \neq 0$，且因为 $\zeta\left(\frac{1}{2}\right)$发散，所以如果$s$和 $1-s$是除负偶数以外的实数，那么s和 $1-s$都不是黎曼 ζ(s) $\big(s \in C$ 且 $Re(s) > 0$ 且$s \neq 1\big)$的零点，也就是说黎曼 ζ(s) $\big(s \in C$ 且$Re(s) > 0$ 且$s \neq 1\big)$没有除负偶数以外的实数零点。当$Re(s) > 0$，如果$Re(s) = 1$,则$Re(1-s) = 0$,那么s和 $1-s$不共轭，所以黎曼 ζ(s) $\big(s \in C$ 且 $s \neq 1\big)$没有实部为 1 或 0 的零点。 当 $Re(s) > 1$, ζ(s) $\big(s \in$ C 并且 $s \neq 1\big)$没有零点,又根据 $\zeta(s) = \zeta(1-s) = 0$ $\big(s \in C$,且 $s \neq 1\big)$成立，那么当 $Re(s) < 0$，则 ζ(s) $\big(s \in C$ 并且 $s \neq 1\big)$也不等于零。又因为当 $\zeta(s) = 0$ $\big(s \in C$, $Re(s) > 0$ 且 $s \neq 1\big)$，如果 $Re(s) = 1$，则 $Re(1-s) = 0$,那么 s 和 $1-s$ 不共轭，又根据 $\zeta(s) = \zeta(1-s) = 0$ $\big(s \in C$,且 $s \neq 1\big)$成立,所以如果 $Re(s) = 0$ 或 $Re(s) = 1$，则 ζ(s) $\big(s \in C$ 并且 $s \neq 1\big)$无零点。所以除了负偶数，黎曼 ζ(s) $\big(s \in C$ 并且 $s \neq 1\big)$有零点的前提条件是 Re(s)的值位于区间(0,1)。黎曼 ζ(s) $\big(s \in C$ 且 $s \neq 1\big)$零点s的实部必须 满足 $0 < Re(s) < 1$ ，也就是$Re(s) \in (0,1)$，这 表明素数定理 成立。 当 $0 < Re(s) < 1$ ，如果s和 $1-s$一个为实数，一个为虚数，则s和 $1-s$不共轭，那么s和 $1-s$不可能同为黎曼 ζ(s) $\big(s \in C$, $Re(s) > 0$ 且 $s \neq 1\big)$的零点，所以 $1-$

s和s只能同为虚数且共轭，s不能为纯虚数，因为如果s为纯虚数，则 $1-$

s和s不共轭，所以 $\zeta(s)(s\in C,0<Re(s)<1)$无纯虚数零点。而且如果

$Re(s)\neq\frac{1}{2}$， 那么$Re(1-s)\neq 0$， 而且必定有$Re(s)\neq$

$Re(1-s)$，那么 $1-s$和s不共轭，所以$Re(s)\neq\frac{1}{2}$不可能成立。所以仅有 $1-s=$

$\overline{s}$ 成立,即仅有 1-σ-ti= σ-ti 成立，所以仅有 $\sigma=\frac{1}{2}$ ，$t\in R$ 且 $t\neq 0$，因此黎曼 $\zeta(s)(s\in C,0<$

Res<1 的非实数零点的实部只能是 12，即仅有 Res=12 成立，等价于 ξs=0s=12+ti 或

s=12−ti ,t∈R 且 t≠0 ,s∈C, 且 s≠1 或 ξ12+ti=0t∈R 且 t≠0 和 ξ12−ti=0t∈R 且 t≠0 成立，

所以在 $Re(s)\in(0,1)$的临界带内，$Re(s)\neq$

$\frac{1}{2}$不可能,不存在实部不等于$\frac{1}{2}$的零点，所以黎曼猜想成立。零点 s 和零点 $1-$

s 的对称性不足以说明黎曼 $\zeta(s)\left(s\in C\right.$ 且 $s\neq$

1 的非平凡零点都位于临界线上**,**零点 s 和零点 1−s 的对称性表示它们关于 Res=12 的临界

线上的一个点$\left(\frac{1}{2},oi\right)$对称，而不是零点 s 和零点 $1-s$ 关于 $Re(s)=$

$\frac{1}{2}$的临界线对称，s 和 $1-$

s 的共轭性才是黎曼 $\zeta(s)\left(s\in C\text{ 且 }s\neq 1\right)$的非平凡零点都位于临界线上的根本原因。

根据$\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\,\zeta(s)(s\in C$ and $s\neq 1)$ (等式 6),那么当 ζ(s)=0 ，就会有

ζ(1-s)= ζ(s)=0(s∈C and $s\neq 1$) 成立。由于$\zeta(s)=\overline{\zeta(\overline{s})}\left(s\in C\text{ 并且 }Re(s)>0\text{ 且 }s\neq 1\right)$，那

么当ζ(s)=0$\left(s\in C,Re(s)>0\text{ 且 }s\neq 1\right)$或 $\zeta(\overline{s})$=0$\left(s\in C,Re(s)>0\text{ 且 }s\neq 1\right)$，那么就必定会

有$\zeta(s)=\zeta(\overline{s})$=0$\left(s\in C,Re(s)>0\text{ 且 }s\neq 1\right)$ 成立。当ζ(s)=0$\left(s\in C,Re(s)>0\text{ 且 }s\neq 1\right)$， 根

据$\zeta(1-s)=\zeta(s)$=0$\left(s\in C\text{ 并且 }Re(s)>0\text{ 且 }s\neq 1\right)$成立，那么 s 和 1-s 也必定共轭。由此

我们就得到 s=$\frac{1}{2}$+ti(t∈ R 且≠ 1)，或 s=$\frac{1}{2}$-ti(t∈ R 且≠ 1)，当 Re(s)>1,欧拉ζ与黎曼ζ函数等

价，同时一个乘积因子都不等于零，所以当 Re(s)>1, ζ(s)就不等于零，又根据

$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)(s\in C$ 且 $s\neq 1)$(等式 7) ,所以正偶数 2n(n 为正整数)虽然能够使 $\sin(\frac{\pi s}{2})$为零，但它不是黎曼ζ(s)的零点。如果 s 为除负偶数和正偶数以外的其它实数，如果它是黎曼ζ(s)$\left(s\in C, Re(s)>0 \text{ 且 } s\neq 1\right)$函数的零点，那么$s$ 和 $1-s$就必定共轭，所以它除了不能使得 $\sin(\frac{\pi s}{2})$为零，而且必须满足 s=1-s，那么 s=$\frac{1}{2}$, 而ζ($\frac{1}{2}$)发散， 所以负偶数以外的实数不是黎曼ζ(s)$\left(s\in C, Re(s)>0 \text{ 且 } s\neq 1\right)$的零点。根据 ζ(s)= ζ(1-s)=0 (s∈C 且 $s\neq 1$)成立，我们知道除了负偶数 ζ(s)(s∈C 且$0<Re(s)<1$)的零点关于点($\frac{1}{2}$,0i)对称。但是仅仅根据ζ(s)(s∈C 且$0<Re(s)<1$)的零点关于点($\frac{1}{2}$,0i)对称，就判定黎曼ζ(s)(s∈C 且 $0<Re(s)<1$)函数的非平凡零点就都位于实部等于$\frac{1}{2}$的临界线上，这可以吗？显然不可以, 当 $Re(s)\in(0,1)$, 假如 s=0.54+ti($t\in R$),Re(s)=0.54，那么 Re(1-s)=0.46, s 和 1-s 关于点($\frac{1}{2}$,0i)对称，但是黎曼认为这样的复数不是黎曼ζ(s)的零点。黎曼的看法是正确的，很显然，当 Re(s)≠ $\frac{1}{2}$, 那么 s 和 1-s 就必定不共轭 ,根据 黎曼 ζ(s)(s∈C 且$0<Re(s)<1$)函数的零点必定共轭，那么, 如果 Re(s)≠ $\frac{1}{2}$, 那么它必定不是ζ(s) (s∈C 且$0<Re(s)<1$)函数的零点。综上所述，黎曼ζ(s)$\left(s\in C \text{ 并且 } s\neq 1\right)$函数的非平凡零点必定都位于复平面 Re(s)= $\frac{1}{2}$的临界线上，黎曼猜想必定成立。

根据黎曼得到的方程 $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ and $s\neq 1)$(公式 6) 以及黎曼已经得到的黎曼 ζ(s)(s∈C and s≠ 1) 函数的零点，表明 ζ(s)(s∈C and s≠ 1) 函数的零点存在。那意味着在 $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)(s\in C$ 且 $s\neq 1)$(公式 6)中, ζ(s)=0(s∈C 且 s≠ 1)中成立。所以仅当 $\sigma=\frac{1}{2}$ 且 ζ(s)=0(s∈C 且 s≠ 1)，那么

L(s, ✗(n))= ✗(n)ζ(s)=0(s∈C 且 s≠ 1, 且 $s\neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)成立。

因为 L(s, ✗(n))= ✗(n)ζ(s)(s∈C 且 s≠ 1, $n\in Z^+$且n 遍取所有正整数，✗(n) ∈ R 且✗(n) ≠ 0)

且 $L(1-s, \chi(n))=\chi(n)\zeta(1-s)$(s∈C 且 $s\neq 1, n\in Z^{+}$ 且n 遍取所有正整数)，因此当 $\sigma=\frac{1}{2}$,

$L(s, \chi(n)) = \overline{L\left(\bar{s}, \chi(n)\right)}$ ($s\in C$, Re(s)>0 且 $s\neq 1$,且 $s\neq -2n$, n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$)必定成立，且 $L(1-s, \chi(n))=L(\bar{s}, \chi(n))$(s∈C 且

$s\neq 1$ ，且 $s\neq -2n$, n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$) 必定成立．通过 $\zeta(1-s)=\zeta(s)=0$(s∈C 且 $s\neq 1$) 且 $\zeta(s)=\zeta(\bar{s})=0$(s∈C,Re(s)>0 且 $s\neq 1$)和 $\zeta(s)=\zeta(1-\bar{s})=0$(s∈C 且 $s\neq 1$),

因此 $L(s, \chi(n))=L(1-s, \chi(n))=0$

(s∈C 且 $s\neq 1, s\neq -2n$,且n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$, $k\in R$),

$\chi(n)\in R$ 且 $\chi(n)\neq 0$),

和 $L(s, \chi(n))=L(\bar{s}, \chi(n))=L(1-\bar{s}, \chi(n))=0$

(s∈C , Re(s)>0 且 $s\neq 1, s\neq -2n$,且n 遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$, $k\in R$),

$\chi(n)\in R$ 且 $\chi(n)\neq 0$)成立,

那么 $s=\bar{s}$ 或 s=1-s 或 $\bar{s}$=1-s，且 $\sin(\frac{\pi s}{2}=0)$，所以 s∈R 且 s=-2n($n\in Z^{+}$),舍去 s=2n($n\in Z^{+}$)，因此 s∈R 且 s=-2n($n\in Z^{+}$)，或σ+ti=1-σ-ti,或 σ-ti=1-σ-ti，因此 $s\in R$ 且 $s=-2n(n\in Z^{+})$,或$\sigma=\frac{1}{2}$且 t=0，或$\sigma=\frac{1}{2}$ 且 t ∈R 且 $t\neq 0$, 因此 $s\in R$ 且 $s=-2n(n\in Z^{+})$ 或 s=$\frac{1}{2}$+0i ,或 s=$\frac{1}{2}$+ti(t ∈R 且 $t\neq 0$) 与 s=$\frac{1}{2}$-ti(t ∈R 且 $t\neq 0$)。

因为$\zeta\left(\frac{1}{2}\right)\to +\infty, \zeta(1)\to +\infty$, $\zeta(1)$发散，$\zeta\left(\frac{1}{2}\right)$更是发散,因此将它们舍去。且因为仅当 $\sigma=\frac{1}{2}$，下面三个等式: $L\left(\sigma+ti, \chi(n)\right)=0$(t∈R 且 $t\neq 0$, $\chi(n)\in R$ 且 $\chi(n)\neq 0$ 且n遍取所有正整数),和 $L\left(1-\sigma-ti, \chi(n)\right)=0$(t∈R 且 $t\neq 0, n\in Z^{+}$,且n遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$)和 $L\left(\sigma-ti, \chi(n)\right)=0$ (t∈CR 且 $t\neq 0, n\in Z^{+}$ 且n遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$) 才全都成立。并且因为 $L(\frac{1}{2}, \chi(n))>0$($n\in Z^{+}$ 且n遍取所有正整数, $\chi(n)\in R$ 且 $\chi(n)\neq 0$)，因此仅 s=$\frac{1}{2}$+ti(t ∈R 且 $t\neq 0$)和 s=$\frac{1}{2}$-ti(t ∈R 且 $t\neq 0$)成立.广义黎曼$L(s, \chi(n)$(s∈C

且 s≠ 1 , $s \neq -2n$,且n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，k ∈ R)函数的性质是基本的，广义黎曼猜想必须是正确的， 以反映 L(s, ✗(n))（s∈C 且

s ≠ 1， $s \neq -2n$,且n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，k ∈ R）函 数 的 性 质 ，L(s, ✗(n))=0（s∈C 且 s≠ 1， $s \neq -2n$,且n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，k ∈ R), 的根只能是 s=$\frac{1}{2}$+ti(t∈R, t≠0)或 s=$\frac{1}{2}$-ti(t∈R,t≠0)，也就是说，Re(s)必须只等于$\frac{1}{2}$，且 Im(s) 必定是实数，因此广义黎曼猜想必定成立。根据 L(1 − s, ✗(n)) = L(s, ✗(n)) =0(s∈C 且 s≠ 1，$s \neq -2n$,且n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0，k ∈ R),所以L(s, ✗(n)))(s∈C 且 s≠ 1 且 s ≠ −2n,且n 遍历所有正整数)函数的零点位于复平面内与实数轴垂直的实数线上且关于点（$\frac{1}{2}$,0i）对称分布，所以当$L\left(1-s, ✗(n)\right) = L\left(s, ✗(n)\right) = 0\left(s \in C\right.$ 且 $s \neq 1$ ，且 $s \neq -2n$,n 遍取所有正整数,✗n∈ R 且✗(n)≠0 ,s 和 1-s 是函数 Ls,✗n(s∈C 且 s≠1 且 s≠$-2n$,n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0）的一对零点，在与复平面实数轴垂直的直线上，相对于点（$\frac{1}{2}$,0i）对称分布。我们得到了 $\overline{L(s, ✗(n))}$ =L($\bar{s}$, ✗(n))(s ∈ C, Re(s) > 0 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0) ,在下面的假设 s=$\frac{1}{2}$+ti(t∈C 且 t ≠ 0)中，t 是一个复数, 那么

$\overline{L\left(s, ✗(n)\right)} = L\left(\bar{s}, ✗(n)\right)$（ s ∈ C, Re(s) > 0 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)中的 s 与下面的假设 s=$\frac{1}{2}$+ti(t∈C and t ≠ 0)相匹配，所以仅$\sigma = \frac{1}{2}$正确.当 $L\left(s, ✗(n)\right)$=$L\left(\bar{s}, ✗(n)\right)$= 0(s ∈ C, Re(s) > 0 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)，因为 s 和 $\bar{s}$ 是一对共轭复数，因此 s 和 $\bar{s}$ 一定是广义L(s, ✗(n))(s ∈ C 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)函数的一对共轭且对称的零点，在复平面上位于垂直于实数轴直线上，关于点（σ,0i）对称。S 是 1-s 的对称零点，也是$\bar{s}$的对称零点。根据复数的定义，广义黎曼函数L(s, ✗(n))(s ∈ C 且 s ≠ 1 且

$s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)中的同一个自变量零点 s 在垂直于复平面实数轴的直线上既关于($\frac{1}{2}$,0i)和 1 –s 共轭对称 ,又关于(σ, 0i)和$\bar{s}$共轭对称？也就是$\sigma = \frac{1}{2}$ ，且仅有 1-s=$\bar{s}$ 正确, 仅 s=$\frac{1}{2}$+ti(t∈R且 t ≠ 0,s∈C) 和 s=$\frac{1}{2}$-ti(t∈R 且 t ≠ 0,s∈C) 成立 ,否则不可能。这是由广义黎曼L(s, ✗(n))(s ∈ C 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且 ✗(n) ≠ 0)函数的零点在经过该点($\frac{1}{2}$,0i)垂直于复平面实数轴的直线上关于该直线与复平面实数轴的垂足($\frac{1}{2}$,0i)共轭对称分布的零点的唯一性决定的，从零点 s 向复平面实数轴作垂线，仅能作出一条,垂足也只有一个。在同一个复平面，ζ(s)函数的同一个零点在经过该点($\frac{1}{2}$,0i)垂直于复平面实数轴的直线上关于该直线与复平面实数轴的垂足($\frac{1}{2}$,0i)共轭且对称分布的零点只会有一个。我已经证明了当狄利克雷特征函数✗(n) (s ∈ Z^+，n 遍历所有正数)是任意的实数,且不等于零 ,则广义黎曼L(s, ✗(n))(s ∈ C 且 s ≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠0) 函数的非平凡零点的都满足 Re(s)= $\frac{1}{2}$ 且 Im(s)≠0 ,都位于与实数轴垂直的临界线上。这些非平凡零点是满足 Re(s)= $\frac{1}{2}$且 Im(s)≠0 的一般复数，因此我证明了当 Dirichlet 特征函数✗(n) (s ∈ Z^+，n 遍历所有正数，✗(n) ∈ R 且✗(n) ≠ 0))是任何不等于零的实数时的广义黎曼猜想。广义黎曼猜想必须满足L(s, ✗(n))(s ∈ C 且 s ≠ 1 且 $s \neq -2n$, n ∈ Z^+, ✗(n) ∈ R 且✗(n) ≠ 0)函数的基本性质，广义黎曼猜想必须是正确的，以反映 L(s, ✗(n))(s ∈ C 且 s ≠ 1, 且 $s \neq -2n$, n ∈ Z^+, ✗(n) ∈ R 且✗(n) ≠ 0) 函数的性质，即 $L\left(s, ✗(n)\right) = 0$(s∈C 且 s≠ 1 且 $s \neq -2n$, n 遍取所有正整数, ✗(n) ∈ R 且✗(n) ≠ 0)的根只能是 s=$\frac{1}{2}$+ti(t∈ R,t≠0)和 s=$\frac{1}{2}$-ti(t∈ R,t≠0) ，也就是说，Re(s)只能等于$\frac{1}{2}$，而且 Im(s)必须是实数，且 Im(s)不等于零。

推论 4:

对于任何复数 s，✗(n 是 Dirichlet 特征并且满足以下性质：

1: 存在一个正整数 q，使得 ✗(n+q)= ✗(n);

2: 当 n 和 q 不互素数时，✗(n)=0;

3: 对任意整数 a 和 b 来说，✗(a).✗(b)=✗(ab);

设 q=2k(k ∈ Z^+)，如果 n 和 n+q 都是素数，并且如果✗(Y)=0 （Y 遍取所有正奇数）和 ✗(n + q) = ✗(n) =0(n 和 n + q 遍取所有正奇数),因为 n （n 遍历所有素数）和 q=2k(k ∈ Z^+) 不是互素，如果 n 和 n+q 都是素数，并且如果✗(Y)=0 （Y 遍取所有正奇数）和✗(n + q) = ✗(n) =0(n 和 n + q 遍取所有正奇数),因为 n(n 遍历所有素数）和 q=2k(k ∈ Z^+)不是互素，那么✗(n)=0（n∈Z^++和 n 和 n+q 遍历所有素数）并且对于任何素数 a 和 b ✗(a)·✗(b)= ✗(ab)（a∈ Z^+,b∈ Z^+，a 遍取所有素数，b 遍取所有素数），那么狄利克雷特征函数✗(n)所描述的三个性质（n∈ Z^+，并且 n 遍取所有素数）满足波利尼亚克猜想的定义，波利尼亚克猜想指出，对于所有自然数 k，存在无限多对素数（p,p+2k）（k ∈ Z^+）对。1849 年，法国数学家 A.波利尼亚克提出了这个猜想。当 k=1 时，波利涅克猜想等价于孪生素数猜想。换句话说，当 L(s, ✗(n)) = 0(s ∈ C 且 s ≠ 1, n ∈ Z^+ 且 n 遍取所有正整数，✗(n) ∈ R 且✗(n) ≠ 0, a(n) = a(p)= ✗(n)),P(p，s)=$\frac{1}{1-a(p)p^{-s}}$)。广义黎曼猜想成立 ,那么波利尼亚克猜想几乎成立，并且如果波利尼亚克猜想成立，那么孪生素数猜想和哥德巴赫猜想也成立。

Reasoning 5:

为了解释为什么朗道-西格尔函数的零点在特殊条件下存在，我们需要从黎曼猜想开始。我已经解决了狄利克雷特征函数✗(n)≡1（n ∈ Z^+ ，且 n 遍历所有正整数）的黎曼猜想和狄利克雷特征函数✗(n)≠0（n ∈ Z^+ ，且 n 遍历所有正整数）的广义黎曼猜想，我提出了狄利克雷特征函数✗(p)≠0（p ∈ Z^+ 且 p 遍历所有素数）的一个特殊形式 L(s,✗(p))(s∈C 且 s ≠ 1, ✗(p) ∈R and ✗(p)≠0, p ∈ Z^+ 且 p 遍历所有素数，包括 1) 函数问题。让我先解释一下什么是朗道-西格尔零点猜想。你们可能知道，朗道-西格尔零点问题，以朗道和他的学生西格尔命名，

归结为解决狄利克雷 L(β,1) (β ∈ R)函数中是否存在异常实零点。我们再来看一下狄利克雷 L 函数是什么。看看上面的证明过程，这是狄利克雷 L(s, ✗(n))(s∈C 且 s≠ 1, n ∈ Z^+ ,且 n 遍历所有正整数)的表达式。我先介绍狄利克雷 L(s, ✗(n))(s∈C且 s ≠ 1 , n ∈ Z^+ ，且 n 遍历所有正整数)函数并解释它与黎曼 ζ(s)(s∈C 且 s ≠ 1)函数的关系。这里，✗(n)(n ∈ Z^+ 且 n 遍历所有正整数)是一个狄利克雷函数的特征值，它是个实数，并且 ✗ (n)(n ∈ Z^+ 且 n 遍历所有正整数) 是 一 个 实 函 数 。 L(s, ✗ (n))(s∈C 且 s ≠ 1 , ✗ (n) ∈ R, n∈ Z^+ 且 n 遍历所有正整数)函数可以解析扩展为整个复平面上的亚纯函数。狄利克雷证明了对于所有 ✗(n)(n∈Z^+ 且 n 遍历所有正整数))，L(1, ✗(n))≠0(s∈C 且 s≠1，✗(n)∈R 且 ✗(n)≠ 0, n∈Z^+ 且 n 遍历所有正整数))，从而证明了狄利克雷定理。狄利克雷定理指出，对于任意正整数 a ,d ,存在无穷多种质数形式 ,如 a+nd ,其中 n 是正整数 ,即等差数列 a+d ,a+2d,a+3d ,... ,中素数有无限个，素数模块 d 和素数模块 a 都有无限个。如果

✗(n)(n ∈ Z^+ 且 n 遍历所有正整数)) 是主特征，那么 L(s, ✗(n))(s∈C 且 s ≠ 1，✗(n)∈R, n∈Z^+且 n 遍历所有正整数) 在 s=1 处有一个单极特征。狄利克雷定义了狄利克雷函数 L(s, ✗(n))(s∈C 且 s≠ 1, ✗(n) ∈R, n∈Z^+且 n 遍历所有正整数)) 中的特征函数的

✗(n)(n∈Z^+且 n 遍历所有正整数))的性质：

1：存在一个正整数 q，使得 ✗(n+q)= ✗(n)(n∈Z^+且 n 遍取所有正整数)；

2:当 n(n∈Z^+ 且 n 遍取所有正整数)和 q 是非互素数时 ,✗(n)≡0(n∈ Z^+ 且 n 遍取所有正整数)；

3：对于任何正整数 a 和 b, ✗(a) ✗(b)= ✗(ab)(a 是一个正整数, b 是一个正整数);

从狄利克雷函数 L(s, ✗(n))(s∈C 且 s≠ 1,X(n)∈R, n∈Z^+ 且 n 遍取所有正整数)的表达式，很容易看出，当 Dirichlet 特征实函数

✗(n)=1(s∈C 且 s≠ 1, n∈Z^+且 n 遍取所有正整数)，这个狄利

克雷函数 L(s,1)(s∈C 且 s≠ 1,✗(n)∈R, n∈Z^+ 且 n 遍取所有正整数) 变成了黎曼 ζ(s)(s∈C 且 s≠ 1) 函数，因此黎曼 ζ(s)(s∈C 且 s≠ 1)是狄利克雷函数 L(s,✗(n))(s∈C 且 s≠ 1,✗(n)∈R,n∈Z^+ 且 n 遍取所有正整数)的一个特殊函数，所以当特征实函数 ✗(n)≡ 0(n∈Z^+ 且 n 遍取所有正整数)时，也被称为狄利克雷函数 L(s,✗(n))(s∈C 且 s≠ 1, ✗(n) ∈R，n∈Z^+ 且 n 遍取所有正整数)的平凡特征函数。当特征实函数✗(n)≡ 0(n∈Z^+ 且 n 遍取所有正整数)时，也被称为狄利克雷函数 L(s,✗(n))(s∈C 且 s≠ 1, ✗(n) ∈R，n∈Z^+ 且 n 遍取所有正整数)的平凡特征函数。当特征实函数✗(n)≠1(n∈ Z^+ 且 n 遍取所有正整数)时，它们被称为狄利克雷函数的非平凡特征函数。L(s,✗(n))(s∈C 且 s≠ 1, ✗(n) ∈R，n∈Z^+ 且 n 遍取所有正整数)表达式中的自变量 s 是一个实数β时，那么对于所有特征函数值✗(n)(n∈Z^+ 且 n 遍取所有正整数)，L(β,✗(n))(β∈R,✗(n) ∈R, n∈Z^+ 且 n 遍取所有正整数) 被称为朗道-西格尔函数。可见朗道-西格尔函数 L(β,✗(n))(β∈R,✗(n)∈R, n∈ Z^+ 且 n 遍取所有正整数)是狄利克雷函数 L(s,✗(n))(s∈C 且 s≠ 1, ✗(n) ∈R，n∈Z^+ 且 n 遍取所有正整数)的一个特殊函数，朗道-西格尔零点猜想是指朗道-西格尔猜测 L(β, (n))(β∈R,✗(n)∈R, n∈ Z^+ 且 n 遍取所有正整数)不为零 ,所以朗道-西格尔关于 L(β, (n))≠0(β∈R,≠-2n, n∈ Z^+ 且 n 遍取所有正整数) 的猜想很容易理解，对吧?好了,现在你们知道了朗道和西格尔零猜想是关于什么的，我们继续看一下如何证明朗道和西格尔零点猜想。看看上面得到的证明过程。因为：

$$GRH\left(s,\ \chi(n)\right)=L\left(s,\ \chi(n)\right)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{s}}=\frac{\chi(n)\eta(s)}{(1-2^{1-s})}=\frac{\chi(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{s}}$$

$$=\frac{\chi(n)}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma+ti}}=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}\chi(n)\left(\frac{1}{n^{\sigma}}\frac{1}{n^{ti}}\right)=$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}\chi(n)(n^{-\sigma})\frac{1}{(\cos\left(\ln((n)\right)+i\sin(\ln(n)))^{t}}$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}\chi(n)(n^{-\sigma}(\cos(\ln(n))+\ i\sin(\ln(n)))^{-t})$$

$$=\frac{(-1)^{n-1}}{(1-2^{1-s})}\sum_{n=1}^{\infty}\chi(n)n^{-\sigma}(\cos(t\ln(n))-i\sin(t\ln(n))\ \ )$$

($t\in C$ 且 $t\neq 0, s\in C$ 且 $s\neq 1, n\in Z^{+}$ 且 n 遍取所有正整数)，因为

ζ(s)=$2^{s}\pi^{s-1}$Sin($\frac{\pi s}{2}$)Γ(1-s)ζ(1-s)(s∈C 且 s≠ 1)(等式 7) ,因此，如果 β∈R and β= $-2n(n\in Z^{+})$ ，

那么 ζ(s)=0。因此

L(β, χ(n))=$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{n=1}^{\infty}\chi(n)(n^{-\beta}(\cos(0\times\ln(n))+i\sin(0\times\ln(n)))$=$\frac{(-1)^{n-1}}{(1-2^{1-\beta})}\sum_{n=1}^{\infty}(\chi(n)n^{-\beta})$
=$\frac{1}{\left(1-2^{1-\beta}\right)}(\chi(1)1^{-\beta}-\chi(2)2^{-\beta}+\chi(3)3^{-\beta}-\chi(4)4^{-\beta}+\cdots)$," × " 是用来表示相乘的符号,

因为以实数为底的实指数函数是一个大于零的函数值，因为 $\chi(n)\in R$ 且 $\chi(1)=\chi(2)=\chi(3)=\chi(3),\ldots$, 所以 $n^{-\beta}>0$ ($n\in Z^{+}$ 且 n 遍取所有正整数) 且 $1^{\beta}-2^{\beta}<0, 3^{\beta}-4^{\beta}<0, 5^{\beta}-6^{\beta}<0,\ldots,(n-1)^{\beta}-(n)^{\beta}<0,\ldots$,或 $1^{\beta}-2^{\beta}>0, 3^{\beta}-4^{\beta}>0, 5^{\beta}-6^{\beta}>0,\ldots,(n-1)^{\beta}-(n)^{\beta}>0$,且 $\frac{1}{\left(1-2^{1-\beta}\right)}\neq 0$,

如果 χ(n) ≠0($\chi(n)\in R, n\in Z^{+}$ 且 n 遍取所有正整数) 且 β∈R and β≠ $-2n(n\in Z^{+})$，那么 L(β, χ(n))≠0(β∈R 且 β≠ $-2n, n\in Z^{+}$, χ(n)∈R 且 χ(n) ≡1, $n\in Z^{+}$ 且 n 遍取所有正整数) 且 L(β, 1)≠0(β∈R 且 β≠ $-2n, n\in Z^{+}$, 且 n 遍取所有正整数), 所以对于黎曼 ζ(s)(s∈C and s≠ 1, $s\neq -2n, n\in Z^{+}$) 函数来说，如果 $s\neq -2n(n\in Z^{+})$，那么它对应的朗道-西格尔函

数 L(β,1)(β∈R 且 β ≠ −2n，n ∈ Z^+，且 n 遍取所有正整数)不存在纯实数零点。如果 s ≠ −2n(n ∈ Z^+)，其它朗道-西格尔函数 L(β, ✗(n)) (β∈R 且 β ≠ −2n，n ∈ Z^+，且 n 遍取所有正整数)也不存在纯实数零点，这意味着如果s ≠ −2n(n ∈ Z^+)，那么黎曼ζ(s) (s∈C and s≠ 1and s ≠ −2n，n ∈ Z^+)函数不存在变量为纯实数的零点，这意味着如果s ≠ −2n(n ∈ Z^+)，那么广义黎曼 L(s, ✗(n))=0(s∈C and s≠ 1, and s ≠ −2n，n ∈ Z^+，且 n 遍取所有正整数)函数也不存在变量 s 为纯实数的零点，同时广义黎曼 L(s, ✗(n))=0(s∈C and s≠ 1, and s ≠ −2n，n ∈ Z^+, ✗(n)∈R 且 n 遍取所有正整数) 函数的零点满足 s=$\frac{1}{2}$+ti(t∈R,t≠0) 和 s=$\frac{1}{2}$-ti(t∈R,t≠0) ，表明孪生素数，波利尼亚克猜想和哥德巴赫猜想几乎成立。

如果 ✗(n) = 0(n ∈ Z^+，且n遍取所有正整数) 或 β ∈R 且 β = −2n (n ∈ Z^+), 那么 L(β, ✗(n))=0(β∈R 且 β≠ −2n，n ∈ Z^+, ✗(n)∈R ，且n遍取所有正整数) 且 L(β, 1)=0(β ∈R and β≠ −2n，n ∈ Z^+，且n遍取所有正整数), 所以对于黎曼 ζ(s)(s∈C and s≠ 1, s ≠ −2n，n ∈ Z^+) 函数来说, 如果s = −2n(n ∈ Z^+)，那么它对应的朗道-西格尔函数 L(β,1)(β∈R 且 β ≠ −2n，n ∈ Z^+，且 n 遍取所有正整数)就存在纯实数零点。这意味着如果s = −2n(n ∈ Z^+)，那么广义黎曼 L(s, ✗ (n))=0(s∈C and s ≠ 1, and s ≠ −2n，n ∈ Z^+, n ∈ Z^+，且 n 遍取所有正整数)函数也存在变量 s 为数纯实数的零点，同时广义黎曼 L(s, ✗ (n))=0(s∈C and s≠ 1, and s = −2n，✗ (n)∈R ,n ∈ Z^+，且 n 遍取所有正整数) 函数的零点满足 s=$\frac{1}{2}$+ti(t∈R,t≠0) 和 s=$\frac{1}{2}$-ti(t∈R,t≠0)，表明孪生素数，波利尼亚克猜想和哥德巴赫猜想都完全成立。

公式 2

让我们定义复数 Z= x+yi(x ∈R,y∈R), 并假设我们有任意复数 s=σ+ui(σ ∈R，u∈R).我们使用 r(r∈R 且 r>0) 来表示复数 Z=x+yi(x∈R，y∈R)的模|Z|,并使用 φ 来表示复数 Z= x+yi(x∈R,y∈R)的复角度 Am(Z).也就是 |Z|=r, 那么 r= $(x^2+y^2)^{\frac{1}{2}}$, 因此 Z=r(Cos(φ)+iSin(φ)) 且 φ=|arccos($\frac{x}{(x^2+y^2)^{\frac{1}{2}}}$)|, 且 φ∈(-π, π], 那么 φ=Am(Z)。根据

$x^{s}=x^{\sigma+ui}=x^{\sigma}x^{ui}$=$x^{\sigma}(\cos(\ln x)+i\sin(\ln x))^{u}$=$x^{\sigma}(\cos(u\ln x)+i\sin(u\ln x))$ 可以得到:

$r^{s}=r^{(\sigma+ui)}=r^{\sigma}r^{ui}=r^{\sigma}(\cos(\ln x)+i\sin(\ln x))^{u}$ =$r^{\sigma}(\cos(u\ln x)+i\sin(u\ln x))$ (r>0)，那么

f(Z,s)=z^{s}=$(r(\cos(\varphi)+i\sin(\varphi))^{\sigma+ui}$=$(r(\cos(\varphi)+i\sin(\varphi))^{\sigma}r(\cos(\varphi)+i\sin(\varphi))^{ui}$ = $r^{\sigma}(\cos(\sigma\varphi)+i\sin(\sigma\varphi))(r(\cos(\varphi)+i\sin(\varphi))^{ui}=r^{\sigma}(\cos(\sigma\varphi)+i\sin(\sigma\varphi))r^{ui}(\cos(\varphi)+i\sin(\varphi))ui$ = rσ(cos(σφ)+isin(σφ))(cos(ulnr)+isin(ulnr))(cos(uφ)+isin(uφ))i

=$r^{\sigma}(\cos(\rho\varphi+u\ln r)+i\sin(\rho\varphi+u\ln r))(\cos(u\varphi)+i\sin(u\varphi))^{i}$ 。

因为

$Z=e^{\ln|Z|+iAm(Z)}$=$e^{\ln|Z|}e^{iAm(Z)}$=$e^{\ln|Z|}$(cos(Am(Z))+isin(Am(Z)))=r(cos(Am(Z))+isin(Am(Z))),so

lnZ=ln|Z|+iAm(Z)(−π<Am(Z)<= π)。

假设 a>0,那么 a^{x}=$e^{\ln(a^{x})}=e^{x\ln a}$ ，那么 z^{s}=$e^{s\ln z}$ 。

假设任意复数 Q= cos(uφ) + isin(uφ)，并且假设

复数 ψ= i,then lnQ=ln|Q|+iAm(Q)（−π<Am(Q)<= π).

因为 0<=|sin(uφ)|<=1,

所以

如果−π<uφ<= π,那么 Am(Q)=uφ 且 − π<Am(Q)<= π；

如果uφ>π，那么 Am(Q)=uφ-2kπ(k∈Z^{+}) 且 − π<Am(Q)<= π；

如果 uφ < $-\pi$，那么 Am(Q)=uφ+2kπ(k∈Z^{+}) 且 − π<Am(Q)<= π. 那么

如果 Am(Q)=uφ，那么

$(\cos(u\varphi)+i\sin(u\varphi))^{i}=Q^{\psi}=e^{\psi\ln Q}=e^{\psi(\ln|Q|+iAm(Q))}$=$e^{i(o+iAm(Q))}$=$e^{-u\varphi}$。

那么 f(Z,s)=z^{s}=$r^{\sigma}(\cos(\sigma\varphi+u\ln r)+i\sin(\sigma\varphi+u\ln r))(\cos(u\varphi)+i\sin(u\varphi))^{i}$

=$e^{-u\varphi}r^{\sigma}(\cos(\rho\varphi+u\ln r)$+i$e^{-u\varphi}r^{\sigma}\sin(\rho\varphi+u\ln r)$。

用$(x^{2}+y^{2})^{\frac{1}{2}}$ 替换 r，将上式化为：

f(Z,s)=z^{s}=$e^{-u\varphi}(x^{2}+y^{2})^{\frac{\sigma}{2}}(\cos(\rho\varphi+u\ln(x^{2}+y^{2})^{\frac{1}{2}}))$

+i$e^{-u\varphi}(x^{2}+y^{2})^{\frac{\sigma}{2}}(\sin(\sigma\varphi+u\ln(x^{2}+y^{2})^{\frac{1}{2}}))$ 。

如果Am(Q)= uφ-2kπ(k∈Z^+),那么

$(\cos(u\varphi)+i\sin(u\varphi))^i=Q^{\psi}=e^{\psi\ln Q}=e^{\psi(\ln|Q|+iAm(Q))}=e^{i(o+i(u\varphi-2k\pi))}=e^{2k\pi-u\varphi}$，那么

$f(Z,s)=z^s=r^{\sigma}(\cos(\sigma\varphi+u\ln r)+i\sin(\sigma\varphi+u\ln r))(\cos(u\varphi)+i\sin(u\varphi))^i$

$=e^{2k\pi-u\varphi}r^{\sigma}(\cos(\sigma\varphi+u\ln r)+ie^{2k\pi-u\varphi}r^{\sigma}\sin(\sigma\varphi+u\ln r)$ 。.

用$(x^2+y^2)^{\frac{1}{2}}$替换 r，将上式化为：

$f(Z,s)=z^s=e^{2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\cos(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$

$+ie^{2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\sin(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$ 。

如果Am(Q)=uφ+2kπ(k∈Z^+), 那么

$(\cos(u\varphi)+i\sin(u\varphi))^i=Q^{\psi}=e^{\psi\ln Q}=e^{\psi(\ln|Q|+iAm(Q))}=e^{i(o+i(u\varphi+2k\pi))}=e^{-2k\pi-u\varphi}$，那么

$f(Z,s)=z^s=r^{\sigma}(\cos(\sigma\varphi+u\ln r)+i\sin(\sigma\varphi+u\ln r))(\cos(u\varphi)+i\sin(u\varphi))^i$

$=e^{-2k\pi-u\varphi}r^{\sigma}(\cos(\sigma\varphi+u\ln r)+ie^{-2k\pi-u\varphi}r^{\sigma}\sin(\sigma\varphi+u\ln r)$ 。

用$(x^2+y^2)^{\frac{1}{2}}$替换 r，将上式化为：

$f(Z,s)=z^s=e^{-2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\cos(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$

$+ie^{-2k\pi-u\varphi}(x^2+y^2)^{\frac{\sigma}{2}}(\sin(\sigma\varphi+u\ln(x^2+y^2)^{\frac{1}{2}}))$。

III. 结论

在黎曼假设和黎曼猜想以及广义黎曼假设和广义黎曼猜想被证明是完全有效之后，对黎曼猜想的研究质数分布和其他与黎曼假设和黎曼猜想有关的研究将起到推动作用。读者在这方面可以做很多事情。

IV.感谢

感谢你阅读本论文。

V.贡献

唯一的作者，提出研究问题，论证和证明问题。

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