GLOBAL WELLPOSEDNESS FOR THE HOMOGENEOUS PERIODIC NAVIER-STOKES EQUATION SMALL INITIAL DATA

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ABSTRACT. We consider the homogeneous incompressible Navier-Stokes equations on periodic domain $\mathbb{T}^d$ with sufficiently small initial datum. For $d \geq 3$ and $s \geq \frac{d}{2} - 1$, the equations are globally wellposed in the energy space $L_t^\infty \dot{H}_x^s(\mathbb{R}^+; \mathbb{T}^d) \cap L_t^2 \dot{H}_x^{s+1}(\mathbb{R}^+; \mathbb{T}^d)$ in the critical sense if the initial data u_0 is divergence free, mean zero and $\|u_0\|_{\dot{H}_x^s(\mathbb{T}^d)}$ is sufficiently small. We use Strichartz estimates for the heat kernel, bilinear Strichartz estimates to obtain an iteration scheme critically depending on the value of $e^{t\triangle}u_0$ in $L_t^2 \dot{H}_x^s([0,T]; \mathbb{T}^d)$ norm. Use such iteration scheme, we can prove $u(t)$ is decreasing in $\dot{H}_x^s(\mathbb{T}^d)$ with time t. The decay property guarantees the global existence and wellposedness.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the incompressible Navier-Stokes (NS) equation

$$\begin{aligned} \partial_t u - \triangle u + \nabla p + \nabla \cdot (u \otimes u) &= 0 \\ \nabla \cdot u &= 0 \\ u(0,x) &= u_0 \end{aligned} \tag{1.1}$$

with periodic boundary conditions in $x \in \mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$ and kinematic viscosity 1. Where the solution is a vector value function $u : \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{R}^d$ and u_0 is a divergence free vector field, i.e., $\xi \cdot \widehat{u_0}(\xi) = 0$. We also consider solutions normalized to have zero spatial mean, i.e., $\int_{\mathbb{T}^d} u(t,x)dx = 0$ or equivalently $\widehat{u}(0) = 0$. The pressure p can be eliminated from the system via Leray projections, and so we view this equation as an evolution equation for u alone. If we take inner product of (1.1) with u and integrate in time, we obtain the fundamental energy identity

$$\frac{1}{2}\int_{\mathbb{T}^d} |u(T,x)|^2 dx + \int_0^T \int_{\mathbb{T}^d} |\nabla u(t,x)|^2 dxdt = \frac{1}{2}\int_{\mathbb{T}^d} |u_0(x)|^2 dx \tag{1.2}$$

for suitable solutions. The solutions to (1.1) obey the Duhamel's formula

$$u(t) = e^{t\triangle}u_0 - \int_0^t e^{(t-s)\triangle}\mathbb{P}\mathrm{div}\,(u(s) \otimes u(s))\, ds, \tag{1.3}$$

here $\mathbb{P}$ is the Leray projector on to divergence free vector fields $\mathbb{P}u = u - \nabla\triangle^{-1}\nabla \cdot u$ and $e^{t\triangle}$ denotes convolution with the heat kernel [2]. For divergence free vector fields u,v, we have $\mathrm{div}\,(u \otimes v) = u \cdot \nabla v$. Hence we can rewrite the equation (1.1) as

$$\partial_t u - \triangle u + \mathbb{P}(u \cdot \nabla u) = 0. \tag{1.4}$$

Theorem 1. *We have the following wellposedness theorem for NS (1.1): Let $d \geq 3$. There exists $\delta > 0$ depending on s and d. For $u_0 \in \dot{H}_x^s(\mathbb{T}^d)$, $\|u_0\|_{\dot{H}_x^s(\mathbb{T}^d)} \leq \delta$, divergence free and mean zero with $s \geq \frac{d}{2} - 1$, the NS (1.1) is globally wellposed in $L_t^\infty \dot{H}_x^s([0,\infty); \mathbb{T}^d) \cap L_t^2 \dot{H}_x^{s+1}([0,\infty); \mathbb{T}^d)$ in the critical sense. Moreover, $\|u(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}$ is decreasing with time t.*

Before we go forward, first define the Fourier coefficient and the Sobolev norm used in the paper. We use $X \lesssim Y$, $Y \gtrsim X$ to denote the estimate $X \leq CY$ for an absolute constant C. If C depends on a parameter α, we denote the inequality by $X \lesssim_\alpha Y$. The Fourier coefficient on a torus$\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ is given by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} u(x) e^{-i\xi\cdot x} dx, \quad u(x) = \sum_{\xi\in\mathbb{Z}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \widehat{u}(\xi) e^{i\xi\cdot x}, \tag{1.5}$$

for all $\xi \in \mathbb{Z}^d$.We also use the notation $\mathcal{F}_x u = \widehat{u}(\xi)$ and $\mathcal{F}_x^{-1}\widehat{u} = u$ to denote Fourier transformation and inverse Fourier transformation respectively in this paper. The homogeneous Sobolev norm $\dot{H}_x^s$ is defined by

$$\|u\|_{\dot{H}_x^s(\mathbb{T}^d)} := \left[\sum_{\xi\in\mathbb{Z}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2\right]^{\frac{1}{2}}. \tag{1.6}$$

Since we only consider divergence free functions u in this paper, we only use homogeneous Sobolev spaces. It is easy to see that $\|u\|_{\dot{H}_x^s(\mathbb{T}^d)} \leq \|u\|_{\dot{H}_x^r(\mathbb{T}^d)}$ whenever $s < r$. Also define the operator $|\nabla|^s$ by

$$|\nabla|^s u := \mathcal{F}_x^{-1} |\xi|^s \widehat{u}(\xi) \tag{1.7}$$

for any $s \in \mathbb{R}$. Also, the heat kernel $e^{t\triangle}u$ can be written as $e^{t\triangle}u := \mathcal{F}_x^{-1} e^{-t|\xi|^2}\widehat{u}(\xi)$. The inner product related to $\dot{H}_x^s(\mathbb{T}^d)$is denoted by

$$\langle u, v\rangle_{\dot{H}_x^s\times\dot{H}_x^s(\mathbb{T}^d)} := \sum_{\xi\in\mathbb{Z}^d} |\xi|^{2s}\, \overline{\widehat{u}(\xi)}\widehat{v}(\xi). \tag{1.8}$$

The NS equation (1.1) has scaling symmetry, for $L > 0$ and $u(t, x)$ is a solution to (1.1) on $\mathbb{T}^d$,

$$u_L(t, x) := \frac{1}{L} u\left(\frac{t}{L^2}, \frac{x}{L}\right)$$

is also a solution to the NS equation with domain scaling to $\mathbb{T}_L^d$. The scaling property of s is given by the equation $\|u_L(t)\|_{\dot{H}_x^s(\mathbb{T}_L^d)} = L^{\frac{d}{2}-1-s}\|u(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}$. When $s = \frac{d}{2} - 1$, the $\dot{H}_x^{\frac{d}{2}-1}$norm of u_L is invariant for all $L > 0$. When $s > \frac{d}{2} - 1$, the equation is sub-critical and we expect the high frequencies of the solution to evolve linearly for all time. The X^s norm has scaling $L^{\frac{d}{2}-1-s}$ for the solution to the NS equation (1.1). Since the conservation law (1.2) is in super-critical and have no use to achieve global regularity. In this paper, we will take advantage from the conservation quantity for the linear heat equation, which can be applied to critical and sub-critical energy spaces: For any $s \geq 0$, from a direct computation there is the conservation quantity

$$\left\|e^{T\triangle}u_0\right\|_{\dot{H}_x^s(\mathbb{T}^d)}^2 + 2\left\|e^{t\triangle}u_0\right\|_{L_t^2\dot{H}_x^{s+1}([0,T];\mathbb{T}^d)}^2 = \|u_0\|_{\dot{H}_x^s(\mathbb{T}^d)}^2 \tag{1.9}$$

for $\forall T > 0$.

The proof of the theorem will follow the manner: First we apply a Strichartz estimate for the heat kernel, and obtain the bound for the integral part of (1.3),

$$\left\|\int_0^t e^{(t-r)\triangle}\mathbb{P}(u\cdot\nabla v)(r)dr\right\|_{X^s(I;\mathbb{T}^d)} \lesssim \|u\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}\|v\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}.$$

The local wellposedness can be obtained by setting up iteration scheme. Let $u_1 = e^{t\triangle}u_0$. For $n \geq 2$, let u_n solve

$$\begin{aligned} \partial_t u_n - \triangle u_n + \mathbb{P}(u_{n-1}\cdot\nabla u_{n-1}) &= 0 \\ u_n(0, x) &= u_0 \end{aligned} \tag{1.10}$$

If the sequence (u_n) converges, the limit is a solution of (1.1) with initial data u_0. The locally wellposedness is hold for large initial data in $H_x^s(\mathbb{T}^d)$ and $s \geq \frac{d}{2} - 1$. For the small initial data, the global wellposedness of the solution heavily rely on the decay property: If the initial data is sufficiently small, for any time $0 < t_1 < t_2$, we have

$$\|u(t_2)\|_{\dot{H}^s_x(\mathbb{T}^d)} \le \|u(t_1)\|_{\dot{H}^s_x(\mathbb{T}^d)}.$$

Moreover, $\|u(t_2)\|_{\dot{H}^s_x(\mathbb{T}^d)} = \|u(t_1)\|_{\dot{H}^s_x(\mathbb{T}^d)}$ if and only if u is a zero solution. To obtain the decay property, here we use the approach in [4] with some modification (See also [1] for a similar setting). Observe that $u_n - u_{n-1}$ is an n-linear operator from the data space to the solution space; denote it by $F_n(u_1, \cdots, u_1)$. Under appropriate convergence assumptions, one gets the following analytic expansion for the solution u,

$$u = u_1 + \sum_{n=2}^{\infty} F_n(u_1, \cdots, u_1). \tag{1.11}$$

By (1.9), if we can prove that $2\langle \sum_{n=2}^{\infty} F_n(u_1, \cdots, u_1), u_1\rangle_{\dot{H}^s_x} + \|\sum_{n=2}^{\infty} F_n(u_1, \cdots, u_1)\|^2_{\dot{H}^s_x}$ is small enough in a short time interval $[0,T]$, the decay property follows. Let $\epsilon(T) := \|u_1\|_{L^2_t\dot{H}^{s+1}_x([0,T];\mathbb{T}^d)}$, by the Strichartz estimate, for $n \ge 2$ and some large constant C,

$$\|F_n(u_1, \cdots, u_1)\|_{L^\infty_t\dot{H}^{s+1}_x([0,T];\mathbb{T}^d)} \le C^{n-1}\epsilon^n(T).$$

Hence choosing T small enough, the summation

$$\left\|\sum_{n=2}^{\infty} F_n(u_1, \cdots, u_1)(T)\right\|^2_{\dot{H}^s_x(\mathbb{T}^d)} + 2\left|\sum_{n=3}^{\infty} \langle F_n(u_1, \cdots, u_1)(T), u_1(T)\rangle_{\dot{H}^s_x\times\dot{H}^s_x}\right| \le \epsilon^2(T), \tag{1.12}$$

which is small enough. The difficulties lies in estimates related to $\langle F_2(u_1,u_1), u_1\rangle$. For the summation of third to infinity iterationin, u_1 can be large in $H^s_x(\mathbb{T}^d)$. Since we can only obtain $\|F_2(u_1,u_1)\|_{L^\infty_t\dot{H}^{s+1}_x([0,T];\mathbb{T}^d)} \le C\epsilon^2(T)$ from Strichartz estimates. Thus, if the initial data in $\dot{H}^s_x$ less than $\delta = 1/2C$ can garantee the decay of $\dot{H}^s_x$ norm.

For the continous embedding inequality on compact subset K in $\mathbb{R}^d$, $\|u\|_{L^\infty_x(K)} \le C_K\|u\|_{\dot{H}^s_x}$. The constant C_K depending on the size of the subset K. Scaling a solution u on unit torus $\mathbb{T}^d$ to a solution on torus with size L, $\mathbb{T}^d_L := \mathbb{R}^d/(2\pi L\mathbb{Z})^d$will affect the constant C in the bilinear estimates. Hence for general large initial data u_0 in $\dot{H}^s_x(\mathbb{T}^d)$, there is no guarantee of decay property and global wellposedness. For other similar results on scaling invariant space ∇BMO, see [9].

2. THE STRICHARTZ ESTIMATE FOR THE HEAT KERNEL

Lemma 2. *For $d \ge 3$, and any time interval $I = [0,T] \subset [0,\infty)$ or $I = [0,\infty)$, and any $u_0(x) \in L^2_x(\mathbb{T}^d)$ we have the homogeneous Strichartz estimates*

$$\left\|e^{t\triangle}u_0\right\|_{L^2_t\dot{H}^1_x(I;\mathbb{T}^d)} \le \|u_0\|_{L^2_x(\mathbb{T}^d)}, \quad \left\|e^{t\triangle}u_0\right\|_{L^\infty_tL^2_x(I;\mathbb{T}^d)} \le \|u_0\|_{L^2_x(\mathbb{T}^d)} \tag{2.1}$$

the inhomogeneous Strichartz estimate for any $f(t,x) \in L^2_{t,x}(I;\mathbb{T}^d)$,

$$\left\|\int_0^t e^{(t-r)\triangle}f(s)dr\right\|_{L^2_t\dot{H}^2_x(I;\mathbb{T}^d)} \le \|f\|_{L^1_t\dot{H}^1_x(I;\mathbb{T}^d)}, \tag{2.2}$$

$$\left\|\int_0^t e^{(t-r)\triangle}f(s)dr\right\|_{L^\infty_t\dot{H}^1_x(I;\mathbb{T}^d)} \le \|f\|_{L^1_t\dot{H}^1_x(I;\mathbb{T}^d)}. \tag{2.3}$$

Proof. See also Tao's work for $u \in C^0_tH^1_0(I;\mathbb{T}^3)$. [11] for this estimate. By Parseval's identity $\sum_{\xi\in\mathbb{Z}^d}|\widehat{u_0}(\xi)|^2 = \|u_0\|_{L^2_x(\mathbb{T}^d)}$, we have the following inequalities for homogeneous Strichartz estimates

$$\left\|e^{t\triangle}u_0\right\|_{L^2_t\dot{H}^1_x(I;\mathbb{T}^d)} = \int_I \sum_{\xi\in\mathbb{Z}^d}|\xi|^2 e^{-2|\xi|^2t}|\widehat{u_0}(\xi)|^2 dt \le \sum_{\xi\in\mathbb{Z}^d}\left(\int_0^\infty |\xi|^2 e^{-2|\xi|^2t}dt\right)|\widehat{u_0}(\xi)|^2,$$

and

$$\left\|e^{t\triangle}u_0\right\|_{L_t^\infty L_x^2(I;\mathbb{T}^d)} = \sup_{t\in I}\sum_{\xi\in\mathbb{Z}^d} e^{-|\xi|^2 t}\left|\widehat{u_0}(\xi)\right|^2 \leq \sum_{\xi\in\mathbb{Z}^d}\left|\widehat{u_0}(\xi)\right|^2.$$

For the inhomogeneous Strichartz estimates, let φ be any function in $L_t^2L_x^2\left(I;\mathbb{T}^d\right)$ and $\|\varphi\|_{L_t^2L_x^2(I;\mathbb{T}^d)} = 1$, (2.2) can be interpret as the following equation

$$\left\|\int_0^t e^{(t-r)\triangle}f(s)dr\right\|_{L_t^2\dot{H}_x^2(I;\mathbb{T}^d)} = \sup_{\varphi}\int_I\int_0^t \operatorname{Re}\sum_{\xi}|\xi|^2 e^{-|\xi|^2(t-r)}\widehat{f}(r,\xi)\,\overline{\widehat{\varphi}(t,\xi)}drdt. \tag{2.4}$$

We can rewrite the boundary of the time integrations into the following formula

$$\int_I\int_0^t e^{-|\xi|^2(t-r)}\widehat{f}(r,\xi)\,\overline{\widehat{\varphi}(t,\xi)}drdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{1}_I(t-s)\,|\xi|^2 e^{-|\xi|^2(t-r)}\mathbf{1}_I(r)\widehat{f}(r,\xi)\mathbf{1}_I(t)\overline{\widehat{\varphi}(t,\xi)}drdt,$$

after applying Young's inequality on functions $\mathbf{1}_I(t-r)\,|\xi|^2 e^{-|\xi|^2(t-r)}$, $\mathbf{1}_I(r)\widehat{f}(r,\xi)$, and $\mathbf{1}_I(t)\overline{\widehat{\varphi}(t,\xi)}$, we have

$$\left|\int_I\int_0^t e^{-|\xi|^2(t-r)}\widehat{f}(r,\xi)\,\overline{\widehat{\varphi}(t,\xi)}drdt\right| \leq \left\|\widehat{f}(t,\xi)\right\|_{L_t^1(I)}\left\|\widehat{\varphi}(t,\xi)\right\|_{L_t^2(I)}\left(\int_I|\xi|^4 e^{-2|\xi|^2 r}dr\right)^{\frac{1}{2}}.$$

Since $\left(\int_I|\xi|^4 e^{-2|\xi|^2 r}dr\right)^{\frac{1}{2}} \leq \left(\int_0^\infty \frac{1}{2}|\xi|^4 e^{-2|\xi|^2 r}dr\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}}|\xi|$, by Hölder inequality, and (2.4), we obtain (2.2). Applying the Minkowski's inequality we can switch the order of integration over t with the summation over ξ,

$$\left\|\int_0^t e^{(t-r)\triangle}f(r)dr\right\|_{L_t^2\dot{H}_x^2(I;\mathbb{T}^d)} \leq \sup_{\varphi}\left(\sum_{\xi\in\mathbb{Z}^d}\frac{1}{\sqrt{2}}|\xi|\left\|\widehat{f}(t,\xi)\right\|_{L_t^1(I)}^2\right)^{\frac{1}{2}}\left(\sum_{\xi\in\mathbb{Z}^d}\left\|\widehat{\varphi}(t,\xi)\right\|_{L_t^2(I)}^2\right)^{\frac{1}{2}} \leq \|f\|_{L_t^1\dot{H}_x^1(I;\mathbb{T}^d)}.$$

For (2.3), for $t-s\geq 0$ it is obvious that $\left|e^{(t-s)\triangle}f(s)\right| \leq |f(s)|$.

$$\left\|\int_0^t e^{(t-r)\triangle}f(r)dr\right\|_{L_t^\infty\dot{H}_x^1(I;\mathbb{T}^d)} \leq \sup_{t\in I}\int_0^t\left\|e^{(t-s)\triangle}f(r)\right\|_{\dot{H}_x^1(\mathbb{T}^d)}dr \leq \|f(t,x)\|_{L_t^1\dot{H}_x^1(I;\mathbb{T}^d)}$$

, we prove (2.3). ☐

Note that it is obvious that all the inequality coefficients in Lemma 2 are constants independent of dimension d and s.

Lemma 3. *Let $d\geq 3$, and any time interval $I=[0,T]\subset[0,\infty)$ or $I=[0,\infty)$, and given $u_0\in\dot{H}_x^s\left(\mathbb{T}^d\right)$, $f\in L^1\dot{H}_x^s\left(I;\mathbb{T}^d\right)$, $s\in R$, f and u_0 are divergence free function. We have the following Strichartz estimates:*

$$\left\|e^{t\triangle}u_0\right\|_{X^s(I;\mathbb{T}^d)} \leq 2\|u_0\|_{\dot{H}_x^s(\mathbb{T}^d)}, \tag{2.5}$$

$$\left\|\int_0^t e^{(t-r)\triangle}f(r)dr\right\|_{L_t^\infty H_x^s(I;\mathbb{T}^d)} \leq \|f\|_{L_t^1\dot{H}_x^s(I;\mathbb{T}^d)}, \tag{2.6}$$

$$\left\|\int_0^t e^{(t-r)\triangle}f(r)dr\right\|_{L_t^2H_x^{s+1}(I;\mathbb{T}^d)} \leq \|f\|_{L_t^1\dot{H}_x^s(I;\mathbb{T}^d)}. \tag{2.7}$$

Proof. If we substitute u_0 and f by $|\nabla|^s u_0$ and $|\nabla|^s f$ in Lemma 2, we can obtain the following Strichartz estimates in $L_t^\infty H_x^s\left(I;\mathbb{T}^d\right)$, $L_t^2H_x^{s+1}\left(I;\mathbb{T}^d\right)$ and $L_t^1\dot{H}_x^s\left(I;\mathbb{T}^d\right)$ spaces without any difficulty. ☐

3. PROOF OF THE THEOREM

Let $\frac{d}{2}-1\leq s\leq\frac{d}{2}$ and $d\geq 4$. Let u_0 be a divergence free, mean zero vector value, and $\|u_0\|_{\dot{H}^s_x(\mathbb{T}^d)}=M<\infty$. By the standard iteration scheme and (1.3), let $u_1=e^{t\triangle}u_0$, and

$$u_n=e^{t\triangle}u_0-\int_0^t e^{(t-s)\triangle}\mathbb{P}\left(u_{n-1}(r)\cdot\nabla u_{n-1}(r)\right)dr. \tag{3.1}$$

It is easy to verify that for divergence free and mean zero vector field u, v, we have that $e^{it\triangle}u$ and $\mathbb{P}\left(u\cdot\nabla v\right)=\mathbb{P}\mathrm{div}\left(u\otimes v\right)$ are also divergence free and mean zero. Therefore if u_n is divergence free and mean zero, u_{n+1} is also divergence free and mean zero. Here define the bilinear form $B\left(u,v\right)$ by

$$B\left(u,v\right)(t):=\int_0^t e^{(t-s)\triangle}\mathbb{P}\left(u\left(s\right)\cdot\nabla v\left(s\right)\right)ds. \tag{3.2}$$

Lemma 4. *Let u, $v\in X^s\left(I;\mathbb{T}^d\right)$ for some time interval $[0,T]$ with $s>\frac{d}{2}-1$ and $d\geq 3$. Then theres is the bound for the bilinear form B,*

$$\left\|B\left(u,v\right)(t)\right\|_{X^s(I;\mathbb{T}^d)}\lesssim_{s,d}\|v\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}\|u\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}. \tag{3.3}$$

Proof. By Bersteins type inequalities, we have $|\nabla^s fg|\lesssim_s|\left(\nabla^s f\right)g|+|f\nabla^s g|$. Using standard dyadic frequency decomposition, the deritivate can be move to the high frequency function. Here we omit the detail proof, from Bersteins type inequalities and taking summation over all frequency decomposition, $|\nabla^s\left(f\nabla g\right)|\lesssim_s\left|f\nabla^{s+1}g\right|+\left|g\nabla^{s+1}f\right|$ for all $s\geq 0$. Hence the estimate holds

$$\|\mathbb{P}\left(u\cdot\nabla v\right)\|_{\dot{H}^s_x(\mathbb{T}^d)}\lesssim\|u\|_{L^\infty_x(\mathbb{T}^d)}\|v\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}+\|v\|_{L^\infty_x(\mathbb{T}^d)}\|u\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}\lesssim\|v\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}\|u\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}. \tag{3.4}$$

The second inequality comes from Sobolev embedding $\|f\|_{L^\infty_x(\mathbb{T}^d)}\lesssim_{s,d}\|v\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}$ for $s>\frac{d}{2}-1$. Applying above inequality to the bilinear form $B\left(u,v\right)$,

$$\begin{aligned}\left\|B\left(u,v\right)\right\|_{X^s(I;\mathbb{T}^d)}&\lesssim\|\mathbb{P}\left(u\cdot\nabla v\right)\|_{L^1_t\dot{H}^s_x(I;\mathbb{T}^d)}\\ \lesssim\int_0^T\|v\left(r\right)\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}\|u\left(r\right)\|_{\dot{H}^{s+1}_x(\mathbb{T}^d)}dr&\lesssim\|v\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}\|u\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}.\end{aligned}$$

The estimate at the scaling critical regularity $s=\frac{d}{2}-1$ will require using bilinear wave estimate technique. The proof is shown in the Appendix. Here we directly quote the result that

$$\|\mathbb{P}\left(u\cdot\nabla v\right)\|_{\dot{H}^{\frac{d}{2}-1}_x(\mathbb{T}^d)}\lesssim\|u\|_{\dot{H}^{\frac{d}{2}}_x(\mathbb{T}^d)}\|v\|_{\dot{H}^{\frac{d}{2}}_x(\mathbb{T}^d)}. \tag{3.5}$$

The same argument can be applied to the special case $s=\frac{d}{2}-1$. □

Proposition 5. *Let $d\geq 3$ and $s\geq\frac{d}{2}-1$. For any $u_0\in\dot{H}^s_x\left(\mathbb{T}^d\right)$, the equation (1.1) is locally wellposed in $X^s\left(I;\mathbb{T}^d\right)$ for some time interval $[0,T]$. The value of T is depending on the value of $\|u_1\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}$.*

Proof. First we prove the local wellposedness in the scaling critical norm. The iteration (3.1) converges in $X^s\left(I;\mathbb{T}^d\right)$ where $I=[0,T]$, $T>0$, and $s\geq\frac{d}{2}-1$. We will choose the value T later. To compute the difference between u_{n+1} and u_n, let $D_n:=\mathbb{P}\left(u_n\cdot\nabla u_n-u_{n-1}\cdot\nabla u_{n-1}\right)$, and separate it into two parts:

$$D_n=\mathbb{P}\left(u_n-u_{n-1}\right)\cdot\nabla u_n+\mathbb{P}u_{n-1}\cdot\nabla\left(u_n-u_{n-1}\right).$$

With bilinear form B defined as in (3.2), the integration $u_{n+1}-u_n=-\int_0^t e^{(t-r)\triangle}D_n\left(r\right)dr$ can be written as $\int_0^t e^{(t-r)\triangle}D_n\left(r\right)ds=B\left(u_n-u_{n-1},u_n\right)+B\left(u_{n-1},u_n-u_{n-1}\right)$. By (3.3),

$$\|u_{n+1}-u_n\|_{X^s(I;\mathbb{T}^d)}\lesssim\left(\|u_n\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}+\|u_{n-1}\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}\right)\|u_n-u_{n-1}\|_{L^2_t\dot{H}^{s+1}_x(I;\mathbb{T}^d)}.$$

Let $\|u_1\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} = \epsilon_0$ and assume that $\|u_n\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} < 2\epsilon_0$ for all $n > 1$. Since $\lim_{t\to 0^+}\|u_1\|_{L_t^2\dot{H}_x^{s+1}([0,t];\mathbb{R}^+)} = 0$, we can choose T small enough that $\epsilon_0 < \frac{1}{16C}$, where C is a constant only depend on d,

$$\|u_{n+1}-u_n\|_{X^s(I;\mathbb{T}^d)} < 4C\epsilon_0\|u_n-u_{n-1}\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} < \frac{1}{4}\|u_n-u_{n-1}\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}. \tag{3.6}$$

By the contraction mapping u_n converges to a solution $u \in X^s\left(I;\mathbb{T}^d\right)$, which also obeys the required Lipschitz property for local wellposedness. Note that by induction we have the following bound

$$\|u_n-u_{n-1}\|_{X^s(I;\mathbb{T}^d)} \le (4C)^{n-1}\|u_1\|^n_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} = (4C)^{n-1}\epsilon_0^n. \tag{3.7}$$

The assumption bound on I holds by applying (3.7)

$$\|u_n\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} \le \sum_{i=2}^{n}\|u_i-u_{i-1}\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} + \|u_1\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} < \frac{1}{3}\epsilon_0+\epsilon_0 < 2\epsilon_0 \tag{3.8}$$

for all $n \ge 1$.

The uniqueness and dependence on initial data can be obtained by the following inequality. Assume u and v are two solutions to (1.1) on time interval $I = [0,T]$ with initial data u_0 and v_0 respectively, there is the bound

$$\|u-v\|_{X^s(I;\mathbb{T}^d)} \lesssim \left\|e^{t\triangle}u_0-e^{t\triangle}v_0\right\|_{X^s(I;\mathbb{T}^d)} + \left(\|u\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}+\|v\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}\right)\|u-v\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)}.$$

If we take subinterval $I' = [0,t'] \subset I$ small enough that $\|u\|_{L_t^2\dot{H}_x^{s+1}(I';\mathbb{T}^d)}, \|v\|_{L_t^2\dot{H}_x^{s+1}(I';\mathbb{T}^d)} \le \frac{1}{4C}$. By subtract $\frac{1}{2}\|u-v\|_{L_t^2\dot{H}_x^{s+1}(I';\mathbb{T}^d)}$ on both side and (2.5), $\|u-v\|_{X^s(I;\mathbb{T}^d)} \lesssim \left\|e^{t\triangle}u_0-e^{t\triangle}v_0\right\|_{X^s(I;\mathbb{T}^d)} \lesssim \|u_0-v_0\|_{\dot{H}_x^s(\mathbb{T}^d)}$. □

3.1. **Decay property of $u(t)$.**

For the global wellposedness, first we prove that on a small time interval the solution u obtained by above iteration scheme is decreasing in $\dot{H}_x^s\left(\mathbb{T}^d\right)$ norm where $s \ge \frac{d}{2}-1$. From the observation of [1][4], the first approximation of the solution to the corresponding linear equation given by $B(u_1,u_1)$ has the worst property. Indeed, the solution u can be written as the summation of linear part u_1, the first approximation part $B(u_1,u_1)$, and the remainder E.

$$u(t) = u_1(t) - B(u_1,u_1)(t) + E(t), \tag{3.9}$$

where $u_1 = e^{t\triangle}u_0$. Recall that $\|u_1\|_{L_t^2\dot{H}_x^{s+1}(I;\mathbb{T}^d)} = \epsilon_0$. Taking $\dot{H}_x^s$ inner product by using (3.9) at a given time $T > 0$,

$$\|u(T)\|^2_{\dot{H}_x^s(\mathbb{T}^d)} \le \|u_1(T)\|^2_{\dot{H}_x^s(\mathbb{T}^d)} + 2\left|\langle u_1(T), B(u_1,u_1)(T)\rangle_{\dot{H}_x^s\times\dot{H}_x^s(\mathbb{T}^d)}\right| + R(T) \quad ,$$

where

$$\begin{aligned} R(t) := {} & \|B(u_1,u_1)(t)\|^2_{\dot{H}_x^s(\mathbb{T}^d)} + \|E(t)\|^2_{\dot{H}_x^s(\mathbb{T}^d)} \\ & + 2\|B(u_1,u_1)(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}\|E(t)\|_{\dot{H}_x^s(\mathbb{T}^d)} + 2\|u_1(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}\|E(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}. \end{aligned}$$

From the iteration scheme (3.7), we have $\|E(T)\|_{\dot{H}_x^s(\mathbb{T}^d)} \lesssim_{s,d} \epsilon_0^3$. By (3.3), $\|B(u_1,u_1)(T)\|_{\dot{H}_x^s(\mathbb{T}^d)} \lesssim_{s,d} \epsilon_0^2$. Let $\|u_0\|_{\dot{H}_x^s(\mathbb{T}^d)} = M \ge 0$. If we choose the time interval to be small enough such that $\epsilon_0 \le (1+C(s,d))^{-1}(1+M)^{-1}$, where $C(s,d)$ is some constant only depending on s and d. Therefore the bound holds for $R(T)$,

$$R(T) \lesssim \epsilon_0^4 + \epsilon_0^6 + 2\epsilon_0^5 + 2M\epsilon_0^3 \le \epsilon_0^2. \tag{3.10}$$

Notice that if u_0 is a non-zero function, there exists $T > 0$ that for all $t \in (0,T)$, $R(t) < \epsilon_0^2$. The equality holds only for zero function if the time interval is carefully choose. It is suffice to show the decay in short time by showing the inner product between $u_1$and the first approximation $B(u_1,u_1)$ in $\dot{H}_x^s \times \dot{H}_x^s\left(\mathbb{T}^d\right)$ is small enough. If $\left|\langle u_1(T), B(u_1,u_1)(T)\rangle_{\dot{H}_x^s\times\dot{H}_x^s(\mathbb{T}^d)}\right| \le \epsilon_0^2$, the following proposition holds.

Proposition 6. *Let $d \ge 3$, $s \ge \frac{d}{2}-1$. If u is a solution to (1.1) and locally wellposed in $X^s\left([0,T];\mathbb{T}^d\right)$ for some $T > 0$ with $\|u(0)\|_{\dot{H}_x^s(\mathbb{T}^d)} \le \frac{1}{2C(s,d)}$, $\|u(t)\|_{\dot{H}_x^s(\mathbb{T}^d)}$ is decreasing on $[0,T]$.*

Proof. Let $u_1 = e^{t\triangle}u_0$,and $\|u_0\|_{\dot{H}^s_x(\mathbb{T}^d)} = M$. By (3.4),

$$\int_0^t \|\mathbb{P}(u_1 \cdot \nabla u_1)(r)\|_{\dot{H}^s_x(\mathbb{T}^d)} dr \lesssim \|u_1\|^2_{L^2_t \dot{H}^{s+1}_x(I;\mathbb{T}^d)} = C(s,d)\,\epsilon_0^2.$$

Hence there is the smallness bound

$$\left|\langle u_1(t), B(u_1,u_1)(t)\rangle_{\dot{H}^s_x \times \dot{H}^s_x}\right| \lesssim \|u_1\|_{L^\infty_t \dot{H}^{s+1}_x(I;\mathbb{T}^d)} \int_0^t \|\mathbb{P}(u_1 \cdot \nabla u_1)(r)\|_{\dot{H}^s_x(\mathbb{T}^d)} dr \le CM\epsilon_0^2. \tag{3.11}$$

Since $M \le \frac{1}{2C}$, the second iteration in (3.9) is bounded by ϵ_0^2. By taking t_1 small enough, $u(t)$ decreasing in $\dot{H}^s_x(\mathbb{T}^d)$ is obtained by (1.9)

$$\|u(t_1)\|^2_{\dot{H}^s_x(\mathbb{T}^d)} \le \|u_1(t_1)\|^2_{\dot{H}^s_x(\mathbb{T}^d)} + 2\epsilon_0^2 = \|u_0\|^2_{\dot{H}^s_x(\mathbb{T}^d)}.$$

For any t in the interval $[0,t_1]$, the same argument gives us $\|u(t)\|_{\dot{H}^s_x(\mathbb{T}^d)} \le \|u_0\|_{\dot{H}^s_x(\mathbb{T}^d)}$. The argument can also be applied to any subinterval of $[0,t_1]$, which gives us the decay in time. By repeating the argument at time t_n, there is the new time interval $[t_n, t_{n+1}]$, $t_{n+1} > t_n$, with the same decay property. By the uniform bound $\|u(t_n)\|_{\dot{H}^s_x(\mathbb{T}^d)} \le \|u_0\|_{\dot{H}^s_x(\mathbb{T}^d)}$, the argument can be applied until we obtain decay property on $[0,T]$. Moreover, the equality $\|u(t_n)\|_{\dot{H}^s_x(\mathbb{T}^d)} = \|u_0\|_{\dot{H}^s_x(\mathbb{T}^d)}$ holds if and only if $u_0 \equiv 0$. □

The maximal time interval of existence can be extended to $[0,\infty)$ due to that it is impossible to have some finite time T, $\lim_{t\to T}\|u(t)\|_{\dot{H}^s_x(\mathbb{T}^d)} = \infty$. Hence the main theorem is proved.

Remark 7. The extend the proof to non-homogeneous NS equations and NS equations on$\mathbb{R}^d$, see [10]. Other regularity and wellposedness theory can also be found in this paper and its references as well.

4. Appendix–bilinear strichartz estimates for wave equations on torus

In this section we use $\tilde{u}$ to denote the Fourier transformation both in time and space,

$$\tilde{u}(\tau,\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{-\infty}^{\infty}\int_{\mathbb{T}^d} u(t,x)\, e^{-i\xi\cdot x} e^{-it\tau} dx dt, \tag{4.1}$$

where $\xi \in \mathbb{Z}^d$, $\tau \in \mathbb{R}$. For the scaling critical exponent $s+1 = \frac{d}{2}$, there is no Sobolev embedding from $L^\infty_x(\mathbb{T}^d)$ into $\dot{H}^{\frac{d}{2}}_x(\mathbb{T}^d)$ when $d > 1$. To obtain the inequality (3.3), we need to take advantage of the null form structure of $\mathbb{P}(u \cdot \nabla v)$. The null form $Q(u,v)$ represent arbitrary linear combinations, with constant real coefficients of the null forms

$$Q(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v. \tag{4.2}$$

The nonlinearity $\mathbb{P}(u \cdot \nabla v)$ for divergence free vector value functions u, v can be represented as $\mathbb{P}(u\cdot\nabla v) = Q\left(|\nabla|^{-1} u, v\right)$. By applying bilinear Strichartz estimates for the wave equation, the inequality (3.5) can be acheived. Here the bilinear estimates following the work by Klainerman and Machedon [6] on $\mathbb{R}\times\mathbb{R}^3$, Klainerman, Selberg [8], Foschi and Klainerman [3] on $\mathbb{R}\times\mathbb{R}^d$ when $d \ge 2$. A similar results for bilinear estimates on compact manifold without boundary can be found in Hani's work [5].

Define the operator w^α_- and w^α_+ on scaling torus $\mathbb{T}^d$ as

$$\mathcal{F}_{t,x} w^\alpha_-(u) := (2\pi)^{-\frac{d}{2}} \sum_{\xi\in\mathbb{Z}^d} \left||\tau| - |\xi|\right|^\alpha \tilde{u}(\tau,\xi), \tag{4.3}$$

and

$$\mathcal{F}_{t,x} w^\alpha_+(u) := (2\pi)^{-\frac{d}{2}} \sum_{\xi\in\mathbb{Z}^d} \left||\tau| + |\xi|\right|^\alpha \tilde{u}(\tau,\xi), \tag{4.4}$$

where $\alpha \in \mathbb{R}$. The proof starts with re-prove a subset of wave operator bilinear STrichartz estimates on torus and use the bilinear Strichartz estimates to obtain (3.5). In this section, the bilinear estimates are performed on the extra time variable t', where $(t', x) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{T}^d$. The original equation $u(t,x)$ on $\mathbb{R}^+ \times \mathbb{T}^d$ is extended to a new domain $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R}^+ \times \mathbb{T}^d$ with the equation

$$\mathfrak{u}(r,t,x) := \cos(r|\nabla|)\, u(t,x), \tag{4.5}$$

where the operator is defined by $\mathcal{F}_x \cos(r|\nabla|)\, u(x) = \cos(r|\xi|)\, \widehat{f}(\xi)$. It is obvious that $\Box_r \mathfrak{u} \equiv 0$ and $\mathfrak{u}(0,t,x) \equiv u(t,x)$, where $\Box_r := \partial_r^2 + \triangle$. Let $\mathfrak{u}_T(r,x) = \mathfrak{u}(r,T,x)$ be the function of r, x at a fixed time T, $\mathfrak{u}_T(r,x)$ is a homogeneous wave function on $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{T}^d$. The initial data for the equation is given by $\mathfrak{u}_T(0) = u(T)$, $\partial_r \mathfrak{u}_T(0) = 0$. Taking integral on both side of the differential equation with respect to r and x

$$|\mathbb{P}(\mathfrak{u}_T \cdot \nabla \mathfrak{u}_T)(0)|^2 = |\mathbb{P}(\mathfrak{u}_T \cdot \nabla \mathfrak{u}_T)(r)|^2 - \int_0^r \partial_{r'} |\mathbb{P}\mathfrak{u}_T \cdot \nabla \mathfrak{u}_T(r')|^2\, dr',$$

a bound for $\mathbb{P}(\mathfrak{u}_T \cdot \nabla \mathfrak{u}_T)$ at $r = 0$ can be obtained.

The following theorem for wave operator bilinear estimates can be found in [3]: Let $d \geq 2$ on $\mathbb{R} \times \mathbb{R}^d$, $\Box f = \Box g = 0$, $\{f(0), f_t(0)\} = \{f_0, f_1\}$, and $\{g(0), g_t(0)\} = \{g_0, g_1\}$. We have

$$\begin{aligned} &\left\| |\nabla|^{\beta_0} w_+^{\beta_+} w_-^{\beta_-}(fg) \right\|_{L_t^2 L_x^2(\mathbb{R};\mathbb{R}^d)} \\ &\lesssim_d \left(\|f_0\|_{\dot{H}_x^{\alpha_1}(\mathbb{R}^d)} + \|f_1\|_{\dot{H}_x^{\alpha_1 - 1}(\mathbb{R}^d)} \right) \left(\|g_0\|_{\dot{H}_x^{\alpha_2}(\mathbb{R}^d)} + \|g_1\|_{\dot{H}_x^{\alpha_2 - 1}(\mathbb{R}^d)} \right) \end{aligned} \tag{4.6}$$

if and only ifα_1, α_2, β_0, β_1, $\beta_2$satisfy the following conditions:

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{d-1}{2}, \tag{4.7}$$

$$\beta_- \geq -\frac{d-3}{4}, \tag{4.8}$$

$$\beta_0 > -\frac{d-1}{2}, \tag{4.9}$$

$$\alpha_i \leq \beta_- + \frac{d-1}{2}, \quad i = 1, 2, \tag{4.10}$$

$$\alpha_1 + \alpha_2 \geq \frac{1}{2}, \tag{4.11}$$

$$(\alpha_i, \beta_-) \neq \left(\frac{d+1}{4}, -\frac{d-3}{4} \right), \quad i = 1, 2 \tag{4.12}$$

$$(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{1}{2}, -\frac{d-3}{4} \right). \tag{4.13}$$

We expect that the bilinear estimates on $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{T}^d$ follow the same inequalities as the well known estimates on $\mathbb{R} \times \mathbb{R}^d$. Since in the later section only the case $\beta^+ = 0$, $\beta_- = \frac{1}{2}$, and $\beta_0 \geq 0$ are considered, we only prove the special cases of the bilinear estimates on $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{T}^d$.

Lemma 8. *For $d \geq 3$, $f = e^{it|\nabla|} f_0$, $g = e^{it|\nabla|} g_0$ and we have*

$$\left\| w_-^{\frac{1}{2}}(fg) \right\|_{L_t^2 L_x^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]; \mathbb{T}_L^d\right)} \lesssim_d \|f_0\|_{L_x^2(\mathbb{T}^d)} \|g_0\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}. \tag{4.14}$$

for all $f_0 \in L_x\left(\mathbb{T}^d\right), g_0 \in \dot{H}_x^{\frac{d}{2}}\left(\mathbb{T}^d\right)$.

Proof. The Fourier transformation $\widetilde{\psi(t)fg}$ can be written as

$$\begin{aligned}\mathcal{F}_{t,x}\psi(t)fg &= (2\pi)^{-\frac{d}{2}}\int\int\sum_{\eta\in\mathbb{Z}^d}\widehat{f_0}(\xi-\eta)\widehat{g_0}(\eta)\delta(\tau'-|\eta|)\delta(\tilde{\tau}-\tau'-|\xi-\eta|)\widehat{\psi}(\tau-\tilde{\tau})d\tau'd\tilde{\tau}\\ &= (2\pi)^{-\frac{d}{2}}\sum_{\eta\in\mathbb{Z}^d}\widehat{f_0}(\xi-\eta)\widehat{g_0}(\eta)\widehat{\psi}(\tau-|\eta|-|\xi-\eta|).\end{aligned}$$

We follow a similar argument in [6], Here define $\Lambda' := \{\tau : \tau = |\xi_1| + |\xi_2|,\ \text{where } \xi_1, \xi_2 \in \mathbb{Z}^d\}$. Also define $S(\tau,\xi) := \{\eta \in \mathbb{Z}^d : ||\xi-\eta| + |\eta| - \tau| = 0\}$, which forms an ellipsoid. Using the inequality

$$\left\|w_-^{\frac{1}{2}}(f,g)\right\|_{L_t^2L_x^2([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \leq \|\psi(t)Q(f,g)\|_{L_t^2L_x^2(\mathbb{R};\mathbb{T}^d)},$$

we have the following estimate:

(4.15)

$$\left\|w_-^{\frac{1}{2}}(fg)\right\|^2_{L_t^2L_x^2([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim \sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{\eta\in S(\tau,\xi)}\left|\widehat{f_0}(\xi-\eta)\right||\widehat{g_0}(\eta)|\,||\tau|-|\xi||^{\frac{1}{2}}\right]^2$$

$$\lesssim \sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)}|\{\eta:\eta\in S(\tau,\xi)\cap S_\eta(\rho)\}|^{\frac{1}{2}}\left[\sum_{\eta\in S(\tau,\xi)\cap S_\eta(\rho)}\left|\widehat{f_0}(\xi-\eta)\right|^2\left(|\eta|^{\frac{d}{2}}|\widehat{g_0}(\eta)|\right)^2\rho^{-d}\,||\tau|-|\xi||\right]^{\frac{1}{2}}\right]^2.$$

Here $S_\eta(\rho) = \{\eta \in \mathbb{Z}^d : |\eta| = \rho\}$, and we define the counting measure for the intersection of the sphere and the ellipsoid as

$$B_1(\tau,\xi,\rho) := |\{\eta : \eta \in S(\tau,\xi)\cap S_\eta(\rho)\}|. \tag{4.16}$$

Also define the set $R(\tau,\xi)$ can be viewed as the ellipsoid $S(\tau,\xi)$ project to a 2 dimensional plane contains the vector ξ, and contains all the possible ρ value, $R(\tau,\xi) := \{\rho\in\mathbb{R}^+ : S(\tau,\xi)\cap S_\eta(\rho)\neq\phi\}$. To simplify the notation, let

$$A(\tau,\xi,\rho) := \sum_{\eta\in S(\tau,\xi)\cap S_\eta(\rho)}\left|\widehat{f_0}(\xi-\eta)\right|^2\left||\eta|^{\frac{d}{2}}\widehat{g_0}(\eta)\right|^2. \tag{4.17}$$

By Cauchy's inequality the equation (4.15) has the bound,

$$\begin{aligned}&\sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)}\rho^{-\frac{d}{2}}\,||\tau|-|\xi||^{\frac{1}{2}}B_1(\tau,\xi,\rho)^{\frac{1}{2}}A(\tau,\xi,\rho)^{\frac{1}{2}}\right]^2\\ \lesssim &\sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)}\rho^{-d}\,||\tau|-|\xi||\,B_1(\tau,\xi,\rho)\right]\left[\sum_{\rho\in R(\tau,\xi)}A(\tau,\xi,\rho)\right].\end{aligned}$$

It is suffice to prove that

$$B_2(\tau,\xi) := \sum_{\rho\in R(\tau,\xi)}\rho^{-d}B_1(\tau,\xi,\rho)\,||\tau|-|\xi|| \lesssim 1 \tag{4.18}$$

for all τ,ξ.

Since the intersection of the ellipsoid $S(\tau,\xi)$ and the $d-1$ dimensional sphere $S_\eta(\rho)$ is a $d-2$ dimensional sphere, and the radius is $\rho\sin\theta(\xi,\eta)$, we have $B_1(\tau,\xi,\rho)\approx_d \rho^{d-2}\sin^{d-2}\theta(\xi,\eta)$,

$$\sum_{\rho\in R(\tau,\xi)}\rho^{-d}B_1(\tau,\xi,\rho)\,||\tau|-|\xi|| \lesssim \sum_{\rho\in R(\tau,\xi)}\rho^{-2}\,||\tau|-|\xi||\sin^{d-2}\theta(\xi,\eta)$$

We may rewrite r as a function of τ,ξ and ω,

$$\rho = \frac{\tau^2-|\xi|^2}{2(\tau-\xi\cdot\omega)}. \tag{4.19}$$

Since all possible η are of distance at least 1, hence the summation can be bounded by the arc length of the ellipse and denote the arc length by s'. To simplify the notation, we denote $\theta(\xi,\eta)$ by θ, i.e. taking the ξ direction to be the positive x-axis direction, we obtain

$$B_2 \lesssim \int_{R(\tau,\xi)} \rho^{-2} \left||\tau| - |\xi|\right| \sin^{d-2}\theta ds'.$$

In (4.19) we have $\xi \cdot \omega = |\xi| \cos\theta$, hence

$$\frac{d\rho}{d\theta} = \frac{\left(\tau^2 - |\xi|^2\right)|\xi| \sin\theta}{2\left(\tau - |\xi|\cos\theta\right)^2}.$$

Hence we have

$$ds' = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2} d\theta = \rho\sqrt{1 + \frac{|\xi|^2 \sin^2\theta}{\left(\tau - |\xi|\cos\theta\right)^2}} d\theta = \rho \frac{\sqrt{\left(\tau - |\xi|\cos\theta\right)^2 + |\xi|^2\sin^2\theta}}{\tau - |\xi|\cos\theta} d\theta.$$

ρ can be view as a function of $a := \frac{\xi}{|\xi|} \cdot \omega$, and it is easy to verify that ρ decreasing as a increasing from -1 to 1. By using the substitution $da = \sin\theta d\theta$

$$\begin{aligned} B_2 &\lesssim \int_{-\pi}^{\pi} \frac{2\left(\tau - |\xi|\cos\theta\right)}{\tau^2 - |\xi|^2} \left||\tau| - |\xi|\right| \frac{\sqrt{\left(\tau - |\xi|\cos\theta\right)^2 + |\xi|^2\sin^2\theta}}{\tau - |\xi|\cos\theta} \sin^{d-2}\theta d\theta \\ &\lesssim \int_{-1}^{1} \frac{\sqrt{\tau^2 + |\xi|^2 - 2\tau|\xi| a}}{\tau + |\xi|} \left(1 - a^2\right)^{\frac{d-3}{2}} da \lesssim 1. \end{aligned}$$

Therefore we obtain the desired inequality

$$\left\| w_-^{\frac{1}{2}}(fg) \right\|^2_{L_t^2 L_x^2\left(\left[-\frac{1}{2},\frac{1}{2}\right];\mathbb{T}^d\right)} \lesssim \sum_{\xi\in\mathbb{Z}^d} \sum_{\tau\in\Lambda'} \left[\sum_{\rho\in R(\tau,\xi)} A(\tau,\xi,\rho) \right] \lesssim \|f_0\|^2_{L_x^2(\mathbb{T}^d)} \|g_0\|^2_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}.$$

□

Lemma 9. *For $d \geq 3$, and $f = e^{it|\nabla|}f_0$, $g = e^{-it|\nabla|}g_0$ we have*

$$\left\| w_-^{\frac{1}{2}}(fg) \right\|_{L_t^2 L_x^2\left(\left[-\frac{1}{2},\frac{1}{2}\right];\mathbb{T}^d\right)} \lesssim \|f\|_{L_x^2(\mathbb{T}^d)} \|g\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}. \tag{4.20}$$

for $f_0 \in L_x\left(\mathbb{T}^d\right), g_0 \in \dot{H}_x^{\frac{d}{2}}\left(\mathbb{T}^d\right)$.

Proof. The Fourier transform is given by

$$\begin{aligned} \mathcal{F}_{t,x}\psi(t) fg &= (2\pi)^{-\frac{d}{2}} \int\int \sum_{\eta\in\mathbb{Z}^d} \widehat{f_0}(\xi-\eta)\widehat{g_0}(\eta)\delta\left(\tau' + |\eta|\right)\delta\left(\tilde{\tau} - \tau' - |\xi-\eta|\right)\widehat{\psi}(\tau - \tilde{\tau}) d\tau' d\tilde{\tau} \\ &= (2\pi)^{-\frac{d}{2}} \sum_{\eta\in\mathbb{Z}^d} \widehat{f_0}(\xi-\eta)\widehat{g_0}(\eta)\widehat{\psi}(\tau + |\eta| - |\xi-\eta|). \end{aligned}$$

Following a similar argument, defining $S(\tau,\xi) := \left\{\eta\in\mathbb{Z}^d : \left||\xi-\eta| - |\eta| - \tau\right| = 0\right\}$, which forms an hyperboloid by rotating the hyperbola in 2 dimensional space. Since for the counting measure of the intersection of the sphere and the hyperboloid has the same bound as in Lemma 8, $B_1(\tau,\xi,\rho) = \left|\left\{\eta : \eta\in S(\tau,\xi)\cap S_\eta(\rho)\right\}\right| \lesssim \rho^{d-2}\sin^{d-2}\theta(\xi,\eta)$. It is suffice to prove the following bound

$$\sum_{\rho\in R(\tau,\xi)} \rho^{-d} B_1(\tau,\xi,\rho) \left||\tau| - |\xi|\right| \lesssim \sum_{\rho\in R(\tau,\xi)} \rho^{-2} \left||\tau| - |\xi|\right| \sin^{d-2}\theta \lesssim 1,$$

where $R(\tau,\xi)$ is the hyperbola from projecting $S(\tau,\xi)$ on 2 dimensional space containing ξ. Following the similar argument, we can rewrite ρ as a function of $\omega := \eta/|\eta|$

$$\rho = \frac{|\xi|^2 - \tau^2}{2(\tau + \xi\cdot\omega)} \tag{4.21}$$

for $\rho > 0$, and

$$\rho = \frac{|\xi|^2 - \tau^2}{2(-\tau + \xi\cdot\omega)} \tag{4.22}$$

for $\rho < 0$, and we can take ω to $-\omega'$. The proof can be obtained by following a similar argument in Lemma 8. □

Lemma 10. *Let $d \geq 3$, let $f = e^{\pm it|\nabla|}f_0$, and $g = e^{it|\nabla|}g_0$, we have*

$$\left\| |\nabla|^{\beta_0} w_-^{\frac{1}{2}}(fg)\right\|_{L_t^2L_x^2([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim_d \|f_0\|_{\dot{H}_x^{\alpha_1}(\mathbb{T}^d)} \|g_0\|_{\dot{H}_x^{\alpha_2}(\mathbb{T}^d)} \tag{4.23}$$

if α_1, α_2, β_0 satisfy the following conditions:

$$\beta_0 + \frac{d}{2} = \alpha_1 + \alpha_2, \tag{4.24}$$

$$\beta_0 \geq 0, \tag{4.25}$$

$$\alpha_i \leq \frac{d}{2}, \quad i = 1,2. \tag{4.26}$$

Proof. For the ellipsoids case it is suffice to prove when $f = e^{it|\nabla|}f_0$, and $g = e^{it|\nabla|}g_0$. Recall that the definitions $S_\eta(\rho) = \{\eta \in \mathbb{Z}^d : |\eta| = \rho\}$, and $B_1(\tau,\xi,r) := |\{\eta : \eta \in S(\tau,\xi) \cap S_\eta(\rho)\}|$. Let $F_0 = |\nabla|^{\alpha_1} f_0$ and $G_0 = |\nabla|^{\alpha_2} g_0$, we have

$$\begin{aligned} \left\| |\nabla|^{\beta_0} w_-^{\frac{1}{2}}(fg)\right\|^2_{L_t^2L_x^2([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} &\lesssim \sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{\eta\in S(\tau,\xi)} |\xi|^{\beta_0}\left|\widehat{f_0}(\xi-\eta)\right| |\widehat{g_0}(\eta)| \, ||\tau|-|\xi||^{\frac{1}{2}}\right]^2 \\ &\lesssim \sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)} B_1(\tau,\xi,\rho)\left[\sum_{\eta\in S(\tau,\xi)\cap S_\eta(\rho)} \left|\widehat{F_0}(\xi-\eta)\right|^2\left|\widehat{G_0}(\eta)\right|^2 \frac{||\tau|-|\xi||\,|\xi|^{2\beta_0}}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\right]^{\frac{1}{2}}\right]^2. \end{aligned} \tag{4.27}$$

On the intersection $S(\tau,\xi)\cap S_\eta(\rho)$, the quantities have the values$|\eta| = \rho$, $|\xi-\eta| = \tau-\rho$, and $B_1(\tau,\xi,\rho) \lesssim_d \rho^{d-2}\sin^{d-2}\theta(\xi,\eta)$. Also due to the symmetry, we can assume that $|\xi| \sim |\xi-\eta| \gtrsim |\eta|$. If $|\xi| \sim |\eta| \gtrsim |\xi-\eta|$, let $\rho' = |\xi-\eta|$, $\tau-\rho' = |\eta|$ and $B_1(\tau,\xi,\rho) = B_1(\tau,\xi,\tau-\rho') \lesssim_d (\rho')^{d-2}\sin^{d-2}\theta(\xi,\xi-\eta)$, the same argument can be applied.

$$A^*(\tau,\xi,\rho) := \sum_{\eta\in S(\tau,\xi)\cap S_\eta(\rho)} \left|\widehat{F_0}(\xi-\eta)\right|^2\left|\widehat{G_0}(\eta)\right|^2. \tag{4.28}$$

By Cauchy's inequality the equation (4.27) has the bound,

$$\begin{aligned} &\sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)} \frac{||\tau|-|\xi||^{\frac{1}{2}}|\xi|^{\beta_0}}{|\tau-\rho|^{\alpha_1}|\rho|^{\alpha_2}} B_1(\tau,\xi,\rho)^{\frac{1}{2}} A^*(\tau,\xi,\rho)^{\frac{1}{2}}\right]^2 \\ &\lesssim \sum_{\xi\in\mathbb{Z}^d}\sum_{\tau\in\Lambda'}\left[\sum_{r\in R(\tau,\xi)} \frac{||\tau|-|\xi||\,|\xi|^{2\beta_0}}{|\tau-\rho|^{2\alpha_1}|\rho|^{2\alpha_2}} B_1(\tau,\xi,\rho)\right]\left[\sum_{\rho\in R(\tau,\xi)} A^*(\tau,\xi,\rho)\right]. \end{aligned}$$

It is suffice to prove that

$$B_2^*(\tau,\xi) := \sum_{r\in R(\tau,\xi), r\le \tau/2} \frac{||\tau|-|\xi||\,|\xi|^{2\beta_0}}{|\tau-\rho|^{2\alpha_1}|\rho|^{2\alpha_2}} B_1(\tau,\xi,\rho) \lesssim 1. \tag{4.29}$$

By the conditions $\beta_0+\frac{d}{2}=\alpha_1+\alpha_2$ and $\alpha_2\le\frac{d}{2}$, $2\alpha_1-2\beta_0\ge 0$ and $d-2-2\alpha_2\ge 0$. Applying the inequalities $\frac{|\rho|}{|\tau-\rho|}\le 1$ and $\frac{|\xi|}{|\tau-\rho|}\sim 1$, we have

$$\frac{|\xi|^{2\beta_0}|\rho|^{d-2-2\alpha_2}}{|\tau-\rho|^{2\alpha_1}} \lesssim \frac{|\rho|^{d-2-2\alpha_2}}{|\tau-\rho|^{2\alpha_1-2\beta_0}} \lesssim |\rho|^{d-2-2\alpha_1-2\alpha_2+2\beta_0} = |\rho|^{-2}.$$

Therefore following the same substitutions in Lemma 8, the following bound holds

$$\begin{aligned} B_2^*(\tau,\xi) &\lesssim \sum_{r\in R(\tau,\xi), r\le\tau/2} \frac{||\tau|-|\xi||\,|\xi|^{2\beta_0}}{|\tau-\rho|^{2\alpha_1}} |\rho|^{d-2-2\alpha_2}\sin^{d-2}\theta(\xi,\eta) \\ &\lesssim \sum_{r\in R(\tau,\xi), r\le\tau/2} |\rho|^{-2}\,||\tau|-|\xi||\sin^{d-2}\theta(\xi,\eta) \lesssim 1. \end{aligned}$$

For the hyperboloid case $f=e^{-it|\nabla|}f_0$, $g=e^{it|\nabla|}g_0$, we separate the estimate into two parts, where $|\eta|\le 2|\xi|$and where $|\eta|>2|\xi|$. For where $|\eta|\le 2|\xi|$, the same argument used in the ellipsoid case can be applied. For $|\xi-\eta|\approx|\eta|>2|\xi|$, if $\beta_0\ge 0$, we can also apply the argument use in the ellipsoid case. □

In the following section, we are going to prove the estimates for the quadratic form. Suppose f,g are two divergence free vector value functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. The quadratic nonlinearities $\mathbb{P}(f\cdot\nabla g)$ can be written symbolically in the form

$$\mathbb{P}(f\cdot\nabla g) = Q\left(|\nabla|^{-1}f,g\right). \tag{4.30}$$

Here $Q(f,g)$represents arbitrary linear combinations of the null forms

$$\tilde{Q}(f,g)(\tau,\xi) = \sum_{\eta\in\mathbb{Z}^d}\int q(\xi-\eta,\eta)\,\tilde{f}(\tau-\tau',\xi-\eta)\,\tilde{g}(\tau',\eta)\,d\tau',$$

where q is a linear combination of the symbols q_{ij}

$$q_{ij}(\xi,\eta) = \xi_i\eta_j - \xi_j\eta_i.$$

See Lemma 2.1 in [7] for details of the following bounds. For any vectors ξ, $\eta\in\mathbb{R}^n$ we have

$$|\xi\wedge\eta| \lesssim |\xi|^{\frac12}|\eta|^{\frac12}|\xi+\eta|^{\frac12}W^{\frac12}(\tau,\xi;\lambda,\eta), \tag{4.31}$$

where $W(\tau,\xi;\lambda,\eta)$ is the maximum of the weights $||\tau|-|\xi||$, $||\lambda|-|\eta||$, $||\tau+\lambda|-|\xi+\eta||$. Since in the paper only linear wave functions are considered, $\tilde{f}(\tau,\xi)$, $\tilde{g}(\tau,\xi)$ are supported on $|\tau|-|\xi|=0$. The equation (4.31) can be reduced to the case

$$|(\xi-\eta)\wedge\eta| \lesssim |\xi-\eta|^{\frac12}|\eta|^{\frac12}|\xi|^{\frac12}||\tau|-|\xi||^{\frac12}.$$

Also there is the bound

$$\left|\frac{\xi}{|\xi|}\wedge\eta\right| \lesssim \frac{|\eta|^{\frac12}|\xi+\eta|^{\frac12}}{(|\tau|+|\xi|)^{\frac12}}W^{\frac12}(\tau,\xi;\lambda,\eta). \tag{4.32}$$

Applying (4.32), and $\tilde{f}(\tau,\xi)$, $\tilde{g}(\tau,\xi)$ supported on $|\tau|-|\xi|=0$ to $Q\left(|\nabla|^{-1}f,g\right)$, the estimate can be reduced to the case

$$\left|\frac{\xi-\eta}{|\xi-\eta|}\wedge\eta\right| \lesssim |\eta|^{\frac12}||\tau|-|\xi||^{\frac12}$$

Lemma 11. *Let $d \geq 3$, and $\Box f = 0$, with $f(0) = f_0$, $f_t(0) = 0$ and $\Box g = 0$, with $g(0) = g_0$, $g_t(0) = 0$. Let the pair $\{\overline{f_0}, \overline{g_0}\}$ be either $\{f_0, g_0\}$ or $\{g_0, f_0\}$. Then we have the following bounds:*

$$\|\mathbb{P}(f \cdot \nabla g)\|_{L_t^2 \dot{H}_x^{\frac{d}{2}-1}([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim_d \|\overline{f_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \tag{4.33}$$

$$\|\mathbb{P}(f \cdot \nabla g)\|_{L_t^2 \dot{H}_x^{\frac{d}{2}-\frac{1}{2}}([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim_d \|\overline{f_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \tag{4.34}$$

and

$$\|\mathbb{P}(f_t \cdot \nabla g)\|_{L_t^2 \dot{H}_x^{\frac{d}{2}-\frac{3}{2}}([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim_d \|\overline{f_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \tag{4.35}$$

$$\|\mathbb{P}(f \cdot \nabla g_t)\|_{L_t^2 \dot{H}_x^{\frac{d}{2}-\frac{3}{2}}([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)} \lesssim_d \|\overline{f_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \tag{4.36}$$

whenever f_0, g_0 satisfying $\|\overline{f_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} < \infty$.

Proof. Let $f(t) = \frac{1}{2}\left(e^{it|\nabla|} + e^{-it|\nabla|}\right) f_0$, $f_t(t) = \frac{i|\nabla|}{2}\left(e^{it|\nabla|} - e^{-it|\nabla|}\right) f_0$, $g(t) = \frac{1}{2}\left(e^{it|\nabla|} + e^{-it|\nabla|}\right) g_0$, $g_t(t) = \frac{i|\nabla|}{2}\left(e^{it|\nabla|} - e^{-it|\nabla|}\right) g_0$. Let $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have

$$\left\|Q\left(|\nabla|^{-1} f, g\right)\right\|_{L_t^2 \dot{H}_x^s(I;\mathbb{T}^d)} \lesssim \left\|w_-^{\frac{1}{2}}\left(f |\nabla|^{\frac{1}{2}} g\right)\right\|_{L_t^2 \dot{H}_x^s(I;\mathbb{T}^d)},$$

The following inequalities for homogeneous wave equations f, g with suitable parameters

$$\left\||\nabla|^{\beta_0} w_-^{\frac{1}{2}}\left(\left(|\nabla|^{\beta_1} f\right) |\nabla|^{\beta_2} g\right)\right\|_{L_t^2 L_x^2(I;\mathbb{T}^d)} \lesssim \|f_0\|_{\dot{H}_x^{s_1}(\mathbb{T}^d)} \|g_0\|_{\dot{H}_x^{s_2}(\mathbb{T}^d)}$$

by applying Lemma 10. Not that $s = \frac{d}{2} - 1$, $\frac{d}{2} - \frac{1}{2}$, $\frac{d}{2} - \frac{3}{2}$, $\frac{d}{2} - \frac{3}{2}$ in the case (4.33), (4.34), (4.35), (4.36) respectively.

For $(\beta_1, \beta_2) = \left(0, \frac{1}{2}\right)$, the parameters are choose as the following table,

equation	(4.33)	(4.34)	(4.35)	(4.36)
β_0	$\frac{d}{2} - 1$	$\frac{d}{2} - \frac{1}{2}$	$\frac{d}{2} - \frac{3}{2}$	$\frac{d}{2} - \frac{3}{2}$
$\alpha_1 + \alpha_2$	$d - 1$	$d - \frac{1}{2}$	$d - \frac{3}{2}$	$d - \frac{3}{2}$
(α_1, α_2)	$\left(\frac{d}{2}, \frac{d}{2} - 1\right)$	$\left(\frac{d}{2}, \frac{d}{2} - \frac{1}{2}\right)$	$\left(\frac{d}{2} - 1, \frac{d}{2} - \frac{1}{2}\right)$	$\left(\frac{d}{2}, \frac{d}{2} - \frac{3}{2}\right)$
(s_1, s_2)	$\left(\frac{d}{2}, \frac{d}{2} - \frac{1}{2}\right)$	$\left(\frac{d}{2}, \frac{d}{2}\right)$	$\left(\frac{d}{2}, \frac{d}{2}\right)$	$\left(\frac{d}{2}, \frac{d}{2}\right)$

Notice that for (4.33) estimates, we use the property $\|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}-\frac{1}{2}}(\mathbb{T}^d)} \leq \|\overline{g_0}\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}$ to finish the proof. ☐

Proposition 12. *Let $d \geq 3$ and the pair $\{\overline{u}, \overline{v}\}$ be either $\{u, v\}$ or $\{v, u\}$, $\overline{u} \in \dot{H}_x^{\frac{d}{2}}\left(\mathbb{T}^d\right)$, $\overline{v} \in \dot{H}_x^{\frac{d}{2}}\left(\mathbb{T}^d\right)$, we have*

$$\|\mathbb{P}(u \cdot \nabla v)\|_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} \lesssim_d \|u\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)} \|v\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}. \tag{4.37}$$

Proof. Note that if $u, v \in L_t^2 \dot{H}_x^{\frac{d}{2}}\left(I; \mathbb{T}^d\right)$ for some time interval I, then for a fixed time $t \in I$, $u(t), v(t) \in \dot{H}_x^{\frac{d}{2}}\left(\mathbb{T}^d\right)$ for a.e. t. Let $\mathfrak{u}(r)$ be a solution to the linear wave equation $\Box_r \mathfrak{u} = 0$ and $\mathfrak{u}(0) = u(t)$, $\mathfrak{u}_r(0) = 0$. Define $\mathfrak{v}(r)$ in the same manner. We want to bound the quantity $\|\mathbb{P}(u \cdot \nabla v)(t)\|_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)}$ by $\|\mathbb{P}(\mathfrak{u} \cdot \nabla \mathfrak{v})\|_{L_r^2 \dot{H}_x^{\frac{d}{2}-1}([-\frac{1}{2},\frac{1}{2}];\mathbb{T}^d)}$ and its derivative in time,

$$\frac{1}{2}\partial_r \|\mathbb{P}(\mathfrak{u} \cdot \nabla \mathfrak{v})\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} = \operatorname{Re}\left\langle \mathbb{P}(\mathfrak{u} \cdot \nabla \mathfrak{v}), \mathbb{P}(\mathfrak{u}_r \cdot \nabla \mathfrak{v}) + \mathbb{P}(\mathfrak{u} \cdot \nabla \mathfrak{v}_r)\right\rangle_{\dot{H}_x^{\frac{d}{2}-1} \times \dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)}$$

Using the $\dot{H}_x^{\frac{1}{2}} - \dot{H}_x^{-\frac{1}{2}}$ duality $\left|\mathrm{Re}\left\langle f,g\right\rangle_{\dot{H}_x^{\frac{d}{2}-1}\times\dot{H}_x^{\frac{d}{2}-1}}\right| = \left|\mathrm{Re}\left\langle f,g\right\rangle_{\dot{H}_x^{\frac{d}{2}-\frac{1}{2}}\times\dot{H}_x^{\frac{d}{2}-\frac{3}{2}}}\right|$,we have

(4.38)

$$\frac{1}{2}\partial_r \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} \leq \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)\right\|_{\dot{H}_x^{\frac{d}{2}-\frac{1}{2}}(\mathbb{T}^d)}\left(\left\|\mathbb{P}\left(\mathfrak{u}_r\cdot\nabla\mathfrak{v}\right)\right\|_{\dot{H}_x^{\frac{d}{2}-\frac{3}{2}}(\mathbb{T}^d)} + \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}_r\right)\right\|_{\dot{H}_x^{\frac{d}{2}-\frac{3}{2}}(\mathbb{T}^d)}\right).$$

Using Cauchy's inequality on the righthand side of (4.38), there is the bound for $r \in I^* = \left[-\frac{1}{2},\frac{1}{2}\right]$

$$\begin{aligned}&\left|\int_0^r \partial_r \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)(r)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)}\,dr\right|\\ &\lesssim \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)\right\|_{L_r^2\dot{H}_x^{\frac{d}{2}-\frac{1}{2}}(I^*;\mathbb{T}^d)}\left(\left\|\mathbb{P}\left(\mathfrak{u}_r\cdot\nabla\mathfrak{v}\right)\right\|_{L_r^2\dot{H}_x^{\frac{d}{2}-\frac{3}{2}}(I^*;\mathbb{T}^d)} + \left\|\mathbb{P}\left(\mathfrak{u}_r\cdot\nabla\mathfrak{v}\right)\right\|_{L_r^2\dot{H}_x^{\frac{d}{2}-\frac{3}{2}}(I^*;\mathbb{T}^d)}\right).\end{aligned}$$

By (4.34), (4.35), and (4.36) we have

$$\left|\int_0^r \partial_r \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)(r)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)}\,dr\right| \lesssim_d \left\|\overline{\mathfrak{v}}(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}\left\|\overline{\mathfrak{u}}(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}. \tag{4.39}$$

Applying (4.41) and (4.39) to the equation

$$\left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} = \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)(t)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} + \int_r^0 \partial_r \left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)}\,dr, \tag{4.40}$$

there is the bound

$$\left\|\mathbb{P}\left(\mathfrak{u}\cdot\nabla\mathfrak{v}\right)(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} \lesssim \left\|\overline{\mathfrak{v}}(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}\left\|\overline{\mathfrak{u}}(0)\right\|^2_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}. \tag{4.41}$$

By substitution $\mathfrak{u}(0) = u(t)$, $\mathfrak{v}(0) = v(t)$ on both side of (4.41),

$$\left\|\mathbb{P}\left(u(t)\cdot\nabla v(t)\right)\right\|_{\dot{H}_x^{\frac{d}{2}-1}(\mathbb{T}^d)} \lesssim \left\|\overline{v}(t)\right\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}\left\|\overline{u}(t)\right\|_{\dot{H}_x^{\frac{d}{2}}(\mathbb{T}^d)}, \tag{4.42}$$

for a.e. time $t \in I$. □

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