My simple guide to free-fall motion in general relativity

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Abstract

The free-fall behaviour of a test object near a static point mass has been calculated for two cases: the first is the usual black-hole solution in general relativity resulting from using Schwarzschild coordinates, and the second is for a model described in previous papers, in which spacetime is completely regular with no event horizon. The predictions are presented and discussed.

1 Introduction

Albert Einstein's theory of general relativity *GR* [1] has been hailed as the greatest intellectual achievement of the twentieth century and one of the two pillars of modern physics [2], and countless books have been written about it, e.g. [3]. It involves a great deal of mathematical physics, which I fear is inaccessible to most people, since some of the theoretical concepts are often shrouded in abstract mathematics. Furthermore, Einstein himself seems to have modified some of its ideas as time passed, while others may have interpreted *GR* differently from how Einstein anticipated. There has indeed been some confusion historically about the exact meaning of *GR*, whether it should be regarded as just a theory of gravity or whether it really is a generally covariant theory as Einstein wanted [4]. At the other end of the spectrum, there are even those who consider it to be entirely wrong, in that it disobeys the laws of pure mathematics [5]. Nevertheless, *GR* does seem to describe some aspects of gravitational physics that Newton's classical law of gravity does not describe correctly, such as the perihelion rotation of the planet Mercury and the bending of starlight, whereas Newton's inverse-square law of gravity is perfectly adequate for describing planetary motion and the ordinary day-to-day gravity we all experience.

For some considerable time I have wanted to understand *GR* from what you could call a simplistic point of view: to boil it down to something I could in principle explain to my grandchildren or someone who has not studied mathematics to a higher degree. However, this is probably impossible, since it does inevitably require a lot of mathematics, including differential calculus and non-Euclidean geometry as a pre-requisite to a full understanding.

Einstein himself was very fond of so-called thought experiments. These are quasi-experiments that do not actually take place in the laboratory or even in outer space, but in the mind. In doing so, however, many unrealistic situations can arise. You can invent measuring devices that do not exist in reality, such as ideal clocks that have no mass, rigid rods for measuring distances, and frames of reference encompassing large volumes of the universe that are equipped with countless idealized, synchronized clocks. It is questionable sometimes, whether deductions made from such thought experiments actually do tell us anything practical about the way the universe works. Nevertheless, I am fascinated by them, and so in this paper I shall devise and describe my own thought experimental set-up to try to understand free-fall in the context of *GR*.

2 My thought experiment

Consider the behaviour of a probe or test object (i.e. an object of negligible mass that does not affect the gravitational field it finds itself in), as it free-falls directly along a radial coordinate towards a point mass causing gravitation.

To do this I need to construct some sort of experimental set-up to measure physical quantities, such as velocity and acceleration. So, I imagine out in space somewhere, where there is no other mass in the region, there is a rail-track made out of a very light rigid material, which cannot deform in any way. At every unit of distance along the track, such as per kilometre, there is an ideal clock and light switch positioned at the side of the track that will record the time at which an object passes by. I am an observer a very long way from this track, so I imagine having to use a telescope to make any measurements. Next, I put another ideal clock onto a trolley, which acts as the above-mentioned test object, and someone gives it a push along the track from one end, which is at an infinite - or let's say very large - distance away. I also imagine there is no friction at all, so the trolley with its own clock, called a proper clock or co-moving clock, will travel along the track at a constant speed. The constant speed results from a postulate usually referred to as Newton's first law of motion (or Galileo's principle), i.e. a body moves at a constant speed if there is no force on it.

Thus far, there is no gravity involved in any of this, but there is already a huge (possibly non-intuitive) issue, for the clock in the proper frame is observed to tick at a different rate from the clock in the observer frame, called the coordinate frame, even though all clocks are deemed to be identical. This insight comes from Einstein's postulate that light travels at the same speed irrespective of the frame of reference - this being the essence of his other theory, special relativity [6]. A time interval dt' between two events in the proper frame is related to the time interval dt between the same two events as observed in the coordinate frame, by the expression:

$$dt'^2 = (1 - v^2/c^2)\, dt^2 \tag{1}$$

where v is the velocity of the test object (or co-moving clock) relative to the coordinate frame, and c is the speed of light. This is likely to be almost the first equation one encounters in every book on relativity, e.g.[7]. If I write the velocity as $v = dr/dt$, which is the radial distance increment dr moved in the coordinate frame divided by the time interval dt on the coordinate clocks, we then have:

$$d\tilde{s}^2 = c^2 dt'^2 = c^2 dt^2 - dr^2 \tag{2}$$

where I have introduced the idea of a spacetime interval $d\tilde{s}$, which is invariant, or the same in both frames.

My idealized rail track runs from minus infinity to the left to plus infinity on the right, with the coordinate origin in the middle, directly in front of us. Via some miracle I now place a very large point mass M at the origin. This is essentially intended to represent a very dense star with spherical symmetry. The (rigid) track itself does not change its dimensions in any way as a consequence of this. This is, after all, only a thought experiment! My friend at plus infinity now takes the trolley and releases it along the track. It starts off with zero speed, but gradually accelerates due to the gravitational attraction caused by M. In classical physics this accelerated motion is described by Newton's inverse-square law of gravitation, which may be written:

$$a = -\frac{GM}{r^2} \tag{3}$$

where $a = d^2r/dt^2$ is the acceleration of the object (trolley and clock) in terms of Newton's gravitational constant G, the mass causing gravitation M, and the radial distance r. The velocity v from a stationary state at $r = \infty$ is found by integrating Equation 3, to give

$$v^2 = \left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r} \tag{4}$$

There is no concept of proper or coordinate time in classical physics, so t here is just "time", as though it were the same everywhere under all conditions and instantaneously measurable.

What causes the gravitational force or acceleration is not specified by Newton; his law is a phenomenological law. In Einstein's theory of general relativity, however, gravity is considered to be a geometrical effect due to the curvature of spacetime, i.e. the distortion of both space and time by the mass M, where time now has the mathematical characteristics of a fourth dimension. A spacetime interval $d\tilde{s}$ - also called a metric line element - for a spherically symmetrical spacetime outside a point mass may be written in terms of spherical polar coordinates $(t, \tilde{r}, \theta, \phi)$ as

$$d\tilde{s}^2 = c^2 dt'^2 = A(\tilde{r})\, c^2 dt^2 - B(\tilde{r})\, d\tilde{r}^2 - \tilde{r}^2\, d\Omega^2 \tag{5}$$

(see, e.g. [8]), where $d\tilde{s}$ as a spacetime increment, dt' an increment of proper time, dt an increment of coordinate time, $d\tilde{r}$ an increment of the radial coordinate $\tilde{r}$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$; A and B are radially dependent functions describing the curvature of the time and radial metric coefficients, respectively, while $\tilde{r}$ is the Schwarzschild radial coordinate, which is not *a priori* exactly the same as the radial distance r in the coordinate frame. The next step is to try to find A and B by solving Einstein's field equations of GR for the vacuum outside a point mass. So-called geodesic equations are found for the four variables from the metric by extremising the path length between two points in spacetime, using a well known method called Lagrangian formalism, which is based on Hamilton's principle of least action. This is tantamount to extremising the proper time. The resulting solution satisfying Einstein's field equations of general relativity for the vacuum outside the point mass is well known, and given by:

$$A(\tilde{r}) = \frac{1}{B(\tilde{r})} = 1 - \frac{\alpha}{\tilde{r}} \tag{6}$$

where α is a constant of integration [8].

Back to my thought experiment: for the case we are considering of radial free-fall, the angular terms in the metric in Equation 5 disappear, since $d\theta = d\phi = 0$, and we have just

$$d\tilde{s}^2 = c^2 dt'^2 = A\, c^2 dt^2 - B\, d\tilde{r}^2 \tag{7}$$

This is the same as Equation 2 in special relativity for motion without gravity, with $A = B = 1$ for a flat spacetime. In a gravitational field in GR the time intervals dt and distance intervals $d\tilde{r}$ have been distorted by the radially dependent factors A and B, respectively. Inserting the solution from Equation 6, we have

$$d\tilde{s}^2 = c^2 dt'^2 = (1 - \alpha/\tilde{r})c^2 dt^2 - (1 - \alpha/\tilde{r})^{-1} d\tilde{r}^2 \tag{8}$$

From the radial geodesic equation, the free-fall equation of motion may be written [8]:

$$\ddot{r} + \frac{A'c^2}{2B}\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 = 0 \tag{9}$$

where $\ddot{r} = d^2\tilde{r}/dt'^2$ is the proper radial acceleration, $\dot{r} = d\tilde{r}/dt$ is the proper velocity, $A' = dA/d\tilde{r}$ and $B' = dB/d\tilde{r}$, and using the metric, the quantity $\dot{t} = dt/dt'$ can be eliminated to give

$$\ddot{r} + \frac{A'}{2AB}c^2 + \left(\frac{A'}{2A} + \frac{B'}{2B}\right)\dot{r}^2 = 0 \tag{10}$$

Note that GR directly delivers proper quantities, i.e. in terms of the proper time. Using part of the solution where $B = 1/A$ one then obtains from the previous equation:

$$\ddot{r} + \frac{A'c^2}{2} = 0 \tag{11}$$

which can be integrated to give

$$\frac{\dot{r}^2}{c^2} = 1 - A \tag{12}$$

where I have used the asymptotic condition: $A \to 1$ and $\dot{r} \to 0$ for $r \to \infty$. Then, using the rest of the solution, $A = 1 - \alpha/\tilde{r}$, we obtain for the proper velocity relative to the speed of light:

$$\frac{\dot{r}^2}{c^2} = \frac{\alpha}{\tilde{r}} \tag{13}$$

This expression predicts that $\dot{r} \to c$ for $\tilde{r} \to \alpha$, and $\dot{r} \to \infty$ for $\tilde{r} \to 0$. By comparing this with Newton's law of gravity and making several approximations (usually referred to as a weak-field approximation), which involves ignoring the difference between r and $\tilde{r}$, and equating Newton's absolute velocity v with $\dot{r}$, we obtain the following correspondence between the parameters in Newton's law and GR:

$$\alpha = \frac{2GM}{c^2} \tag{14}$$

As α turns out to be a positive quantity, called the Schwarzschild (or gravitational) radius, this leads to the idea of a black hole and event horizon, because the functions A and B become negative, if $\tilde{r} < \alpha$, and a coordinate discontinuity would then occur at this distance $\tilde{r} = \alpha$ from the point mass.

Another expression for velocity can be obtained by rearranging the radial metric in Equation 7 to give

$$\frac{u^2}{c^2} = \frac{A\dot{r}^2/c^2}{(1 + B\dot{r}^2/c^2)} \tag{15}$$

where $u = d\tilde{r}/dt$ is the radial coordinate velocity. Using Equation 12 and $B = 1/A$ then gives

$$\frac{u^2}{c^2} = (1 - A)A^2 \tag{16}$$

from which one obtains

$$\frac{u^2}{c^2} = \frac{\alpha}{\tilde{r}}\left(1 - \frac{\alpha}{\tilde{r}}\right)^2 \tag{17}$$

This quantity shows the characteristics, $u \to 0$ for $\tilde{r} \to \alpha$ and $u \to \infty$ for $\tilde{r} \to 0$.

The above analysis has introduced some tricky conceptual issues, apart from some moderately difficult mathematics. According to *SR*, i.e. in the absence of gravity, the proper clock (co-moving with the test object along the imagined track) ticks more slowly than stationary coordinate clocks (placed beside the track). But when a mass is present, the coordinate clocks themselves tick locally at a reduced rate that depends on the distance from M, since A becomes less than unity in a gravitational field, i.e. coordinate time intervals become $\sqrt{A}\,dt$ in the presence of a mass, whereas a long way from the mass, coordinate time intervals are still dt. We thus have in essence three different time quantities: proper time, local coordinate time, and coordinate time at infinity. In addition, space is stretched by the factor $\sqrt{B}\,dr$ as the trolley approaches the central mass, while the track itself remains rigid with radial spatial intervals dr. According to *GR*, then, time is "squashed" as r decreases, i.e. A falls from 1 to zero as $\tilde{r}$ decreases from infinity to α. This effect is known to be experimentally correct: a clock above the Earth at a higher gravitational potential ticks more quickly than on Earth itself, due to the effect called gravitational time dilation. Thus, a clock approaching a massive body slows down, and will stop altogether at $\tilde{r} = \alpha$.

3 Regular spacetime

The analysis in the previous section is correct mathematically, but I believe it to be misleading, due to the non-specific nature of the Schwarzschild radial coordinate – so misleading, in fact, that generations of scientists and non-scientists have come to believe in the actual existence of black holes, rather than thinking of them as just a mathematical curiosity. We should remember that the quantity $\tilde{r}$ is a mathematical expediency that was introduced in order to enable Einstein's *GR* field equations to be solved - but it isn't specified or determined by *GR*. In other words, the solution for A and B in Equation 4 is expressed in terms of $\tilde{r}$, but this is not necessarily the same quantity as the true radial coordinate distance r from the point mass.

The question therefore remains, whether or not we can find a unique relationship between $\tilde{r}$ and r, or is it really unspecifiable? Other authors have indeed addressed this issue. Abrams [9] and Crothers [10] have shown that $\tilde{r}$ is related to r via an infinite set of possible solutions:

$$\tilde{r} = [|r - r_0|^n + k^n]^{1/n} \tag{18}$$

where n is an integer, and r_0 and k are arbitrary constants. For example, Schwarzschild's original solution for the field due to a point mass has $n = 3$ [11].

Although there may be an infinite number of possible mathematical forms for $\tilde{r}(r)$, I would say they are not all physically equivalent, and that there can surely be only one solution that corresponds correctly to physical reality? The key to solving the problem lies in the realisation that Newton's law relates strictly to the curvature of time, while space curvature plays no part in that law. This is an undeniable fact, but if the reader wishes to disagree, I will simply call it a new axiom that I am introducing. Applying this axiom then enables a completely precise solution to be obtained, as I have already shown in a previous paper [8], leading to the very simple linear relationship:

$$\tilde{r} = r + \alpha \tag{19}$$

which is the first and simplest of the set of solutions posited by Abrams and Crothers, with $n = 1$. Using Equations 6 and 19 we then have

$$A = \frac{1}{B} = 1 - \frac{\alpha}{\tilde{r}} = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{20}$$

which gives

$$\frac{\dot{r}^2}{c^2} = 1 - A = \frac{\alpha}{(r + \alpha)} \tag{21}$$

and

$$\frac{u^2}{c^2} = (1-A)A^2 = \frac{\alpha\, r^2}{(r+\alpha)^3} \tag{22}$$

I have called this a regular model, since the function A does not show a discontinuity for any value of r from 0 to ∞. There is therefore no event horizon, and the mass M is not a black hole, i.e. the spacetime is completely regular for all values of r. The derived proper and coordinate velocities for both models (black hole and regular) are shown plotted in Figure 1 for comparison.

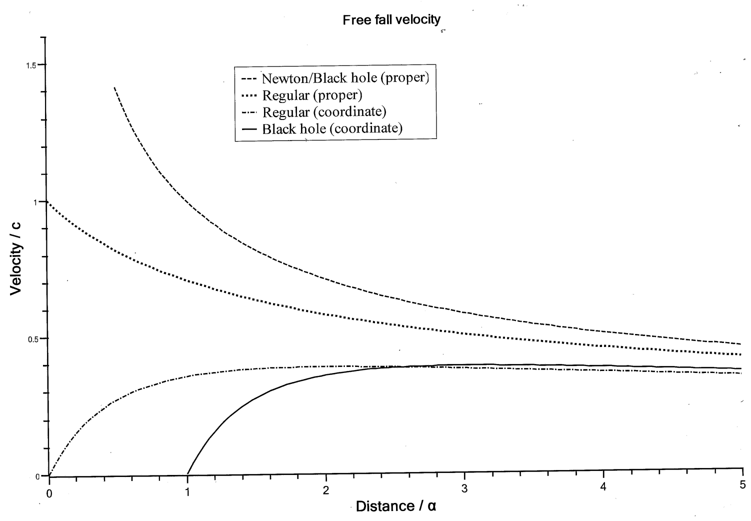

Figure 1: Free fall velocity for black hole and regular solutions: proper and coordinate velocities

4 Discussion

We can now proceed further and analyse some predictions and consequences. As mentioned, the regular model in my paper is based on the fact that Newton's inverse-square law is a manifestation of time curvature. When space curvature is added as an ingredient, this modifies the law governing the radial free-fall velocity and acceleration of a test mass. Using Equation 21 we obtain for the proper acceleration:

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} = \frac{-\frac{1}{2}\,c^2\alpha}{(r+\alpha)^2} \tag{23}$$

as opposed to $-\frac{1}{2}\, c^2\alpha/r^2$ in the black-hole model. As $r \rightarrow 0$, $\ddot{r}$ now tends to a constant value given by $\ddot{r} \rightarrow -\frac{1}{2}\, c^2/\alpha$, rather than to infinity. This behaves like Newton's law for $r >> \alpha$, but predicts that gravity falls off as masses approach each other closely and r is of the order of α. But what do these proper quantities actually describe? They can be understood as the speed and acceleration of a test object from the point of view of an observer that is co-moving with the test object: in terms of the observer's own clock, with distances determined from the markings on the rigid track as it passes by. The speed increases up to the speed of light c, but is limited to c when the object impacts the point mass at $r = 0$. The black-hole solution is different, of course. It predicts the proper velocity reaches the speed of light at $r = \alpha$, and infinite speed at $r = 0$ - which is profoundly incorrect, in my mind.

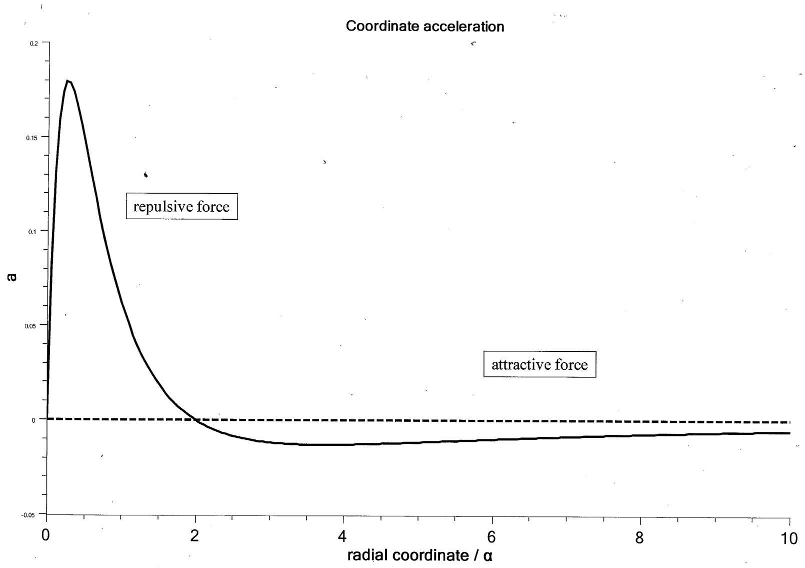

Figure 2: Coordinate acceleration.

Next, the coordinate velocity is the speed of the falling object measured according to times and distances recorded by an observer at infinity (or a long way from M where spacetime can be regarded as flat). In my model, u is given by Equation 22, and the commensurate coordinate acceleration a_c is:

$$a_c = \frac{d^2r}{dt^2} = u\frac{du}{dr} = -\frac{1}{2}\, c^2\alpha \left[\frac{r(r-2\alpha)}{(r+\alpha)^4}\right] \tag{24}$$

This expression is plotted in Figure 2 as a function of the radial coordinate r for small values of r. For large distances $r >> \alpha$ from the point mass, Newton's law is recovered, i.e. $a_c \sim 1/r^2$, but as r decreases, a discrepancy occurs that progressively widens, e.g. for $r/\alpha = 10^4$ we have only a relative difference from Newton's law of 6×10^{-4}; for $r/\alpha = 10^3$, the difference is 6×10^{-3}; for $r/\alpha = 10^2$, it is 6.8×10^{-2} or 6.8%, and for $r/\alpha = 10$, it is 47% lower. Then, at $r = 2\alpha$ the attractive gravitational force or acceleration disappears completely, and becomes repulsive. The falling object is then observed to decelerate until it finally comes to rest when it reaches the point mass M. Thus, there is a soft landing from the point of view of a distant observer. In the black-hole model, on the other hand, the prediction is extremely strange, in that $u \to 0$ for $\tilde{r} \to \alpha$. In the early days of relativity, this led to the idea of a frozen star, because that model suggests test objects would come to rest in space, queue up at the event horizon at $r = \alpha$, and never actually disappear from view.

Finally, in this paper I have proposed a modified law of gravitation that deviates from Newton's law for very small distances from the mass causing gravitation. The modification to gravity is not in the same scale as to be able to account for the presence of dark matter in spiral galaxies. However, it provides a much more realistic and intuitive solution to the way bodies may behave when they come in close proximity to each other. Gravity actually decreases as the separation decreases to zero, and becomes repulsive from the viewpoint of a distant observer. This remarkable prediction also removes the singularity that is conventionally thought to occur at the origin of coordinates.

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