A conjecture regarding the primes and π

Shreyansh Jaiswal
Atomic Energy Central School - 6 Mumbai, India

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Abstract

In this article we introduce a new function:

$$f(\phi) = \sum_{p}^{\infty} \left(\frac{1}{\phi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P} \right)^p$$

Where, p and P represent the prime numbers, both starting from 2
We also conjecture that:

$$f(\pi) = \sum_{p}^{\infty} \left(\frac{1}{\pi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P} \right)^p = \pi$$

We also rigorously show that

$$3.85 > f(\pi) > 3.1240$$

1 Introduction

The function:

$$f(\phi) = \sum_{p}^{\infty} \left(\frac{1}{\phi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P} \right)^p$$

was produced mainly by curious analysis. Still, the conjecture, if true, hints towards an hidden connection between π and the primes.

2 Proving the absolute convergence of $f(\pi)$

For proving the convergence of the function $f(\phi)$ at $\phi = \pi$, we first look at the series:

$$\sum_{P}^{\infty} \frac{1}{e^P}$$

Using the simple comparision test, we first look at the series:

$$\sum_{n=1}^{\infty} \frac{1}{e^n}$$

if this series is convergent, then:

$$\sum_{P}^{\infty} \frac{1}{e^P}$$

is also convergent, due to simple observations that there are n for which:

$$\frac{1}{e^P} < \frac{1}{e^n}$$

as for each prime p_k, there is one n such that $p_k > n > p_{k-1}$. From there, it's easy to prove that the $\frac{1}{e^n}$ series is convergent due to concept of geometric series. From there we also derive loose upper bounds for $\frac{1}{e^P}$ as < 0.58. For,

$$f(\pi) = \sum_{p}^{\infty} \left(\frac{1}{\pi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P} \right)^p$$

and let's define:

$$A := \frac{1}{\pi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P}$$

We can compute A's values for some primes, and from there we see that it converges rather quickly and also that A < 1: Also, this single value below the table solidifies the claim that A < 1.

Number of primes used	A
10	0.87898
1,000	0.8789805171
10,000	0.8789805171

Table 1: Number of primes vs A

$$1/e^{10000} \approx 1.13 \cdot 10^{-4343}$$

. From these facts, we can also assert:

$$\lim_{p \to \infty} A^p = 0 \tag{1}$$

using:
Proposition: For $0 < A < 1$ and $b > 1$, it holds that $A^b < A < 1$.
So, now we move on to proving the fact that:

$$f(\pi) = \sum_{p}^{\infty} \left(\frac{1}{\pi} + \frac{1}{e} + \sum_{P}^{\infty} \frac{1}{e^P} \right)^p = \sum_{p}^{\infty} A^p$$

is absolutely convergent. We use Cauchy's root test. Using the proposition, and the root test,we have to prove that:

$$(A_n^p)^{\frac{1}{n}} \leq \alpha \ < \ 1$$

We now use the Prime Number Theorem's statement that the nth prime (P_n) satisfies:

$$P_n \sim n \cdot ln(n)$$

thus,

$$(A_n^p)^{\frac{1}{n}} \approx A^{\ln(n)}$$

and thus by the fact that $ln(n) > 1$ for $n > 0$ we successfully establish the existence, of α. Thus, we now know for certain that $f(\pi)$ is convergent.

3 Empirical analysis of $f(\pi)$

We've taken the upper limit for the number of primes as 300.(in the figure) This upper limit is also very good for understanding about the general idea of $f(\pi)$ as this series converges rather quickly.

We also present a table with elaborate data:

No. of Primes Used	$f(\pi)$	$\pi - f(\pi)$
2	1.420669964	1.720922689589793
10	3.083712747	0.057879906589793
20	3.124018511	0.017574142589793
30	3.124179985	0.017412668589793
60	3.124180155	0.017412498589793
100	3.1241801561402285895	0.017412497449564
300	3.124180156140228589441358	0.017412497449564
1000	3.124180156140228589441358	0.017412497449564
10000	3.12418015614022858944135909589	0.017412497449564

Table 2: Table of data with $\pi - f(\pi)$ values.

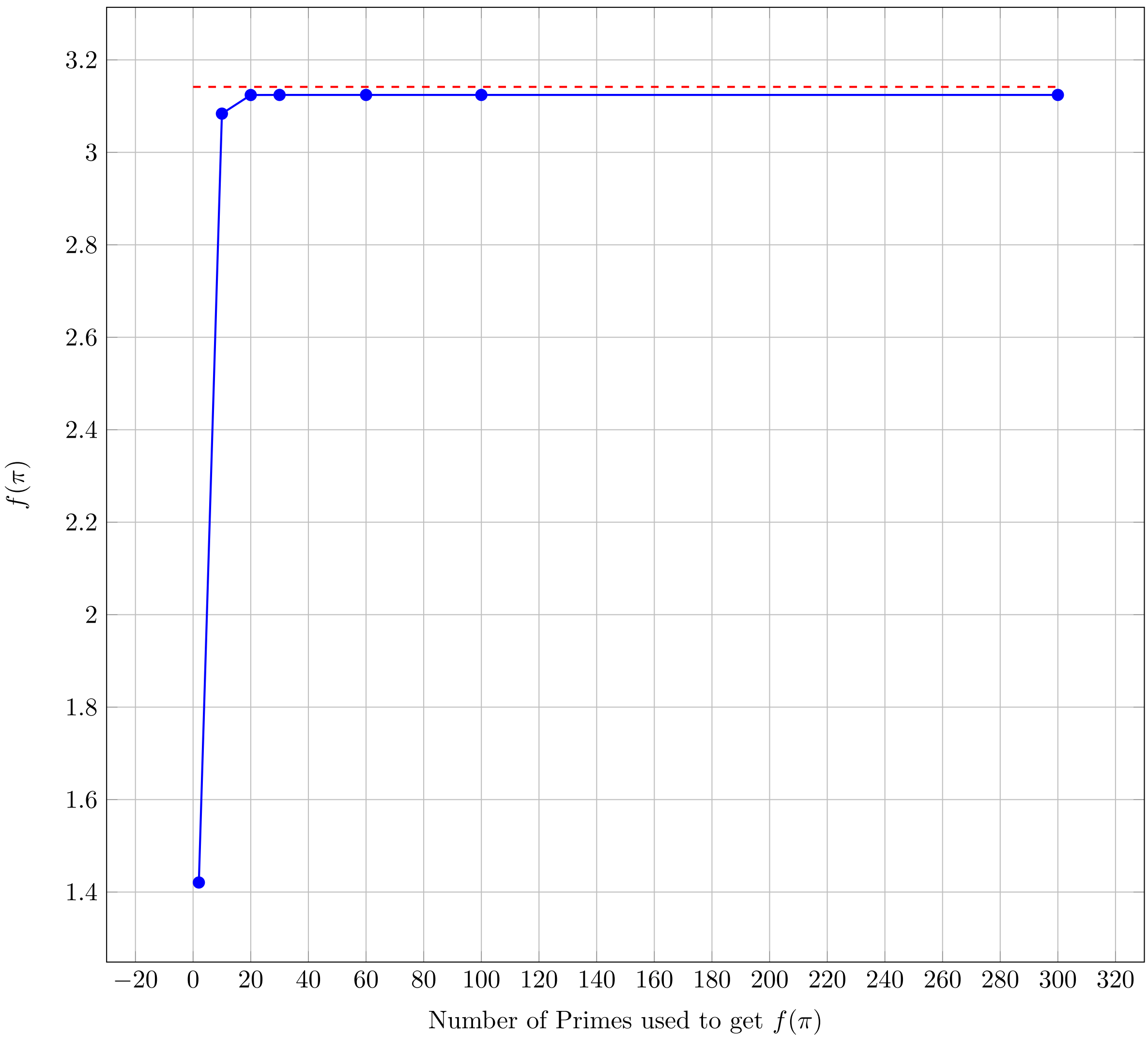

Figure 1: Convergence of expression $f(\pi)$ with a horizontal line at $y = \pi$.

For reference, the 10,000th term of this series E, would be around:

$$\approx 0.8789805171^{10000} \approx 0$$

But still, giving a definite closed form convergence of $f(\pi)$ with just empirical data is not feasible. But, empirical data, still supports the notion of the conjecture.

4 Bounds for the value of $f(\pi)$

It's easy to say, that a strong lower bound for $f(\pi)$ is:

$$f(\pi) > 3.1240$$

based on our empirical data. So, we'll focus on upper bounds. Take the relation:

$$\sum_{n=1}^{\infty} A^{(2n+1)} > \sum_{p}^{\infty} A^{p}$$

And based on the geometric series formula:

$$\frac{A^3}{1 - A^2} = \sum_{n=1}^{\infty} A^{(2n+1)} > \sum_{p}^{\infty} A^{p}$$

to get ourselves a rather "secure" bound. we can simply take A = 0.9, which gives us:

$$3.85 > \sum_{p}^{\infty} A^p$$

Hence, our final bounds are

$$3.85 > \sum_{p}^{\infty} A^p = f(\pi) > 3.1240$$

which ends the section.

5 Using Ramanujan's series for $\frac{1}{\pi}$

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

Using this brilliant series for $\frac{1}{\pi}$ originally given by Srinivasa Ramanujan, we aim to make our conjecture a bit more "non referential looking". This formula also increases the beauty of the supposed conjecture. Using the formula for $\frac{1}{\pi}$, we can rewrite our conjecture as:

$$\sum_{p}^{\infty} \left(\frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}} + \sum_{P}^{\infty} \frac{1}{e^P} + \frac{1}{e} \right)^p = \pi$$

6 Potential implication emerging from the conjecture's truth

If our conjecture was true, then this would lead to an interesting case of self reference regarding π. It would also hint towards a connection between primes and π. This connection is still very well evident between primes and π. For example, the Riemann Zeta function at positive even integers is evaluated as:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}$$

And the zeta function is related to the primes by the Euler product formula:
The Euler product formula expresses the Riemann zeta function $\zeta(s)$ for $\Re(s) > 1$ as an infinite product over all primes:

$$\zeta(s) = \prod_{p=\text{ prime}} \frac{1}{1 - p^{-s}}.$$

This shows the deep connection between the zeta function and prime numbers, where each prime p contributes a factor $\frac{1}{1-p^{-s}}$, encoding the distribution of primes.
Even if the conjecture could be false, we still have the fact that:

$$f(\pi) \approx \pi$$

Which is beautiful by itself.

7 Conclusion

In this paper, we introduced a new conjecture connecting the primes and π through the function $f(\phi)$. We rigorously proved the absolute convergence of $f(\pi)$, provided empirical data supporting the bounds $3.85 > f(\pi) > 3.1240$, and established that this function rapidly converges with increasing primes. The conjecture suggests an intriguing relationship between π and the distribution of primes, further evidenced by its empirical behavior and convergence properties.

While we have not yet provided a formal closed-form solution for $f(\pi)$, the numerical results strongly support our conjecture. If proven true, this conjecture could deepen our understanding of the hidden connections between primes and fundamental constants like π. This work opens the door for further exploration into these relationships and highlights the importance of prime numbers in understanding the structure of mathematical constants.

References

[1] Chieh-Lei Wong, *On the Elegance of Ramanujan's Series for* π, arXiv:2104.12412 [math.NT], 2021.