Level filtration

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Abstract

Let X be a complex projective manifold of dimension n, and denote the total cohomology $\sum_{i=-\infty}^{+\infty} H^i(X;\mathbb{Q})$ over $\mathbb{Q}$ by $H(X;\mathbb{Q})$ where our convention is that the cohomology $H^i(X;\mathbb{Q}) = 0$ for i beyond the range $[0, 2n]$. Let u be a Hodge class. Then the cup-product "$\smile$" yields a homomorphism

$$\begin{array}{rcl} [u] : H(X;\mathbb{Q}) & \to & H(X;\mathbb{Q}) \\ \omega & \to & u \smile \omega. \end{array} \tag{0.1}$$

Let $N^i H^{2i+k}(X)$ be the subgroups of coniveau i in the coniveau filtration of cohomology $H^{2k+i}(X;\mathbb{Q})$. For each fixed integer $k \in [-2n, 2n]$, let

$$L_k H(X) = \sum_{i=0}^{n} N^i H^{2i+k}(X) \tag{0.2}$$

that form the decreasing filtration

$$L_{-2n} H(X) \subset \cdots \subset L_0 H(X) \subset \cdots \subset L_{2n} H(X;\mathbb{Q}) = H(X;\mathbb{Q}) \tag{0.3}$$

called the level filtration. In this paper, we prove that for non-negative level k,

$$[u]\Big(L_k H(X) \Big) \subset L_k H(X) \tag{0.4}$$

provided a supportive intersection exists. It implies that the millennium Hodge conjecture ([1]) is true provided a supportive intersection exists.

Key words: coniveau filtration, level filtration, millennium Hodge conjecture
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Contents

1 Introduction

Let X be a complex projective manifold of complex dimension n. Denote the cohomology group of degree i with rational coefficients by $H^i(X;\mathbb{Q})$, the total cohomology $\sum_{i=-\infty}^{+\infty} H^i(X;\mathbb{Q})$ by $H(X;\mathbb{Q})$. Denote the union of the subgroups

$$ker\bigg(H^i(X;\mathbb{Q}) \to H^i(X\backslash V;\mathbb{Q})\bigg) \tag{1.1}$$

over all subvarieties V of $cod(V) \geq p$ by $N^pH^i(X;\mathbb{Q})$ where the map on cohomology groups is the restriction map, and the individual subgroup

$$ker\bigg(H^i(X;\mathbb{Q}) \to H^i(X\backslash V;\mathbb{Q})\bigg) \tag{1.2}$$

will be denoted by $H^i_V(X;\mathbb{Q})$. In this paper, the notation $H^i_S(X;\mathbb{Q})$ in (1.2) is extended to any closed set S. These subgroups form a decreasing filtration

$$N^nH^i(X;\mathbb{Q}) \subset \cdots \subset N^pH^i(X;\mathbb{Q}) \subset \cdots \subset N^0H^i(X;\mathbb{Q}) = H^i(X;\mathbb{Q}), \tag{1.3}$$

on the cohomology with the fixed degree i. It is called coniveau filtration. Note that the coniveau p is in the range $[0,n]$. Now for each integer $k \in [-2n, 2n]$ we define a subgroup

$$L_kH(X) := \sum_{i=0}^{n} N^iH^{2i+k}(X;\mathbb{Q}) \tag{1.4}$$

where $H^j(X;\mathbb{Q}) = 0$ for $j \notin [0,2n]$. They also form a decreasing filtration

$$L_{-2n}H(X) \subset \cdots \subset L_0H(X) \subset \cdots \subset L_{2n}H(X;\mathbb{Q}) = H(X;\mathbb{Q}) \tag{1.5}$$

But it is on the total cohomology. We call it the level filtration where

$$level = degree - 2coniveau.$$

Let $u \in H^q(X;\mathbb{Q})$ be a Hodge class, i.e. it is of $(\frac{q}{2},\frac{q}{2})$ type in the Hodge decomposition. The class defines a homomorphism

$$\begin{array}{rcl} [u] : H(X;\mathbb{Q}) & \rightarrow & H(X;\mathbb{Q}) \\ \omega & \rightarrow & u \smile \omega. \end{array} \tag{1.6}$$

In another field, we'll define a supportive intersection for currents. Let's work with the same complex projective manifold X. Denote the space of currents with real coefficients by $\mathscr{D}'(X)$. The $\mathcal{S}(X)$ denotes the $\mathbb{Q}$ module freely generated by regular cells. We should note that a regular cell is a pair of an oriented polyhedron Π and a differential embedding of a neighborhood of Π to X. An element in $\mathcal{S}(X)$ is called a regular chain. A closed regular chain is called a regular cycle. It is well-known that a singular cohomology class with rational coefficients is represented by a regular cycle. The singular cohomology can also be represented by closed currents. On the other hand, a holomorphic p-chain is a particular type of a current $\mathcal{T}_V$ of dimension $2p$ of the integration current on the regular points of a complex analytic cycle V of dimension p in the complex manifold $X\backslash M$ for some compact manifold M ([4], [5]). In the literature, the same V as above induces two different types of currents: one in $X\backslash M$ which is closed; the other in X which is not closed. We'll refer the holomorphic chain to the one in X which is not closed. Let $\mathscr{H}(X)$ be the subspace of $\mathscr{D}'(X)$ over $\mathbb{Q}$ freely generated by holomorphic chains. Since $\mathcal{S}(Y)$ is identified with a subspace of $\mathscr{D}'(Y)$ over $\mathbb{Q}$, we let

$$\mathscr{L}(X) := \mathcal{S}(X) + \mathscr{H}(X) \tag{1.7}$$

be the subspace over $\mathbb{Q}$ and call it the space of quasi-chains. *

Definition 1.1. *A bilinear map*

$$\begin{array}{rcl} [\bullet \wedge \bullet] : \mathscr{L}(Y) \times \mathscr{L}(Y) & \rightarrow & \mathscr{D}'(Y) \\ (T_1, T_2) & \rightarrow & [T_1 \wedge T_2] \end{array}$$

is called a supportive intersection on the quasi-chains if the map satisfies

1) *(Cohomologicality) $[T_1 \wedge T_2]$ descends to the cup-product on cohomology, i.e. if T_1, T_2 are closed, so is $[T_1 \wedge T_2]$ and it satisfies*

$$\big\langle [T_1 \wedge T_2] \big\rangle = \langle T_1 \rangle \smile \langle T_2 \rangle \tag{1.8}$$

where $\langle \bullet \rangle$ denotes the Poincaré dual of a singular cycle in cohomology, and $\smile$ the cup-product,

2) *(Supportivity)*

$$supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2). \tag{1.9}$$

where supp stands for support.

* $\mathscr{D}'(Y)$ is not over $\mathbb{Q}$. But for the convenience, we take $\mathscr{L}(X)$ to be a vector space over $\mathbb{Q}$.

It is quite obvious that if the class u is algebraic, the map $[u]$ preserves the level filtration. The converse is the content of the millennium Hodge conjecture. Our Main theorem below states that u being a Hodge is sufficient in preserving the level filtration provided a supportive intersection exists. Therefore as expected, the millennium Hodge conjecture follows from this preservation. Precisely,

Main theorem 1.2. *For $k \geq 0$, if a supportive intersection on quasi-chains exists, then*

$$[u]\bigg(L_k H(X)\bigg) \subset L_k H(X).$$

Corollary 1.3. *Main theorem implies that the millennium Hodge conjecture is true, provided a supportive intersection exists.*

Proof of Corollary 1.3: If $deg(u) = 0$, the proof is trivial. So, we let

$$deg(u) = 2p \neq 0.$$

Denote the Poincaré dual of a singular chain by the class symbol $\langle \bullet \rangle$. Then the fundamental class $\langle X \rangle$ is in $L_0 H(X)$. By the main theorem, $u \smile \langle X \rangle$ also lies in $L_0 H(X)$. Notice $u \smile \langle X \rangle = u$. Hence u is class supported on an algebraic cycle V of the dimension p ($u \in H_V^{2n-2p}(X; \mathbb{Q})$). We may assume V is prime (i.e an irreducible variety). Let $\tilde{V}$ be the smooth resolution of V. By the condition $2p - deg(p) = 0$, we can apply Delign's corollary 8.2.8, [2] which states that the Gysin map

$$J_! : H^0(\tilde{V}; \mathbb{Q}) \to H_V^{2n-2p}(X; \mathbb{Q}) \tag{1.10}$$

is surjective. Hence u is the Gysin image of a linear combination of the connected components of $\langle \tilde{V} \rangle$. So it is represented by an algebraic cycle.

□

2 The support

The unusual but effective ingredient is the support that allows us to have an intersection outside of the environment of cohomology. In this section, we introduce two technical lemmas on it. Let X be a complex projective manifold of an arbitrary dimension. Let $H^i(X; \mathbb{G})$ be the singular cohomology of degree i with coefficients in the Abelian group

$$\mathbb{G} = \mathbb{Q} \ or \ \mathbb{R}.$$

Inspired by Hodge's work, the coniveau filtration was formulated in [3] in terms of classes. However, Hodge's original expression is in their representatives via homology. See [6]. So, let's continue it in representatives. Let $\mathcal{Z}(X)$ be the space of closed currents and $\mathcal{E}(X)$ be the space of exact currents. Then de Rham theory gives the equality

$$\frac{\mathcal{Z}(X)}{\mathcal{E}(X)} \simeq \sum_i H^i(X;\mathbb{R}) \tag{2.1}$$

which means closed currents are also representatives of cohomology. Let $T^pH^i(X;\mathbb{Q})$ be the subgroup of $H^i(X;\mathbb{Q})$, whose elements are represented by some closed currents supported on some subvarieties of codimension at least p, i.e.

$$T^pH^i(X;\mathbb{Q}) := \frac{\bigcup_{cod(V)\geq p} Ker\Big(\mathcal{Z}^i(X) \to \mathcal{Z}^i(X-V)\Big) + \mathcal{E}(X)}{\mathcal{E}(X)} \cap H^i(X;\mathbb{Q}),$$

where the superscript i is the degree of the currents, and V is a subvariety. We call $T^pH^i(X;\mathbb{Q})$ the current-supported subgroup. We call the original definition of the coniveau filtration in homological algebra notation (1.1) the class-supported subgroup. Then we claim that the truncated current-supported filtration is the same as the class-supported coniveau filtration for non-negative levels, i.e. for the case $i-2p \geq 0$.

Lemma 2.1. *Let X be a complex projective manifold. Then*

$$T^pH^i(X;\mathbb{Q}) = N^pH^i(X;\mathbb{Q}), \quad for\ i-2p \geq 0. \tag{2.2}$$

Proof. We recall the coniveau filtration's subgroup $N^pH^i(X;\mathbb{Q})$ is defined as the subgroup

$$\bigcup_{cod(V)\geq p} ker\Big\{H^i(X;\mathbb{Q}) \to H^i(X-V;\mathbb{Q})\Big\}.$$

For each $\alpha \in T^pH^i(X;\mathbb{Q})$, let t_α be a current that represents α and is supported on an algebraic subvariety V of codimension $\geq p$. Since V is a closed set, the restricted current $t_\alpha|_{X-V}$ is well-defined and is equal to zero. Hence

$$t_\alpha \in Ker\Big(\mathcal{Z}^i(X) \to \mathcal{Z}^i(X-V)\Big).$$

Then the de Rham theory (2.1) implies

$$\alpha \in Ker\Big(H^i(X;\mathbb{Q}) \to H^i(X-V;\mathbb{Q})\Big).$$

Hence

$$T^pH^i(X;\mathbb{Q}) \subset N^pH^i(X;\mathbb{Q}). \tag{2.3}$$

Conversely, let $\alpha \in N^pH^i(X;\mathbb{Q})$. Since $i-2p \geq 0$, the cohomological degree's requirement for the duality in Corollary 8.2.8, [2] † is met. Then the corollary holds and it directly implies that α has a singular representative c_α lying on some algebraic subvariety V of $codim(V) \geq p$. Since c_α can be regarded as a current, then the class α lies in $T^pH^i(X;\mathbb{Q})$. We complete the proof.

□

The re-interpretation in Lemma 2.1 provides the basis for the transition to support. Following is a simple lemma for the support of a non-closed current – a holomorphic chain $\mathcal{T}_V$.

Lemma 2.2.

$$supp(\mathcal{T}_V) = \overline{V} \tag{2.4}$$

where $\overline{V}$ is an algebraic cycle in X obtained by taking the closure of each irreducible component in V.

Proof. Since M is compact, each defining equation of V can be extended across M. Hence the closure of each component of V is algebraic. Let W_i be an irreducible subvariety in the cycle V. So $W_i \subset X\backslash M$. If $a \in supp(T_{W_i})$, there is neighborhood U of a that supports a test form ϕ such that $\int_{W_i} \phi \neq 0$. Hence U must meet W_i. So, $a \in \overline{W_i}$ where $\overline{W_i}$ is the closure in X. Conversely, if $a \in \overline{W_i}$, there is such a test form ϕ that $\int_{W_i} \phi \neq 0$. Hence $a \in supp(T_{W_i})$. Then

$$supp(T_{W_i}) = \overline{W_i}.$$

Take the linear combination of W_i, we obtain the lemma.

□

3 Proof

The Main theorem should follow from the proposition below.

Proposition 3.1. *For a non-negative k,*

$$[u]\bigg(H^k(X;\mathbb{Q})\bigg) \subset L_kH(X).$$

Proof. The proof is essentially the continuation of Harvey-Lawson's description of holomorphic chains but in direction of supportive intersection. Let $deg(u) = 2(n-p)$ where $0 \leq p \leq 2n$. Let c_u be a regular cycle representing the class u. Then $c_u = \sum_{i=1}^{m} r_i e_i$ where r_i are rational numbers and e_i is diffeomorphic to the interior of a $2p$ dimensional disk. Let $B = \cup_i e_i$ be the submanifold of the

†The requirement $i-2p \geq 0$ for the corollary is quite implicit in the original paper.

real manifold X, that has disconnected components e_i. Then there is a tubular neighborhood $P : U \to B$ that has a $\mathbb{R}^{2n-2p}$-bundle structure. Consider the non-compact manifold U, the integration on $\sum_{i=1}^{m} a_i e_i$ defines a closed current $\mathcal{T}$ in U. By the duality for non-compact manifold U, $\mathcal{T}$ as a closed current is homologous to a closed form ω_U which is compactly supported in the vertical direction of the bundle. By the Thom isomorphism for the $\mathbb{R}^{2(n-p)}$-bundle, $P_*(\omega_U) = a_i \neq 0$ where a_i is a number determined by the connected component of the base of the vector bundle U. Notice this ω_U is the restriction of the differential form representing the class ω. Hence ω_U is chosen to be a $(n-p, n-p)$ form. In the neighbohood $W_i \subset e_i$ of a point in the base e_i, each fibre $P^{-1}(v)$ for $v \in W_i$ needs to be a complex submanifold (otherwise the $P_*(\omega_U) = 0$). Thus at the point v, the complexified tangent spaces $T_{v,\mathbb{C}} e_i$ and $T_{v,\mathbb{C}}(P^{-1}(v))$ form a complex orthogonal decomposition of the $T_{v,\mathbb{C}} U \simeq \mathbb{C}^n$, i.e.

$$T_{v,\mathbb{C}} e_i \oplus T_{v,\mathbb{C}}(P^{-1}(v)) \simeq \mathbb{C}^n.$$

Since the decomposition holds in a neighborhood of $v \in e_i$, each neighborhood of e_i is a complex analytic manifold of dimension p. Therefore e_i is a complex submanifold of dimension p. So, we let

$$c_u = \sum_{i=1}^{l} a_i e_i - \sum_{i=l+1}^{m} a_i e_i$$

where each e_i for $i = 1, \cdots, l, l+1, \cdots, m$ is a complex submanifold of dimension p, and a_i are positive rational numbers. For each e_i (all i), let $s_i = \partial e_i$ be the boundary which is a regular chain of the real dimension $2p - 1$. Then $a_i e_i$ represents the class $a_i \langle e_i \rangle$ in the relative homology

$$H_{2(n-p)}(X, s_i; \mathbb{Q}).$$

We apply the theorem 1.11, [4] to obtain a positive holomorphic p-chain h_i and a positive, closed current k_i such that $dh_i = a_i T_{s_i}$ and for classes $a_i \langle e_i \rangle = \langle h_i + k_i \rangle$ in

$$H_{2p}(X, s_i; \mathbb{Q}).$$

Expressing it in currents in X, we have

$$a_i T_{e_i} = h_i + k_i + d\Gamma_i + j_*(\eta_i) \tag{3.1}$$

where Γ_i is a current of dimension $2p + 1$, η_i is a dimensional $2p$ current in s_i and $j : s_i \hookrightarrow C \times X$ is the inclusion map. Since the dimension of s_i is lower, η_i must be 0. Since s_i which is the boundary of h_i is compact, we can extend the complex analytic chain associated to h_i to the entire X to obtain an algebraic cycle A_i of X. Thus the current has a decomposition expression,

$$T_{c_u} = \sum_{i=1}^{l} (h_i + k_i + d\Gamma_i) - \sum_{i=l+1}^{m} (h_i + k_i + d\Gamma_i) \tag{3.2}$$

Let δ be a regular cycle representing an arbitrary cohomological class of degree k in X (including the fundamental class $\langle X\rangle$ whose level is 0). In the following we compute the supportive intersection.

By the supportivity in Definition 1.1,

$$supp([h_i \wedge \delta]) \subset supp(h_i). \tag{3.3}$$

By Lemma 2.2,

$$supp(h_i) \subset |A_i|. \tag{3.4}$$

For the other term in the decomposition (3.2), since $k_i + d\Gamma_i$ is a closed current which represents a cohomology class, we can make an argument in cohomology. Recall the cohomology class of a closed current or Poincaré dual of a singular cycle are all denoted by $\langle \bullet \rangle$. Notice $k_i + d\Gamma_i$ is a current of the quasi-chain $e_i - h_i$. Thus $[(k_i + d\Gamma_i) \wedge \delta]$ is well-defined. By the supportivity of Definition 1.1 and Lemma 2.2,

$$supp[(k_i + d\Gamma_i) \wedge \delta] \subset supp(k_i + d\Gamma_i) \subset |A_i| \cup \bar{e}_i \tag{3.5}$$

where $\bar{e}_i$ is the closure of the cell. Furthermore by Lemma 2.1 and cohomologicality in Definition 1.1,

$$\langle k_i + d\Gamma_i \rangle \in H^{2(n-p)}_{|A_i| \cup \bar{e}_i}(X; \mathbb{Q}) \tag{3.6}$$

i.e. $k_i + d\Gamma_i$ is class-supported on the closed set $|A_i| \cup \bar{e}_i$. Since $\bar{e}_i$ is contracted to a point $o \in X$,

$$H^{2(n-p)}_{|A_i| \cup \bar{e}_i}(X; \mathbb{Q}) = H^{2(n-p)}_{|A_i| \cup \{o\}}(X; \mathbb{Q}). \tag{3.7}$$

Next to show

$$H^{2(n-p)}_{|A_i| \cup \{o\}}(X; \mathbb{Q}) = H^{2(n-p)}_{|A_i|}(X; \mathbb{Q})$$

it suffices to show

$$H^{2(n-p)}_{|A_i| \cup \{o\}}(X; \mathbb{Q}) \subset H^{2(n-p)}_{|A_i|}(X; \mathbb{Q}).$$

If $p = n$, then $H^{2(n-p)}(X; \mathbb{Q}) \simeq \mathbb{Q}$. Then any class is represented by a constant. Hence

$$H^{2(n-p)}_{|A_i| \cup \{o\}}(X; \mathbb{Q}) = H^{2(n-p)}_{|A_i|}(X; \mathbb{Q}).$$

If $p < n$, let ϕ be a closed form on X whose integral on any chains in

$$X \backslash (|A_i| \cup \{o\})$$

is 0. Let C be a singular cycle in $X \backslash |A_i|$ that has the dimension $deg(\phi)$ (which is less than $2n$). If o is on the $supp(C)$, then there is a homotopic cycle $C' \simeq C$ in $X \backslash |A_i|$ such that o does not lie on C'. Then by the assumption, $\int_{C'} \phi = 0$.

So $\int_C \phi = 0$. Hence ϕ is restricted to the 0 class in $H^{2(n-p)}(X\backslash|A_i|)$. By the expression (1.2), cohomology,

$$H^{2(n-p)}_{|A_i|\cup\{o\}}(X;\mathbb{Q}) = H^{2(n-p)}_{|A_i|}(X;\mathbb{Q}). \tag{3.8}$$

Hence $k_i + d\Gamma_i$ is class-supported on $|A_i|$. Since (3.8) asserts h_i is current-supported on $|A_i|$, $\sum_i h_i$ by Lemma 2.1 is class-supported on $|A_i|$. Thus we conclude c_u is class-supported on the algebraic set $\cup_{i=1}^m |A_i|$ whose real dimension is $2p$. Since $dim([c_u \wedge \delta]) = k + 2n - 2p$, the class $\langle [c_u \wedge \delta] \rangle$ lies in $L_k H(X)$. We complete the proof. □

Proof of Main theorem 1.2: Let $\omega \in L_k H(X)$. Let $deg(\omega) = i$. Then there is an algebraic set V of $cod(V) = 2r$ such that ω is class-supported on V and $2r \geq i - k$. Now we let $\tilde{V}$ be the smooth resolution of V and $J : \tilde{V} \to X$ the morphism. By Corollary 8.2.8, [2], there is $\omega' \in H^{i-2r}(\tilde{V};\mathbb{Q})$ such that $J_!(\omega') = \omega$. Let $u' = J^*(u)$ which is still a Hodge class. Now we apply the proposition 3.1 to the setting in the complex projective manifold $\tilde{V}$ where $u' = J^*(u)$ is the Hodge class, and ω' is the degree $i - 2r$ cohomological class in $\tilde{V}$. We obtain $[u'](\omega') \in L_{i-2r}H(\tilde{V})$. Notice that for the Gysin homomorphism, $J_!$ has the projection formula which asserts

$$J_!([u'](\omega')) = [u](\omega) \tag{3.9}$$

Also the morphsim of the Hodge structure preserves the level. Hence

$$[u](\omega) \in L_{i-2r}H(X) \tag{3.10}$$

Since level filtration is a decreasing filtration and $k \geq i - 2r$. We obtain

$$[u](\omega) \in L_k H(X). \tag{3.11}$$

We complete the proof.

□

Remark: Main theorem assumes the existence of a supportive intersection whose proof for regular chains are given in [7]. Its natural extension to the quasi-chains in the complex case is an easy consequence of the dominant convergence theorem.

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