Formulas for SU(3) Matrix Generators

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Abstract

The Lie algebra of a Lie group is a set of commutation relations, equations satisfied by the group's generators. For SU(2) and many other Lie groups, the equations have been solved and matrix generators are realized as algebraic expressions. This article derives formulas for a basis of matrix generators for the irreducible representations of the Lie group SU(3). A special sequence of eigenvectors is deduced to assist in the derivation. As algebraic functions, the formulas are suited to numerical evaluation, algebraic manipulations, and analytic operations.

PACS: 02.20.Qs General properties, structure, and representation of Lie groups

1 Introduction

The special unitary group SU(3) has extensive applications in physics, from particle physics[1, 2, 3, 4, 5, 6] and nuclear physics[7, 8] to the ubiquitous harmonic oscillator, specifically the 3D isotropic harmonic oscillator [9].

For these investigations, it may be convenient to have quick and easy access to matrix representations of the group. To possibly aid such endeavors, this article derives formulas for basis generators of the group SU(3).

Many Lie groups have formulas for their basis generators. Matrix representations of the group SU(2), one step down from SU(3), can be found for any integer or half-integer spin t by substituting t into well-known formulas.[10, 11, 12, 13] The homogeneous Lorentz group has matrix realizations determined by compound SU(2) formulas.[2] Having formulas

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for these groups is fundamental to many numerical, algebraic and analytic explorations of topics related to SU(2) and the Lorentz group. The same may become true for SU(3).

The formulas derived here from the Lie algebra are more compact and simpler than the formulas of the earlier experimental work,[14] which were inferred, in part, from numerical small-dimensional irreducible representations (irreps) obtained irrep-by-irrep by computer.

This article finds formulas for finite-dimensional matrices that realize a set of basis generators. The basis T^3, Y, $T^{\pm}$, $U^{\pm}$, $V^{\pm}$, is herein called the basis $TYUV$. The equations to solve are the commutation relations (CR) of the $\mathfrak{su}(3)$ Lie algebra for the $TYUV$ basis. One must specify the basis since the CRs of the algebra depend on the choice of basis.

We include an additional equation. There exist two Casimir operators, which are functions of the generators that commute with the generators and hence with all elements of the algebra.[3] One, the one called the cubic Casimir, plays no part in the derivation. A well-known equation for the other, often called the quadratic Casimir operator, is added to the CRs of the algebra and becomes an essential part of the derivation here. The quadratic Casimir equation is known to be a consequence of the $\mathfrak{su}(3)$ Lie algebra, so the derivation remains founded on the Lie algebra.

The $TYUV$ basis has useful properties. For one, the eight basis generators can be realized as matrices with real components. Thus, the derivation and solutions involve real-valued functions. In addition, the CRs for the three T generators, T^3,$T^{\pm}$, satisfy the $\mathfrak{su}(2)$ Lie algebra which has known algebraic formulas for the matrix realizations of its irreps. It follows that the T matrices T^3, $T^{\pm}$ can be completely reduced to a direct sum of irreps of $\mathfrak{su}(2)$, with known formulas for the components of each irreducible representation.

Two of the eight generators T^3 and Y are diagonal matrices, by assumption and by convention. Then the eigenvectors of an $\mathfrak{su}(3)$ irrep are the columns of the unit matrix, and the eigenvalues (t, y) of T^3 and Y are their diagonal components. An eigenvector has a pair of simultaneous eigenvalues. When plotted, the simultaneous eigenvalues (t, y) make a pattern called a multiplet.[15, 3, 11, 4, 10]

Applying one of the six generators $T^{\pm}$, $U^{\pm}$, $V^{\pm}$ to an eigenvector gives another eigenvector with raised or lowered eigenvalues. The six generators $T^{\pm}$, $U^{\pm}$, $V^{\pm}$ raise and lower eigenvalues. Starting with one of the points on the multiplet pattern, all the points can be reached by successive application of $T^{\pm}$, $U^{\pm}$, or $V^{\pm}$.

Multiplets have been well studied, and we make extensive use of their known properties to organize eigenvectors. We find that we can also determine Y. Including the three T generators and Y, we have four generators that we can find formulas for from previously known results.

Except that the order of SU(2)-irreps in the direct sum making the T-matrices is arbitrary. We spend Sections 3, 4, 5, producing a special sequence of $\mathfrak{su}(2)$ irreps that is helpful in the derivation. It turns out that the index k of the k^{th} $\mathfrak{su}(2)$ irrep depends on two multiplet quantities a and b, both non-negative integers.

Then the place n of an eigenvector in the special sequence is a function $n(a,b;\alpha)$ of three parameters, with a,b to select an $\mathfrak{su}(2)$ irrep and α to determine its location in that irrep.

Make a one-to-one correspondence between the row index r and the place n, $r \leftrightarrow n(a_r,b_r,\alpha_r)$, and a one-to-one correspondence for the column index c, $c \leftrightarrow n(a_c,b_c,\alpha_c)$. Then the indices r,c that locate a matrix component M^{rc} are a function of the six quantities $(a_r,b_r,\alpha_r,a_c,b_c,\alpha_c)$.

The six parameters that locate a matrix component become available parameters for the domain of the function that gives the value of that matrix component. The parameters that determine the r^{th} row and the c^{th} column can also determine the value of the component M^{rc}.

The equations to solve and some initial conditions are found in Section 2. The special sequence of eigenvectors is constructed in Sections 3, 4, 5. In Section 6, formulas are determined that relate the row/column indices r,c of a matrix component M^{rc} to the parameters $(a_r,b_r,\alpha_r,a_c,b_c,\alpha_c)$.

The four given matrix generators T,Y are assembled in Section 7. Sections 8 and 9 have the derivation of formulas for the four remaining U,V matrices. Section 8 considers the CRs that are linear in the U,V matrices. After evaluating several linear CRs, only two unknown functions remain to be determined. In Section 9, the CRs quadratic in the U,V matrices and the quadratic Casimir operator equation are solved for the two unknown functions. Inverting the steps of the derivation yields formulas for the matrices $U^{\pm}$ and $V^{\pm}$ in Section 10.

The generators are shown to satisfy three of the algebra's 28 CRs in Section 11. The verification of the remaining 25 CRs and of the quadratic Casimir equation is included as Supplemental Material in the form of a Wolfram Mathematica Notebook program.[16, 17] The Casimir equation is verified even though the equation is a consequence of the 28 CRs of the algebra and is automatically satisfied whenever the 28 CRs are satisfied. The program also calculates the $TYUV$ basis of a user-selected $\mathfrak{su}(3)$ irrep.

In the discussion Section 12, the practical distinctions that separate generators $TYUV$ into various subsets are linked to the intrinsic properties of the $\mathfrak{su}(3)$ Lie algebra. The issue of real-valued solutions to quadratic equations is raised. In the same section, a transformation to a different basis is presented and possible topics for further investigations are suggested.

Appendix A provides numerical examples of many quantities that arise in the derivation of the formulas and are referred to in the text.

Appendix B presents a FORTRAN computer program that calculates matrix generators for an irreducible representation (irrep) of the user's choice[18, 19] The author has used the program to verify that the matrix generators for $1 \le p+q \le 20$ obey the 28 CRs of the $\mathfrak{su}(3)$ Lie algebra, as well as obeying the quadratic Casimir equation.

2 Lie algebra

In this section, the Lie algebra is derived and the equations to solve are set. For details of the derivation of the algebra, it is recommended to consult the literature, where the same derivation can be found, but with more complete discussions [10, 15].

Unitary 3×3 matrices with determinants equal to one can be shown to form a Lie group under matrix multiplication, which is also called the dot product. The group is called the special unitary group of degree 3, abbreviated SU(3).

Let the matrix $D(\theta)$ be an element of SU(3). One can show that the matrix can be written as a function of eight real parameters θ_i, $i \in \{1, 2, ..., 8\}$ with eight generators F^i,

$$D(\theta) \equiv \exp i\theta_i F^i \quad . \tag{1}$$

The generators F^i are traceless 3×3 Hermitian matrices.

One basis of generators consists of the well-known Gell-Mann matrices multiplied by a factor of one-half,[20, 3, 4]

$$F^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \tag{2}$$

$$F^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad F^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad F^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \tag{3}$$

$$F^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad F^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad . \tag{4}$$

By inspection, these F^is are hermitian and traceless.

The F^is have complex-valued components. A change of basis to matrices with real-valued components is possible in many ways. We choose a new basis of eight matrices $\{T^3, Y, T^\pm, U^\pm, V^\pm\}$, which is given by the following formulas,

$$T^3 = F^3 \; ; \; Y = \frac{2}{\sqrt{3}}F^8 \; ; \; T^\pm = F^1 \pm iF^2 \; ; \; U^\pm = F^6 \pm iF^7 \; ; \; V^\pm = F^4 \pm iF^5 \quad . \tag{5}$$

We call this the $TYUV$ basis. From equations (2) to (5), one finds that

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad Y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \; ; \; T^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$T^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad U^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad , U^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \tag{7}$$

$$V^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad V^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad . \tag{8}$$

The transformation is invertible, so one can determine the basis of F^is from the $TYUV$ basis. Since this is a change of basis, we consider the $TYUV$ basis to be a basis of generators for the $\mathfrak{su}(3)$ Lie algebra.

The eight $TYUV$ matrices in (6), (7), (8), produce $8 \times 7/2 = 28$ non-trivial commutation relations (CR). The CRs are listed below in (9) to (14) and can be verified directly from (6), (7), (8).

We want to keep the same notation for the $TYUV$ basis of every irrep of the $\mathfrak{su}(3)$ Lie algebra. One must judge from the context whether the matrix T^+ is the matrix in (5), the fundamental representation, or a basis generator for some other $\mathfrak{su}(3)$ irrep.

We can describe the problem considered in this article. The unknowns are the real-valued components of eight $d \times d$ square matrices T^3, Y, $T^{\pm}$, $U^{\pm}$, $V^{\pm}$, where d is the dimension of the matrices. Linear combinations of the eight matrices form a vector space $\mathfrak{g}$ under matrix addition and scalar multiplication by real numbers. Note that we are using the same notation for the unknown matrices as was used for the fundamental rep (6), (7), (8).

The 28 commutation relations (CR) assumed for the eight unknown matrices are listed below in (9) to (14), which are the CRs determined by the fundamental rep. The CRs show that the commutator of any two of the eight matrices is a linear combination of the eight, so the commutator of two elements of $\mathfrak{g}$ is an element of $\mathfrak{g}$. If $A, B \in \mathfrak{g}$, then $[A, B] \in \mathfrak{g}$. The commutator keeps $\mathfrak{g}$ closed. One can show that the eight matrices form the basis for a Lie algebra.[10, 15]

Since the Lie algebra has been determined from the fundamental rep of the SU(3) Lie group, we call it the $\mathfrak{su}(3)$ Lie algebra with the $TYUV$ basis, sometimes called a Cartan-Weyl basis.[15, 3, 11]

The Lie algebra's CRs can be grouped into CRs that are not dependent on U, V matrices, CRs that are linear in the U, V matrices, and CRs that are quadratic in the U, V matrices.

CRs not dependent on U, V matrices:

$$[T^+, T^-] = 2\,T^3 \quad ; \quad [T^3, T^{\pm}] = \pm\,T^{\pm} \quad ; \tag{9}$$

$$[Y, T^{\pm}] = 0 \quad ; \quad [Y, T^3] = 0 \quad . \tag{10}$$

CRs linear in U, V matrices:

$$[T^3, U^{\pm}] = \mp \frac{1}{2} U^{\pm} \; ; \; [T^3, V^{\pm}] = \pm \frac{1}{2} V^{\pm} \; ; \; [Y, U^{\pm}] = \pm \, U^{\pm} \; ; \; [Y, V^{\pm}] = \pm \, V^{\pm} \tag{11}$$

$$[T^{\pm}, U^{\mp}] = [T^{\pm}, V^{\pm}] = 0 \quad ; \quad [T^{\pm}, U^{\pm}] = \pm \, V^{\pm} \quad ; \quad [T^{\pm}, V^{\mp}] = \mp \, U^{\mp} \quad . \tag{12}$$

CRs quadratic in U, V matrices:

$$[U^+, U^-] = \frac{3}{2} Y - T^3 \quad ; \quad [V^+, V^-] = \frac{3}{2} Y + T^3 \quad ; \tag{13}$$

$$[U^{\pm}, V^{\mp}] = \pm \, T^{\mp} \quad ; \quad [U^{\pm}, V^{\pm}] = 0 \quad , \tag{14}$$

The commutator $[A, B]$ of two matrices is the difference of their dot products, $[A, B] \equiv AB - BA$.

The Casimir operator C that we utilize can be written as a function that is quadratic in the generators. It is not a linear combination of generators and, so, it is not an element of the algebra. The defining feature of Casimir operators is that they commute with all elements of the algebra. One finds that, for a given matrix irrep, the effect of C acting on a generator's matrix is equivalent to multiplication of the matrix by a constant,

$$\begin{aligned} C &\equiv \left(\{T^+, T^-\} + \{U^+, U^-\} + \{V^+, V^-\}\right)/2 + {T^3}^2 + 3Y^2/4 \\ &= \left(p^2 + pq + q^2 + 3p + 3q\right)/3 \, \mathbf{1} \; , \end{aligned} \tag{15}$$

where the anti-commutator is $\{A, B\} \equiv AB + BA$, and the identity matrix $\mathbf{1}$ is appropriately dimensioned. The quantities p and q are integers that identify the irreducible representation of the $\mathfrak{su}(3)$ Lie algebra, the (p, q) $\mathfrak{su}(3)$ irrep.

Note that, since the quadratic Casimir equation (15) is a consequence of the algebra, an alternative derivation could be devised that is based directly and solely on the 28 CRs of the algebra. The alternative derivation may be presented elsewhere.

In summary, the unknowns are the eight matrices T^3, Y, $T^{\pm}$, $U^{\pm}$, $V^{\pm}$, that are assumed to have real-valued components. They are required to satisfy the 28 CRs of the Lie algebra $\mathfrak{su}(3)$ with the $TYUV$ basis and to satisfy the quadratic Casimir equation. By satisfying the 28 CRs of the algebra, the matrices form a basis of generators for the algebra. The basis is restricted to a single irrep of the algebra.

3 Eigenvectors

As stated in the Introduction, the four matrices $T^{\pm}$, T^3, Y, can be found with the known formulas for SU(2)-irreps and from the properties of a multiplet. In this section, we begin the discussion of the multiplet for the (p, q) $\mathfrak{su}(3)$ irrep.

By the CRs in (9), the T-matrices satisfy the Lie algebra $\mathfrak{su}(2)$. Therefore, we can assume that the T-matrices are completely reduced to direct sums of irreps of SU(2). However, a similarity transformation can rearrange the order of the SU(2) irreps. We consider multiplet properties to find a preferred sequence of the SU(2) irreps.

We choose T^3 and Y to be diagonal matrices, which is the standard choice. However, we find that the eigenvalues of Y are not convenient to index the components of the matrix. So we define a matrix Y^0 with eigenvalues that are shifted from those of Y. The new matrix Y^0 and its eigenvalues y^0 are

$$Y^0 \equiv Y + 2(p-q)/3\,\mathbf{1} \quad \text{and} \quad y^0 = y + 2(p-q)/3 \quad . \tag{16}$$

The eigenvalues of Y^0 and Y differ by $2(p-q)/3$. The two matrices Y and Y^0 are equivalent in the sense that one can be found given the other. But unlike Y, the matrix Y^0 is neither a generator nor a linear combination of generators because the identity matrix $\mathbf{1}$ is not a combination of generators.

Let α and y^0 be the r^{th} component along the diagonals of T^3 and Y^0, resp, *i.e.* $T^3_{rr} = \alpha$ and $Y^0_{rr} = y^0$. Therefore, α and y^0 are simultaneous eigenvalues for the eigenvector, in Dirac's 'bra-ket' notation, $|\alpha, y^0\rangle$. We have

$$T^3|\alpha, y^0\rangle = \alpha|\alpha, y^0\rangle \quad , \quad Y^0|\alpha, y^0\rangle = y^0|\alpha, y^0\rangle \quad , \tag{17}$$

with the r^{th} column of the identity matrix $\mathbf{1}$ as $|\alpha, y^0\rangle$,

$$|\alpha, y^0\rangle = (0, ..., 1, ...0)^{\text{T}}, \tag{18}$$

with 1 in the r^{th} place. For convenience, the transpose T is used to display the column vector. Multiplying the eigenvector by an arbitrary nonzero real number A gives an eigenvector with the same eigenvalue(s). We make note of eigenvector normalization when pertinent. We can assume that the eigenvectors are columns of the identity matrix.

For the (p, q)-irrep, the matrix generators are $d \times d$ square matrices with real-valued components, where the dimension d is known to be[3, 4, 10]

$$d = \frac{1}{2}(p+1)(q+1)(p+q+2) \quad . \tag{19}$$

Thus, the identity matrix $\mathbf{1}$ has dimension d and the eigenvector $|\alpha, y\rangle$ is a column vector with d components.

CRs: $[T^3, T^{\pm}], [T^3, U^{\pm}], [T^3, V^{\pm}], [Y, T^{\pm}], [Y, U^{\pm}], [Y, V^{\pm}]$. These equations govern the action of the raising and lowering matrices, $T^{\pm}, U^{\pm}, V^{\pm}$, on the eigenvectors $|\alpha, y^0\rangle$.

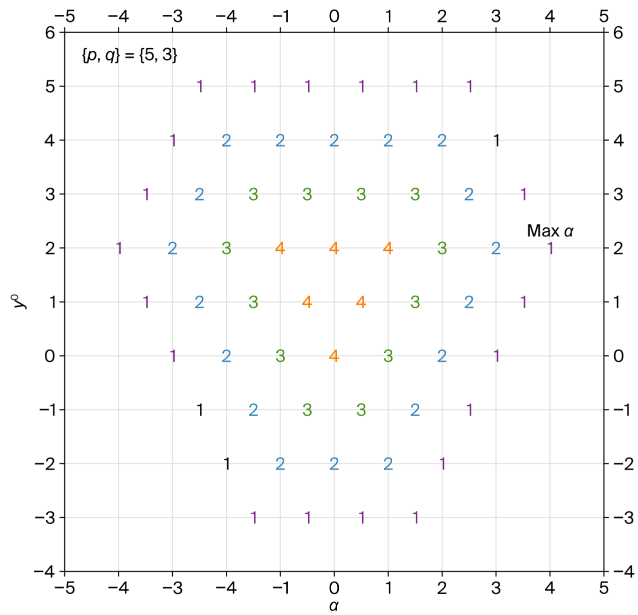

Figure 1: *The* $(p, q) = (5, 3)$ *multiplet.* The eigenvalues α and y^0 of the simultaneous eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0 make a six-sided figure whose sides have p or q spaces. The point (α, y^0) is marked with the number of eigenvectors, the 'multiplicity,' that share the eigenvalues α and y^0. The multiplicity is one on the rim and increases inward on smaller six-sided figures until reaching the central triangle with multiplicity 4. The point on the rim with the maximum T^3 eigenvalue $\alpha = 4$ is labeled 'Max α', often referred to as max weight.

Consider the CR with T^3 and one of the raising or lowering matrices R. We have $[T^3, R]$ $= rR$, where the constant r is found in the CR. Applying the CR to an eigenvector gives the following.

$$T^3 R|\alpha, y^0\rangle = RT^3|\alpha, y^0\rangle + rR|\alpha, y^0\rangle = (\alpha + r)\, R|\alpha, y^0\rangle \Rightarrow R|\alpha, y^0\rangle \propto |\alpha + r, y^0\rangle \quad . \tag{20}$$

Thus, the action of applying the matrix R to an eigenvector $|\alpha, y^0\rangle$ produces an eigenvector $|\alpha + r, y^0\rangle$, which has raised or lowered the eigenvalue α depending on the sign of r. For the Rs and rs in the CRs listed above, we get

$$T^{\pm}|\alpha, y^0\rangle \propto |\alpha \pm 1, y^0\rangle \ , \ U^{\pm}|\alpha, y^0\rangle \propto |\alpha \mp 1/2, y^0 \pm 1\rangle \ , \ V^{\pm}|\alpha, y^0\rangle \propto |\alpha \pm 1/2, y^0 \pm 1\rangle \ , \tag{21}$$

which includes the results of raising and lowering the eigenvalues α and y^0. It is known that all eigenvectors of a (p, q)-irrep can be obtained by the action of raising and lowering matrices applied to any eigenvector of the irrep.

4 Multiplets

The plot of the eigenvalues (α, y^0) of the eigenvectors of a (p, q)-irrep produces a three- or six-sided figure called a multiplet. See Figure 1. We list some well-known multiplet properties (MPs).[3, 4, 10, 15]

MP1. The multiplets have a six-sided pattern with $p+1$ points in the row with maximum y^0 and $q+1$ points in the row with minimum y^0. The sides alternately have p spaces and q spaces.

In Figure 1, the top row has six points with $p = 5$ spaces; the bottom row has $q = 3$ spaces.

MP2. The points (α, y^0) on the outer rim each represent one eigenvector, so they have multiplicity one. Each step inward from the rim has points representing an additional eigenvector, until the inner triangle whose points have the maximum multiplicity.

In Figure 1, the outer rim has multiplicity '1,' while the inner triangle is filled with points with multiplicity '4,' meaning each point represents four distinct eigenvectors that happen to have the same eigenvalues.

MP3. The point on the right with the largest eigenvalue $\alpha_{\max}$, which is often called the point with maximum 'weight,' has coordinates $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$.

Theorem 1 (MP4)**.** *At least one eigenvector has null eigenvalues $\alpha = y^0 = 0$. Thus, a multiplet always has at least one eigenvector $|0,0\rangle$ at the $\alpha = y^0 = 0$ vertex of the inner triangle.*

Proof. Start on the multiplet at the point of maximum weight, $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$. By (21), after q applications of the raising matrix U^+ and p applications of V^-, we get an eigenvector at the origin $(\alpha, y^0) = (0,0)$, which completes the proof. See Figure 1. □

Theorem 2 (MP5)**.** *At any point of the multiplet, take double the spin component, i.e. 2α, and take y^0 to make the pair $(2\alpha, y^0)$, both integers. The pair never has one odd integer and one even integer. It is always true that the pair are both even integers or they are both odd integers.*

Proof. Since p and q are integers, it follows that their sum and difference $(p+q)$ and $(p-q)$, are even integers or are both odd integers. They are both even when p and q are both even or odd; they are both odd integers when just one of (p, q) is even and the other is odd. Thus, MP5 is true for the eigenvector with maximum weight, the point $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$, since we take double the spin component $2\alpha_{\max} = (p+q)$. But all eigenvectors of the multiplet can be reached by the action of the raising and lowering matrices, (21), so MP5 is true for all points of the multiplet. □

5 Eigenvector Sequencing

As stated previously, we determine a special sequence of SU(2) irreps for the reduced T matrices. The process takes two steps. First, the eigenvectors are collected into SU(2) irreps. After collecting SU(2) irreps, the SU(2)-irreps are placed in a special order.

Collect the eigenvectors into SU(2)-irreps. An SU(2) irrep with spin t has $2t+1$ eigenvectors $|\alpha, y^0\rangle$ with spin components $\alpha = t, t-1, ..., -t$. We proceed by example.

For example, consider the eigenvectors whose eigenvalues are plotted in the $y^0 = 3$ row of Figure 1. The leftmost point of the row has $\alpha = -7/2$, and the rightmost point has α $= +7/2$, all with $y^0/2 = 3/2$. By the MP2 multiplet property, the multiplicities increase by one from "1" outside to a constant "3" inside the row.

See Table 1 which records the successive removal of eigenvectors in batches of SU(2)-irreps. Note that, as a consequence of MP2, each SU(2) irrep collected has a unique combination of spin t and half-eigenvalue $y^0/2$. There is only one $(t, y^0/2)$-SU(2) irrep for a given combination $(t, y^0/2)$.

Table 1: Collecting the eigenvectors in the $y^0 = 3$ row of Figure 1. The first line has the original multiplicities of the eigenvectors with T^3 eigenvalues α. The second and third line have the multiplicities after the removal of eigenvectors collected in the $(t, y^0/2)$-SU(2) irrep in the right column.

α	$-7/2$	$-5/2$	$-3/2$	$-1/2$	$1/2$	$3/2$	$5/2$	$+7/2$	$(t, y^0/2)$
y^0=3 row	1	2	3	3	3	3	2	1	$(7/2, 3/2)$
		1	2	2	2	2	1		$(5/2, 3/2)$
			1	1	1	1			$(3/2, 3/2)$

The row $y^0 = 3$ has $1+2+3+3+3+3+2+1 = 18$ eigenvectors that have been collected into three $(t, y^0/2)$-SU(2) irreps $(7/2, 3/2)$, $(5/2, 3/2)$, $(3/2, 3/2)$. These SU(2)-irreps have $2t + 1 = 8, 6, 4$, eigenvectors each.

The generalization to the other rows of Figure 1 and to the multiplet of any (p, q)-irrep should be apparent.

Theorem 3 (MP6)**.** *Each of the collected irreducible representations* $(t, y^0/2)$*-SU (2) is unique. Exactly one SU(2)-irrep has that combination of* t *and* $y^0/2$.

Proof. Consider the set of all eigenvectors with eigenvalue y^0, a horizontal row of the multiplet. By MP2, only one eigenvector has the maximum eigenvalue α as the eigenvector is on the rim of the multiplet. Again by MP2, there is at least one eigenvector with eigenvalues in unit steps from $-\alpha$ to $+\alpha$. Collecting one eigenvector for each eigenvalue makes exactly one $(t, y^0/2)$ -SU (2) irrep of spin $t = \alpha$. By MP2, the remaining set of eigenvectors has only one eigenvector with the maximum eigenvalue $\alpha - 1$. The collection process produces exactly one $(t, y^0/2)$ -SU (2) irrep of spin $t = \alpha - 1$. By MP2, the process continues until the remaining eigenvectors have a constant multiplicity of one and collecting them makes the final SU(2)-irrep for the y^0 row. Since each collection step produces exactly one SU(2)-irrep and the following steps give successively smaller spins t,, it follows that none of the $(t, y^0/2)$ -SU (2) irreps have duplicate t. Eigenvectors that have different eigenvalues y^0 are in other constant sets y^0. Thus, either t or y^0 are different for the other collected $(t, y^0/2)$ -SU (2) irreps and that proves the statement. □

That completes the collection of the eigenvectors into SU(2)-irreps.

Put the SU(2)-irreps in order. Spin t and $y^0/2$ worked well in collecting eigenvectors in SU (2) irreps. However, we retire them in favor of their sum and difference, which are more useful quantities for sequencing the SU(2)-irreps.

Let a and b be the difference and sum defined in

$$a \equiv t - y^0/2 \quad ; \quad b \equiv t + y^0/2 \quad . \tag{22}$$

Solving the equations for t and y^0 gives

$$t = (a + b)/2 \quad ; \quad y^0 = b - a \quad . \tag{23}$$

We recognize that a plot of points (a, b) involves a rotation of $\pi/4$ about the origin of a plot of points (t, y^0) and a rescaling. Thus, when plotted, as in Figure 2 for the $(p, q) = (5, 3)$-multiplet in Fig 1, the points (a, b) form a rectangle aligned with the axes a and b.

In view of the new parameters $(t, y^0/2) \to (a, b)$, we give the $(t, y^0/2)$-SU(2) irreps a second name, the '(a, b)-SU(2) irrep.' Since the eigenvectors collected in $(t, y^0/2)$-SU(2) are the same eigenvectors in the (a, b)-SU(2) irrep, we have a corollary to Theorem 3.

Corollary 1 (MP6)**.** *There is exactly one (a, b)-SU(2) irrep for each combination of integers $a = 0, ..., q$ and $b = 0, ..., p$.*

We now show that the points (a, b) are pairs of integers: In a spin t SU(2) irrep, one of the spin components α is $\alpha = t$. From MP5, $2t$ and y^0 are even integers or odd integers. By their definitions as a sum and difference in (22), it follows that a and b are integers.

Plotting the points (a, b) for the (a, b)-SU(2) irreps of the $(p, q) = (5, 3)$-multiplet in Figure 1 produces the rectangular array in Figure 2. There are $(p + 1)(q + 1) = 24$ points in the rectangle with one SU(2) irrep represented at each point.

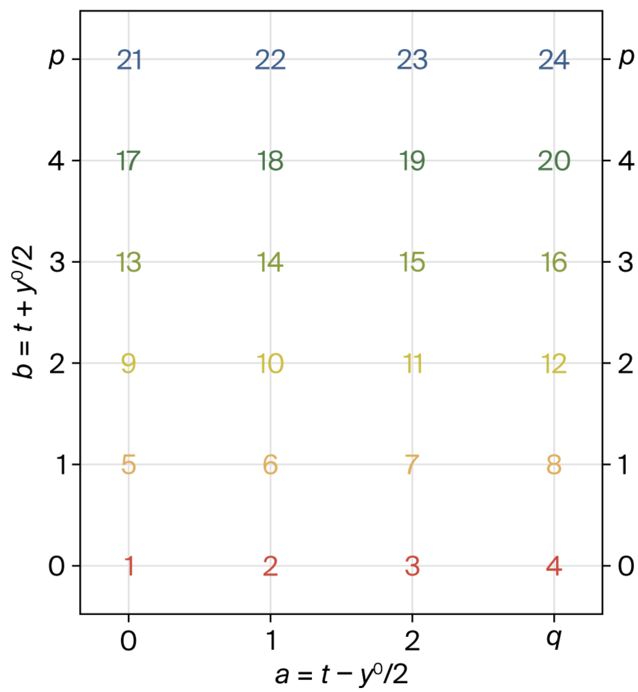

Figure 2: *The sequence of SU(2) irreps in the direct sum of SU(2) irreps making the T-matrices.* For the $(p,q) = (5,3)$ multiplet in Figure 1, simultaneous eigenvectors $|\alpha, y^0\rangle$ are collected in $(t, y^0/2)$-SU(2) irreps, as detailed in Table 1. By (22), transforming $(t, y^0/2)$ to (a,b), renames the su(2) irreps as (a,b)-SU(2) irreps. By Corollary 1, each of the $(q{+}1)(p{+}1)$ points of the rectangle represents exactly one (a,b)-SU(2) irrep. The sequence of SU(2) irreps k marks the points of the rectangle. The $k = 1$ irrep is the $(a,b) = (0,0)$-SU(2) irrep and the $k = 24$ irrep is the $(a,b) = (p,q) = (5,3)$-SU(2) irrep.

Since there is exactly one SU(2)-irrep represented at each point, we can assign an order to the SU(2)-irreps by assigning an order to the points of the rectangle. The points are placed in row-by-row order, starting with the bottom row of the rectangle. One can see that the (a,b)-SU(2) irrep is assigned the k^{th} place in the sequence, where

$$k = 1 + a + b\,(q+1) \quad . \tag{24}$$

The range of the SU(2)-irrep index k is $k = 1, ..., (q+1)(p+1)$.

For example, in Figure 2, the $k = 15^{\text{th}}$ SU(2) irrep is the $(a,b) = (2,3)$-SU(2) irrep which was earlier called the $(t, y^0/2) = (5/2, 1/2)$-SU(2) irrep.

The reduction of T-matrices into the direct sum of SU(2)-irreps is complete. By (24), the (a,b)-SU(2) irrep is the k^{th} SU(2) irrep in the direct sum, where $a = 0, ..., q$, $b = 0, ..., p$.

We call k the 'single index' for the list of SU(2) irreps. The pair of integers (a, b) is called the 'double index' for the SU(2)-irrep list.

In each SU(2)-irrep, the eigenvectors are placed in descending order. Each (a, b)-SU(2) irrep has spin $t = (a+b)/2$ with a number $N(a, b) = 2t + 1$ of eigenvectors $|\alpha, y^0\rangle$. we have

$$N(a, b) = 2t + 1 = a + b + 1 \quad . \tag{25}$$

These eigenvectors are placed in order of the spin component α, largest first, $\alpha = t, ..., -t$. Which makes $|\alpha, y^0\rangle$ the m^{th} eigenvector in the (a, b)-SU(2) irrep, where

$$m = t - \alpha + 1 = (a + b)/2 - \alpha + 1 \quad . \tag{26}$$

The place number m is a positive integer, $m = 1, ..., a + b + 1$.

In general, the point (a, b) for the (a, b)-SU(2) irrep is somewhere along the row b in the rectangle. The eigenvectors to the left of (a, b) precede our selected eigenvector $|\alpha, y^0\rangle$. In the b row, those eigenvectors in the $(\bar{a}, b)$-SU(2) irreps with $\bar{a} = 0, ..., a - 1$ precede the eigenvector $|\alpha, y^0\rangle$ in the sequence of eigenvectors.

The number of eigenvectors in the partial row to the left of (a, b) is

$$\sum_{\bar{a}=0}^{a-1} N(\bar{a}, b) = \sum_{\bar{a}=0}^{a-1} (\bar{a} + b + 1) = a(a + 1)/2 + ab \quad . \tag{27}$$

This does not include the eigenvectors preceding $|\alpha, y^0\rangle$ in the (a, b)-SU(2) irrep itself, which we will count separately below.

For a complete row, $a - 1 = q$, so substitute $a = q + 1$ in (27). Thus, the points $(\bar{a}, \bar{b})$ with $0 \leq \bar{a} \leq q$ make up the row with $b = \bar{b}$. The SU(2)-irreps for the row have a number of eigenvectors $n_{\text{row}}(\bar{b})$ given by the formula

$$n_{\text{row}}(\bar{b}) = (q + 1)(q + 2)/2 + (q + 1)\bar{b} \quad . \tag{28}$$

All complete rows below the row with the point (a, b) have eigenvectors that precede our eigenvector $|\alpha, y^0\rangle$ in the sequence of eigenvectors.

Our eigenvector $|\alpha, y^0\rangle$ is in the (a, b)-SU(2) irrep which is represented by the point (a, b). Combining the numbers of eigenvectors in the complete rows below (a, b) with the number in the partial row b to the left of (a, b) and placing the eigenvector as the m^{th} eigenvector in the (a, b)-SU(2) irrep, gives us the place n of the eigenvector $|\alpha, y^0\rangle$ in the sequence of d eigenvectors.

Therefore, the place n is given by the formula

$$n(a,b,\alpha) = \left[\sum_{\bar{b}=0}^{b-1} n_{\rm row}(\bar{b}) + \sum_{\bar{a}=0}^{a-1} N(\bar{a},b)\right] + m$$
$$= \left[(q+1)\,b\,(b+1)\,/2 + b\,(q+1)\,(q+2)\,/2 + a(a+1)/2 + ab\right] + (t-\alpha+1)$$

or

$$n(a,b,\alpha) = \frac{1}{2}\left[(a+b)\,(a+b+1) + qb\,(q+b+2)\right] + \left[t-\alpha+1\right] \quad , \tag{29}$$

where the first square-bracketed term on the right counts the eigenvectors of the complete rows and the partial row $(\bar{a},\bar{b})$-SU(2) irreps that precede the (a,b)-SU(2) irrep. The second square-bracketed term is the place of the eigenvector $|\alpha, y^0\rangle$ in the (a,b)-SU(2) irrep, which has $t = (a+b)/2$.

For example, in Figure 2, consider the (a,b)= $(3,2)$-SU(2) irrep, which has spin $t = 5/2$ and $\alpha = 5/2, 3/2, 1/2, -1/2, -3/2. -5/2$. One of its eigenvectors is $|1/2,-1\rangle$. With α =1/2, the eigenvector is the number $m = 3$ eigenvector in its irrep. The partial row to the left of $(3,2)$, has $3+4+5 = 12$ eigenvectors. And the complete rows below $(3,2)$ have a total of $10+14 = 24$ eigenvectors. Therefore, the eigenvector $|1/2,-1\rangle$ in the $(3,2)$-SU(2) irrep is the $24+12+3 = 39^{\rm th}$ eigenvector in the sequence of $d = 120$ eigenvectors for the $(p,q) = (5,3)$ irrep of SU(3).

The sequence of eigenvectors is determined. The eigenvector $|\alpha, y^0\rangle$ in the (a,b)-SU(2) irrep is the $n(a,b,\alpha)^{\rm th}$ in the sequence of d eigenvectors. This is a one-to-one correspondence of the integers n, where $1 \leq n \leq d$, with the d eigenvectors in the collection of simultaneous eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0.

6 The Eigenvector Sequence and Matrix Indices

Let M^{rc} denote one of the matrix generators. We say that the indices r,c are a pair of 'single' indices, $r,c = 1,...,d$.

The number of rows in M^{rc} is the same as the number of ones on the diagonal of the unit matrix and the number of eigenvectors in the SU(3)-irrep; all three numbers being the dimension d of the representation. Hence, the row index r can be put in a one-to-one correspondence with the place number n of the eigenvectors of the SU(3)-irrep. See Appendix A. The same is true for the column index c. We have

$$r = n(a_r, b_r, \alpha) \quad ; \quad c = n(a_c, b_c, \beta) \quad , \tag{30}$$

where $0 \leq a_r, a_c \leq q$, $0 \leq b_r, b_c \leq p$ and the spin indices have ranges $-t_r \leq \alpha \leq t_r$, $-t_c \leq \beta \leq t_c$, with SU(2)-irrep spins $t_r = (a_r + b_r)/2$ and $t_c = (a_c + b_c)/2$.

In this way, the row-column address of a matrix component in the matrix inherently contains parameters, *e.g.* a, b, α, that may be used in the formulas for the component's value. Thus, the indices of the component as well as the value of the component are functions of the parameters (a, b) and α.

It follows that we can identify the row and column with either the pair of integers r, c or with the two sets of three parameters $\{a_r, b_r, \alpha\}$ and $\{a_c, b_c, \beta\}$. In view of this, we introduce a new symbol for the component M^{rc}. We have

$$M^{rc} = M^{(a_r,b_r),(a_c,b_c)}_{\alpha,\beta} \quad . \tag{31}$$

One could say that the combinations (a_r, b_r, α), (a_c, b_c, β) are a pair of 'triple' indices, since all three values are needed to determine the row or column indices r or c.

It is convenient to talk about '(i, j)-blocks.' For fixed 'double' indices (a_i, b_i), (a_j, b_j), the rectangular matrix with rows and columns indexed by α, β is called an '(i, j)-block' defined by

$$M^{(i,j)}_{\alpha,\beta} = M^{(a_i,b_i),(a_j,b_j)}_{\alpha,\beta} \quad , \tag{32}$$

where $-t_i \leq \alpha \leq t_i$, $-t_j \leq \beta \leq t_j$, with $t_i = (a_i + b_i)/2$ and $t_j = (a_j + b_j)/2$. By (24), $i = 1 + a_i + b_i(q + 1)$ and $j = 1 + a_j + b_j(q + 1)$, with $i, j = 1, ..., (p + 1)(q + 1)$.

The block structure of the matrix M is a square array, $(p + 1)(q + 1)$ on one side, of (i, j)-blocks with each block containing a rectangular matrix of components $(2t_i + 1)$ by $(2t_j + 1)$.

The block for the commutator of two matrices M^1 and M^2 can be expressed in terms of dot products of the blocks of M^1 and M^2 as follows

$$\left[M^1, M^2\right]^{(a_i,b_i),(a_j,b_j)}_{\alpha,\beta} = {}^1M^{(a_i,b_i),(a_n,b_n)}_{\alpha,\eta}\, {}^2M^{(a_n,b_n),(a_j,b_j)}_{\eta,\beta} - {}^2M^{(a_i,b_i),(a_{\bar{n}},b_{\bar{n}})}_{\alpha,\bar{\eta}}\, {}^1M^{(a_{\bar{n}},b_{\bar{n}}),(a_j,b_j)}_{\bar{\eta},\beta} \quad , \tag{33}$$

with summation understood for repeated block indices and repeated spin components. The labels '1' and '2' are moved to the left to avoid crowding the superscripts on the right. When we write the CRs of the algebra, the commutator of two matrices M^1 and M^2 is written in terms of block-block dot products.

7 The T-matrices and Y

After a long development based largely on well-known properties of multiplets, it is time to write the formulas for the given matrices, the T-matrices and Y.

The T-matrices satisfy the Lie algebra $\mathfrak{su}(2)$ and are reduced to a direct sum of SU(2)-irreps in the order determined in Sections 5 and 6. Invoking standard formulas for the SU(2)-irreps, the components of the T-matrices can be written[2]

$$^{\pm}T^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = \sqrt{(t_i \pm \alpha)(1 + t_i \mp \alpha)}\, \delta_{\beta,\alpha\mp 1}\, \delta_{i,j} \quad , \tag{34}$$

$$^{3}T^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = \alpha\, \delta_{\alpha,\beta}\delta_{i,j} \quad , \tag{35}$$

where, by (23), $t_i = (a_i+b_i)/2$, $t_j = (a_j+b_j)/2$, $\alpha = t_i, t_i-1, ..., -t_i$, $\beta = t_j, t_j-1, ..., -t_j$. In these equations, the double indices on the right crowd out the identifiers '$\pm$' and '3' which move to the left of 'T.'

By (16) and(23), we can write the matrices Y^0 and Y,

$$^{0}Y^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = (b_i - a_i)\, \delta_{\alpha,\beta}\delta_{i,j} \quad , \tag{36}$$

$$Y^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = [b_i - a_i - 2\,(p-q)\,/3]\; \delta_{\alpha,\beta}\delta_{i,j} \quad . \tag{37}$$

The matrix Y^0 is useful in the preceding work and the work to follow, but it is not a generator. The matrix Y is one of the eight basis generators.

CRs: $[T^+, T^-]$, $[T^3, T^\pm]$, $[Y, T^\pm]$. These five CRs listed from (9) and (10) involve only the four generators $T^\pm$, T^3, Y. By (34), (35), (37), one can show that CRs (9) and (10) are satisfied.

We have formulas for four of the eight generators, $T^\pm$, T^3, Y. That leaves four unknown generators, $U^\pm$ and $V^\pm$.

8 Commutation relations linear in unknown generators

Let us begin to constrain the components of the four unknown generators, $U^\pm$ and $V^\pm$, by looking at CRs linear in the unknowns. The commutators of these CRs combine a given T, Y generator with an unknown U, V generator, (11) and (12).

CR: $[T^3, U^\pm] = \mp U^\pm/2$. By T^3 in (35), the CR with T^3 and U^+ gives

$$T^3U^+ - U^+T^3 = -U^+/2 \tag{38}$$

$$^{3}T^{(i,n)}_{(\alpha,\,\sigma)}\; ^{+}U^{(n,j)}_{(\sigma,\,\beta)} - {}^{+}U^{(i,\bar{n})}_{(\alpha,\,\bar{\sigma})}\; ^{3}T^{(\bar{n},\beta)}_{(\bar{\sigma},\,\beta)} = -\; ^{+}U^{(i,j)}_{(\alpha,\,\beta)}/2 \tag{39}$$

$$\alpha\, ^{+}U^{(i,j)}_{(\alpha,\,\beta)} - {}^{+}U^{(i,j)}_{(\alpha,\,\beta)}\beta = -\; ^{+}U^{(i,j)}_{(\alpha,\,\beta)}/2 \quad , \tag{40}$$

which implies that either

$$ {}^{+}U^{(i,j)}_{(\alpha,\beta)} = 0 \quad \text{or} \quad {}^{+}U^{(i,j)}_{(\alpha,\beta)} = {}^{+}U^{(i,j)}_{(\alpha,\alpha+1/2)} \quad . \tag{41} $$

Thus, the only components of the (i,j) block of U^{+} that may not vanish have spin components related by $\beta = \alpha + 1/2$. The possibly nonzero components of a U^{+} block occupy a diagonal of the block.

CR: $[T^3, V^{\pm}] = \pm V^{\pm}/2$. Calculations for $V^{\pm}$ are similar to those just completed for U^{+}. The results for all four of these CRs show that only the following components of the unknown matrices can be nonzero,

$$ {}^{\pm}U^{(i,j)}_{(\alpha,\alpha\pm1/2)} \quad ; \quad {}^{\pm}V^{(i,j)}_{(\alpha,\alpha\mp1/2)} \quad . \tag{42} $$

For all four U, V-matrices, nonzero components can occur only on one diagonal in a block, a diagonal of components with spin indices α and β that differ by plus or minus one half.

The difference of a half in spin indices implies that, of the two spins t_i, t_j, one spin is an integer and the other is a half-integer. Recall that spin t is a half-integer when $2t$ is an odd integer. Thus, $2(t_j - t_i)$ is some odd integer $2m+1$. It is tempting to simply assume that $m = 0$, and $2(t_j - t_i) = \pm 1$, which is what we do.

Assumption 1. *For the nonzero (i,j) blocks of the unknown matrices $U^{\pm}$ and $V^{\pm}$ the column-row spin difference is one-half,*

$$ t_j - t_i = \pm 1/2 \quad . \tag{43} $$

Upper, lower blocks. Since the spins of the block t_i, t_j are not equal, the nonzero blocks of $U^{\pm}$ and $V^{\pm}$ are not diagonal blocks. Those (i,j) blocks that have $j > i$ are above the main diagonal and are called 'upper' blocks. The (i,j) blocks below the diagonal have $i > j$, and are called lower blocks.

By (23), we have $t = (a+b)/2$. Then Assumption 1 constrains the (a,b) double indices for nonzero (i,j)-blocks. We have

$$ a_j + b_j = a_i + b_i \pm 1 \quad , \tag{44} $$

where the choice $+1$ for the sign applies to nonzero upper blocks, $j > i$, and the choice -1 works with nonzero lower blocks, $i > j$.

CRs: $[Y, U^{\pm}] = \pm U^{\pm}$ and $[Y, V^{\pm}] = \pm V^{\pm}$. These calculations follow the process shown in (38) to (41): the CR is written out, the matrix Y is substituted and the result is simplified. One finds that nonzero components in $U^{\pm}$ and $V^{\pm}$ occur only in (i,j) blocks with

$$ y_j - y_i = y^0_j - y^0_i = \mp 1 \quad , \tag{45} $$

where the choice of signs on the right is correlated with the sign in $U^{\pm}$ and $V^{\pm}$. Thus, y_j - $y_i = -1$ for U^{+} and V^{+}. The constraint applies to the entire (i,j) block.

The conditions for non-zero (i,j) blocks have consequences for their (a,b) double indices $(a_i,b_i)(a_j,b_j)$. By (23), we have $y^0 = b-a$ and the difference in y^0s in (45) gives

$$b_j - a_j = b_i - a_i \mp 1 \quad , \tag{46}$$

for $U^{\pm}$ and $V^{\pm}$. Note the $\mp$ sign in (46) is correlated with the the $\pm$ sign in $U^{\pm}$ and $V^{\pm}$ and is the same for both upper and lower blocks of a given U,V-matrix.

There are constraints on the (a,b) double indices, (44) and (46), and constraints on the spin components in (42). These constraints imply that the only potentially nonzero components of the upper blocks of $U^{\pm}$ and $V^{\pm}$ are the following,

$$^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\,\beta)} \quad ; \quad ^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\,\beta)} \quad ; \quad ^{-}U^{(a,b-1),(a,b)}_{(\beta+1/2,\,\beta)} \quad ; \quad ^{-}V^{(a,b-1),(a,b)}_{(\beta-1/2,\,\beta)} \quad , \tag{47}$$

and the potentially nonzero components of their lower blocks are

$$^{+}U^{(a,b),(a,b-1)}_{(\alpha,\,\alpha+1/2)} \quad ; \quad ^{+}V^{(a,b),(a,b-1)}_{(\alpha,\,\alpha-1/2)} \quad ; \quad ^{-}U^{(a,b),(a-1,b)}_{(\alpha,\,\alpha-1/2)} \quad ; \quad ^{-}V^{(a,b),(a-1,b)}_{(\alpha,\,\alpha+1/2)} \quad . \tag{48}$$

These eight are the only nonzero component functions of the U,V matrices.

Each of the eight components in (47) and (48) is determined by three quantities, a, b, and α or β. The values of a and b are those of the rectangle of Figure 2, except when the appearance of $a-1$ or $b-1$ in an index removes $a=0$ or $b=0$. Therefore, there are $(p+1)(q+1)-1$ nonzero blocks for each of the eight components. Each nonzero block has only one nonzero diagonal with spin indices $(\alpha,\alpha \pm 1/2)$ or $(\beta \pm 1/2, \beta)$.

Thus, the block-block dot product of two matrix generators is the product of two blocks with only one non-zero diagonal each. By (33), the dot product of two such blocks is a block with just one non-zero diagonal. This property simplifies the calculations.

By (34), the raising and lowering T -matrices T^{-} and T^{+}, are the transpose of each other. This property gives $F^1 = (T^{+} + T^{-})/2$ and $F^2 = -i(T^{+} - T^{-})/2$, which are hermitian matrices. To have similar constructs for the U- and the V-matrices, one requires that the raising and lowering matrices are each other's transpose. The full basis of hermitian generators is discussed in Section 12.

Assumption 2. *The unknown raising and lowering matrices are assumed to be transposes of one another,*

$$^{+}U = {^{-}U}^{\mathrm{T}} \quad \text{and} \quad ^{+}V = {^{-}V}^{\mathrm{T}} \quad . \tag{49}$$

By Assumption 2, we need only deal with one of a pair of CRs that are each other's transpose because the transpose of a CR is satisfied when the CR is satisfied.

The possibly non-zero components are listed in (47) and (48). Assumption 2 requires the pairwise identifications,

$$ {}^{-}U^{(a,b),(a-1,b)}_{(\beta,\,\beta-1/2)} = {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\,\beta)} \quad , \tag{50} $$

where $-t+1 \leq \beta \leq t$,

$$ {}^{-}U^{(a,b-1),(a,b)}_{(\alpha+1/2,\,\alpha)} = {}^{+}U^{(a,b),(a,b-1)}_{(\alpha,\,\alpha+1/2)} \quad , \tag{51} $$

where $-t \leq \alpha \leq t-1$,

$$ {}^{-}V^{(a,b),(a-1,b)}_{(\beta,\,\beta+1/2)} = {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\,\beta)} \quad , \tag{52} $$

where $-t \leq \beta \leq t-1$,

$$ {}^{-}V^{(a,b-1),(a,b)}_{(\alpha-1/2,\,\alpha)} = {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\,\alpha-1/2)} \quad , \tag{53} $$

where $-t+1 \leq \alpha \leq t$. In (50) - (53), we have $t = (a+b)/2$.

In view of these transpose properties, all unknown components of ${}^{-}U$ or ${}^{-}V$ can be replaced with their counterparts in ${}^{+}U$ or ${}^{+}V$. Thus the number of unknowns is halved.

CR: $[T^{-}, U^{+}] = 0$. This CR gives recursion relations for components of the matrix generator U^{+}. The T matrix formulas in (34) give recursion relations for components of the matrix generator U^{+}. One has the following,

$$ \left[(t_i - \alpha + 1)\,(t_i + \alpha)\right]^{1/2}\, {}^{+}U^{(a_i,b_i),(a_j,b_j)}_{(\alpha,\,\beta)} = {}^{+}U^{(a_i,b_i),(a_j,b_j)}_{(\alpha-1,\,\beta-1)} \left[(t_j - \beta + 1)\,(t_j + \beta)\right]^{1/2} \quad , \tag{54} $$

where $t_i = (a_i + b_i)/2$, $t_j = (a_j + b_j)/2$, and the ranges of the spin indices are $\alpha = t_i, t_i - 1, ..., -t_i + 1$ and $\beta = t_j, t_j - 1, ..., -t_j + 1$. The equation holds for both upper and lower blocks. We first treat the upper blocks.

For (54) with upper blocks and applying the expression in (47), we have $\alpha = \beta - 1/2$ and double indices $(a_i, b_i) = (a-1, b)$ and $(a_j, b_j) = (a, b)$. By (23), it follows that $t_i = t - 1/2$ and $t_j = t$, where $t = (a+b)/2$. The upper block U^{+} recursion becomes

$$ (t + \beta - 1)^{1/2}\, {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\,\beta)} = {}^{+}U^{(a-1,b),(a,b)}_{(\beta-3/2,\,\beta-1)}\, (t+\beta)^{1/2} \quad . \tag{55} $$

Starting with $\beta = t$ at the top of the nonzero diagonal in the block, successive application of the recursion in (55), produces an expression for all the nonzero components along an upper U^{+} block's diagonal,

$$ {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\,\beta)} = \left(\frac{t+\beta}{2t}\right)^{1/2} {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} \quad . \tag{56} $$

where $a = 1, ..., q$, $b = 0, ..., p$, $t = (a+b)/2$ and $\beta = t, t-1, ..., -t+1$.

For (54) with the lower blocks of U^{+}, substitute the double indices and spin component requirements from (48), *i.e.* $\alpha + 1/2 = \beta$ and $(a_i, b_i), (a_j, b_j) = (a, b), (a, b-1)$. Then apply the resulting recursion repeatedly to obtain an expression for all components of the block's nonzero diagonal. One finds an expression for the nonzero components of a lower U^+ block's diagonal

$$ {}^{+}U^{(a,b),(a,b-1)}_{(\alpha,\,\alpha+1/2)} = (t-\alpha)^{1/2}\,{}^{+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} \quad , \tag{57} $$

where $a = 0, , ..., q$, $b = 1, ..., p$, $t = (a+b)/2$ and $\alpha = t-1, t-2, ..., -t$.

Thus, CR $[T^-, U^+] = 0$ yields recursions that reduce the number of unknowns in an upper or lower non-zero block of U^+. A block's nonzero components lie on a diagonal and just one component suffices to get the other components along the diagonal.

CR: $[T^+, V^+] = 0$. Follow the steps for CR $[T^-, U^+] = 0$: In the commutator $[T^+, V^+]$, substitute T^+ from (34) to find an expression for both upper and lower blocks of V^+. For the upper and lower blocks of V^+, (47) and (48) give restrictions on the double block indices and spin component indices.

For the nonzero components along an upper V^+ block's diagonal, one gets

$$ {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\,\beta)} = (t-\beta)^{1/2}\,{}^{+}V^{(a-1,b),(a,b)}_{(t-1/2,\,t-1)} \quad , \tag{58} $$

where $a = 1, ..., q$, $b = 0, ..., p$, $t = (a+b)/2$ and $\beta = t-1, t-2, ..., -t+1$. For the nonzero components of a lower V^+ block's diagonal, we find

$$ {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\,\alpha-1/2)} = \left(\frac{t+\alpha}{2t}\right)^{1/2}\,{}^{+}V^{(a,b),(a,b-1)}_{(t,\,t-1/2)} \quad , \tag{59} $$

where $a = 0, , ..., q$, $b = 1, ..., p$, $t = (a+b)/2$ and $\alpha = t, t-2, ..., -t+1$.

Thus, one component determines all components on the diagonal of nonzero components in a block of V^+.

CRs: $[T^\pm, V^\mp] = \mp U^\mp$, $[T^\pm, U^\pm] = \pm V^\pm$. These CRs bridge the gap between the matrices U and V.

Start with the CR $[T^-, V^+] = U^+$. Substituting the formula for T^- from (34) gives

$$ \left[(t_i-\alpha)\,(t_i+\alpha+1)\right]^{1/2}\,{}^{+}V^{(i,j)}_{\alpha+1,\beta} - {}^{+}V^{(i,j)}_{\alpha,\beta-1}\left[(t_j+\beta)\,(t_j-\beta+1)\right]^{1/2} = {}^{+}U^{(i,j)}_{\alpha,\beta} \quad , \tag{60} $$

which holds for both upper and lower blocks. We treat upper and lower blocks separately.

The nonzero components of the upper blocks of U^+ and V^+, $i < j$, are displayed in (47). Therefore, we substitute for (i,j) the double indices $(a-1,b),(a,b)$ and we require $\alpha = \beta - 1/2$. By (23) and with $t = (a+b)/2$, we have $t_i = t-1/2$ with $t_j = t$. The CR (60) is now

$$ \left[(t-\beta)\,(t+\beta)\right]^{1/2}\,{}^{+}V^{(a-1,b),(a,b)}_{\beta+1/2,\beta} - {}^{+}V^{(a-1,b),(a,b)}_{\beta-1/2,\beta-1}\left[(t+\beta)\,(t-\beta+1)\right]^{1/2} = {}^{+}U^{(a-1,b),(a,b)}_{\beta-1/2,\beta} \quad , \tag{61} $$

for the upper blocks.

Next, substitute expressions for the components from the recursions in (54),(58). The result is a relationship between the U^+ and V^+ upper block unknowns,

$$^{+}V^{(a-1,b),(a,b)}_{(t-1/2,\,t-1)} = -\,(2t)^{-1/2}\;{}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} \quad , \tag{62}$$

where simplification included canceling a common factor of $(t+\beta)$, so $\beta \neq -t$. The equation holds because upper blocks have at least one component with $\beta > -t$, since $a \geq 1$. The equation reduces the number of upper block unknowns from two to one for the matrices U^+ and V^+.

For the lower blocks, substitutions follow from (48), we replace (i,j) by $(a,b),(a,b-1)$ and assume that $\beta = \alpha + 1/2$. With $t = (a+b)/2$, we have $t_i = t$, $t_j = t - 1/2$. The CR for lower blocks $i > j$ is then

$$[(t-\alpha)\,(t+\alpha+1)]^{1/2}\;{}^{+}V^{(a,b),(a,b-1)}_{\alpha+1,\alpha+1/2} - {}^{+}V^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2}\;[(t-\alpha)\,(t+\alpha)]^{1/2} = {}^{+}U^{(a,b),(a,b-1)}_{\alpha,\alpha+1/2} \quad , \tag{63}$$

A relationship between the two unknowns in the lower block follows using the recursions (57),(59). One finds that

$$^{+}V^{(a,b),(a,b-1)}_{(t,\,t-1/2)} = (2t)^{1/2}\;{}^{+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} \quad , \tag{64}$$

where $t = (a+b)/2$. There is only one lower block unknown for U^+ and V^+.

The remaining CRs in the set, $[T^+, V^-] = -\,U^-$ and $[T^\pm, U^\pm] = \pm\,V^\pm$, are satisfied by applying constraints that have been derived up to this point.

We can introduce an abbreviated notation. One unknown is the upper U^+ block diagonal's endpoint

$$U^{(a,b)} \equiv {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} \tag{65}$$

and the other unknown is the lower block diagonal's endpoint

$$U_{(a,b)} \equiv {}^{+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} \quad , \tag{66}$$

where $U^{(a,b)}$ and $U_{(a,b)}$ depend only on a pair of integers (a,b) whose values are restricted to a $(p+1)$ by $(q+1)$ rectangle, as in Figure 2. The placement of the (a,b) indices as superscript or subscript is meant to be a mnemonic for the upper block and lower block, respectively.

The CRs that are linear in the unknowns $U^\pm$ and $V^\pm$ have supplied many constraints on the unknown U, V matrices that reduce the number of unknowns to just two, $U^{(a,b)}$ and $U_{(a,b)}$. This prepares us for dealing with equations that are quadratic in the unknown matrices.

9 Equations quadratic in unknown generators

The equations that are quadratic in the unknown matrices are the CRs in (13) and (14) as well as the expression for the Casimir operator (15).

We have an action plan to reduce the number of unknowns in the quadratic equations. The plan has three steps.

Step 1. The matrices U^- and V^- are rewritten using their transposes U^+ and V^+ by (50) - (53).

Step 2. The nonzero components of the blocks of U^+ and V^+ are functions of one unknown per block via recursions (56), (57), (58),(59). The upper and lower block unknowns of U^+ and V^+ are ${}^+U^{(a-1,b),(a,b)}_{(t-1/2,t)}$, ${}^+V^{(a-1,b),(a,b)}_{(t-1/2,t-1)}$, ${}^+U^{(a,b),(a,b-1)}_{(t-1,t-1/2)}$, and ${}^+V^{(a,b),(a,b-1)}_{(t,t-1/2)}$, four unknowns for a given (a,b).

Step 3. By (62) and (64), the V^+ unknowns ${}^+V^{(a,b),(a,b-1)}_{(t-1,t-1/2)}$, and ${}^+V^{(a,b),(a,b-1)}_{(t,t-1/2)}$ can be rewritten in terms of the U^+ unknowns $U^{(a,b)}$ and $U_{(a,b)}$ in (65) and (66).

After applying these three steps to an equation, the only unknowns have two forms $U^{(a,b)}$ and $U_{(a,b)}$, with the possibility that they appear with various parameters (a,b).

The nonzero blocks of the unknown matrices $U^{\pm}$ and $V^{\pm}$ occur off-diagonal, as upper blocks and lower blocks. We make a few remarks about the dot products of two matrices, each with off-diagonal blocks.

The dot product of one upper block with a second upper block gives results in a block that is twice removed from the diagonal blocks. The same is true for lower-lower block dot products, except that the result is twice lowered. Only an upper-lower block dot product or a lower-upper block dot product can contribute to a diagonal block.

The systematics of the off-diagonal blocks for the unknown matrices makes three equations for each CR that is quadratic in unknown matrices. With one equation for upper-upper block commutators, one for lower-lower block commutators, and one for upper-lower together with lower-upper block commutators, this makes three equations for each CR that is quadratic in the generators U, V.

CR: $[U^+, U^-] = 3Y/2 - T^3$. By selecting just the dot products of the upper-lower blocks of U^+ and U^- and the lower-upper blocks, we get contributions to the diagonal blocks on the left side. On the right side, the matrices Y and T^3, have nonzero diagonal blocks. One has

$$[U^+_{\text{Upper}}, U^-_{\text{Lower}}] + [U^+_{\text{Lower}}, U^-_{\text{Upper}}] = 3Y/2 - T^3 \tag{67}$$

As just remarked above, the upper-upper and lower-lower block dot products do not make nonzero diagonal blocks.

With (47) and (48), writing out the dot products on the left gives the following.

$$ {}^{+}U^{(a,b),(a+1,b)}_{\alpha,\,\alpha+1/2}\,{}^{-}U^{(a+1,b),(a,b)}_{\alpha+1/2,\,\alpha} - {}^{-}U^{(a,b),(a-1,b)}_{\alpha,\,\alpha-1/2}\,{}^{+}U^{(a-1,b),(a,b)}_{\alpha-1/2,\,\alpha} + {}^{+}U^{(a,b),(a,b-1)}_{\alpha,\,\alpha+1/2}\,{}^{-}U^{(a,b-1),(a,b)}_{\alpha+1/2,\,\alpha} - {}^{-}U^{(a,b),(a,b+1)}_{\alpha,\,\alpha-1/2}\,{}^{+}U^{(a,b+1),(a,b)}_{\alpha-1/2,\,\alpha} \quad . \tag{68} $$

Applying Steps 1,2,3, we obtain the following

$$ \left(\frac{t+\alpha+1}{2t+1}\right) {U^{(a+1,b)}}^2 - \left(\frac{t+\alpha}{2t}\right) {U^{(a,b)}}^2 + (t-\alpha)\,{U_{(a,b)}}^2 - (t-\alpha+1)\,{U_{(a,b+1)}}^2 = 3y^0/2 - p + q - \alpha \quad , \tag{69} $$

where $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$.

CR: $[V^+, V^-] = 3Y/2 + T^3$. The upper-lower dot products combined with those of the lower-upper blocks produce nonzero diagonal blocks. One has

$$ [V^+_{\text{Upper}}, V^-_{\text{Lower}}] + [V^+_{\text{Lower}}, V^-_{\text{Upper}}] = 3Y/2 + T^3 \tag{70} $$

By (47) and (48), the two commutators on the left become

$$ {}^{+}V^{(a,b),(a+1,b)}_{\alpha,\,\alpha-1/2}\,{}^{-}V^{(a+1,b),(a,b)}_{\alpha-1/2,\,\alpha} - {}^{-}V^{(a,b),(a-1,b)}_{\alpha,\,\alpha+1/2}\,{}^{+}V^{(a-1,b),(a,b)}_{\alpha+1/2,\,\alpha} + {}^{+}V^{(a,b),(a,b-1)}_{\alpha,\,\alpha-1/2}\,{}^{-}V^{(a,b-1),(a,b)}_{\alpha-1/2,\,\alpha} - {}^{-}V^{(a,b),(a,b+1)}_{\alpha,\,\alpha+1/2}\,{}^{+}V^{(a,b+1),(a,b)}_{\alpha+1/2,\,\alpha} \quad . \tag{71} $$

Applying Steps 1,2,3, produces

$$ \left(\frac{t-\alpha+1}{2t+1}\right) {U^{(a+1,b)}}^2 - \left(\frac{t-\alpha}{2t}\right) {U^{(a,b)}}^2 + (t+\alpha)\,{U_{(a,b)}}^2 - (t+\alpha+1)\,{U_{(a,b+1)}}^2 = 3y^0/2 - p + q + \alpha \quad , \tag{72} $$

where $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$. This equation differs from (69) by the signs of some of its terms.

We can eliminate the dependence on α in (69) and (72) by first adding the equations to obtain

$$ \frac{2(t+1)}{2t+1} {U^{(a+1,b)}}^2 - {U^{(a,b)}}^2 + 2t{U_{(a,b)}}^2 - 2\,(t+1)\,{U_{(a,b+1)}}^2 = 3y^0 - 2p + 2q \tag{73} $$

Subtracting (72) from (69) gives

$$ \frac{1}{2t+1} {U^{(a+1,b)}}^2 - \frac{1}{2t} {U^{(a,b)}}^2 - {U_{(a,b)}}^2 + {U_{(a,b+1)}}^2 = -1 \quad , \tag{74} $$

where we cancel a factor of α and that requires at least one spin component $\alpha \neq 0$. In these equations, we have $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$.

Equations (73) and (74) are two equations in four unknowns, the squares ${U^{(a+1,b)}}^2$, ${U^{(a,b)}}^2$, ${U_{(a,b)}}^2$, and ${U_{(a,b+1)}}^2$. The two unknown functions $U^{(a,b)}$ and $U_{(a,b)}$ appear with three sets of parameters: (a,b), $(a+1,b)$, and $(a,b+1)$.

The Casimir equation (15), is quadratic in the matrices $U^{\pm}$ and $V^{\pm}$ and depends otherwise on block diagonal matrices, the T-matrices, Y, and the unit matrix $\mathbf{1}$. As noted previously, the dot products of the upper-lower combined with the lower-upper blocks produce nonzero diagonal blocks. Putting the unknowns on the left and the knowns on the right, we have

$$\{U^{+}_{\text{Upper}}, U^{-}_{\text{Lower}}\} + \{U^{+}_{\text{Lower}}, U^{-}_{\text{Upper}}\} + \{V^{+}_{\text{Upper}}, V^{-}_{\text{Lower}}\} + \{V^{+}_{\text{Lower}}, V^{-}_{\text{Upper}}\} = \\ -\{T^{+}, T^{-}\} - 2{T^{3}}^{2} - 3Y^2/2 + \left[2\left(p^2+pq+q^2+3p+3q\right)/3\right]\mathbf{1} \quad . \quad (75)$$

Rather than displaying the anti-commutators in detail, we refer to the commutators displayed in (68) and (71). By carefully replacing the minus signs with pluses, we obtain the appropriate expression for the anti-commutators on the left.

On the right side of (75), the matrices are known. The T matrices can be evaluated with (34) and (35). By (37), the Y-matrix is known. Substituting these known matrices in (75) gives

$$\frac{2(t+1)}{2t+1}{U^{(a+1,b)}}^2 + {U^{(a,b)}}^2 + 2t{U_{(a,b)}}^2 + 2\left(t+1\right){U_{(a,b+1)}}^2 = \\ -2t\left(t+1\right) - \frac{3}{2}\left[y^0 - 2(p-q)/3\right]^2 + \frac{2}{3}\left(p^2+pq+q^2+3p+3q\right) \quad , \quad (76)$$

where $t = (a+b)/2$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$. This equation has the same unknowns as (73) and (74).

The three equations (73),(74),(76) involve the four squares, the two upper block unknowns ${U^{(a+1,b)}}^2$, ${U^{(a,b)}}^2$, and the two lower block unknowns ${U_{(a,b)}}^2$, ${U_{(a,b+1)}}^2$. We shall show that linear combinations of these three equations produce two recursions.

First, let us separate the two upper block unknowns from the two lower block unknowns. Multiply (73),(74),(76) by $(t+1/2)$, $2t(t+1)$, $1/2$, resp. Adding, rearranging, and replacing t by $(a+b)/2$ yields

$$\left[(a+1)+b+1\right]{U^{(a+1,b)}}^2 = \left(a+b+1\right){U^{(a,b)}}^2 - 3a^2 + a\left(-2p+2q-3\right) + 2q + pq \quad , \quad (77)$$

which gives us a recursion in the double index parameter a for the quantity $(a+b+1)\times {U^{(a,b)}}^2$. Since the parameter a in the equation has integer values $a = 0, ..., q-1$, it follows that

$$(a+b+1)\,{U^{(a,b)}}^2 = (b+1)\,{U^{(0,b)}}^2 + \sum_{\bar{a}=0}^{\bar{a}=a} \left[-3\bar{a}^2 + \bar{a}\,(-2p+2q-3) + 2q + pq\right] \quad . \tag{78}$$

We find that

$$(a+b+1)\,{U^{(a,b)}}^2 = (b+1)\,{U^{(0,b)}}^2 + a\,(p+a+1)\,(q-a+1) \quad . \tag{79}$$

This equation shows how the combination $(a+b+1){U^{(a,b)}}^2$ depends on the double index parameter a.

Next, multiply (73),(74),(76) by $-(t+1/2)$, $2t(t+1)$, $1/2$, resp. Adding, rearranging, and replacing t by $(a+b)/2$ yields the lower block equation

$$\begin{aligned}&[a+(b+1)]\,[a+(b+1)+1]\,{U_{(a,b+1)}}^2 \\ &\qquad = (a+b)\,(a+b+1)\,{U_{(a,b)}}^2 - 3b^2 + b\,(2p-2q-3) + 2p + pq\;.\end{aligned} \tag{80}$$

This is a recursion in the double index parameter b for the quantity $(a+b)(a+b+1){U_{(a,b)}}^2$. Since the parameter b takes the integer values $b = 0, ..., p-1$ in the equation, it follows that

$$(a+b)\,(a+b+1)\,{U_{(a,b)}}^2 = a\,(a+1)\,{U_{(a,0)}}^2 + \sum_{\bar{b}=0}^{\bar{b}=b} \left[-3\bar{b}^2 + \bar{b}\,(2p-2q-3) + 2p + pq\right] \quad . \tag{81}$$

We find that

$$(a+b)\,(a+b+1)\,{U_{(a,b)}}^2 = a\,(a+1)\,{U_{(a,0)}}^2 + b\,(p-b+1)\,(q+b+1) \quad . \tag{82}$$

This equation shows how the combination $(a+b)(a+b+1){U_{(a,b)}}^2$ depends on the double index parameter b.

For the third linear combination of (73),(74),(76), we ignore (73) and multiply (74) by $2(t+1)$ and subtract (76). One finds

$$\begin{aligned}&(a+b+1)\,{U^{(a,b)}}^2 + (a+b)\,(a+b+1)\,{U_{(a,b)}}^2 = \\ &\qquad - a^3 - b^3 - (p-q)\left(a^2 - b^2\right) + (a+b)\,(pq+p+q+1) \quad .\end{aligned} \tag{83}$$

Substitute (79) and (82) in (83) and simplify. One gets

$$(b+1)\,{U^{(0,b)}}^2 + a\,(a+1)\,{U_{(a,0)}}^2 = 0 \quad , \tag{84}$$

which holds for $a = 0, ..., q-1$ and $b = 0, ..., p-1$.

For $a = 0$, one has $(b+1)\, {U^{(0,b)}}^2 = 0$, where $b = 0, ..., p-1$. Putting that back in (84) gives $a\,(a+1)\, {U_{(a,0)}}^2 = 0$ for $a = 0, ..., q-1$. Though not all cases of a and b are covered, we drop the constants in (79) and (82), *i.e.*

$$(b+1){U^{(0,b)}}^2 = 0 \quad ; \quad a(a+1){U_{(a,0)}}^2 = 0 \quad . \tag{85}$$

The removal of these constants simplifies the formulas.

By (79), the upper block unknown $U^{(a,b)}$ satisfies

$$(a+b+1)\, {U^{(a,b)}}^2 = a\,(p+a+1)\,(q-a+1) \tag{86}$$

and, by (82), the lower block unknown $U_{(a,b)}$ obeys

$$(a+b)\,(a+b+1)\, {U_{(a,b)}}^2 = b\,(p-b+1)\,(q+b+1) \quad . \tag{87}$$

Taking positive square roots to get $U^{(a,b)}$ and $U_{(a,b)}$ from their squares is an assumption. However, one can show that any of the four choices for the two sign factors would produce generators that satisfy the CRs. To avoid the clutter of introducing two sign variables ϵ_1 and ϵ_2, we simply take the positive roots.

Assumption 3. *In (86) and (87), take the positive square roots to obtain the two unknowns* $U^{(a,b)}$ *and* $U_{(a,b)}$*.*

By Assumption 3, we have the following,

$$U^{(a,b)} = {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} = \left[a\,(p+a+1)\,(q-a+1)\,/\,(a+b+1)\right]^{1/2} \tag{88}$$

and

$$U_{(a,b)} = {}^{+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} = \left[b\,(p-b+1)\,(q+b+1)\,/\,((a+b)\,(a+b+1))\right]^{1/2} \quad . \tag{89}$$

Thus, we have found the one remaining unknown quantity for an upper block and the one remaining unknown for a lower block. With all the unknowns determined, we can construct a set of basis generator formulas.

10 Matrix generator formulas

In (6), (7), (8), we have 4×4 matrix generators T^3, Y, T^+, T_{-}, U^+, U^-, V^+, V^- that form a basis for the fundamental representation of the $\mathfrak{su}(3)$ Lie algebra. In this section, formulas are presented for the components of matrices that realize the generators of the (p, q) irrep of the algebra, where p and q are nonnegative integers that identify the irrep.

Based on the properties of multiplets, a special sequence of integers $n(a, b, \alpha)$ in (29) produces an integer n, $n = 1, ..., d$, given a set of parameters a, b, α defined by multiplets. Here d is the dimension of the matrix representation.

If we are given two sets of parameters, we get two integers between 1 and d, say $n(a_r, b_r, \alpha)$ and $n(a_c, b_c, \beta)$. These may be used as row r and column c indices to locate a component M^{rc} of a matrix M.

For a matrix M, the value of a component M^{rc} and its row and column indices r, c are parameterized by two sets of three quantities a, b, α. There is one set for the row index and one set for the column index. By (30), one has

$$M^{rc} = M^{(a_r,b_r)(a_c,b_c)}_{\alpha,\beta} \quad ; \quad r = n(a_r, b_r, \alpha) \quad ; \quad c = n(a_c, b_c, \beta) \quad , \tag{90}$$

where, by (29),

$$n(a, b, \alpha) = \frac{1}{2}\left[(a + b)(a + b + 1) + qb(q + b + 2)\right] + [t - \alpha + 1] \tag{91}$$

and $t = (a + b)/2$.

We now turn to the formulas. All components vanish that are not listed below.

T^3. The matrix T^3 is one of the two diagonal matrix generators in the basis. By (35), we have the following,

$$^3T^{rc} = \alpha \quad ; \quad r = n(a, b, \alpha) \quad ; \quad c = n(a, b, \alpha) \tag{92}$$

where $\alpha = t, ..., -t$, $t = (a + b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$.

Y. For the other diagonal matrix, by (37), we have

$$Y^{rc} = b - a - 2\,(p - q)\,/3 \quad ; \quad r = n(a, b, \alpha) \quad ; \quad c = n(a, b, \alpha) \tag{93}$$

where $\alpha = t, ..., -t$, $t = (a + b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. Since the value Y^{rc} does not depend on the spin component α, each diagonal block of Y is proportional to an identity matrix of the appropriate dimension, which is $2t + 1 = a + b + 1$.

T^+. By (34), the nonzero components of T^+ are

$$^+T^{rc} = [(t + \alpha)(1 + t - \alpha)]^{1/2} \quad ; \quad r = n(a, b, \alpha) \quad ; \quad c = n(a, b, \alpha - 1) \tag{94}$$

where $\alpha = t, ..., -t + 1$, $t = (a + b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. Since the spin components are in decreasing order, the condition $\beta = \alpha - 1$ makes $c > r$, which places the non-zero components of T^+ just above the diagonal.

T^-. By (34), a nonzero block of T^- has components

$$^-T^{rc} = [(t-\alpha)(1+t+\alpha)]^{1/2} \quad ; \quad r = n(a,b,\alpha) \quad ; \quad c = n(a,b,\alpha+1) \tag{95}$$

where $\alpha = t-1, ..., -t$, $t = (a+b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. The nonzero components of T^- are just below its diagonal. T^- is the last of the four matrices that are non-zero in diagonal blocks.

We now outline the procedure for determining the four generator matrices $U^{\pm}$ and $V^{\pm}$. One substitutes the formulas for $U^{(a,b)}$ and $U_{(a,b)}$ in (88) and (89) back through the derivation. There are the U-V bridge equations, (61), (63). Then there are the recursions (56), (57), (58), (59), which fill up the non-zero diagonal in each U^+ and V^+ block. That completes U^+ and V^+. The final two matrices U^- and V^- are determined because they are the transposes of the U^+ and V^+ matrices, respectively, by (50) - (53).

By construction, each U, V matrix formula is dependent on either $U^{(a,b)}$ or $U_{(a,b)}$. The presence of U in the formulas would be confusing, and hence we introduce an alternate notation. We write the formulas for the matrices U, V in terms of two functions g and h, which are defined as

$$g(a,b) \equiv [a(p+a+1)(q-a+1)] \,/\, [(a+b)(a+b+1)] \tag{96}$$

$$h(a,b) \equiv [b\,(p-b+1)\,(q+b+1)\,/\,((a+b)\,(a+b+1))] \quad .$$

Note that g is the square of the function $U^{(a,b)}$ divided by $(a+b)$ and that h is the square of $U_{(a,b)}$.

The four matrix generators U^+, U^-, V^+, V^-, each have non-zero components in their upper and lower blocks, the ones listed in (47) and (48). So, there are two formulas, upper and lower, for the nonzero components of each of these generators.

U^+, upper blocks. By (56) and (88), we get

$$^+U^{rc} = [g(a,b)\,(t+\beta)]^{1/2} \quad ; \quad r = n(a-1,b,\beta-1/2) \quad ; \quad c = n(a,b,\beta) \tag{97}$$

where $\beta = t, ..., -t+1$, $a = 1, ..., q$, $b = 0, 1, ..., p$, $t = (a+b)/2$.

U^+, lower blocks. By (57) and (89), we get

$$^+U^{rc} = [h(a,b)\,(t-\alpha)]^{1/2} \quad ; \quad r = n(a,b,\alpha) \quad ; \quad c = n(a,b-1,\alpha+1/2) \tag{98}$$

where $\alpha = t-1, ..., -t$, $a = 0, ..., q$, $b = 1, ..., p$, $t = (a+b)/2$.

U^-, upper blocks. By (51), (57) and (89), we get

$$^-U^{rc} = [h(a,b)\,(t-\beta)]^{1/2} \quad ; \quad r = n(a,b-1,\beta+1/2) \quad ; \quad c = n(a,b,\beta) \tag{99}$$

where $\beta = t-1, ..., -t$, $a = 0, ..., q$, $b = 1, ..., p$, $t = (a+b)/2$.

U^-, lower blocks. By (50), (56) and (88), we get the following

$$^{-}U^{rc} = [g(a,b)\,(t+\alpha)]^{1/2} \quad ; \quad r = n(a,b,\alpha) \quad ; \quad c = n(a-1,b,\alpha-1/2) \tag{100}$$

where $\alpha = t, ..., -t+1$, $a = 1, ..., q$, $b = 0, 1, ..., p$, $t = (a+b)/2$.

V^+, upper blocks. By (58), (62) and (88), we get the following

$$^{+}V^{rc} = -\,[g(a,b)\,(t-\beta)]^{1/2} \quad ; \quad r = n(a-1,b,\beta+1/2) \quad ; \quad c = n(a,b,\beta) \tag{101}$$

where $\beta = t-1, ..., -t$, $a = 1, ..., q$, $b = 0, 1, ..., p$, $t = (a+b)/2$.

V^+, lower blocks. By (59), (64) and (89), we get the following

$$^{+}V^{rc} = [h(a,b)\,(t+\alpha)]^{1/2} \quad ; \quad r = n(a,b,\alpha) \quad ; \quad c = n(a,b-1,\alpha-1/2) \tag{102}$$

where $\alpha = t, ..., -t+1$, $a = 0, ..., q$, $b = 1, ..., p$, $t = (a+b)/2$.

V^-, upper blocks. By (53), (59), (64) and (89), we get

$$^{-}V^{rc} = [h(a,b)\,(t+\beta)]^{1/2} \quad ; \quad r = n(a,b-1,\beta-1/2) \quad ; \quad c = n(a,b,\beta) \tag{103}$$

where $\beta = t, ..., -t+1$, $a = 0, ..., q$, $b = 1, ..., p$, $t = (a+b)/2$.

V^-, lower blocks. By (52), (58), (62) and (88), we get

$$^{-}V^{rc} = -\,[g(a,b)\,(t-\alpha)]^{1/2} \quad ; \quad r = n(a,b,\alpha) \quad ; \quad c = n(a-1,b,\alpha+1/2) \tag{104}$$

where $\alpha = t-1, ..., -t$ $a = 1, ..., q$, $b = 0, 1, ..., p$, $t = (a+b)/2$.

The eight formulas for the U, V matrices have some common characteristics. Each formula has a factor of $g^{1/2}$ or $h^{1/2}$. The upper blocks have $(t \pm \beta)^{1/2}$, while the lower blocks have $(t \pm \alpha)^{1/2}$.

By inspection, one sees that U^- is the transpose of U^+ and V^- is the transpose of V^+. For example, consider the upper block of V^-, (103) and the lower block of V^+, (102). They are related by $^{-}V \leftrightarrow {}^{+}V$, $(a, b-1) \leftrightarrow (a, b)$, and $(\beta - 1/2,\, \beta) \leftrightarrow (\alpha,\, \alpha - 1/2)$.

For example, the matrix U^+ for the $(p, q) = (2, 1)$ irrep is shown in (121) in Appendix A. The double indices (a, b) run from $a = 0, ..., q = 0, 1$ and $b = 0, ..., p = 0, 1, 2$. Choose (a, b) $= (1, 2)$. Then $t = (a+b)/2 = 3/2$ and the index β runs through $\beta = 3/2, 1/2, -1/2, -3/2$. From these, let us choose the first, $\beta = 3/2$. For these parameters and by (96) and (97), the corresponding component $^{+}U^{rc}$ in an upper block of U^+ has a value of $[g(a,b)\,(t+\beta)]^{1/2} =$ 1. By (29), (91) or Table 2, the component is in row $r = n(a-1, b, \beta-1/2) = n(0, 2, 1) =$ 9 and column $c = n(a, b, \beta) = n(1, 2, 3/2) = 12$. One confirms the value and location of the component by (121) where one sees that $^{+}U^{9,12} = 1$.

In terms of blocks, we see from (121) that the component ${}^{+}U^{9,12}$ is a component in the block in the row $i = 5$ and the column $j = 6$ of the block array. This is the block with double indices $(a-1,b),(a,b) = (0,2),(1,2)$. The components of the block are

$$ {}^{+}U^{(0,2),(1,2)}_{(\alpha,\beta)} = \boxed{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{array}} \, , \tag{105} $$

where we have carried the block's outlines from (121). The component ${}^{+}U^{9,12} = 1$ is in the first row and first column of the block. Since $\alpha = 1, 0, -1$ and $\beta = 3/2, 1/2, -1/2, -3/2$ are the ordered values of the indices α and β in the block, the component ${}^{+}U^{(0,2),(1,2)}_{(\alpha,\beta)}$ has spin components $\alpha, \beta = 1, 3/2$ in the block. The values, rows and columns for the other two components on the nonzero diagonal of the block, *i.e.* the components $\sqrt{2/3}$ and $1/\sqrt{3}$, can be found in the same way.

The FORTRAN program in Appendix B implements formulas (92-104) in the program's Sections 6-13 and Section 16. For each generator, for the allowed values of the parameters a, b and α or β, the program calculates the row and column indices r, c and then calculates the value of the associated component, thus building a matrix for the generator.

11 Formula Verification

In this section, we verify that the formulas satisfy three sample CRs of the $\mathfrak{su}(3)$ Lie algebra. The three calculations show what is involved in verifying the entire set of 28 CRs for the algebra as well as the verification of the quadratic Casimir expression. Verifying that the formulas in Section 10 satisfy the 28 CRs of the Lie algebra is one way to show that the generators are a basis for the algebra.

The complete verification of the formulas is relegated to a computer program in the Supplementary Material.[16] The formulas in Section 10 are found to produce generators that satisfy all 28 CRs of the algebra plus the quadratic Casimir equation.

The matrix for a generator M has a block structure $M^{(i,j)}$, where $i, j = 1, ..., (p+1)(q+1)$. The CRs have the form

$$ \left[M^1, M^2\right]^{(i,j)} = \left(f^{12c} M^c\right)^{(i,j)} \quad , \tag{106} $$

where f^{12c} are coefficients with real values and the repeated index c is summed from 1 to 8. If the CR is satisfied block by block for each of its (i,j) blocks, then the CR is satisfied for the generators obtained with the formulas.

Writing the block indices with the double index notation as in (33), we have the block-by-block version of a CR,

$$ {}^{1}M^{(a_i,b_i),(a_n,b_n)}_{\alpha,\eta}\,{}^{2}M^{(a_n,b_n),(a_j,b_j)}_{\eta,\beta} - {}^{2}M^{(a_i,b_i),(a_{\bar{n}},b_{\bar{n}})}_{\alpha,\bar{\eta}}\,{}^{1}M^{(a_{\bar{n}},b_{\bar{n}}),(a_j,b_j)}_{\bar{\eta},\beta} = f^{12c}\,{}^{c}M^{(a_i,b_i),(a_j,b_j)}_{\alpha,\beta} \quad , \tag{107} $$

where, to give space to the indices, the matrix labels $1, 2, c$ are moved to the left.

When we wrote the formulas in Section 10, we were careful to provide limits for parameters such as a, b, α. In this section, we prefer to note that the a-type indices must be in the range $0, ..., q$. Therefore, if an expression has $a + 1$ in its double indices, as occurs in (111), then the largest value for a is $q - 1$. The appearance of $a - 1$ forces $a = 1$ to be the minimum value for a. For indices of type b, the range is $0, ..., p$. However, the appearance of $b - 1$ or $b + 1$ increases the minimum b to $b = 1$ or decreases the maximum b to $b = p - 1$, respectively.

CR: $[T^+, T^-] = 2\,T^3$. By (94), (95) and (107), for a single possibly nonzero block of $[T^+, T^-]$, we have, with no sums over a, b, α and $t = (a + b)/2$,

$$ {}^{+}T^{(a,b),(a,b)}_{\alpha,\alpha-1}\,{}^{-}T^{(a,b),(a,b)}_{\alpha-1,\alpha} - {}^{-}T^{(a,b),(a,b)}_{\alpha,\alpha+1}\,{}^{+}T^{(a,b),(a,b)}_{\alpha+1,\alpha} = \tag{108} $$

$$ [(t+\alpha)(1+t-\alpha)]^{1/2}\,[(t+\alpha)(1+t-\alpha)]^{1/2} - \\ [(t-\alpha)(1+t+\alpha)]^{1/2}\,[(t-\alpha)(1+t+\alpha)]^{1/2} = \tag{109} $$

which simplifies,

$$ (t+\alpha)(1+t-\alpha) - (t-\alpha)(1+t+\alpha) = 2\alpha = 2\,{}^{3}T^{(a,b)(a,b)}_{\alpha,\alpha} \quad , \tag{110} $$

the last step by (92). Thus, the generator formulas (92), (94), (95) produce generators $T^{\pm}$ and T^3 that satisfy the CR $[T^+, T^-] = 2\,T^3$.

CR: $[T^+, U^+] = V^+$. The CR is an example with a commutator that has one of the diagonal block matrices T^3, Y, $T^{\pm}$, and one of the matrices with nonzero non-diagonal blocks $U^{\pm}$ and $V^{\pm}$.

The dot product of a diagonal block with an upper block is an upper block and the dot product of a diagonal block with a lower block is a lower block, so this CR involves more complications than the previous example with the commutator of two diagonal block matrices, $[T^+, T^-]$.

By (107), we have an expression for the nonzero blocks of $[T^+, U^+]$,

$$ {}^{+}T^{(a,b),(a,b)}_{\alpha,\alpha-1}\,{}^{+}U^{(a,b),(a+1,b)}_{\alpha-1,\alpha-1/2} - {}^{+}U^{(a,b),(a+1,b)}_{\alpha,\alpha+1/2}\,{}^{+}T^{(a+1,b),(a+1,b)}_{\alpha+1/2,\alpha-1/2} + \\ {}^{+}T^{(a,b),(a,b)}_{\alpha,\alpha-1}\,{}^{+}U^{(a,b),(a,b-1)}_{\alpha-1,\alpha-1/2} - {}^{+}U^{(a,b),(a,b-1)}_{\alpha,\alpha+1/2}\,{}^{+}T^{(a,b-1),(a,b-1)}_{\alpha+1/2,\alpha-1/2} \\ = A^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2} + B^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2} \quad , \tag{111} $$

where the first two terms on the left contribute to a component A in a block above the main diagonal of blocks and the last two terms contribute to a component B in a block below the main diagonal of blocks.

For A, the formulas (94) and (97) give, without sums over a, b, α and with $t = (a+b)/2$,

$$A^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2} = [(t+\alpha)(1+t-\alpha)]^{1/2} \left[g(a+1,b) \left(\frac{a+b+1}{2} + \alpha - \frac{1}{2} \right) \right]^{1/2} -$$
$$\left[g(a+1,b) \left(\frac{a+b+1}{2} + \alpha + \frac{1}{2} \right) \right]^{1/2} \left[(\frac{a+b+1}{2} + \alpha + \frac{1}{2})(1 + \frac{a+b+1}{2} - \alpha - \frac{1}{2}) \right]^{1/2}$$
$$= [g(a+1,b)(1+t-\alpha)]^{1/2} [(t+\alpha) - (1+t+\alpha)] = -[g(a+1,b)(1+t-\alpha)]^{1/2} \ . \quad (112)$$

The dummy indices a and α can be replaced by $a = \bar{a}-1$ and $\alpha = \beta+1/2$, with $\bar{t} = (\bar{a}+b)/2$ $= (a+b)/2 + 1/2 = t + 1/2$. These changes in (112) and the formula for V^{+} in (101) yield the following result,

$$A^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2} = A^{(\bar{a}-1,b),(\bar{a},b)}_{\beta+1/2,\beta} = -[g(\bar{a},b)(\bar{t}-\beta)]^{1/2} = {}^{+}V^{(\bar{a}-1,b),(\bar{a},b)}_{(\beta+1/2,\beta)} \ . \quad (113)$$

We have shown that the first two terms A of (111) give us the upper block of V^{+}.

The last two terms of (111) form B. Formulas (94) and (98) give, without sums over a, b, α and with $t = (a+b)/2$,

$$B^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2} = [(t+\alpha)(1+t-\alpha)]^{1/2} [h(a,b)(t-\alpha+1)]^{1/2} -$$
$$[h(a,b)(t-\alpha)]^{1/2} \left[(\frac{a+b-1}{2} + \alpha + \frac{1}{2})(1 + \frac{a+b-1}{2} - \alpha - \frac{1}{2}) \right]^{1/2}$$
$$= [h(a,b)(t+\alpha)]^{1/2} [(1+t-\alpha) - (t-\alpha)] = [h(a,b)(t+\alpha)]^{1/2} = {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\alpha-1/2)} \ , \quad (114)$$

the last step using the formula for V^{+} in (102). Thus, the last two terms B of the commutator $[T^{+}, U^{+}]$ in (111) give us the lower block of V^{+}.

By (113) and (114), the commutator $[T^{+}, U^{+}]$ produces the upper and lower blocks of the matrix V^{+}, thus verifying that the generator formulas satisfy the CR $[T^{+}, U^{+}] = +V^{+}$. That is the second of three sample verifications.

CR: $[V^{+}, V^{-}] = 3Y/2 + T^{3}$. This is an example with a commutator that involves two generators that each have an upper and a lower block. By (107), we have an expression for the nonzero blocks of $[V^+, V^-]$,

$$
\begin{aligned}
{}^{+}V^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2}\,{}^{-}V^{(a+1,b),(a+1,b+1)}_{\alpha-1/2,\alpha} - {}^{-}V^{(a,b),(a,b+1)}_{\alpha,\alpha+1/2}\,{}^{+}V^{(a,b+1),(a+1,b+1)}_{\alpha+1/2,\alpha} \\
+ {}^{+}V^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2}\,{}^{-}V^{(a+1,b),(a,b)}_{\alpha-1/2,\alpha} - {}^{-}V^{(a,b),(a-1,b)}_{\alpha,\alpha+1/2}\,{}^{+}V^{(a-1,b),(a,b)}_{\alpha+1/2,\alpha} \\
+ {}^{+}V^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2}\,{}^{-}V^{(a,b-1),(a,b)}_{\alpha-1/2,\alpha} - {}^{-}V^{(a,b),(a,b+1)}_{\alpha,\alpha+1/2}\,{}^{+}V^{(a,b+1),(a,b)}_{\alpha+1/2,\alpha} \\
+ {}^{+}V^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2}\,{}^{-}V^{(a,b-1),(a-1,b-1)}_{\alpha-1/2,\alpha} - {}^{-}V^{(a,b),(a-1,b)}_{\alpha,\alpha+1/2}\,{}^{+}V^{(a-1,b),(a-1,b-1)}_{\alpha+1/2,\alpha} \\
= A^{(a,b),(a+1,b+1)}_{\alpha,\alpha} + B^{(a,b),(a,b)}_{\alpha,\alpha} + C^{(a,b),(a-1,b-1)}_{\alpha,\alpha} \quad , \qquad (115)
\end{aligned}
$$

where terms contributing to the same component are collected together. The first two terms on the left contribute to a component A in a block above the main diagonal of blocks, the middle four terms contribute to a component B in a diagonal block, and the last two terms contribute to a component C in a block below the main diagonal of blocks.

The first two terms of (115) form the term A. We have, without sums over a, b, α,

$$
\begin{aligned}
A^{(a,b),(a+1,b+1)}_{\alpha,\alpha} = \\
-\left[g(a+1,b)\left(\frac{(a+1)+b}{2} - \alpha + \frac{1}{2}\right) h(a+1,b+1)\left(\frac{(a+1)+(b+1)}{2} + \alpha\right)\right]^{1/2} \\
-(-1)\left[h(a,b+1)\left(\frac{a+(b+1)}{2} + \alpha + \frac{1}{2}\right) g(a+1,b+1)\left(\frac{(a+1)+(b+1)}{2} - \alpha\right)\right]^{1/2} \\
= 0 \ . \qquad (116)
\end{aligned}
$$

The expression vanishes because the two terms have the same factors dependent on α. Also, by the definitions of g and h in (96), the numerator of $g(a+1,b)$ is the same as the numerator of $g(a+1,b+1)$, the numerator of $h(a,b+1)$ is the same as the numerator of $h(a+1,b+1)$, and the denominator of $g(a+1,b)h(a+1,b+1)$ is the same as the denominator of $h(a,b+1)g(a+1,b+1)$. The terms have opposite signs, so they cancel.

The four middle terms of (115) form the term B. We have, without sums over a, b, α and with $t = (a+b)/2$,

$$B^{(a,b),(a,b)}_{\alpha,\alpha} =$$

$$(-1)^2 \left[g(a+1,b)\left(\frac{(a+1)+b}{2} - \alpha + \frac{1}{2}\right) g(a+1,b)\left(\frac{(a+1)+b)}{2} - \alpha + \frac{1}{2}\right)\right]^{1/2}$$

$$- (-1)^2 \left[g(a,b)(\frac{a+b}{2} - \alpha) g(a,b)(\frac{a+b}{2} - \alpha)\right]^{1/2}$$

$$+ \left[h(a,b)(\frac{a+b}{2} + \alpha) h(a,b)(\frac{a+b}{2} + \alpha)\right]^{1/2}$$

$$- \left[h(a,b+1)\left(\frac{a+(b+1)}{2} + \alpha + \frac{1}{2}\right) h(a,b+1)\left(\frac{a+(b+1)}{2} + \alpha + \frac{1}{2}\right)\right]^{1/2}$$

$$= g(a+1,b)\,(1+t-\alpha) - g(a,b)(t-\alpha) + h(a,b)(t+\alpha) - h(a,b+1)\,(1+t+\alpha)$$

$$= -\frac{3}{2}a + \frac{3}{2}b - p + q + \alpha = \frac{3}{2}Y^{(a,b),(a,b)}_{\alpha,\alpha} + {}^3T^{(a,b),(a,b)}_{\alpha,\alpha} \quad , \quad (117)$$

where we identify α with ${}^3T^{(a,b),(a,b)}_{\alpha,\alpha}$ by (92) and $-3a/2 + 3b/2 - p + q$ with $3Y^{(a,b),(a,b)}_{\alpha,\alpha}/2$ by (93).

The last two terms of (115) form the term C. We have, without sums over a, b, α and with $t = (a+b)/2$,

$$C^{(a,b),(a-1,b-1)}_{\alpha,\alpha} = -\left[h(a,b)(t+\alpha) g(a,b-1)\left(\frac{a+(b-1)}{2} - \alpha + \frac{1}{2}\right)\right]^{1/2}$$

$$- (-1)\left[g(a,b)(t-\alpha) h(a-1,b)(\frac{(a-1)+b}{2} + \alpha + \frac{1}{2})\right]^{1/2} = 0\ . \quad (118)$$

The expression vanishes because the two terms have the same factors. The factors dependent on α are the same in both terms. According to the definitions of g and h in (96), the numerator of $g(a, b-1)$ is the same as the numerator of $g(a, b)$, the numerator of $h(a, b)$ is the same as the numerator of $h(a-1, b)$, and the denominator of $h(a, b)g(a, b-1)$ is the same as the denominator of $g(a, b)h(a-1, b)$. The terms have opposite signs, so they cancel.

Combining the results (116), (117), (118), for the terms A, B and C in the expression (115) for the commutator $[V^+, V^-]$ shows that the formulas in Section 10 for V^+, V^-, T^3, Y satisfy the CR, $[V^+, V^-] = 3Y/2 + T^3$.

There are 28 CRs that need to be verified with the formulas in Section 10. In this section, we have verified three of them. To verify the remaining 25 CRs, we trust computer software. A link is provided to a Wolfram Mathematica notebook that tests each CR using symbolic algebra computer software.[16] The equation for the quadratic Casimir operator (15) is also tested. The program shows that the 29 equations are satisfied by the formulas in Section 10.

12 Discussion

The derivation distinguishes subsets of the $TYUV$ basis. By the three CRs (9), the T generators are the basis for an $\mathfrak{su}(2)$ sub-algebra. As a consequence, the T matrices can be a direct sum of $\mathfrak{su}(2)$ irreps, which creates the block array. What is left unsaid is the CRs of the T, Y generators satisfy the Lie algebra of $\mathfrak{u}2$. This is an intrinsic property independent of basis that distinguishes the four generators T^3, $T^{\pm}$, Y whose matrix realizations can be determined from the known characteristics of multiplets.

The Lie algebra $\mathfrak{su}(3)$ has a maximal subalgebra $\mathfrak{u}(2)$, so the basis splits into a natural decomposition of the generators. The generators T, Y are the basis of $\mathfrak{u}(2)$, and the other generators, the U, V generators. The CRs quadratic in U, V show that the commutators of linear combinations of U, V generators are elements of the $\mathfrak{u}(2)$ subalgebra. The decomposition is an intrinsic property. Those interested in such considerations are invited to explore the literature.

Any other basis of $\mathfrak{su}(3)$ can be written as a linear combination of the generators in the $TYUV$ basis. For example, we can invert the transformation (5) and apply it to the basis $TYUV$ of any (p, q) $\mathfrak{su}(3)$ irrep and obtain a basis F^i for that irrep. We find

$$F^1 = (T^+ + T^-)/2 \; ; \; F^2 = -i(T^+ - T^-)/2 \; ; \; F^3 = T^3 \; ; \; F^4 = (V^+ + V^-)/2 \; ;$$
$$F^5 = -i(V^+ - V^-)/2 \; ; \; F^6 = (U^+ + U^-)/2 \; ; \; F^7 = -i(U^+ - U^-)/2 \; ; \; F^8 = \sqrt{3}Y/2 \; . \quad (119)$$

The F^i matrices are the Hermitian and traceless counterparts of the matrices of the fundamental rep (2) to (4).

Let us turn now to a different topic. Quadratic equations with real coefficients may not have real solutions. The standard example is $x^2 + 1 = 0$, which has no real-valued solutions for x since the square of a real number is positive. The formulas in Section 10 produce real-valued results, showing explicitly that the CRs of the $\mathfrak{su}(3)$ algebra are quadratic equations with real-valued solutions. Any irrep of $\mathfrak{su}(3)$ has a $TYUV$ basis consisting of matrices with real-valued components.

13 FYI

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A Aspects of the (2,1)-irrep

Why $(2,1)$? The dimensions of irreducible representations of SU(3) (irreps) grow rapidly with p and q. The matrix generators for the $(p,q) = (2,1)$ SU(3)-irrep are 15×15 matrices, since the dimension d is $d = (p+1)(q+1)(p+q+2)/2 = 15$. The dimension is $d = 120$ for the $(p,q) = (5,3)$-irrep of Figures 1 and 2 in the text and is too large for this appendix. So, this appendix has the $d = 15$ dimensional $(2,1)$ SU(3)-irrep.

Parameters				Indices		
b	a	α	y^0	k	m	n
0	0	0	0	1	1	1
0	1	$\frac{1}{2}$	-1	2	1	2
0	1	$-\frac{1}{2}$	-1	2	2	3
1	0	$\frac{1}{2}$	1	3	1	4
1	0	$-\frac{1}{2}$	1	3	2	5
1	1	1	0	4	1	6
1	1	0	0	4	2	7
1	1	-1	0	4	3	8
2	0	1	2	5	1	9
2	0	0	2	5	2	10
2	0	-1	2	5	3	11
2	1	$\frac{3}{2}$	1	6	1	12
2	1	$\frac{1}{2}$	1	6	2	13
2	1	$-\frac{1}{2}$	1	6	3	14
2	1	$-\frac{3}{2}$	1	6	4	15

Table 2: *The Sequence n of SU(3) Eigenvectors.* The eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0 were sequenced by their eigenvalues α and y^0. The eigenvalues α, y^0 are used to sort the eigenvectors into SU(2)-irreps, each SU(2)-irrep identified by the index k with $k = 1, ..., (p+1)(q+1) = 6$. In one SU(2)-irrep, the eigenvalue α of the m^{th} eigenvector decreases as m increases. There are $a+b+1$ spin components α in each SU(2)-irrep, so $m = 1, ..., a+b+1$. The resulting sequence n of the eigenvectors is the last column.

The allowed values of the (a,b) parameters form a rectangle. See Figure 2. The points (a,b) are ordered by row b and then column a as in Table 2. Each row has $q+1$ points with $a = 0, ..., q$ and there are $p+1$ rows, $b = 0, ..., p$. In Table 2, there are $p+1 = 3$ rows $b = 0, 1, 2$ and $q+1 = 2$ columns $a = 0, 1$.

Each point (a,b) represents an SU(2)-irrep with $a+b+1$ eigenvectors. An SU(2)-irrep has eigenvectors with one Y^0 eigenvalue $y^0 = b - a$ and multiple T^3 eigenvalues α. In an

SU(2)-irrep, the αs decrease from $t = (a+b)/2$ to $-t = -(a+b)/2$, where t is the spin of the SU(2)-irrep.

The index k distinguishes different SU(2)-irreps, and the index m distinguishes different eigenvectors in each SU(2)-irrep. The index n distinguishes different eigenvectors. See Table 2.

The formulas for the generators, (92)-(104), are formulas for the components of blocks. The array of blocks makes up the matrix. The array of blocks for the matrix T^+ is

$$T^+ = \left(\begin{array}{|c|cc|cc|ccc|ccc|cccc|}
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{3}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{3}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
\end{array}\right) , \qquad (120)$$

with each block outlined. Note that non-zero components occur only in diagonal blocks.

For example, consider the component ${}^{+}T^{14,15}$, the component in the right column 15 of row 14, which is the row just above the bottom row. The double indices $(a_i, b_i)(a_j, b_j)$ and the spin components α, β can be read from Table 2. We have $(a_i, b_i, \alpha) = (1, 2, -1/2)$ since $i = 14$ $(= n)$ and $(a_j, b_j, \beta) = (1, 2, -3/2)$ since $j = 15$ $(= n)$. Now, the value of the component can be calculated from (94). With $t_i = (a_i + b_i)/2 = 3/2$, one finds that ${}^{+}T^{14,15}$ $= [(t_i + \alpha)(1 + t_i - \alpha)]^{1/2}\ \delta_{\beta,\alpha-1} = \sqrt{3}$, which is the value of ${}^{+}T^{14,15}$ in (120).

The U,V-matrices are nonzero in the upper and lower off-diagonal blocks. The array of blocks for the generator U^+ is

$$U^+ = \left(\begin{array}{|c|cc|cc|ccc|ccc|cccc|}
\hline
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}\right), \tag{121}$$

where all of the nonzero components occur in the upper and lower blocks just above and below the diagonal blocks.

For the matrix U^+, let us look at the component ${}^+U^{14,7}$, a component in a lower block of U^+. In Table 2, we have $(a_i, b_i, \alpha) = (1, 2, -1/2)$ since $i = 14$ $(= n)$ and $(a_j, b_j, \beta) = (1, 1, 0)$ since $j = 7$ $(= n)$. By (98), with $a = a_i = 1$, $b = b_i = 2$, $\alpha = -1/2$, $t = (a_i + b_i)/2 = 3/2$, we have

$${}^+U^{14,7} = \left[\frac{b(p-b+1)(q+b+1)}{(a+b)(a+b+1)}\right]^{1/2} (t-\alpha)^{1/2} = \frac{2}{\sqrt{3}} \,,$$

which confirms the value of the component ${}^+U^{14,7}$ in the matrix (121).

B Computer Program

The following Fortran-90 program calculates a basis of matrix generators for a given (p, q)-irrep.[18] Access to Fortran compilers is available to the public.[19]

Users should refer to Section 1 of the program for technical details.

```
!
! This FORTRAN program calculates a basis of matrix generators for an
! irreducible representation of the Lie algebra of the SU(3) Lie group.
!
```

```
!
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!--14.  Check 28 commutation relations and the Casimir equation
!--15.  Save the results to a file, end program
!--16.  External functions, end-of-file
!
!---------------------------0. Preamble---------------------------------
! The program appears as Appendix B of the article
! Formulas for SU(3) Matrix Generators,
! by Richard Shurtleff, Wentworth Institute of Technology, Boston,
! MA, USA - retired
! email: shurtleffr(at)wit.edu, momentummatrix(at)yahoo.com
!
!   Acknowledgment
! I would like to express my appreciation for Patrick Koh's debugging of a
! previous program which guided the writing of this program.
!
!   Copyright: CC BY-SA. Public use and modification of this code are
! allowed provided that the preprint[1] or any subsequent published
! version is cited.
!
!   References
!
!   [1] R. Shurtleff, "Formulas for SU(3) Matrix Generators"
!
!   [2] This program can be downloaded by following the link:
```

```
! https://www.dropbox.com/scl/fi/i9wdzu251vgpatg0wpfgw/ReviseSU3GENpq.
! MDPIb.f90?rlkey=cj2ai8a1rlgbv2nkef0kv5fts&st=wa5vn0rk&dl=0
!
!
!-------------------------1. Readme-----------------------------------
!
! This program calculates 8 matrices. The matrices form a set of basis
! generators for the (p,q) irrep of the su(3) Lie algebra.
! See the text for details.
!
! This FORTRAN 90 program ran successfully on a Windows 11 computer with a
!  GNU fortran compiler Code::Blocks 20.03, Created: 2010/25/05 11:52
!  Updated: 2010/25/05 11:52, Author: HighTec EDV-Systeme GmbH,
!  Copyright 2010 HighTec EDV-Systeme GmbH, available at
!  https://gcc.gnu.org/fortran/ and http://www.codeblocks.org
!
! INSTRUCTIONS FOR USING THE PROGRAM:
! 1. Input your value of p and q in Sec. 4
! 2. Set the tolerance for error, MaxErrLimit, if you wish.
!    Initially, the MaxErrLimit is 1.D-10.
!
! OUTPUT to the standard screen:
! Some data is sent to the standard output. The data includes the integers
! p and q that identify the irrep, the dimension of the irrep's
! matrices, the error tolerance, the maximum error found in 29 equations
! that the generators are required to solve.
!
! OUTPUT to a DATA File named "p#q#SU3GenII.dat" where # stands for the
! values of p,q. The DATA File's records are
! Record 1: the integers p, q, and the max error in the 28 commutation
! relations of the Lie algebra plus the Casimir expression.
! The Format of Record 1: FORMAT(2I3,1ES16.7).
!
! Records 2,3,...: the components of the eight matrices, a total
! of 8n**2 real numbers, where n = dimREP, the matrix dimension.
! Each of the 8 TYUV matrices X has n**2 components arranged by rows, i.e.
!   X(1,1), X(1,2),X(1,3),...,X(1,n),X(2,1),X(2,2),X(2,3),...,X(n,n).
! The sequence of TYUV matrices is X = T3, Y, Tp, Tm, Up, Um, Vp, Vm.
! The Format of Records 2,3,...: FORMAT(5F16.12).
! Thus the matrix components appear 5 real numbers per line until the
```

```
! last line.
!
! Sample records from (p,q) = (3,1) irrep file "p3q1SU3GenII.dat"
! Record #1:
! 3  1   3.5527137E-15
! Record #12:
! -0.500000000000  0.000000000000  0.000000000000 0.000000000000 0.0000000
! Record #541:
! 0.000000000000  1.118033988750  0.000000000000  0.000000000000 0.0000000
! Record #923
! 0.000000000000  0.000000000000  0.000000000000
!
! Given n = dimREP = 24 for (p,q) = (3,1), the four Records shown tell
! us that
! Record 1: p = 3, q = 1, MaxErr = 3.5527137E-15
! Record 12: 5*(11-1)+1 = 51 = 2*n + 3 implies T3(3,3) = -0.5
! Record 541: 5*(540-1)+2 = 4*24**2 + 16*24 + 9 implies that
! Up(17,9) = 1.11803399
! Record 923: 5*(922-1) + 3 = 4608 = 8*24**2 = 8*n**2,
! which is the expected total # of components.
!
!
PROGRAM SU3Generators
!
    IMPLICIT NONE
!---------------------2. Program Start, Interface Blocks -----------------
!
INTERFACE
    FUNCTION nSEQUENCE(p,q,a,b,alpha) RESULT  (w)    !
    IMPLICIT NONE        ! The nth place in the sequence belongs to
        INTEGER :: p,q     ! the T3 eigenvector with eigenvalue alpha
        INTEGER :: a,b,m    ! in the (a,b)-SU2 irrep
        REAL(8) :: alpha,w !the T3 eigenvector has the mth place in the
!                           ! irrep, where m = (a+b)/2 -alpha + 1
    END FUNCTION
END INTERFACE
INTERFACE
    FUNCTION g(p,q,a,b) RESULT  (w)    !
    IMPLICIT NONE        ! Four of the U,V formulas have
        INTEGER :: p,q     ! the factor SQRT(g), where
```

```
        INTEGER :: a,b    !g = a*(p+a+1)*(q-a+1)/[(a+b)(a+b+1)]
        REAL(8) :: w      ! w = g
    END FUNCTION
END INTERFACE
INTERFACE
    FUNCTION h(p,q,a,b) RESULT  (w)    !
    IMPLICIT NONE        ! Four of the U,V formulas have
        INTEGER :: p,q     ! the factor SQRT(h), where
        INTEGER :: a,b    !h = b*(p-b+1)*(q+b+1)/[(a+b)(a+b+1)]
        REAL(8) :: w      ! w = h
    END FUNCTION
END INTERFACE
!
!-----------------------3. Type declarations ----------------------
!
INTEGER ::  p,q         ! integers p,q identify the (p,q) SU(3) irrep
INTEGER ::  a,b,c !(a,b) identifies an SU2-irrep in the reduced T-matrices
INTEGER ::  ai,bi,aj,bj ! i for row and j for column; used with blocks
!  ALLOCATABLE allows an altered code to calculate more pairs of (p,q)
REAL(8), ALLOCATABLE :: unitMatrix(:,:)
REAL(8),ALLOCATABLE :: T3(:,:),Y(:,:),Tp(:,:),Tm(:,:),Up(:,:),Um(:,:)
REAL(8),ALLOCATABLE :: Vp(:,:),Vm(:,:)
!
REAL(8) :: MaxErr,MaxErrLimit !Max error found in the equations, tolerance
INTEGER ::  dimREP    !  TYUV matrix dimension; dimREP = #rows = #cols
REAL(8) ::  alpha,beta ! spin indices of components in a block
REAL(8) ::  t ! spin of an SU2 irrep
INTEGER  :: i,j,k,m,n,r,w   ! dummy indices; r row, w column, n dimREP
!
REAL(8),ALLOCATABLE:: T3Tpcomm(:,:),T3Tmcomm(:,:),T3Upcomm(:,:)
REAL(8),ALLOCATABLE:: T3Umcomm(:,:),T3Ycomm(:,:),T3Vpcomm(:,:)
REAL(8),ALLOCATABLE:: T3Vmcomm(:,:),TpTmcomm(:,:),TpUpcomm(:,:)
REAL(8),ALLOCATABLE:: TpUmcomm(:,:),TpYcomm(:,:),TpVpcomm(:,:)
REAL(8),ALLOCATABLE:: TpVmcomm(:,:),TmUpcomm(:,:),TmUmcomm(:,:)
REAL(8),ALLOCATABLE:: TmYcomm(:,:),TmVpcomm(:,:),TmVmcomm(:,:)
REAL(8),ALLOCATABLE:: YUpcomm(:,:),YUmcomm(:,:),YVpcomm(:,:)
REAL(8),ALLOCATABLE:: YVmcomm(:,:),UpUmcomm(:,:),UpVpcomm(:,:)
REAL(8),ALLOCATABLE:: UpVmcomm(:,:),UmVpcomm(:,:),UmVmcomm(:,:)
REAL(8),ALLOCATABLE:: VpVmcomm(:,:),Casimir(:,:)
!
```

```
LOGICAL:: MaxErrNotTooBig  !  check that error is within tolerance: T-Yes
CHARACTER (len=20) :: file_name    ! file name for output data file
!
!
!---------------------4. Set (p,q), MaxErrLimit ----------------------
!
! Set p and q so that the (p,q)-irrep is calculated.
p = 3        ! the irrep's identifiers (p,q)
q = 2       !
!  Set the tolerance:
   MaxErrLimit = 1.D-10 ! the largest allowed error in
   !                        any component of the 29 eqns
!
!   Calculate the dimension of the irrep:
dimREP = (p+1)*(q+1)*(p+q+2)/2 !Matrices are nxn square, n = dimREP
!
!WRITE(*,*) 'p,q = ', p,q
WRITE(*,*)    'The program calculates basis generators &
                for the p,q = ',p,q,' irrep of SU(3)'
WRITE(*,*) 'The dimension of the matrices is dimREP = ', dimREP
!
!------------------------5. Preliminaries ----------------------
!
 ALLOCATE( unitMatrix(dimREP,dimREP)) !unit matrix, a.k.a. identity matrix
    DO  i = 1,dimREP
        DO  j = 1,dimREP
    unitMatrix(i,j) = 0.0_8       ! The unit matrix has zeros everywhere,
        END DO
            unitMatrix(i,i) = 1.0_8    ! except ones along the diagonal.
    END DO
!
!
!----------------6. Make T3 ----------------------
!
ALLOCATE( T3(dimREP,dimREP))
!
DO  i = 1,dimREP ! Initially null
    DO  j = 1,dimREP
        T3(i,j) = 0._8
    END DO
```

```
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            r = nSEQUENCE(p,q,a,b,alpha)
            T3(r,r) = alpha
        END DO
    END DO
END DO
!
!-----------------7. Make Y -----------------------
!
ALLOCATE( Y(dimREP,dimREP))
!
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Y(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            r = nSEQUENCE(p,q,a,b,alpha)
            Y(r,r) = DBLE(b-a-2._8*(p-q)/3._8)
        END DO
    END DO
END DO
!
!-----------------8. Make Tp -----------------------
!
ALLOCATE( Tp(dimREP,dimREP))
!
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Tp(i,j) = 0._8
    END DO
```

```
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha - 1._8
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
              t = (a+b)/2._8
        Tp(r,c) = DSQRT((t+alpha)*(1._8+t-alpha))
        END DO
    END DO
END DO
!
!-----------------9. Make Tm -----------------------
!
ALLOCATE( Tm(dimREP,dimREP))
!
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Tm(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 2,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha + 1._8
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
              t = (a+b)/2._8
        Tm(r,c) = DSQRT((t-alpha)*(1._8+t+alpha))
        END DO
    END DO
END DO
!
!-----------------10. Make Up -----------------------
!
```

```
ALLOCATE( Up(dimREP,dimREP))
!
! Make the upper blocks of Up
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Up(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p                              !upper block
    DO a = 1,q
        DO m = 1,a+b+1
            ai=a-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta - 0.5_8
    IF ((alpha .LT. -(ai+b)/2._8).OR.(alpha .GT. (ai+b)/2._8)) THEN
    CYCLE
    END IF
             r = nSEQUENCE(p,q,ai,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
         Up(r,c) = DSQRT(g(p,q,a,b)*(t+beta))
        END DO
    END DO
END DO
! Make the lower blocks of Up
DO b = 1,p                              !lower block
    DO a = 0,q
        DO m = 1,a+b+1  ! replace 1 by 2 and drop IF
            bj = b - 1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha + 0.5_8
    IF ((beta .LT. -(a+bj)/2._8).OR.(beta .GT. (a+bj)/2._8)) THEN
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,bj,beta)
            t = (a+b)/2._8
        Up(r,c) = DSQRT(h(p,q,a,b)*(t-alpha))
        END DO
```

```
    END DO
END DO
!
!-----------------11. Make Um -----------------------
!
ALLOCATE( Um(dimREP,dimREP))
!
! Make the upper blocks of Um
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Um(i,j) = 0._8
    END DO
END DO
!
! Make the upper blocks of Um
DO b = 1,p                             !upper block
    DO a = 0,q
        DO m = 2,a+b+1
            bi = b-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta + 0.5_8
    IF ((alpha .LT. -(a+bi)/2._8).OR.(alpha .GT. (a+bi)/2._8)) THEN
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,bi,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
        Um(r,c) = DSQRT(h(p,q,a,b)*(t-beta))
        END DO
    END DO
END DO
! Make the lower blocks of Um
DO b = 0,p                             !lower block
    DO a = 1,q
        DO m = 1,a+b+1  !for m = 1, alpha = t = (a+b)/2 MOVE TO Up
            aj = a - 1  ! tj = (a+b-1)/2
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha - 0.5_8 !For m = 1, beta = (a+b-1)/2
    IF ((beta .LT. -(aj+b)/2._8).OR.(beta .GT. (aj+b)/2._8)) THEN
    CYCLE
```

```
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,aj,b,beta)
            t = (a+b)/2._8
        Um(r,c) = DSQRT(g(p,q,a,b)*(t+alpha))
        END DO
    END DO
END DO
!
!----------------12. Make Vp ----------------------
!
ALLOCATE( Vp(dimREP,dimREP))
!
! Make the upper blocks of Vp
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Vp(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p                              !upper block
    DO a = 1,q
        DO m = 1,a+b+1
            ai = a-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta + 0.5_8
    IF ((alpha .LT. -(ai+b)/2._8).OR.(alpha .GT. (ai+b)/2._8)) THEN
    CYCLE
    END IF
             r = nSEQUENCE(p,q,ai,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
        Vp(r,c) = -DSQRT(g(p,q,a,b)*(t-beta))
        END DO
    END DO
END DO
! Make the lower blocks of Vp
DO b = 1,p                              !lower block
    DO a = 0,q
        t = DBLE((a+b)/2._8)
```

```
        DO m = 1,a+b+1  ! 1 replaced by 2
            bj = b - 1
            alpha = DBLE(t - m + 1._8)
            beta = alpha - 0.5_8
    IF ((beta .LT. -(a+bj)/2._8).OR.(beta .GT. (a+bj)/2._8)) THEN
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,bj,beta)
            t = (a+b)/2._8
        Vp(r,c) = DSQRT(h(p,q,a,b)*(t+alpha))
        END DO
    END DO
END DO
!
!-----------------13. Make Vm -----------------------
!
ALLOCATE( Vm(dimREP,dimREP))
!
! Make the upper blocks of Vm
DO  i = 1,dimREP    ! Initially null
    DO  j = 1,dimREP
        Vm(i,j) = 0._8
    END DO
END DO
DO b = 1,p                              !upper block
    DO a = 0,q
        t = (a+b)/2._8
        DO m = 1,a+b+1
            bi = b-1
            beta = DBLE(t - m + 1._8)
            alpha = beta - 0.5_8
    IF ((alpha .LT. -(a+bi)/2._8).OR.(alpha .GT. (a+bi)/2._8)) THEN
    !WRITE(*,*)  "a,bi,ti,r,alpha = ", a,bi,(a+bi)/2._8,r,alpha
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,bi,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
        Vm(r,c) = DSQRT(h(p,q,a,b)*(t+beta))
        END DO
```

```
    END DO
END DO
! Make the lower block of Vm
!
DO b = 0,p                              !lower block
    DO a = 1,q
        t = (a+b)/2._8
        DO m = 1,a+b+1
            aj = a - 1
            alpha = DBLE(t - m + 1._8)
            beta = alpha + 0.5_8
    IF ((beta .LT. -(aj+b)/2._8).OR.(beta .GT. (aj+b)/2._8)) THEN
    CYCLE
    END IF
              r = nSEQUENCE(p,q,a,b,alpha)
            c = nSEQUENCE(p,q,aj,b,beta)
        Vm(r,c) = -DSQRT(g(p,q,a,b)*(t-alpha))
        END DO
    END DO
END DO
!
! The 8 matrices T3,Y,Tp,Tm,Up,Um,Vp,Vm have been calculated. Check them.
!
!
! ----14. Check 28 commutator relations and the Casimir equation --------
!
n = dimREP ! Use n to abbreviate dimREP; a single character versus six
!
ALLOCATE(T3Tpcomm(n,n),T3Tmcomm(n,n),T3Upcomm(n,n),T3Umcomm(n,n))
ALLOCATE(T3Ycomm(n,n),T3Vpcomm(n,n),T3Vmcomm(n,n),TpTmcomm(n,n))
ALLOCATE(TpUpcomm(n,n),TpUmcomm(n,n),TpYcomm(n,n),TpVpcomm(n,n))
ALLOCATE(TpVmcomm(n,n),TmUpcomm(n,n),TmUmcomm(n,n),TmYcomm(n,n))
ALLOCATE(TmVpcomm(n,n),TmVmcomm(n,n),YUpcomm(n,n),YUmcomm(n,n))
ALLOCATE(YVpcomm(n,n),YVmcomm(n,n),UpUmcomm(n,n),UpVpcomm(n,n))
ALLOCATE(UpVmcomm(n,n),UmVpcomm(n,n),UmVmcomm(n,n),VpVmcomm(n,n))
ALLOCATE(Casimir(n,n))
!
    T3Tpcomm = MATMUL(T3,Tp) - MATMUL(Tp,T3) - Tp                  ! 1
    MaxErr = MAXVAL(ABS( T3Tpcomm ))
        T3Tmcomm = MATMUL(T3,Tm) - MATMUL(Tm,T3) + Tm
```

```
        MaxErr = MAXVAL(ABS( (/MaxErr, T3Tmcomm/) ))
            T3Upcomm = MATMUL(T3,Up) - MATMUL(Up,T3) + Up/2._8
            MaxErr = MAXVAL(ABS( (/MaxErr, T3Upcomm/) ))
                T3Umcomm = MATMUL(T3,Um) - MATMUL(Um,T3) - Um/2._8
                MaxErr = MAXVAL(ABS( (/MaxErr, T3Umcomm/) ))
                    T3Ycomm = MATMUL(T3,Y) - MATMUL(Y,T3)           ! 5
                    MaxErr = MAXVAL(ABS( (/MaxErr, T3Ycomm/) ))
                        T3Vpcomm = MATMUL(T3,Vp) - MATMUL(Vp,T3) - Vp/2.
                        MaxErr = MAXVAL(ABS( (/MaxErr, T3Vpcomm/) ))
                          T3Vmcomm = MATMUL(T3,Vm) - MATMUL(Vm,T3) + Vm/2.
                          MaxErr = MAXVAL(ABS( (/MaxErr, T3Vmcomm/) ))
 !
    TpTmcomm = MATMUL(Tp,Tm) - MATMUL(Tm,Tp) - 2._8*T3
    MaxErr = MAXVAL(ABS( (/MaxErr, TpTmcomm/) ))
        TpUpcomm = MATMUL(Tp,Up) - MATMUL(Up,Tp) - Vp
        MaxErr = MAXVAL(ABS( (/MaxErr, TpUpcomm/) ))
            TpUmcomm = MATMUL(Tp,Um) - MATMUL(Um,Tp)                ! 10
            MaxErr = MAXVAL(ABS( (/MaxErr, TpUmcomm/) ))
                TpYcomm = MATMUL(Tp,Y) - MATMUL(Y,Tp)
                MaxErr = MAXVAL(ABS( (/MaxErr, TpYcomm/) ))
                    TpVpcomm = MATMUL(Tp,Vp) - MATMUL(Vp,Tp)
                    MaxErr = MAXVAL(ABS( (/MaxErr, TpVpcomm/) ))
                        TpVmcomm = MATMUL(Tp,Vm) - MATMUL(Vm,Tp) + Um
                        MaxErr = MAXVAL(ABS( (/MaxErr, TpVmcomm/) ))
        TmUpcomm = MATMUL(Tm,Up) - MATMUL(Up,Tm)
        MaxErr = MAXVAL(ABS( (/MaxErr, TmUpcomm/) ))
            TmUmcomm = MATMUL(Tm,Um) - MATMUL(Um,Tm) + Vm           ! 15
            MaxErr = MAXVAL(ABS( (/MaxErr, TmUmcomm/) ))
                TmYcomm = MATMUL(Tm,Y) - MATMUL(Y,Tm)
                MaxErr = MAXVAL(ABS( (/MaxErr, TmYcomm/) ))
                    TmVpcomm = MATMUL(Tm,Vp) - MATMUL(Vp,Tm) - Up
                    MaxErr = MAXVAL(ABS( (/MaxErr, TmVpcomm/) ))
                        TmVmcomm = MATMUL(Tm,Vm) - MATMUL(Vm,Tm)
                        MaxErr = MAXVAL(ABS( (/MaxErr, TmVmcomm/) ))
!
        YUpcomm = MATMUL(Y,Up) - MATMUL(Up,Y) - Up
        MaxErr = MAXVAL(ABS( (/MaxErr, YUpcomm/) ))
            YUmcomm = MATMUL(Y,Um) - MATMUL(Um,Y) + Um              ! 20
            MaxErr = MAXVAL(ABS( (/MaxErr, YUmcomm/) ))
                YVpcomm = MATMUL(Y,Vp) - MATMUL(Vp,Y) - Vp
```

```
                MaxErr = MAXVAL(ABS( (/MaxErr, YVpcomm/) ))
                    YVmcomm = MATMUL(Y,Vm) - MATMUL(Vm,Y) + Vm
                    MaxErr = MAXVAL(ABS( (/MaxErr, YVmcomm/) ))
        UpUmcomm = MATMUL(Up,Um) - MATMUL(Um,Up) - (3._8*Y/2._8 - T3)
        MaxErr = MAXVAL(ABS( (/MaxErr, UpUmcomm/) ))
            UpVpcomm = MATMUL(Up,Vp) - MATMUL(Vp,Up)
            MaxErr = MAXVAL(ABS( (/MaxErr, UpVpcomm/) ))
                UpVmcomm = MATMUL(Up,Vm) - MATMUL(Vm,Up) - Tm        ! 25
                MaxErr = MAXVAL(ABS( (/MaxErr, UpVmcomm/) ))
        UmVpcomm = MATMUL(Um,Vp) - MATMUL(Vp,Um) + Tp
        MaxErr = MAXVAL(ABS( (/MaxErr, UmVpcomm/) ))
            UmVmcomm = MATMUL(Um,Vm) - MATMUL(Vm,Um)
            MaxErr = MAXVAL(ABS( (/MaxErr, UmVmcomm/) ))
        VpVmcomm = MATMUL(Vp,Vm) - MATMUL(Vm,Vp) - (3._8*Y/2._8 + T3) ! 28
        MaxErr = MAXVAL(ABS( (/MaxErr, VpVmcomm/) ))
!
        Casimir = (MATMUL(Tp,Tm)+MATMUL(Tm,Tp)+MATMUL(Up,Um)+ &
            MATMUL(Um,Up)+MATMUL(Vp,Vm)+MATMUL(Vm,Vp))/2.+MATMUL(T3,T3) &
        +3._8*MATMUL(Y,Y)/4._8 - (dble(p**2+p*q+q**2)/3._8+ &
         dble(p + q))*unitMatrix                                     ! 29
        MaxErr = MAXVAL(ABS( (/MaxErr, Casimir/) ))
!
    WRITE(*,*)  'The 28 commutation relations plus the quadratic Casimir &
                expression are obeyed: ',(MaxErr.LE.MaxErrLimit)
WRITE(*,*) 'The largest error in the 29 equations is ',MaxErr,','
WRITE(*,*)  'which should be smaller than the tolerance MaxErrLimit = ',&
            MaxErrLimit
WRITE(*,*)      ! a blank line
!
    IF(MaxErr.LE.MaxErrLimit) THEN !Is the largest error within tolerance?
        MaxErrNotTooBig = .TRUE.        ! TRUE means Yes.
    ELSE IF (MaxErr > MaxErrLimit)    THEN
        MaxErrNotTooBig = .FALSE.  ! FALSE means No, not within tolerance.
    END IF
!
!----------15. Save the results to a file, end program -------------------
!
!MaxErrNotTooBig = .FALSE. ! Use this statement to test the failure option
!MaxErrNotTooBig = .TRUE.    ! Use this statement to make an output
!
```

```
    IF (MaxErrNotTooBig) THEN             ! No Failure(s) found
        WRITE (file_name,"('p',i0,'q',i0,'SU3GenII.dat')")p,q
        OPEN(Unit=5,file=file_name)
        WRITE(5,3000) p,q, MaxErr   ! Start with p,q,MaxErr.
        CLOSE(5)
    OPEN(Unit=5,file=file_name,STATUS='OLD', POSITION='APPEND') !
!    Next, upload the 8*dimREP**2 components of the 8 basis matrices:
   WRITE(5,4000) TRANSPOSE(T3),TRANSPOSE(Y),TRANSPOSE(Tp), TRANSPOSE(Tm),&
                 TRANSPOSE(Up), TRANSPOSE(Um), TRANSPOSE(Vp), TRANSPOSE(Vm)
      CLOSE(5)
    END IF
!
3000 FORMAT(2I3,1ES16.7)  ! Record 1 has two integers and a real number
4000 FORMAT(5F16.12)  !Components of the 8 matrices, 5 numbers per line
WRITE(*,*) "The output file p#q#SU3GenII.dat has the integers p = ",p, &
            " and q = ",q," for the #s in the file's name."
!
                        END PROGRAM Su3Generators
!
!-----------------16. External functions, end-of-file---------------------
!
FUNCTION nSEQUENCE(p,q,a,b,alpha) RESULT  (w)
    IMPLICIT NONE        ! The nth place in the sequence belongs to
        INTEGER :: p,q     ! the T3 eigenvector with eigenvalue alpha
        INTEGER :: a,b,m    ! in the (a,b)-SU2 irrep
        REAL(8) :: alpha,w !the T3 eigenvector has the mth place in the
!                           ! irrep, where m = (a+b)/2 -alpha + 1
    w = ((a+b)*(a+b+1._8)+q*b*(q+b+2._8))/2._8 + ((a+b)/2._8-alpha+1._8)
    END FUNCTION
FUNCTION g(p,q,a,b) RESULT  (w)
    IMPLICIT NONE        ! Four of the U,V formulas have
    INTEGER :: p,q     ! the factor SQRT(g), where
        INTEGER :: a,b    !g = a*(p+a+1)*(q-a+1)/[(a+b)(a+b+1)]
        REAL(8) :: w      ! w = g
    w = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
    END FUNCTION
FUNCTION h(p,q,a,b) RESULT  (w)
    IMPLICIT NONE        ! Four of the U,V formulas have
INTEGER :: p,q           ! the factor SQRT(h), where
        INTEGER :: a,b   ! h = b*(p-b+1)*(q+b+1)/[(a+b)(a+b+1)]
```

```
        REAL(8) :: w     ! w = h
    w = b*(p-b+1._8)*(q+b+1._8)/((a+b)*(a+b+1._8))
    END FUNCTION
! end-of-file
```

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