Theoretical Structures as Networks from the Perspective of Automated Creation and Proving of Theorems

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Abstract: We envisage theoretical structures (especially in pure mathematics and theoretical physics) as networks made of elementary propositions (representing nodes) interconnected through deductive relationships (representing throats). This vision can be exploited as a basis for employing traditional network modeling techniques in the automated search for new theorems as well as for automated proving of proposed theorems and conjectures. This deductive, deterministic and intuitive approach can replace some of the conventional approaches (which are generally more sophisticated and elaborate and hence they are more expensive) in certain areas of automated and assisted theorem proving in addition to its benefit in the automated search for novel theorems. However, we admit that it has a number of limitations and shortcomings although this similarly applies to other methods in this field; moreover some of these limitations and shortcomings can be overcome by the reformulation of certain theoretical structures where we rely for the viability of this reformulation on our perception of theoretical structures as elaborate high-level linguistic systems.

Keywords: Automated theorem proving, automated theorem creation, automated search for novel theorems, theoretical structures, mathematical structures, axiomatic structures, pure mathematics, theoretical physics, traditional network modeling.

1 Introduction

Automated and assisted theorem proving is a fundamental subject in computer science. It dates back to the early days of the invention of modern computers; a fact that reflects the traditional view that computers are logical devices governed by very strict and well defined logical rules which are at the foundation of any reasoning process that underlies proving (and deductive reasoning in general) especially in the mathematical disciplines which computers are primarily designed to deal with. Accordingly, computers should be capable of mimicking these intellectual processes by automating the logical procedures and protocols that govern and regulate theorem proving (and deductive reasoning to be more general).

In fact, the theoretical foundations of this view (which justifies and theoretizes the automation within this paradigm) were laid down much earlier than the invention of modern computers (e.g. in the work of Whitehead and Russell [1] although this is not the only or the first start of this school of thought about the theoretical foundations of automation of logical reasoning in general and proving in particular).

Despite the limitation of this "classical" view and the approaches that are based on it (noting its deductive deterministic nature and considering that not all forms of reasoning and proving yield themselves to these approaches) it remains one of the most common views in the field of automated theorem proving (see for instance [2–5]) although the tide is generally turning towards other views and approaches especially with the recent rise of artificial intelligence[1] with its largely statistical and empirical nature although this does not necessarily contradict the essence of this "classical" view.

Anyway, the subject of automated and assisted theorem proving is very wide and deep and the literature in this regard is very extensive; a fact that reflects the great diversity of the schools of thought and the methods and techniques employed and exploited in this field (see for instance [2–14]) and hence we are not going to go through this in this short paper because it is out of its scope and size and because this will not add any novel substance

[1] Some scholars consider automated theorem proving as a branch of artificial intelligence or a form of it. Although we do not object this view in the extensive sense of "artificial intelligence" (which makes any computer activity a form of artificial intelligence), we object this view within the conventional modern paradigm of "artificial intelligence". Our view in this respect is that artificial intelligence is a very powerful and useful tool for dealing with automated theorem proving theoretically and practically and it is closely related to it but automated theorem proving is not necessarily a part of artificial intelligence or an instance of it, i.e. automated theorem proving can be theoretized and implemented with and without artificial intelligence although artificial intelligence is one of the best and most natural tools for tackling automated theorem proving and dealing with its challenges.

or ideas to this field.

Our approach in this paper is actually based on the aforementioned "classical" view of automated theorem proving in its most basic and generic form which reflects its simplicity and intuitiveness and hence we avoid unnecessarily complicated approaches that compete with it in some of the areas of automated theorem proving. In brief, we envisage theoretical structures (notably common branches of pure mathematics and theoretical physics distinguished by their axiomatic frameworks and structures) as directed networks made of simple (or elementary) propositions connected through deductive relationships[2] where these propositions play the role of nodes while the relationships play the role of throats.

Based on this vision, an entire theoretical structure can be built and grown deductively by automated inference through the search for routes that connect separated nodes (i.e. nodes not connected directly). Accordingly, this vision can be taken as a basis for automated creation and proving of theorems and propositions through the construction of appropriate databases of elementary propositions with the associated information about their connectivity relationships. This approach (which we label as "traditional network modeling" approach)[3] is simple and does not require the use of artificial intelligence or any similar sophisticated methods and techniques although the approach in itself is not in contradiction with these sophisticated methods (i.e. in principle it can lend itself to the use of these sophisticated methods).

2 Description of the Network Modeling Approach

A general description of the proposed network modeling approach has been given in the last two paragraphs of the Introduction. To put it in a more practical and technical language, let for instance have a database of 100 elementary propositions indexed as $1, 2, \ldots, 100$. This database represents the nodes of the network (which represents the concerned theoretical structure). Let also have a relational database of elementary connections where each one of these connections links one of the 100 elementary nodes to another elementary node. This database (which essentially provides the information about the throats of

[2] *Deductive* relationships are the most simple and natural type of relationships that to be considered in this context although other types of relationships may also be considered in a more elaborate variant of this approach.

[3] "Traditional network modeling" approach refers to the network modeling approach that we used extensively in the past in our fluid mechanics research (see for instance [15–18]). This approach has nothing to do with the neural networks (in their various shapes and forms) which are used in artificial intelligence (particularly in deep learning methodologies).

Table 1: Part of the connectivity database for the network of 100 nodes (indexed as $1, 2, \ldots, 100$) which is described in the text (see § 2).

$1 \Rightarrow 5$	$2 \Rightarrow 7$	...	$59 \Rightarrow 6$	...	$99 \Rightarrow 29$	$100 \Rightarrow 33$
$1 \Rightarrow 13$	$2 \Rightarrow 22$	...	$59 \Rightarrow 99$	...	$99 \Rightarrow 43$	$100 \Rightarrow 88$
$1 \Rightarrow 59$	$2 \Rightarrow 61$	...		...	$99 \Rightarrow 100$	$100 \Rightarrow 95$
$1 \Rightarrow 73$		...		...		$100 \Rightarrow 98$
$1 \Rightarrow 95$		...		...		

the network) represents the actual structure of the network and its connectivity. In fact, each throat is essentially an "if" (or conditional) statement (noting that an "*iff*" statement actually represents two "if" statements and hence it should be split as such).

Now, any connected route from a node representing a given proposition (call it P) to another node representing another given proposition (call it Q) represents a proven (or discovered) theorem, i.e. $P \Rightarrow Q$ where P reaches Q through a number of other nodes connected by elementary throats (e.g. $P \Rightarrow A \Rightarrow B \Rightarrow C \Rightarrow Q$ where A, B, C are other nodes). Hence, all we need to prove (or find) the theorem $P \Rightarrow Q$ is to see if we can find such a route (i.e. $\Rightarrow A \Rightarrow B \Rightarrow C \Rightarrow$), and this can be easily achieved by systematic and automated search using simple network searching techniques.

In Table 1 we present a part of such a network where it can be easily seen that we can obtain for example the theorems $1 \Rightarrow 100$ (i.e. $1 \Rightarrow 59 \Rightarrow 99 \Rightarrow 100$) and $1 \Rightarrow 88$ (i.e. $1 \Rightarrow 59 \Rightarrow 99 \Rightarrow 100 \Rightarrow 88$) as well as several other theorems although most of these theorems are usually trivial.

Despite the conceptual simplicity of the proposed network modeling approach (as well as the relative ease of programming and implementing it in a network modeling computer code), it requires substantial effort and novel ideas in formulating the theoretical structure as elementary propositions connected by elementary relationships. So, the main effort and ingenuity are actually required in building the nodes and throats databases that formulate the theoretical structure as a network in the described way. However, we should admit that some theoretical structures may not lend themselves (easily or at all) to such simple network formulation while other theoretical structures may lend themselves to such network formulation only partially. This issue will be investigated further later on (see § 3).

3 Assessment of the Network Modeling Approach

In this section we briefly assess the proposed network modeling approach highlighting its main pros and cons and proposing some ideas about how to deal with and overcome its main limitations.

The main pros of the proposed network modeling approach are its intuitiveness and relative simplicity (conceptually and practically). Moreover, it applies to the creation and searching for new theorems as well as to proving known theorems (or dealing with conjectures and open problems). Furthermore, it applies to general theoretical structures and not restricted to mathematics although mathematical structures are the natural field of its application.

However, it is obvious that the applicability and validity of the proposed network modeling approach is restricted to axiomatic structures based on deduction and inference that lend themselves to be cast in the form of elementary propositions and statements connected through elementary deductive relationships. It is obvious that not all types of theoretical structures and theorems lend themselves to such deductive network approach. Nevertheless, we believe that (at least) in some cases the theoretical structures and theorems that do not lend themselves to this approach as they are can be reformulated to be in a proper form that can be tackled by the proposed network modeling approach. This is based on our view that theoretical structures are actually elaborate high-level linguistic systems[4] and hence they in principle can be formulated and reformulated in different shapes and forms and this should allow (at least in some types of theoretical structures) reformulating these problematic theoretical structures to put them in a form that lends itself to the manipulation by the proposed approach of network modeling.

[4] We discussed this issue in detail in chapter two of our book "Critique of Logical Foundations of Induction and Epistemology" (which is in Arabic and is available on the Internet). We also touched on this issue in our book about quantum physics [19].

4 Conclusions

We summarize the main issues which we discussed in the present paper in the following points:

1. Automated and assisted theorem proving (as well as automated theorem creation and searching) are not necessarily an artificial intelligence issue and this allows the proposition of more simple approaches to tackle these problems including our proposition of using network modeling approach in its relatively simple form.
2. The proposed network modeling approach is based on viewing and formulating the concerned theoretical structure as a network of nodes (representing elementary propositions) connected by throats (representing elementary deductive relationships). The main effort in this approach is the formulation of the theoretical structure in this simple network form and building the required databases that provide the information about the elementary propositions (i.e. nodes) and their connectivity relationships (i.e. throats).
3. The proposed network modeling approach is not restricted to automated theorem proving but it extends to finding new propositions and theorems and hence building and elaborating entire theoretical structures from rather basic pieces of information and data.
4. Although the proposed network modeling approach is limited by the fact that not all theoretical structures and theorems lend themselves to this form of network modeling, we believe that some of these limitations can be overcome by reformulating the theoretical structures and theorems to put them in the proper form that facilitates the employment of the proposed network modeling approach. This belief is based on our view that theoretical structures are essentially elaborate high-level linguistic systems and hence the reality that these structures represent and depict can be expressed and depicted in different shapes and forms (i.e. although the essence of the reality behind these theoretical structures is unique it can be expressed and depicted non-uniquely and in various shapes and forms).

References

[1] A.N. Whitehead; B. Russell. *Principia Mathematica.* Cambridge at the University Press, first edition, 1910-1913.

[2] M. Newborn. *Automated Theorem Proving Theory and Practice.* Springer-Verlag New York, Inc., first edition, 2001.

[3] F. Pfenning. *Automated Theorem Proving.* Carnegie Mellon University, Draft of Spring 2004 edition, 2004.

[4] W. Bibel. Early History and Perspectives of Automated Deduction. *In: J. Hertzberg, M. Beetz and R. Englert (eds) KI 2007: Advances in Artificial Intelligence. Lecture Notes in Computer Science. Springer, Berlin, Heidelberg*, 4667:2–18, 2007.

[5] D.A. Plaisted. History and Prospects for First-Order Automated Deduction. *In: A. Felty and A. Middeldorp (eds) Automated Deduction - CADE-25, CADE 2015, Lecture Notes in Computer Science*, 9195, 2015.

[6] W. Bibel. *Automated Theorem Proving.* Springer, second edition, 1987.

[7] M. Fitting. *First-Order Logic and Automated Theorem Proving.* Springer-Verlag New York, Inc., first edition, 1990.

[8] L. Wos; R. Overbeek; R. Lusk; J. Boyle. *Automated Reasoning Introduction and Applications.* McGraw-Hill, second edition, 1992.

[9] J. Harrison. A Short Survey of Automated Reasoning. *In: H. Anai, K. Horimoto and T. Kutsia (eds) Algebraic Biology, AB 2007, Lecture Notes in Computer Science, Volume 4545 (Springer)*, 2007.

[10] J. Harrison. *Handbook of Practical Logic and Automated Reasoning.* Cambridge University Press, first edition, 2009.

[11] P. Lu; L. Qiu; W. Yu; S. Welleck; K-W. Chang. A Survey of Deep Learning for Mathematical Reasoning. *Proceedings of the 61st Annual Meeting of the Association for Computational Linguistics*, 1:14605–14631, 2023.

[12] R. Ndungi; S. Uyun. A Review of Automated Reasoning and Its Applications in the 21st Century. *The Indonesian Journal of Computer Science*, 12(2):483–491, 2023.

[13] Z. Li; J. Sun; L. Murphy; Q. Su; Z. Li; X. Zhang; K. Yang; X. Si. A Survey on Deep Learning for Theorem Proving. *arXiv:2404.09939 (Published as a conference paper at COLM 2024)*, 2024.

[14] F. Portoraro; E.N. Zalta (ed); U. Nodelman (ed). *Automated Reasoning, The Stanford Encyclopedia of Philosophy.* Spring 2024 edition, 2024.

[15] T. Sochi. *Pore-Scale Modeling of Non-Newtonian Flow in Porous Media.* PhD thesis, Imperial College London, 2007.

[16] T. Sochi; M.J. Blunt. Pore-scale network modeling of Ellis and Herschel-Bulkley fluids. *Journal of Petroleum Science and Engineering*, 60(2):105–124, 2008.

[17] T. Sochi. Pore-scale modeling of viscoelastic flow in porous media using a Bautista-Manero fluid. *International Journal of Heat and Fluid Flow*, 30(6):1202–1217, 2009.

[18] T. Sochi. Modelling the Flow of Yield-Stress Fluids in Porous Media. *Transport in Porous Media*, 85(2):489–503, 2010.

[19] T. Sochi. *The Epistemology of Quantum Physics.* Amazon Kindle Direct Publishing, first edition, 2022.