Gaussian Integral

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September 8, 2024

ABSTRACT: The Gaussian integral is defined as the integral of the function e^{-x^2} over the entire real line. Mathematically, it is expressed as: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. In this note we give some formulas related to the Gaussian integral.

I. Introduction

1. Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \tag{1}$$

2. Proof

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2}\,r\,dr\,d\theta = \pi \tag{2}$$

3. Double Integral

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \implies \frac{\pi}{4} = \int_{0}^{\infty}\int_{0}^{\infty} e^{-x^2-y^2}\,dx\,dy \tag{3}$$

In this note we give some formulas related to (3).

II. Some Relations

Entry 1.

for $s = \sqrt{\frac{1}{2}\,W\left(\frac{\pi^2}{8}\right)}$ we have

$$\frac{\pi}{4} = s + \int_{R(s)} e^{-x^2-y^2}\,dx\,dy \tag{4}$$

where

$$R(s) = \left\{x^2 + y^2 \le s^2\,,\ x \ge 0,\ y \ge 0\right\} \tag{5}$$

and $W(x)$ is the Lambert Function.

Entry 2.

for $s = \sqrt{\frac{1}{2}\,W\left(\frac{\pi^2}{8}\right)}$ we have

$$s = \int_{T(s)} e^{-x^2-y^2}\,dx\,dy \tag{6}$$

where

$$T(s) = \left\{x^2 + y^2 \ge s^2\,,\ x \ge 0,\ y \ge 0\right\} \tag{7}$$

and $W(x)$ is the Lambert Function.

Entry 3.

$$s=\sqrt{\frac{1}{2}\,W\left(\frac{\pi^2}{8}\right)}\Longrightarrow \pi=4\,s\,e^{s^2} \tag{8}$$

$$s=0.5684990471\ldots \tag{9}$$

Entry 4.

$$s_1=1/2\,,\ s_{n+1}=\int_{T(s_n)} e^{-x^2-y^2}\,dx\,dy\Longrightarrow s_n\longrightarrow s=\sqrt{\frac{1}{2}\,W\left(\frac{\pi^2}{8}\right)}=0.5684\ldots \tag{10}$$

where

$$T(s_n)=\left\{x^2+y^2\geq s_n^2\,,\ x\geq 0, y\geq 0\right\}\,,\ n=1,2,3,\ \ldots \tag{11}$$

Entry 5.

$$s_1=1/2\,,\ s_{n+1}=\frac{1}{2}\left(s_n+\int_{T(s_n)} e^{-x^2-y^2}\,dx\,dy\right)\Longrightarrow s_n\longrightarrow s=0.5684\ldots \tag{12}$$

where

$$T(s_n)=\left\{x^2+y^2\geq s_n^2\,,\ x\geq 0, y\geq 0\right\}\,,\ n=1,2,3,\ \ldots \tag{13}$$

Entry 6.

for $s=0.6171377584\ldots$ we have

$$s=\int_{T(s)} e^{-x^2-y^2}\,dx\,dy \tag{14}$$

where

$$T(s)=\{x+y\geq s\,,\ x\geq 0,\ y\geq 0\} \tag{15}$$

Entry 7.

for $s=0.6171377584\ldots$ we have

$$\frac{\pi}{4}=s+\int_{R(s)} e^{-x^2-y^2}\,dx\,dy \tag{16}$$

where

$$R(s)=\{x+y\leq s\,,\ x\geq 0,\ y\geq 0\} \tag{17}$$

Entry 8.

for $s=0.6171377584\ldots$ we have

$$\frac{\pi}{4}=s+\sum_{n=0}^{\infty}\frac{(-1)^n\,s^{2n+2}}{(2n+1)\,(2n+1)\,n!}\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{2n}{2k}} \tag{18}$$

Entry 9.

$$s_1=1/2\,,\ s_{n+1}=\int_{T(s_n)} e^{-x^2-y^2}\,dx\,dy\Longrightarrow s_n\longrightarrow s=0.6171\ldots \tag{19}$$

where

$$T(s_n)=\{x+y\geq s_n\,,\ x\geq 0, y\geq 0\}\,,\ n=1,2,3,\ \ldots \tag{20}$$

Entry 10.

$$s_1 = 1/2\,,\ \ s_{n+1} = \frac{12}{37}\, s_n + \frac{25}{37} \int_{T(s_n)} e^{-x^2-y^2}\, dx\, dy \implies s_n \longrightarrow s = 0.6171 \ldots \tag{21}$$

where

$$T(s_n) = \{x + y \geq s_n\,,\ x \geq 0, y \geq 0\}\ ,\ n = 1, 2, 3,\ \ldots \tag{22}$$

Entry 11.

for $s = 0.5420337521 \ldots$ we have

$$s = \int_{T(s)} e^{-x^2-y^2}\, dx\, dy \tag{23}$$

where

$$T(s) = [0,\ \infty] \times [0,\ \infty] - [0,\ s] \times [0,\ s] \tag{24}$$

Entry 12.

for $s = 0.5420337521 \ldots$ we have

$$\frac{\pi}{4} = s + \int_{R(s)} e^{-x^2-y^2}\, dx\, dy\ = s + \int_0^s \int_0^s e^{-x^2-y^2}\, dx\, dy = s + \left(\int_0^s e^{-x^2}\, dx \right)^2 \tag{25}$$

where

$$R(s) = [0,\ s] \times [0,\ s] \tag{26}$$

Entry 13.

for $s = 0.5420337521 \ldots$ we have

$$\frac{\pi}{4} = s + \sum_{n=0}^{\infty} (-1)^n\, s^{2n+2} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!\,(2k+1)\,(2n-2k+1)} \tag{27}$$

Entry 14.

$$s_1 = 1/2\,,\ \ s_{n+1} = \int_{T(s_n)} e^{-x^2-y^2}\, dx\, dy \implies s_n \longrightarrow s = 0.5420 \ldots \tag{28}$$

where

$$T(s_n) = [0,\ \infty] \times [0,\ \infty] - [0,\ s_n] \times [0,\ s_n]\,,\ \ n = 1,\ 2,\ 3,\ \ \ldots \tag{29}$$

Entry 15.

$$s_1 = 1/2\,,\ \ s_{n+1} = \frac{1}{2} \left(s_n + \int_{T(s_n)} e^{-x^2-y^2}\, dx\, dy \right) \implies s_n \longrightarrow s = 0.5420 \ldots \tag{30}$$

where

$$T(s_n) = [0,\ \infty] \times [0,\ \infty] - [0,\ s_n] \times [0,\ s_n]\,,\ \ n = 1,\ 2,\ 3,\ \ \ldots \tag{31}$$

III. Endnote

Entry 16.

$$\pi = 4\, e^{s^2} \int_{T(s)} e^{-x^2-y^2}\, dx\, dy\ \ ,\ \ s > 0 \tag{32}$$

where

$$T(s) = \left\{ x^2 + y^2 \geq s^2,\ \ x \geq 0,\ \ y \geq 0 \right\} \tag{33}$$

Entry 17.

for $s = \sqrt{-\frac{1}{2}\, \mathrm{W}\left(-\frac{2}{\pi^2}\right)} = 0.3631930863 \ldots$ we have

$$\pi = \frac{1}{s} + \int_{R(s)} e^{-x^2-y^2}\, d x\, d y \tag{34}$$

where

$$R(s) = \left\{x^2 + y^2 \le s^2,\ \{x,\ y\} \in \mathbb{R}^2\right\} \tag{35}$$

and $W(x)$ is the Lambert function.

Entry 18.

$$s_1 = 1/3\ ,\ s_{n+1} = \left(\int_{T(s_n)} e^{-x^2-y^2}\, d x\, d y\right)^{-1} \Longrightarrow s_n \longrightarrow s = \sqrt{-\frac{1}{2}\, W\!\left(-\frac{2}{\pi^2}\right)} = 0.3631\ ... \tag{36}$$

where

$$T(s_n) = \left\{x^2 + y^2 \ge s_n^2,\ \{x,\ y\} \in \mathbb{R}^2\right\}\ ,\ n = 1,\ 2,\ 3,\ ... \tag{37}$$

and $W(x)$ is the Lambert function.

IV. References

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