Power-spectral Numbers

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Abstract

Given $n = p_1^{e_1} \cdots p_k^{e_k}$, there is a canonical isomorphism $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$. The spectral basis of $\mathbb{Z}_n$ is an explicit realization of this isomorphism within $\mathbb{Z}_n$ itself. This paper is a study of those numbers whose spectral basis consists of primes and powers. For example, if M_p is a Mersenne prime with exponent p, then $2M_p$ has spectral basis $\{M_p, 2^p\}$, while $2^p M_p$ has spectral basis $\{M_p^2, 2^p\}$. Isospectral and isotropic numbers are introduced and many other numbers with interesting spectral bases are presented.

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1 The spectral basis

This section is adapted from Ireland [3, p. 194] and Sobczyk [4].

Let $n_1, n_2, \ldots, n_k$ be pairwise relatively prime integers and let $\psi_i : \mathbb{Z} \to \mathbb{Z}_{n_i}$, $\mathbb{Z}_{n_i} := \mathbb{Z}/n_i\mathbb{Z}$, denote the natural homomorphism $x \mapsto x \bmod n_i$. Define

$$\begin{aligned} \psi &:= \mathbb{Z} \to \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \\ \psi(x) &= (\psi_1(x), \psi_2(x), \ldots, \psi_k(x)) \end{aligned}$$

It is easy to check that ψ is a ring homomorphism. Let us find the image and kernel of ψ. Given $y = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k)$, there exists $x \in \mathbb{Z}$ such that $\psi(x) = y$ if and only if $\psi_i(x) = y_i$ for $i = 1, 2, \ldots, k$, that is, if and only if $x \equiv y_i \bmod n_i$ for $i = 1, 2, \ldots, k$. The Chinese Remainder Theorem guarantees that this system of congruences has a solution, that is, there exists $x \in \mathbb{Z}$ such that $\psi(x) = y$, and so ψ is a ring epimorphism. The kernel is all $x \in \mathbb{Z}$ such that $\psi(x) = 0$, and this happens if and only if $x \equiv 0 \bmod n_i$, $i = 1, 2, \ldots, k$, that is, if and only if x is divisible by $n := n_1 n_2 \cdots n_k$, and so the kernel of ψ is the ideal $n\mathbb{Z}$ in $\mathbb{Z}$. Therefore, the map ψ induces an isomorphism,

$$\tilde{\psi} : \mathbb{Z}_n \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k},$$

where $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. The *spectral basis* of $\mathbb{Z}_n$ will be an explicit realization of $\tilde{\psi}^{-1}$, the inverse of the isomorphism $\tilde{\psi}$, within $\mathbb{Z}_n$ itself.

Assume that n has at least two prime factors. The case of one factor is dealt with by p-adic expansion, and we will not discuss that in this paper. Consider $n_i = p_i^{e_i}$ so that $n = p_1^{e_1} \cdots p_k^{e_k}$. The homomorphism ψ takes the form

$$\begin{aligned} \psi &:= \mathbb{Z} \to \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}} \\ \psi(x) &= (\psi_1(x), \psi_2(x), \ldots, \psi_k(x)) \end{aligned}$$

where $\psi_i : \mathbb{Z} \to \mathbb{Z}_{p_i^{e_i}}$ is the canonical homomorphism $x \mapsto x \bmod p_i^{e_i}$.

The direct sum $\mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$ has canonical projections

$$\begin{aligned} \pi_i &: \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}} \to \mathbb{Z}_{p_i^{e_i}}, \\ \pi_i&(x_1, \ldots, x_k) = x_i. \end{aligned} \tag{1}$$

The projections $\{\pi_i\}$ have the following properties.

$$\text{Idempotence:} \quad \pi_i^2 = \pi_i, \tag{2a}$$

$$\text{Orthogonality:} \quad \pi_i \pi_j = 0, \quad i \neq j, \tag{2b}$$

$$\text{Decomposition of the identity:} \quad \pi_1 + \cdots + \pi_k = \mathrm{Id} \tag{2c}$$

where Id is the identity map of $\mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$.

What elements s_i of $\mathbb{Z}_n$ correspond to the projections π_i? They are multiplication maps by integers s_i, denoted by s_i for convenience, that must satisfy

$$s_i^2 = s_i \tag{3a}$$

$$s_i s_j = 0, \quad i \neq j, \tag{3b}$$

$$s_1 + \cdots + s_k = 1. \tag{3c}$$

Properties (3) mean that $\{s_1, \ldots, s_k\}$ comprise a decomposition of the identity into mutually orthogonal projections.

Define the integers h_i by $h_i = n/p_i^{e_i}$. Clearly, they are collectively relatively prime, that is, $\gcd(h_1, \ldots, h_k) = 1$. Thus, there must exist integers a_i such that

$$a_1 h_1 + \cdots + a_k h_k = 1 \quad \text{in} \quad \mathbb{Z}_n. \tag{4}$$

We will now show that $s_i := a_i h_i$ are the elements we're looking for. Property (3b), namely $s_i s_j = 0$, $i \neq j$, is immediate. If we apply s_i to (3c) we get

$$s_i s_1 + \cdots s_i^2 + \cdots + s_i s_k = s_i,$$

and so $s_i^2 = s_i$, and property (3a) holds. Apply h_i to (3c) to obtain

$$h_i s_1 + \cdots h_i s_i + \cdots + h_i s_k = h_i.$$

But then $h_i s_j = 0$, $i \neq j$, so we have

$$h_i s_i = h_i \quad \text{in} \quad \mathbb{Z}_n.$$

from which it follows that

$$s_i = (h_i^{-1} \bmod p_i^{e_i})\, h_i. \tag{5}$$

The elements $\{s_i\}$ comprise the *spectral basis* of $\mathbb{Z}_n$ and each s_i acts as a projection from $\mathbb{Z}_n$ onto $\mathbb{Z}_{p_i^{e_i}}$ just like the projection π_i of (2).

Any element of x in $\mathbb{Z}_n$ can now be written uniquely as

$$x = x_1 s_1 + \cdots + x_k s_k,$$

where

$$x_i = x \bmod p_i^{e_i}.$$

Thus, we have

$$\tilde{\psi}^{-1}(x_1, \ldots, x_k) = x_1 s_1 + \cdots + x_k s_k.$$

Definition 1. We call $\mathrm{SC}(n) = \{a_1, \ldots, a_k\}$ defined by (4) the *spectral coefficients* of $\mathbb{Z}_n$ and $s_i = a_i h_i$ defined by (5) the *spectral idempotents* or *spectral elements* of $\mathbb{Z}_n$. The collection $\mathrm{SB}(n) = \{s_1, \ldots, s_k\}$ is called the *spectral basis* of $\mathbb{Z}_n$. We shall say "n has spectral basis" instead of "$\mathbb{Z}_n$ has spectral basis" when convenient.

We present for future reference the basic properties of a spectral basis. Recall that $n = p_1^{e_1} \cdots p_k^{e_k}$, $k > 1$.

$$h_i = n/p_i^{e_i} \tag{6a}$$
$$s_i = (h_i^{-1} \bmod p_i^{e_i})\, h_i \tag{6b}$$
$$s_i^2 = s_i, \tag{6c}$$
$$s_i s_j = 0 \quad (i \neq j), \tag{6d}$$
$$s_1 + \cdots + s_k = 1, \tag{6e}$$
$$x = (x \bmod p_1^{e_1})\, s_1 + \cdots + (x \bmod p_k^{e_k})\, s_k, \tag{6f}$$

where all equations holds in $\mathbb{Z}_n$.

We shall call an integer *composite* if it has more than one prime divisor and refer to it specifically as a *prime* or *prime power* when it is a prime or prime power, respectively.

Definition 2 (Index of a composite integer)**.** The index of a composite integer n is the integer idx such that $s_1 + \cdots + s_k = idx \cdot n + 1$ in $\mathbb{Z}$, where $\{s_1, \ldots, s_k\}$ is the spectral basis of $\mathbb{Z}_n$.

See Subsection 11.1 for tables of first index and Subsection 11.2 for tables of factorials and primorials and their indices.

2 Mersenne I

The general case in this section is $2^s M_p^t$ where $M_p = 2^p - 1$ is a Mersenne prime and p is the (necessarily prime) Mersenne exponent.

Theorem 3. *The integer $2p^k$, $p > 2$, $k \geq 1$, has spectral basis $\{p^k, p^k + 1\}$.*

Proof. Left to the reader. □

Theorem 4. *Let M_p be a Mersenne prime with Mersenne exponent p. Then*

1. *$2M_p$ has spectral basis $\{M_p, 2^p\}$ or, equivalently, $\{M_p, M_p + 1\}$.*
2. *$2^p M_p$ has spectral basis $\{M_p^2, 2^p\}$ or, equivalently, $\{M_p^2, M_p + 1\}$.*
3. *$2^{p+1} M_p$ has spectral basis $\{M_p^2, 2^{2p}\}$ or, equivalently, $\{M_p^2, (M_p + 1)^2\}$*
4. *$2^{2p+1} M_p^2$ has spectral basis $\{M_p^2 (M_p + 2)^2, (M_p^2 - 1)^2\}$.*

The proofs for each case are similar and proceed in the same way: verifying the idempotence of each spectral element and verification of the decomposition of the identity.

Proof of Theorem 4.1. Use $k = 1$ in Theorem 3. □

Proof of Theorem 4.2. 2^pM_p has spectral basis $\{M_p^2, 2^p\}$ or, equivalently, $\{M_p^2, M_p + 1\}$. Table 1 shows the first six 2^pM_p and their spectral coefficients.

p	n	2^pM_p	a	b
2	12	$(2)^2(3)$	(3)	(1)
3	56	$(2)^3(7)$	(7)	(1)
5	992	$(2)^5(31)$	(31)	(1)
7	16256	$(2)^7(127)$	(127)	(1)
13	67100672	$(2)^{13}(8191)$	(8191)	(1)

Table 1: Spectral coefficients of 2^pM_p.

Observe that $a = M_p$ and $b = 1$ so the spectral basis is claimed to be $\{M_p^2, 2^p\}$. Let us verify the projection properties and decomposition of the identity. Observe that

$$\begin{aligned} M_p^2 + 2^p &= (2^p - 1)^2 + 2^p \\ &= (2^{2p} - 2 \cdot 2^p + 1 + 2^p \\ &= 2^{2p} - 2^p + 1 \\ &= 2^p(2^p - 1) + 1 \\ &= 2^pM_p + 1 \\ M_p^2 + 2^p &\equiv 1 \pmod{2^pM_p}. \end{aligned}$$

Idempotence for M_p^2 in 2^pM_p.

$$\begin{aligned} (M_p^2)^2 - M_p^2 &= M_p^2(M_p^2 - 1) \\ &= M_p^2(M_p - 1)(M_p + 1) \\ &= M_p(M_p - 1) \cdot 2^pM_p \\ (M_p^2)^2 &\equiv M_p^2 \pmod{2^pM_p}. \end{aligned}$$

Idempotence for 2^p in 2^pM_p.

$$\begin{aligned} (2^p)^2 - 2^p &= 2^p(2^p - 1) \\ &= 2^pM_p \\ (2^p)^2 &\equiv 2^p \pmod{2^pM_p}. \end{aligned}$$

□

The sequence $\{2^pM_p\}$ is A139256: $a_n = \sigma(P_n)$ where P_n is the nth perfect number.

Proof of Theorem 4.3. $2^{p+1}M_p$ has spectral basis $\{M_p^2, 2^{2p}\}$. Table 2 shows the first six $2^{p+1}M_p$ and their spectral coefficients.

It is clear that if we take $a = M_p$ and $b = 2^{p-1}$ then the spectral basis is claimed to be $\{M_p^2, 2^{2p}\}$.

$$M_p^2 + 2^{2p} = (2^p - 1)^2 + 2^{2p}$$

p	n	$2^{p+1}M_p$	a	b
2	24	$(2)^3(3)$	(3)	(2)
3	112	$(2)^4(7)$	(7)	$(2)^2$
5	1984	$(2)^6(31)$	(31)	$(2)^4$
7	32512	$(2)^8(127)$	(127)	$(2)^6$
13	134201344	$(2)^{14}(8191)$	(8191)	$(2)^{12}$

Table 2: Spectral coefficients of $2^{p+1}M_p$.

$$\begin{aligned} &= 2^{2p} - 2^{p+1} + 1 + 2^{2p} \\ &= 2^{2p+1} - 2^{p+1} + 1 \\ &= 2^{p+1}(2^p - 1) + 1 \\ &= 2^{p+1}M_p + 1 \\ M_p^2 + 2^{2p} &\equiv 1 \pmod{2^{p+1}M_p}. \end{aligned}$$

Idempotence of M_p^2 in $2^{p+1}M_p$.

$$\begin{aligned} (M_p^2)^2 - M_p^2 &= M_p^2(M_p - 1)(M_p + 1) \\ &= M_p(2^p - 2) \cdot 2^p M_p \\ &= M_p(2^{p-1} - 1) \cdot 2^{p+1} M_p \\ (M_p^2)^2 &\equiv M_p^2 \pmod{2^{p+1}M_p}. \end{aligned}$$

Idempotence of 2^{2p} in $2^{p+1}M_p$.

$$\begin{aligned} (2^{2p})^2 - 2^{2p} &= 2^{2p}(2^{2p} - 1) \\ &= 2^{2p}(2^p + 1)(2^p - 1) \\ &= 2^{p-1}(2^p + 1) \cdot 2^{p+1} M_p \\ (2^{2p})^2 &\equiv 2^{2p} \pmod{2^{p+1}M_p}. \end{aligned}$$ □

The sequence $\{2^{p+1}M_p\}$ is A064591: Nonunitary perfect numbers: n is the sum of its nonunitary divisors; i.e., $\sigma(n) - \sigma^u(n) = n$. Recall that a divisor d of n is nonunitary if both d and n/d have a common factor. Unitary perfect numbers are in A002827.

Proof of Theorem 4.4. $2^{2p+1}M_p^2$ has spectral basis $\{M_p^2(M_p+2)^2, (M_p^2-1)^2\}$. Table 3 shows the first six $2^{2p+1}M_p^2$ and their spectral coefficients.

p	n	$2^{2p+1}M_p^2$	a	b
2	288	$(2)^5(3)^2$	$(5)^2$	(2)
3	6272	$(2)^7(7)^2$	$(3)^4$	$(2)(3)^2$

5	1968128	$(2)^{11}(31)^2$	$(33)^2$	$(2)(3)^2(5)^2$
7	528515072	$(2)^{15}(127)^2$	$(129)^2$	$(2)(3)^4(7)^2$
13	9005000365703168	$(2)^{27}(8191)^2$	$(8193)^2$	$(2)(3)^4(5)^2(7)^2(13)^2$

Table 3: Spectral coefficients of $2^{2p+1}M_p^2$.

Note that $a = (M_p + 2)^2 = (2^p + 1)^2$, so we take the equation

$$a \cdot M_p^2 + b \cdot 2^{2p+1} = 2^{2p+1}M_p^2 + 1,$$

and solve for b. Thus,

$$\begin{aligned}
(M_p + 2)^2 M_p^2 + b \cdot 2^{2p+1} &= 2^{2p+1}M_p^2 + 1 \\
2^{2p+1}b &= 2^{2p+1}M_p^2 - (M_p + 2)^2 M_p^2 + 1 \\
&= 2^{2p+1}(2^p - 1)^2 - (2^p + 1)^2(2^p - 1)^2 + 1 \\
&= 2^{2p+1}(2^p - 1)^2 - (2^{2p} - 1)^2 + 1 \\
&= 2^{2p+1}(2^p - 1)^2 - (2^{2p} - 1)^2 + 1 \\
&= 2^{2p+1}(2^{2p} - 2^{p+1} + 1) - (2^{4p} - 2^{2p+1} + 1) + 1 \\
&= 2^{4p+1} - 2^{3p+2} + 2^{2p+1} - 2^{4p} + 2^{2p+1} - 1 + 1 \\
&= 2^{4p+1} - 2^{4p} - 2^{3p+2} + 2^{2p+2} \\
2^{2p+1}b &= 2^{2p+1}(2^{2p} - 2^{2p-1} - 2^{p+1} + 2) \\
b &= 2^{2p} - 2^{2p-1} - 2^{p+1} + 2 \\
&= 2^{2p-1}(2 - 1) - 2^{p+1} + 2 \\
&= 2^{2p-1} - 2^{p+1} + 2 \\
&= 2(2^{2p-2} - 2 \cdot 2^{p-1} + 1) \\
&= 2((2^{p-1})^2 - 2 \cdot 2^{p-1} + 1) \\
b &= 2(2^{p-1} - 1)^2.
\end{aligned}$$

The spectral coefficients of $2^{2p+1}M_p^2$ are

$$\{(2^p + 1)^2, 2(2^{p-1} - 1)^2\}$$

or, equivalently,

$$\{(M_p + 2)^2, (M_p - 1)^2/2\}.$$

Since $2^{2p+1} = 2(M_p + 1)^2$, the spectral basis is

$$\{(M_p + 2)^2 \cdot M_p^2, (M_p - 1)^2/2 \cdot 2(M_p + 1)^2\},$$

or, equivalently, $\{M_p^2(M_p + 2)^2, (M_p^2 - 1)^2\}$ is the spectral basis for $2^{2p+1}M_p^2$. □

See Section 6 for power-spectral numbers with three prime factors involving the Mersenne and Fermat primes.

i	$f_i = 2^i$	$F_i = 2^{f_i} + 1$
0	1	3
1	2	5
2	4	17
3	8	257
4	16	65537

Table 4: The only known Fermat primes.

3 Fermat I

It is easy to prove the following:

Theorem 5. *The integer $n^a + 1$ is possibly prime only for $n = 2$ and $a = 2^i$, $i \geq 0$.*

Definition 6 (Fermat number, Fermat prime)**.** A *Fermat number* is a positive integer of the form $F_i = 2^{f_i} + 1$, $f_i = 2^i$, where i is a nonnegative integer. A *Fermat prime* is a Fermat number that is also prime. See Table 4 for the list of the only known Fermat primes.

Observe that Fermat primes can be regarded as dual to the Mersenne primes $M_p = 2^p - 1$, p an odd prime. The general case in this section is $2^s F_i^t$ where F_i is a Fermat prime.

The following identities will be useful in what follows.

Theorem 7. *The Fermat numbers satisfy the following recurrence relations*

$$F_i = (F_{i-1} - 1)^2 + 1 \qquad (i \geq 1), \tag{7a}$$
$$F_i = F_0 \cdots F_{i-1} + 2 \qquad (i \geq 1), \tag{7b}$$
$$F_i = F_{i-1} + 2^{2^{i-1}} F_0 \cdots F_{i-2} \qquad (i \geq 2), \tag{7c}$$
$$F_i = F_{i-1}^2 - 2(F_{i-2} - 1)^2 \qquad (i \geq 2). \tag{7d}$$

Theorem 8. *If $F_i = 2^{f_i} + 1$ is a Fermat prime with exponent $f_i = 2^i$, $i \geq 0$, then*

1. *$2^{f_i} F_i$ has spectral basis $\{F_i, 2^{2f_i}\}$.*
2. *$2^{f_i+1} F_i$ has spectral basis $\{F_i^2, 2^{2f_i}\}$.*
3. *$2^{2f_i+1} F_i^2$ has spectral basis $\{(F_i - 2)^2 F_i^2, (F_i^2 - 1)^2\}$.*

We will treat each case separately.

Proof of Theorem 8.1. $\{F_i, 2^{2f_i}\}$ is the spectral basis for $2^{f_i} F_i$. Table 5 shows the spectral coefficients of $2^{f_i} F_i$.

i	n	$2^{f_i}F_i$	a	b
0	6	$(2)(3)$	(1)	(2)
1	20	$(2)^2(5)$	(1)	$(2)^2$
2	272	$(2)^4(17)$	(1)	$(2)^4$
3	65792	$(2)^8(257)$	(1)	$(2)^8$
4	4295032832	$(2)^{16}(65537)$	(1)	$(2)^{16}$

Table 5: Spectral coefficients of $2^{f_i}F_i$.

Idempotence of F_i with respect to $2^{f_i}F_i$:

$$\begin{aligned} F_i^2 - F_i &= (F_i - 1)F_i = 2^{f_i}F_i \\ F_i^2 &\equiv F_i \pmod{2^{f_i}F_i}. \end{aligned} \tag{8}$$

Idempotence of 2^{2f_i} with respect to $2^{f_i}F_i$:

$$\begin{aligned} (2^{2f_i})^2 - 2^{2f_i} &= 2^{2f_i}(2^{2f_i} - 1) \\ &= 2^{2f_i}(2^{f_i} - 1)(2^{f_i} + 1) \\ &= 2^{f_i}(2^{f_i} - 1) \cdot 2^{f_i}F_i \\ (2^{2f_i})^2 &\equiv 2^{2f_i} \pmod{2^{f_i}F_i}. \end{aligned} \tag{9}$$

Decomposition of the identity for $2^{f_i}F_i$:

$$\begin{aligned} F_i + 2^{2f_i} &= (2^{f_i} + 1) + 2^{2f_i} \\ &= 2^{2f_i} + 2^{f_i} + 1 \\ &= 2^{f_i}(2^{f_i} + 1) + 1 \\ &= 2^{f_i}F_i + 1 \\ F_i + 2^{2f_i} &\equiv 1 \pmod{2^{f_i}F_i}. \end{aligned}$$

□

Proof of Theorem 8.2. $\{F_i^2, 2^{2f_i}\}$ is the spectral basis for $2^{f_i+1}F_i$. Table 6 shows the spectral coefficients of $2^{f_i+1}F_i$.

Idempotence of F_i^2 with respect to $2^{f_i+1}F_i$:

$$\begin{aligned} (F_i^2)^2 - F_i^2 &= F_i^2(F_i^2 - 1) \\ &= F_i^2(F_i + 1)(F_i - 1) \\ &= F_i(2^{f_i} + 2) \cdot 2^{f_i}F_i \\ &= F_i(2^{f_i-1} + 1) \cdot 2^{f_i+1}F_i \end{aligned}$$

Therefore,

$$(F_i^2)^2 \equiv F_i^2 \pmod{2^{f_i+1}F_i}.$$

i	n	$2^{f_i+1}F_i$	a	b
0	12	$(2)^2(3)$	(3)	(1)
1	40	$(2)^3(5)$	(5)	(2)
2	544	$(2)^5(17)$	(17)	$(2)^3$
3	131584	$(2)^9(257)$	(257)	$(2)^7$
4	8590065664	$(2)^{17}(65537)$	(65537)	$(2)^{15}$

Table 6: Spectral coefficients of $2^{f_i+1}F_i$.

Idempotence of 2^{2f_i} with respect to $2^{f_i+1}F_i$:

$$\begin{aligned}(2^{2f_i})^2 - 2^{2f_i} &= 2^{2f_i}(2^{2f_i} - 1)\\ &= 2^{2f_i}(2^{f_i} - 1)(2^{f_i} + 1)\\ &= 2^{f_i}(2^{f_i} - 1) \cdot 2^{f_i}F_i\\ &= 2^{f_i-1}(2^{f_i} - 1) \cdot 2^{f_i+1}F_i\end{aligned}$$

Therefore,

$$(2^{2f_i})^2 \equiv 2^{2f_i} \pmod{2^{f_i+1}F_i}.$$

Decomposition of the identity with respect to $2^{f_i+1}F_i$:

$$\begin{aligned}F_i^2 + 2^{2f_i} &= (2^{f_i} + 1)^2 + 2^{2f_i}\\ &= 2^{2f_i} + 2^{f_i+1} + 1 + 2^{2f_i}\\ &= 2^{2f_i+1} + 2^{f_i+1} + 1\\ &= 2^{f_i+1}(2^{f_i} + 1) + 1\\ &= 2^{f_i+1}F_i + 1\end{aligned}$$

Therefore,

$$F_i^2 + 2^{2f_i} \equiv 1 \pmod{2^{f_i+1}F_i}.$$

Proof of Theorem 8.3. $2^{2f_i+1}F_i^2$ has spectral basis $\{(F_i-2)^2F_i^2, (F_i^2-1)^2\}$. Table 7 shows the spectral coefficients for $2^{2f_i+1}F_i^2$.

i	n	$2^{2f_i+1}F_i^2$	a_i	b_i
0	72	$(2)^3(3)^2$	(1)	$(2)^3$
1	800	$(2)^5(5)^2$	$(3)^2$	$(2)(3)^2$
2	147968	$(2)^9(17)^2$	$(3)^2(5)^2$	$(2)(3)^4$
3	8657174528	$(2)^{17}(257)^2$	$(3)^2(5)^2(17)^2$	$(2)(3)^2(43)^2$
4	36894614055915880448	$(2)^{33}(65537)^2$	$(3)^2(5)^2(17)^2(257)^2$	$(2)(3)^4(11)^2(331)^2$

Table 7: Spectral coefficients of $2^{2f_i+1}F_i^2$.

Observe that $a_0 = 1$ and $a_i = F_0^2 \cdots F_{i-1}^2$ for $i > 0$. The decomposition of the identity for $2^{2f_i+1}F_i^2$ is

$$\begin{aligned} F_0^2 \cdots F_i^2 + B_i &= 2^{2f_i+1}F_i^2 + 1 \\ B_i &= 2^{2f_i+1}F_i^2 - F_0^2 \cdots F_i^2 + 1 \end{aligned}$$

It is easily checked that $B_0 = 2^6 = (3^2 - 1)^2$. Assume $i \geq 1$. Then

$$\begin{aligned} B_i &= (2 \cdot (2^{f_i})^2 - F_0^2 \cdots F_{i-1}^2)F_i^2 + 1 \\ &= (2 \cdot (F_i - 1)^2 - (F_i - 2)^2)F_i^2 + 1 \quad \text{(using (7b))} \\ &= (2F_i^2 - 4F_i + 2 - F_i^2 + 4F_i - 4)F_i^2 + 1 \\ &= (F_i^2 - 2)F_i^2 + 1 \\ &= F_i^4 - 2F_i^2 + 1 \\ B_i &= (F_i^2 - 1)^2 \end{aligned}$$

The spectral basis of $2^{2f_i+1}F_i^2$ is

$$\left\{F_0^2 \cdots F_i^2, (F_i^2 - 1)^2\right\}.$$

Using (7b) this can be written as $\{(F_i - 2)^2F_i^2, (F_i^2 - 1)^2\}$. □

Recall from Theorem 4.4 that $2^{2p+1}M_p^2$ has spectral basis

$$\left\{M_p^2(M_p + 2)^2, (M_p^2 - 1)^2\right\}.$$

This is an example of what will now be known as *Mersenne-Fermat duality* with respect to power-spectral bases. The fact that there are only five known Fermat primes limits its appearance.

4 Odd numbers of the form pq

The general case in this section is $n = pq$ with p and q odd primes. See Section 5 for the general case $p^s q^t$ $s, t \geq 1$.

Theorem 9. *Given p an odd prime and $k \geq 3$ an integer, the integer pq has power-spectral basis $\{q, p^k\}$ if and only if $q = (p^k - 1)/(p - 1)$ is prime.*

Remark 10. If $q = (p^k - 1)/(p - 1)$, then $q = 1$ for $k = 1$ and $q = p + 1$ for $k = 2$, so that $k \geq 3$ whenever p is an odd prime. If $p = 2$, then $q = 2^k - 1$ is a Mersenne prime and k a Mersenne exponent. See Theorem 4.1.

Proof. If pq has spectral basis $\{q, p^k\}$, then it has the decomposition of the identity

$$\begin{aligned} q + p^k &= pq + 1 \\ p^k - 1 &= pq - q \\ (p-1)q &= p^k - 1 \\ q &= \frac{p^k - 1}{p - 1}, \quad (k \geq 3). \end{aligned}$$

Now suppose that $q = (p^k - 1)/(p - 1)$, $k \geq 3$, is prime. Then we automatically have the equation

$$q + p^k = pq + 1. \tag{10}$$

Now let us prove the idempotent properties of a spectral basis. Observe that

$$\begin{aligned} p^{2k} - p^k &= p^k(p^k - 1) \\ &= p^k(q - 1)q \\ &= p^{k-1}(p - 1) \cdot pq \\ p^k &\equiv p^k \pmod{pq}. \end{aligned}$$

Further observe that

$$\begin{aligned} q^2 - q &= q(q - 1) \\ &= (\frac{p^k - 1}{p - 1} - 1)q \\ &= \frac{p^k - p}{p - 1} \cdot q \\ &= \frac{p^{k-1} - 1}{p - 1} \cdot pq \quad (k \geq 3) \\ q^2 &\equiv q \pmod{pq}. \end{aligned}$$

Thus, $\{q, p^k\}$ is the spectral basis for pq. ∎

Table 8 shows all power-spectral numbers of the form pq less than 10^6, including Sections 2 and 4.

	n	$(p)(q)$	(q)	p^{r^e}
1.	6	(2)(3)	(3)	$(2)^2$
2.	14	(2)(7)	(7)	$(2)^3$
3.	39	(3)(13)	(13)	$(3)^3$
4.	62	(2)(31)	(31)	$(2)^5$
5.	155	(5)(31)	(31)	$(5)^3$
6.	254	(2)(127)	(127)	$(2)^7$
7.	3279	(3)(1093)	(1093)	$(3)^7$
8.	5219	(17)(307)	(307)	$(17)^3$
9.	16382	(2)(8191)	(8191)	$(2)^{13}$
10.	19607	(7)(2801)	(2801)	$(7)^5$
11.	70643	(41)(1723)	(1723)	$(41)^3$
12.	97655	(5)(19531)	(19531)	$(5)^7$
13.	208919	(59)(3541)	(3541)	$(59)^3$
14.	262142	(2)(131071)	(131071)	$(2)^{17}$
15.	363023	(71)(5113)	(5113)	$(71)^3$
16.	402233	(13)(30941)	(30941)	$(13)^5$
17.	712979	(89)(8011)	(8011)	$(89)^3$

Table 8: Power spectral numbers of the form pq.

See Table 38 in Subsection 11.3 for a list of cyclotomic primes.

5 Cyclotomic Primes

The following is from Ireland-Rosen, [3, p. 193-4].

Let n be a positive integer and let $\zeta_n = e^{2\pi i/n}$, denoted by ζ when there is no ambiguity. The number ζ is called an nth root of unity since it clearly is a root of the equation $x^n - 1 = 0$, as are all powers of ζ. Thus, we have

$$x^n - 1 = (x-1)(x-\zeta)\cdots(x-\zeta^{n-1}).$$

Definition 11. The nth cyclotomic polynomial $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{(a,n)=1} (z - \zeta^a), \quad (1 \le a \le n).$$

The roots of $\Phi_n(x)$ are precisely the primitive n roots of unity, i.e., those nth roots of unity of order n. Clearly the degree of $\Phi_n(x)$ is $\phi(n)$, where ϕ is Euler ϕ-function. Theorem 12 might be called the Fundamental Theorem of Cyclotomic Polynomials.

Theorem 12. *Let n be a positive integer. Then $x^n - 1 = \prod_{d|n} \Phi_d(x)$, $\Phi_n(x) \in \mathbb{Z}[x]$, and $\Phi_n(x)$ is irreducible in $\mathbb{Z}[x]$.*

Proof. See Proposition 13.2.2 and its corollary, and Theorem 13.1 in [3]. □

If we denote by $\tau(n)$ the number of divisors of n, then it can be shown that $\tau(p^e) = e+1$, $\tau(1) = 1$, and that τ is multiplicative, that is, $\tau(p_1^{e_1} \cdots p_k^{e_k}) = \tau(p_1^{e_1}) \cdots \tau(p_k^{e_k})$. Thus, $\tau(n) - \tau(m) = 1$, where m is a divisor of n, precisely when $n = p^e$ and $m = p^{e-1}$. Now let m be a divisor of n and observe that

$$\frac{x^n - 1}{x^m - 1} = \frac{\prod_{d|n} \Phi_d(x)}{\prod_{d|m} \Phi_d(x)} = \prod_{d|n,(d,m)=1} \Phi_d(x).$$

But $\prod_{d|n,(d,m)=1} \Phi_d(x)$ can be prime only if there is just one divisor of n but not m, and this happens only if $n = r^e$ and $m = r^{e-1}$, r prime. Hence, we have

$$\frac{x^{r^e} - 1}{x^{r^{e-1}} - 1} = \Phi_{r^e}(x),$$

and for each prime power r^e we are interested in those primes p such that $\Phi_{r^e}(p)$ is prime. We shall refer to $\Phi_{r^e}(p)$ as a *cyclotomic prime*. The following theorem generalizes Theorem 9.

Theorem 13. *Let p, q, r be primes and e a nonnegative integer. Then the integer $p^{r^e}q$ is power spectral with spectral basis $\{q, p^{r^{e+1}}\}$ if and only if $q = \Phi_{r^{e+1}}(p)$ is prime, where $\Phi_{r^{e+1}}(x)$ is the r^{e+1}-th cyclotomic polynomial.*

Remark 14. Observe that $\Phi_r(2) = (2^r - 1)/(2 - 1) = 2^r - 1$ prime for r prime yields the Mersenne primes. Thus, $\Phi_{r^e}(p)$, $e \geq 1$, being prime for p prime provide a generalization of Mersenne primes to odd primes.

Remark 15. This theorem also accounts for the Fermat prime examples. For example, by 8.1 we have $2^{f_i}F_i$, $f_i = 2^i$, $i = 0...4$, with spectral basis $\{F_i, 2^{2f_i}\}$, where $F_i = \Phi_{2f_i}(2)$, since $\Phi_{2f_i}(x) = x^{f_i} + 1 = x^{2^i} + 1$.

Remark 16. It should come as no surprise that primes of the form $\Phi_{r^e}(p)$ have been considered before. In Bateman [1] a table of primes of the form $(p^r - 1)/(p^d - 1)$ where p is a prime and r and d are positive integers (their notation and terminology) has been compiled. Their TABLE II (p154-5) has columns q and p^r where r is prime so that necessarily $d = 1$. Thus, by Theorem 13 the number pq has spectral basis $\{q, p^r\}$. However, no mention of power-spectral numbers occurs in their paper. See Table 38 in Subsection 11.3 for a list of cyclotomic primes.

6 Mersenne II

Theorem 17. *Let M_p is a Mersenne prime with Mersenne exponent $p > 2$. Then*

1. $2^{2p-1} \cdot 3 \cdot M_p^2$ *has power-spectral basis*
$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$$
of index 2.

2. $2^{2p} \cdot 3 \cdot M_p^2$ *has power-spectral basis*
$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

3. $2^{2p+1} \cdot 3 \cdot M_p^2$ *has power-spectral basis*
$$\left\{M_p^2\left(M_p+2\right)^2, 4M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

The numbers 1 and 2 comprise an isospectral pair. See Section 8.

Remark 18. Recall that $2^{odd} \equiv 8 \pmod{12}$ so $2^{odd} + 1 \equiv 9 \pmod{12}$. Therefore, $2^p + 1$ is divisible by 3. Recall that $2^{even} \equiv 4 \pmod{12}$ for *even* ≥ 2 so $2^{even} - 1 \equiv 3 \pmod{12}$. Therefore, $2^{p-1} - 1$ is divisible by 3.

Proof of 17.1. Table 9 shows the spectral coefficients of $2^{2p-1} \cdot 3 \cdot M_p^2$.

p	n	$2^{2p-1} \cdot 3 \cdot M_p^2$	a	b	c
3	4704	$(2)^5(3)(7)^2$	$(3)^3$	(2)	$(2)^3(3)$
5	1476096	$(2)^9(3)(31)^2$	$(3)(11)^2$	(2)	$(2)^3(3)(5)^2$
7	396386304	$(2)^{13}(3)(127)^2$	$(3)(43)^2$	(2)	$(2)^3(3)^3(7)^2$
13	6753750274277376	$(2)^{25}(3)(8191)^2$	$(3)(2731)^2$	(2)	$(2)^3(3)^3(5)^2(7)^2(13)^2$

Table 9: Power-spectral coefficients of $2^{2p-1} \cdot 3 \cdot M_p^2$, $p > 2$.

The perceived pattern in the spectral coefficients is $a = 3(r+1)^2$, $M_p = 3r+1$, $b = 2$, and c is to be determined. The decomposition of the identity for $2^{2p-1} \cdot 3 \cdot M_p^2$ with index 2 is

$$\begin{aligned} a \cdot 3 \cdot M_p^2 + b \cdot 2^{2p-1} M_p^2 + c \cdot 2^{2p-1} \cdot 3 &= 2 \cdot 2^{2p-1} \cdot 3 \cdot M_p^2 + 1 \\ 3(r+1)^2 \cdot 3 \cdot M_p^2 + 2 \cdot 2^{2p-1} M_p^2 + c \cdot 2^{2p-1} \cdot 3 &= 2^{2p} \cdot 3 \cdot M_p^2 + 1 \\ 9(r+1)^2 M_p^2 + c \cdot 2^{2p-1} \cdot 3 &= 2^{2p} \cdot 2 \cdot M_p^2 + 1 \\ (M_p+2)^2 M_p^2 + c \cdot 2^{2p-1} \cdot 3 &= 2^{2p+1} M_p^2 + 1 \end{aligned}$$

$$3 \cdot 2^{2p-1}c = 2^{2p+1}M_p^2 - (M_p+2)^2 M_p^2 + 1$$

Substituting $M_p = 2^p - 1$ and simplifying we obtain

$$\begin{aligned} 3 \cdot 2^{2p-1}c &= 2^{2p+1}\left(2^{2p} - 2^{2p-1} - 2^{p+1} + 2\right) \\ c &= 2^2\left(2^{2p} - 2^{2p-1} - 2^{p+1} + 2\right) \\ &= 2^2 \cdot 2\left(2^{p-1} - 1\right)^2 \\ 3c &= 2^3\left(2^{p-1} - 1\right)^2 \\ c &= 24\left(\frac{2^{p-1} - 1}{3}\right)^2. \end{aligned}$$

Thus, the spectral coefficients of $2^{2p-1} \cdot 3 \cdot M_p^2$ are

$$\begin{aligned} &\left\{\frac{(2^p+1)^2}{3}, 2, 24\left(\frac{2^{p-1}-1}{3}\right)^2\right\}, \\ &\left\{\frac{(M_p+2)^2}{3}, 2, 24\left(\frac{2^{p-1}-1}{3}\right)^2\right\}, \end{aligned}$$

and the power-spectral basis is

$$\begin{aligned} &\left\{\frac{(M_p+2)^2}{3} \cdot 3 \cdot M_p^2, 2 \cdot 2^{2p-1}M_p^2, 24\left(\frac{2^{p-1}-1}{3}\right)^2 \cdot 2^{2p-1} \cdot 3\right\}, \\ &\left\{M_p^2(M_p+2)^2, 2^{2p}M_p^2, 2^3 \cdot 2^{2p-1}\left(2^{p-1}-1\right)^2\right\}, \\ &\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, 2^{2p+2}\left(2^{p-1}-1\right)^2\right\}, \\ &\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, 2^{2p}\left(2^p-2\right)^2\right\}, \\ &\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p+1)^2\left(M_p-1\right)^2\right\}. \end{aligned}$$

Therefore, the power-spectral basis of $2^{2p-1} \cdot 3 \cdot M_p^2$ is

$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

□

Proof of 17.2. Table 10 shows the spectral coefficients of $2^{2p} \cdot 3 \cdot M_p^2$.

p	n	$2^{2p} \cdot 3 \cdot M_p^2$	a	b	c
3	9408	$(2)^6(3)(7)^2$	$(3)^3$	(1)	$(2)^2(3)$
5	2952192	$(2)^{10}(3)(31)^2$	$(3)(11)^2$	(1)	$(2)^2(3)(5)^2$
7	792772608	$(2)^{14}(3)(127)^2$	$(3)(43)^2$	(1)	$(2)^2(3)^3(7)^2$
13	13507500548554752	$(2)^{26}(3)(8191)^2$	$(3)(2731)^2$	(1)	$(2)^2(3)^3(5)^2(7)^2(13)^2$

Table 10: Power-spectral coefficients of $2^{2p} \cdot 3 \cdot M_p^2$, $p > 2$.

Note that the spectral coefficient a of Table 10 is the same as in Table 9. Thus, with $a = 3(r+1)^2$, $M_p = 3r+1$, $b = 1$, and c to be determined, the decomposition of the identity for $2^{2p} \cdot 3 \cdot M_p^2$ with index 1 is

$$a \cdot 3 \cdot M_p^2 + b \cdot 2^{2p-1} M_p^2 + c \cdot 2^{2p} \cdot 3 = 2^{2p} \cdot 3 \cdot M_p^2 + 1$$

The calculation now proceeds similarly to the previous one, and we obtain

$$c = 12 \left(\frac{2^{p-1} - 1}{3} \right)^2 .$$

The power-spectral coefficients of $2^{2p} \cdot 3 \cdot M_p^2$ are

$$\left\{ \frac{(2^p + 1)^2}{3}, 1, 12 \left(\frac{2^{p-1} - 1}{3} \right)^2 \right\}$$

and the power-spectral basis of $2^{2p} \cdot 3 \cdot M_p^2$ is

$$\left\{ M_p^2 (M_p + 2)^2, M_p^2 (M_p + 1)^2, (M_p^2 - 1)^2 \right\} .$$

Observe that this power-spectral basis is identical to that of $2^{2p-1} \cdot 3 \cdot M_p^2$. That is, $2^{2p-1} \cdot 3 \cdot M_p^2$ and $2^{2p} \cdot 3 \cdot M_p^2$ are an isospectral pair. See Section 8 for a discussion on isospectral chains. □

Proof of Theorem 17.3. The spectral coefficients of $2^{2p+1} \cdot 3 \cdot M_p^2$ are given in Table 11.

p	n	$2^{2p+1} \cdot 3 \cdot M_p^2$	a	b	c
3	18816	$(2)^7(3)(7)^2$	$(3)^3$	(2)	$(2)(3)$
5	5904384	$(2)^{11}(3)(31)^2$	$(3)(11)^2$	(2)	$(2)(3)(5)^2$
7	1585545216	$(2)^{15}(3)(127)^2$	$(3)(43)^2$	(2)	$(2)(3)^3(7)^2$
13	27015001097109504	$(2)^{27}(3)(8191)^2$	$(3)(2731)^2$	(2)	$(2)(3)^3(5)^2(7)^2(13)^2$

Table 11: Power-spectral coefficients of $2^{2p+1} \cdot 3 \cdot M_p^2$, $p > 2$.

The spectral coefficients are $a = 3(r+1)^2$, $M_p = 3r+1$, $b = 2$, and c is to be determined. The calculation proceeds similarly to the previous two and so we obtain the power-spectral coefficients to be

$$\left\{3\left(\frac{2^p+1}{3}\right)^2, 2, 6\left(\frac{2^{p-1}-1}{3}\right)^2\right\}.$$

Therefore, the power-spectral basis of $2^{2p+1} \cdot 3 \cdot M_p^2$ is

$$\left\{M_p^2\left(M_p+2\right)^2, 4M_p^2(M_p+1)^2, \left(M_p^2-1\right)^2\right\}$$ □

Theorem 19. *Let M_p be a Mersenne prime with Mersenne exponent $p > 2$. Then*

1. $2^{2p-3} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, \frac{1}{4}M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$$

of index 2.

2. $2^{2p-2} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, \frac{1}{4}M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

3. $2^{2p+1} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, 16M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

Furthermore, the numbers 1 and 2 comprise an isospectral pair. See Section 8.

Proof of Theorem 19.1. Table 12 shows the spectral coefficients of $2^{2p-3} \cdot 3^2 \cdot M_p^2$.

p	n	$2^{2p-3} \cdot 9 \cdot M_p^2$	a	b	c
3	3528	$(2)^3(3)^2(7)^2$	(1)	(2)	$(2)^5$
5	1107072	$(2)^7(3)^2(31)^2$	$(11)^2$	(2)	$(2)^5(5)^2$
7	297289728	$(2)^{11}(3)^2(127)^2$	$(43)^2$	(2)	$(2)^5(3)^2(7)^2$
13	5065312705708032	$(2)^{23}(3)^2(8191)^2$	$(2731)^2$	(2)	$(2)^5(3)^2(5)^2(7)^2(13)^2$

Table 12: Power-spectral coefficients of $2^{2p-3} \cdot 9 \cdot M_p^2$, $p > 2$.

A bit of fiddling reveals that the spectral coefficients of $2^{2p-3} \cdot 3^2 \cdot M_p^2$ for $p > 2$ are given by $\{(r+1)^2, 2, c\}$, where r is defined by $M_p = 3r + 1$ and c is to be determined. The decomposition of the identity with index 2 takes the form

$$\left(\frac{2^p+1}{3}\right)^2 \cdot 9 \cdot M_p^2 + 2 \cdot 2^{2p-3} \cdot M_p^2 + C = 2 \cdot 2^{2p-3} \cdot 3^2 \cdot M_p^2 + 1$$

where C is to be determined. Thus,

$$\begin{aligned}
(2^p+1)^2 M_p^2 + C &= 16 \cdot 2^{2p-3} M_p^2 + 1 \\
(M_p+2)^2 M_p^2 + C &= 2^{2p+1} M_p^2 + 1 \\
(M_p+2)^2 M_p^2 + C &= 2(M_p+1)^2 M_p^2 + 1 \\
C &= 2(M_p+1)^2 M_p^2 - (M_p+2)^2 M_p^2 + 1 \\
&= [2(M_p+1)^2 - (M_p+2)^2] M_p^2 + 1 \\
&= [2M_p^2 + 4M_p + 2 - M_p^2 - 4M_p - 4] M_p^2 + 1 \\
&= [M_p^2 - 2] M_p^2 + 1 \\
&= M_p^4 - 2M_p^2 + 1 \\
C &= (M_p^2 - 1)^2.
\end{aligned}$$

The spectral decomposition of $2^{2p-3} \cdot 3^2 \cdot M_p^2$ is

$$\left\{(M_p+2)^2 M_p^2, \frac{1}{4}(M_p+1)^2 M_p^2, (M_p^2-1)^2\right\}$$

In terms of 2^p, p a Mersenne exponent, the spectral basis is

$$\left\{(2^{2p}-1)^2, 2^{2p-2}, (2^{2p}-2^{p+1})^2\right\}.$$

Let us verify that $(M_p+2)^2 M_p^2$ is an idempotent for $2^{2p-3} \cdot 3^2 \cdot M_p^2$. Thus,

$$\begin{aligned}
\left((M_p+2)^2 M_p^2\right)^2 - (M_p+2)^2 M_p^2 &= \left((M_p+2)^4 M_p^2 - (M_p+2)^2\right) M_p^2 \\
&= M_p^2 (M_p+2)^2 \left((M_p+2)^2 M_p^2 - 1\right) \\
&= M_p^2 (M_p+2)^2 \left((M_p+2)M_p - 1\right)\left((M_p+2)M_p + 1\right) \\
&= M_p^2 (M_p+2)^2 \left((2^p+1)(2^p-1) - 1\right)\left(M_p^2 + 2M_p + 1\right) \\
&= M_p^2 (M_p+2)^2 \left(2^{2p} - 2\right)(M_p+1)^2 \\
&= 2(2^p+1)^2 \left(2^{2p} - 2\right) \cdot 2^{2p} M_p^2
\end{aligned}$$

Since $2^{\text{odd}} \equiv 2 \pmod 6$, $2^{\text{odd}} + 1 \equiv 3 \pmod 6$, so we have $(2^p+1)^2 = 3^2 L_p^2$ and

$$\left((M_p+2)^2 M_p^2\right)^2 - (M_p+2)^2 M_p^2 = 8L_p^2\left(2^{2p}-2\right) \cdot 2^{2p-3} \cdot 9 \cdot M_p^2$$

Therefore,

$$\left((M_p+2)^2 M_p^2\right)^2 - (M_p+2)^2 M_p^2 \equiv 0 \pmod{2^{2p-3} \cdot 9 \cdot M_p^2}. \qquad \square$$

Proof of Theorem 19.2. Table 13 shows the spectral coefficients of $2^{2p-2} \cdot 9 \cdot M_p^2$.

p	n	$2^{2p-2} \cdot 9 \cdot M_p^2$	a	b	c
3	7056	$(2)^4(3)^2(7)^2$	$(3)^2$	(1)	$(2)^4$
5	2214144	$(2)^8(3)^2(31)^2$	$(11)^2$	(1)	$(2)^4(5)^2$
7	594579456	$(2)^{12}(3)^2(127)^2$	$(43)^2$	(1)	$(2)^4(3)^2(7)^2$
13	10130625411416064	$(2)^{24}(3)^2(8191)^2$	$(2731)^2$	(1)	$(2)^4(3)^2(5)^2(7)^2(13)^2$

Table 13: Power-spectral coefficients of $2^{2p-2} \cdot 9 \cdot M_p^2$, $p > 2$.

A bit of fiddling reveals that the spectral coefficients of $2^{2p-2} \cdot 3^2 \cdot M_p^2$ for $p > 2$ are given by $\{a^2, b, c^2\}$, where $a = r+1$, $b = 1$, $c = 2r$ and r is defined by $M_p = 3r+1$. Thus,

$$PSC : \left\{ \left(\frac{M_p+2}{3}\right)^2, 1, 2^2\left(\frac{M_p-1}{3}\right)^2 \right\}$$

$$PSB : \left\{ (M_p+2)^2 M_p^2, \quad \frac{1}{4}(M_p+1)^2 M_p^2, \quad \left(M_p^2-1\right)^2 \right\}$$

The PSB of $2^{2p-2} \cdot 3^2 \cdot M_p^2$ is also the PSB for $2^{2p-3} \cdot 3^2 \cdot M_p^2$. See section 8 for a discussion of isospectral chains. $\square$

Proof of Theorem 19.3. Table 14 shows the spectral coefficients of $2^{2p+1} \cdot 9 \cdot M_p^2$.

p	n	$2^{2p+1} \cdot 9 \cdot M_p^2$	a	b	c
3	56448	$(2)^7(3)^2(7)^2$	$(3)^2$	$(2)^3$	(2)
5	17713152	$(2)^{11}(3)^2(31)^2$	$(11)^2$	$(2)^3$	$(2)(5)^2$
7	4756635648	$(2)^{15}(3)^2(127)^2$	$(43)^2$	$(2)^3$	$(2)(3)^2(7)^2$
13	81045003291328512	$(2)^{27}(3)^2(8191)^2$	$(2731)^2$	$(2)^3$	$(2)(3)^2(5)^2(7)^2(13)^2$

Table 14: Power-spectral coefficients of $2^{2p+1} \cdot 9 \cdot M_p^2$, $p > 2$.

The spectral coefficient a of $2^{2p+1} \cdot 3^2 \cdot M_p^2$ are $(r+1)^2$, where $M_p = 3r+1$. Thus,

$$(r+1)^2 \cdot 3^2 \cdot M_p^2 + 2^3 \cdot 2^{2p+1} M_p^2 + C = 2^{2p+1} \cdot 3^2 \cdot M_p^2 + 1$$

and after some algebra we obtain

$$C = (M_p^2 - 1)^2.$$

The power-spectral basis of $2^{2p+1} \cdot 3^2 \cdot M_p^2$ is then

$$\left\{ \left(\frac{2^p + 1}{3} \right)^2 \cdot 3^2 \cdot M_p^2, 2^3 \cdot 2^{2p+1} \cdot M_p^2, (M_p^2 - 1)^2 \right\}.$$

After some simplification, we have

$$\left\{ (M_p + 2)^2 M_p^2, 16 M_p^2 (M_p + 1)^2, (M_p^2 - 1)^2 \right\}.$$

It is left to the reader to verify the projection properties. □

7 Fermat Primes II

Theorem 20. *Let F_i be a Fermat prime with exponent $f_i = 2^i$. Then the following numbers are power-spectral.*

1. $2^{2f_i - 1} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 \cdot F_i^2, (F_i^2 - 1)^2\}.$$

   with index 2.

2. $2^{2f_i} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\}.$$

3. $2^{2f_i + 1} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i - 2)^2 F_i^2, 4(F_i - 1)^2 \cdot F_i^2, (F_i^2 - 1)^2\}.$$

Furthermore, 1 and 2 form an isospectral pair. See Section 8.

Proof of Theorem 20.1. Table 15 shows the spectral coefficients of $2^{2f_i - 1} \cdot 3 \cdot F_i^2$.

n	$2^{2f_i-1} \cdot 3 \cdot F_i^2$	a	b	c

600	$(2)^3(3)(5)^2$	(3)	(2)	$(2)^3(3)$
110976	$(2)^7(3)(17)^2$	$(3)(5)^2$	(2)	$(2)^3(3)^3$
6492880896	$(2)^{15}(3)(257)^2$	$(3)(5)^2(17)^2$	(2)	$(2)^3(3)(43)^2$
27670960541936910336	$(2)^{31}(3)(65537)^2$	$(3)(5)^2(17)^2(257)^2$	(2)	$(2)^3(3)^3(11)^2(331)^2$

Table 15: Power-spectral coefficients of $2^{2f_i-1} \cdot 3 \cdot F_i^2$, $i = 1, 2, 3, 4$.

It is clear that we have

$$a = F_0 \cdots F_{i-1}^2 = \frac{1}{3}(F_i - 2)^2 \quad \text{(by (7b))},$$
$$b = 2,$$

The decomposition of the identity with index 2 is then

$$a \cdot 3 \cdot F_i^2 + b_i \cdot 2^{2f_i-1} \cdot F_i^2 + C = 2 \cdot 2^{2f_i-1} \cdot 3 \cdot F_i^2 + 1$$

where C is to be determined. Thus,

$$\begin{aligned}
\frac{1}{3}(F_i - 2)^2 \cdot 3 \cdot F_i^2 + 2 \cdot 2^{2f_i-1} \cdot F_i^2 + C &= 3 \cdot 2^{2f_i} \cdot F_i^2 + 1 \\
(F_i - 2)^2 F_i^2 + 2^{2f_i} \cdot F_i^2 + C &= 3 \cdot 2^{2f_i} \cdot F_i^2 + 1 \\
(F_i - 2)^2 F_i^2 + C &= 2 \cdot 2^{2f_i} F_i^2 + 1 \\
C &= 2(F_i - 1)^2 F_i^2 - (F_i - 2)^2 F_i^2 + 1 \\
&= 2(F_i^4 - 2F_i^3 + F_i^2) - (F_i^4 - 4F_i^3 + 4F_i^2) + 1 \\
&= F_i^4 - 2F_i^2 + 1 \\
C &= (F_i^2 - 1)^2
\end{aligned}$$

Consequently, the power-spectral basis of $2^{2f_i-1} \cdot 3 \cdot F_i^2$ is

$$\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 \cdot F_i^2, (F_i^2 - 1)^2\} \qquad \square$$

Proof of Theorem 20.2. Table 16 shows the spectral coefficients of $2^{2f_i} \cdot 3 \cdot F_i^2$.

n	$2^{2f_i} \cdot 3 \cdot F_i^2$	a	b	c
1200	$(2)^4(3)(5)^2$	(3)	(1)	$(2)^2(3)$
221952	$(2)^8(3)(17)^2$	$(3)(5)^2$	(1)	$(2)^2(3)^3$
12985761792	$(2)^{16}(3)(257)^2$	$(3)(5)^2(17)^2$	(1)	$(2)^2(3)(43)^2$
55341921083873820672	$(2)^{32}(3)(65537)^2$	$(3)(5)^2(17)^2(257)^2$	(1)	$(2)^2(3)^3(11)^2(331)^2$

Table 16: Power-spectral coefficients of $2^{2f_i} \cdot 3 \cdot F_i^2$, $i = 1, 2, 3, 4$.

It is clear that we have

$$a = F_0 \cdots F_1^2 \cdot F_{i-1}^2 = \frac{1}{3}(F_i - 2)^2 \quad \text{(by (7b))},$$
$$b = 1.$$

The decomposition of the identity of $2^{2f_i} \cdot 3 \cdot F_i^2$ with index 1 is

$$a \cdot 3 \cdot F_i^2 + b \cdot 2^{2f_i} \cdot F_i^2 + C = 2^{2f_i} \cdot 3 \cdot F_i^2 + 1$$

where C to be determined. Thus,

$$\begin{aligned} \frac{1}{3}(F_i - 2)^2 \cdot 3 \cdot F_i^2 + 1 \cdot 2^{2f_i} \cdot F_i^2 + C &= 3 \cdot 2^{2f_i} \cdot F_i^2 + 1 \\ (F_i - 2)^2 F_i^2 + C &= 2(F_i - 1)^2 F_i^2 + 1 \\ C &= 2(F_i - 1)^2 F_i^2 - (F_i - 2)^2 F_i^2 + 1 \\ &= (F_i - 1)^2. \end{aligned}$$

Consequently, the power-spectral basis of $2^{2f_i} \cdot 3 \cdot F_i^2$ is

$$\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\}$$

Note that this power-spectral basis the same as 20.1, so that 20.1 and 20.2 form an isospectral pair (8). Furthermore, by 17.2, the power-spectral basis of $2^{2p} \cdot 3 \cdot M_p^2$ is

$$\left\{M_p^2(M_p + 2)^2, M_p^2(M_p + 1)^2, (M_p^2 - 1)^2\right\}$$

This is an example of Mersenne-Fermat duality as in Theorem 20. □

Proof of 20.3. Table 17 shows the spectral coefficients of $2^{2f_i+1} \cdot 3 \cdot F_i^2$.

n	$2^{2f_i+1} \cdot 3 \cdot F_i^2$	a	b	c
2400	$(2)^5(3)(5)^2$	(3)	(2)	$(2)(3)$
443904	$(2)^9(3)(17)^2$	$(3)(5)^2$	(2)	$(2)(3)^3$
25971523584	$(2)^{17}(3)(257)^2$	$(3)(5)^2(17)^2$	(2)	$(2)(3)(43)^2$
110683842167747641344	$(2)^{33}(3)(65537)^2$	$(3)(5)^2(17)^2(257)^2$	(2)	$(2)(3)^3(11)^2(331)^2$

Table 17: Power-spectral coefficients of $2^{2f_i+1} \cdot 3 \cdot F_i^2$, $i = 1, 2, 3, 4$.

$$a = F_0 \cdot F_1^2 \cdot F_{i-1}^2, = \frac{1}{3}(F_i - 2)^2 \quad \text{(by (7b))},$$

$$b = 2.$$

The decomposition of the identity of $2^{2f_i+1} \cdot 3 \cdot F_i^2$ with index 1 is

$$a \cdot 3 \cdot F_i^2 + b \cdot 2^{2f_i+1} \cdot F_i^2 + C = 2^{2f_i+1} \cdot 3 \cdot F_i^2 + 1$$

where C to be determined. Thus,

$$\begin{aligned}
\frac{1}{3}(F_i-2)^2 \cdot 3 \cdot F_i^2 + 2 \cdot 2^{2f_i+1} \cdot F_i^2 + C &= 2^{2f_i+1} \cdot 3 \cdot F_i^2 + 1 \\
(F_i-2)^2 F_i^2 + 4(F_i-1)^2 F_i^2 + C &= 6(F_i-1)^2 F_i^2 + 1 \\
C &= 2(F_i-1)^2 F_i^2 - (F_i-2)^2 F_i^2 + 1 \\
C &= (F_i^2-1)^2.
\end{aligned}$$

Consequently, the power-spectral basis of $2^{2f_i+1} \cdot 3 \cdot F_i^2$ is

$$\{(F_i-2)^2 F_i^2, 4(F_i-1)^2 F_i^2, (F_i^2-1)^2\}.$$

By comparison, by Theorem 17.3 the power-spectral basis of $2^{2p+1} \cdot 3 \cdot M_p^2$ is

$$\left\{M_p^2\left(M_p+2\right)^2, 4M_p^2(M_p+1)^2, \left(M_p^2-1\right)^2\right\} \quad (p > 2).$$

This is an example of what can be called *Mersenne-Fermat duality.* □

Theorem 21. *Let F_i be a Fermat prime with Fermat exponent $f_i = 2^i$. Then*

1. $2^3 \cdot 9 \cdot 5^2$ *has power-spectral basis*

$$\left\{3^2 5^2, 2^3 5^3, 2^6 3^2\right\}.$$

2. $2^{2f_i-3} \cdot 9 \cdot F_i^2$ *has power-spectral basis*

$$\left\{(F_i-2)^2 F_i^2, \frac{1}{4}(F_i-1)^2 F_i^2, (F_i^2-1)^2\right\}.$$

   with index 2.

3. $2^{2f_i-2} \cdot 9 \cdot F_i^2$ *has power-spectral basis*

$$\left\{(F_i-2)^2 F_i^2, \frac{1}{4}(F_i-1)^2 F_i^2, (F_i^2-1)^2\right\}.$$

4. $2^{2f_i+1} \cdot 9 \cdot F_i^2$, *has power-spectral basis*

$$\left\{(F_i-2)^2 F_i^2, 16(F_i-1)^2 F_i^2, (F_i^2-1)^2\right\}.$$

Furthermore, the numbers of Theorem 2 and Theorem 3 form an isospectral pair for $i = 2, 3, 4$. *See Section 8 for a discussion of isospectral chains.*

Proof of 21.2. The power-spectral coefficients of $2^{2f_i-3} \cdot 9 \cdot F_i^2$, $i = 2, 3, 4$ are given in Table 18.

n	$2^{2f_i-3} \cdot 9 \cdot F_i^2$	a	b	c
83232	$(2)^5(3)^2(17)^2$	$(5)^2$	(2)	$(2)^5(3)^2$
4869660672	$(2)^{13}(3)^2(257)^2$	$(5)^2(17)^2$	(2)	$(2)^5(43)^2$
20753220406452682752	$(2)^{29}(3)^2(65537)^2$	$(5)^2(17)^2(257)^2$	(2)	$(2)^5(3)^2(11)^2(331)^2$

Table 18: Power-spectral coefficients of $2^{2f_i-3} \cdot 9 \cdot F_i^2$, $i = 2, 3, 4$.

It can be shown that if one defines r by $F_i = 3r - 1$ and $a = r - 1 = (2^{f_i} - 1)/3$ then the power-spectral coefficients are $\{a^2, 2, c\}$ where c is to be determined. The decomposition of the identity with index 2 is

$$\left(\frac{2^{f_i} - 1}{3}\right)^2 \cdot 9 \cdot F_i^2 + 2 \cdot 2^{2f_i-3} \cdot F_i^2 + C = 2 \cdot 2^{2f_i-3} \cdot 9 \cdot F_i^2 + 1$$

Solving for C one finds

$$C = (F_i^2 - 1)^2.$$

The power-spectral basis is

$$\left\{\left(2^{2f_i} - 1\right)^2, 2^{2f_i-2} \cdot F_i^2, (F_i^2 - 1)^2\right\}.$$

and in terms of the Fermat primes F_i is

$$\left\{(F_i - 2)^2 F_i^2, \frac{1}{4}(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$

□

Proof of 21.3. Power-spectral basis of $2^{2f_i-2} \cdot 9 \cdot F_i^2$, index 1.

n	$2^{2f_i-2} \cdot 9 \cdot F_i^2$	a	b	c
900	$(2)^2(3)^2(5)^2$	(1)	(1)	$(2)^4$
166464	$(2)^6(3)^2(17)^2$	$(5)^2$	(1)	$(2)^4(3)^2$
9739321344	$(2)^{14}(3)^2(257)^2$	$(5)^2(17)^2$	(1)	$(2)^4(43)^2$
41506440812905365504	$(2)^{30}(3)^2(65537)^2$	$(5)^2(17)^2(257)^2$	(1)	$(2)^4(3)^2(11)^2(331)^2$

Table 19: Power-spectral coefficients of $2^{2f_i-2}\cdot 9\cdot F_i^2$, $i=1,2,3,4$.

Since this is a special case of $9p^{2s}q^{2t}$ where $p=2$, with $s=f_i-1$, we let $2^{f_i-1}=3r-1$ so that $r=(2^{f_i-1}+1)/3$, $a=2r-1$, $c=4r$. Thus,

$$a=2\left(\frac{2^{f_i-1}+1}{3}\right)-1=\frac{2^{f_i}-1}{3}=\frac{F_i-2}{3}$$

and

$$c=4\left(\frac{2^{f_i-1}+1}{3}\right)=2\left(\frac{2^{f_i}+2}{3}\right)=2\left(\frac{F_i+1}{3}\right).$$

Thus, the power-spectral basis of $2^{2f_i-2}\cdot 9\cdot F_i^2$ is

$$\left\{\left(\frac{F_i-2}{3}\right)^2\cdot 3^2\cdot F_i^2, 2^{2f_i-2}\cdot F_i^2, \left(2\cdot\frac{F_i+1}{3}\right)^2\cdot 2^{2f_i-2}\cdot 3^2\right\}$$
$$\{(F_i-2)^2F_i^2, 2^{2f_i-2}F_i^2, 2^{2f_i}(F_i+1)^2\}$$
$$\left\{(F_i-2)^2F_i^2, \frac{1}{4}(F_i-1)^2F_i^2, (F_i^2-1)^2\right\}.$$

It is left to the reader to verify the projection properties and decomposition of the identity. □

Proof of Theorem 21.4. Power-spectral basis of $2^{2f_i+1}\cdot 9\cdot F_i^2$, index 1.

n	$2^{2f_i+1}\cdot 9\cdot F_i^2$	a	b	c
7200	$(2)^5(3)^2(5)^2$	(1)	$(2)^3$	(2)
1331712	$(2)^9(3)^2(17)^2$	$(5)^2$	$(2)^3$	$(2)(3)^2$
77914570752	$(2)^{17}(3)^2(257)^2$	$(5)^2(17)^2$	$(2)^3$	$(2)(43)^2$
332051526503242924032	$(2)^{33}(3)^2(65537)^2$	$(5)^2(17)^2(257)^2$	$(2)^3$	$(2)(3)^2(11)^2(331)^2$

Table 20: Power-spectral coefficients of $2^{2f_i+1}\cdot 9\cdot F_i^2$, $f_i=2^i$, $i=,2,3,4$.

By comparison with the power-spectral basis of Theorem 21.3 it is clear that we can take $a=(F_i-2)/3$, $b=2^3$. The decomposition of the identity is

$$a^2\cdot 9\cdot F_i^2+2^3\cdot 2^{2f_i+1}\cdot F_i^2+C=2^{2f_i+1}\cdot 9\cdot F_i^2+1$$

with C to be determined. Thus,

$$\left(\frac{F_i-2}{3}\right)^2\cdot 9\cdot F_i^2+C=2^{2f_i+1}\cdot 1\cdot F_i^2+1$$

$$
\begin{aligned}
C &= 2^{2f_i+1}F_i^2 - (F_i-2)^2 F_i^2 + 1 \\
&= 2\cdot(F_i-1)^2F_i^2 - (F_i-2)^2 F_i^2 + 1 \\
&= \left[2(F_i-1)^2 - (F_i-2)^2\right] F_i^2 + 1 \\
&= \left(2F_i^2 - 4F_i + 2 - F_i^2 + 4F_i - 4\right) F_i^2 + 1 \\
&= \left(F_i^2 - 2\right) F_i^2 + 1 \\
&= F_i^4 - 2F_i^2 + 1 \\
C &= (F_i^2-1)^2
\end{aligned}
$$

Consequently, the power-spectral basis of $2^{2f_i+1}\cdot 9\cdot F_i^2$ is

$$
\left\{\left(\frac{F_i-2}{3}\right)^2\cdot 9\cdot F_i^2, 2^3\cdot 2^{2f_i+1}\cdot F_i^2, (F_i^2-1)^2\right\}
$$
$$
\left\{(F_i-2)^2 F_i^2, 2^{2f_i+4}F_i^2, (F_i^2-1)^2\right\}
$$
$$
\left\{(F_i-2)^2 F_i^2, 16(F_i-1)^2F_i^2, (F_i^2-1)^2\right\}.
$$

It is left to the reader to verify the projection properties and decomposition of the identity.

□

8 Isospectral chains

Recall from Section 1 that if n has at least two prime factors then the decomposition of the identity has the form

$$
s_1 + \cdots + s_m = kn + 1, \tag{11}
$$

where k is the index of n. Consider Table 21. The numbers in each row are called *isospectral pairs* since $n_1 = 2n_2$, n_1 and n_2 have the same spectral basis, and index$(n_i) = i$.

	Index=2	Index=1	Spectral basis
Theorem 17:	$2^{2p-1}\cdot 3\cdot M_p^2$	$2^{2p}\cdot 3\cdot M_p^2$	$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$
Theorem 19:	$2^{2p-3}\cdot 3^2\cdot M_p^2$	$2^{2p-2}\cdot 3^2\cdot M_p^2$	$\left\{(M_p+2)^2M_p^2, (1/4)(M_p+1)^2M_p^2, (M_p^2-1)^2\right\}$
Theorem 20:	$2^{2f_i-1}\cdot 3\cdot F_i^2$	$2^{2f_i}\cdot 3\cdot F_i^2$	$\{(F_i-2)^2F_i^2, (F_i-1)^2\cdot F_i^2, (F_i^2-1)^2\}$
Theorem 21:	$2^{2f_i-3}\cdot 9\cdot F_i^2$	$2^{2f_i-2}\cdot 9\cdot F_i^2$	$\{(F_i-2)^2F_i^2, (1/4)(F_i-1)^2F_i^2, (F_i^2-1)^2\}$

Table 21: Isopectral pairs that are also power-spectral.

Definition 22 (Isopectral chain)**.** An *isospectral chain* of length k is a finite sequence of integers $n_1, \ldots, n_k$, all with the same spectral basis, such that

$$
n_1 = 2n_2 = \cdots = kn_k,
$$

and n_k has index k. A chain of length k is *maximal* if it is not part of a chain of length $k+1$. See Table 22 for the first numbers of the isospectral chains of length k.

Conjecture 23. There exist isospectral chains of any length.

	n_1	Factored	k
1.	6	(2)(3)	1
2.	84	$(2)^2(3)(7)$	2
3.	10980	$(2)^2(3)^2(5)(61)$	3
4.	488880	$(2)^4(3)^2(5)(7)(97)$	4
5.	5385063600	$(2)^4(3)^3(5)^2(7)(19)(23)(163)$	5
6.	1400839158600	$(2)^3(3)^5(5)^2(7)(37)(109)(1021)$	6

Table 22: First maximal isopectral chain of length k.

9 Isotropic numbers

Theorem 24. *If p and $q = p + 2$ are prime, then $n = pq$ has both spectral coefficients equal to $a = (p + q)/4 = (p + 1)/2$.*

Proof. Observe that

$$\begin{aligned} aq + ap &= \frac{p+q}{4} \cdot q + \frac{p+q}{4} \cdot p \\ &= \frac{p+1}{2}(2p+2) \\ &= (p+1)^2 \\ &= p^2 + 2p + 1 \\ aq + ap &= pq + 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left(\frac{p+1}{2} \cdot q\right)^2 - \left(\frac{p+1}{2} \cdot q\right) &= \left(\frac{p+1}{2} \cdot q\right)\left(\frac{p+1}{2} \cdot q - 1\right) \\ &= \frac{1}{4}\left((p+1)q\right)\left((p+1)q - 2\right) \\ &= \frac{1}{4}\left((p+1)q\right)\left((p+1)(p+2) - 2\right) \\ &= \frac{1}{4}\left((p+1)q\right)\left(p^2 + 3p\right) \\ &= \frac{(p+1)(q+1)}{4} \cdot pq \end{aligned}$$

and so

$$\left(\frac{p+1}{2}\cdot q\right)^2 \equiv \left(\frac{p+1}{2}\cdot q\right) \pmod{pq}.$$

Similarly, one can show that

$$\left(\frac{p+1}{2}\cdot p\right)^2 - \left(\frac{p+1}{2}\cdot p\right) = \frac{p^2-1}{4}\cdot pq$$

and so

$$\left(\frac{p+1}{2}\cdot p\right)^2 \equiv \left(\frac{p+1}{2}\cdot p\right) \pmod{pq}.$$

Therefore, pq is isotropic. □

Table 23 shows the first twelve examples of Theorem 24.

	n	$(p)(q)$	a
1.	15	(3)(5)	2
2.	35	(5)(7)	3
3.	143	(11)(13)	6
4.	323	(17)(19)	9
5.	899	(29)(31)	15
6.	1763	(41)(43)	21
7.	3599	(59)(61)	30
8.	5183	(71)(73)	36
9.	10403	(101)(103)	51
10.	11663	(107)(109)	54
11.	19043	(137)(139)	69
12.	22499	(149)(151)	75

Table 23: Products of twin primes are isotropic.

Definition 25 (Isotropic numbers)**.** A number n with at least two prime factors is *isotropic* if all of its spectral coefficients are equal.

Table 24 shows the isotropic numbers up to $1728 = 12^3$.

	n	Factored	a
1.	15	(3)(5)	2
2.	30	(2)(3)(5)	1
3.	35	(5)(7)	3

4.	44	$(2)^2(11)$	3
5.	63	$(3)^2(7)$	4
6.	95	$(5)(19)$	4
7.	99	$(3)^2(11)$	5
8.	104	$(2)^3(13)$	5
9.	119	$(7)(17)$	5
10.	143	$(11)(13)$	6
11.	209	$(11)(19)$	7
12.	279	$(3)^2(31)$	7
13.	287	$(7)(41)$	6
14.	319	$(11)(29)$	8
15.	323	$(17)(19)$	9
16.	377	$(13)(29)$	9
17.	429	$(3)(11)(13)$	2
18.	527	$(17)(31)$	11
19.	539	$(7)^2(11)$	9
20.	559	$(13)(43)$	10
21.	575	$(5)^2(23)$	12
22.	639	$(3)^2(71)$	8
23.	675	$(3)^3(5)^2$	13
24.	779	$(19)(41)$	13
25.	783	$(3)^3(29)$	14
26.	858	$(2)(3)(11)(13)$	1
27.	861	$(3)(7)(41)$	2
28.	899	$(29)(31)$	15
29.	923	$(13)(71)$	11
30.	989	$(23)(43)$	15
31.	1007	$(19)(53)$	14
32.	1189	$(29)(41)$	17
33.	1199	$(11)(109)$	10
34.	1325	$(5)^2(53)$	17
35.	1343	$(17)(79)$	14
36.	1349	$(19)(71)$	15
37.	1519	$(7)^2(31)$	19
38.	1722	$(2)(3)(7)(41)$	1
39.	1728	$(2)^6(3)^3$	19

Table 24: Isotropic numbers.

9.1 Balanced numbers

Theorem 26 (Balanced numbers)**.** *If x is a prime or prime power such that $y = x + 2$ is also a prime or prime power, then xy is isotropic and $a = (x + y)/4$. The number xy is called* balanced*. See Table 25.*

Proof. Similar to the proof of Theorem 24. □

	n	Factored	a
1.	63	$(3)^2(7)$	4
2.	99	$(3)^2(11)$	5
3.	575	$(5)^2(23)$	12
4.	675	$(3)^3(5)^2$	13
5.	783	$(3)^3(29)$	14
6.	2303	$(7)^2(47)$	24
7.	6399	$(3)^4(79)$	40
8.	6723	$(3)^4(83)$	41
9.	15875	$(5)^3(127)$	63
10.	28223	$(13)^2(167)$	84
11.	58563	$(3)^5(241)$	121
12.	129599	$(19)^2(359)$	180
13.	529983	$(3)^6(727)$	364
14.	705599	$(29)^2(839)$	420
15.	1871423	$(37)^2(1367)$	684
16.	3415103	$(43)^2(1847)$	924
17.	4875263	$(47)^2(2207)$	1104
18.	5759999	$(7)^4(2399)$	1200
19.	13838399	$(61)^2(3719)$	1860
20.	25401599	$(71)^2(5039)$	2520
21.	43059843	$(3)^8(6563)$	3281
22.	47032163	$(19)^3(6857)$	3429
23.	62726399	$(89)^2(7919)$	3960
24.	594872099	$(29)^3(24391)$	12195

Table 25: Balanced numbers.

See Subsection 11.7 for more tables on balanced numbers.

9.2 Hypoisotropic numbers

Definition 27. An isotropic number is called *hypoisotropic* if it is isotropic with all spectral coefficients equal to 1. See Table 26 for the first six squarefree hypoisotropic numbers, also known as Giuga numbers (OEIS A007850).

	n	Factored
1.	30	(2)(3)(5)
2.	858	(2)(3)(11)(13)
3.	1722	(2)(3)(7)(41)
4.	66198	(2)(3)(11)(17)(59)
5.	2214408306	(2)(3)(11)(23)(31)(47057)
6.	24423128562	(2)(3)(7)(43)(3041)(4447)

Table 26: Squarefree hypoisotropic numbers, also known as Giuga numbers.

If $\{s_1, \ldots, s_k\}$ is the spectral basis of $n = p_1^{e_1} \cdots p_k^{e_k}$, then one has the projection equation

$$\begin{aligned}
s_i^2 &\equiv s_i \pmod{n} \\
(a_i h_i)^2 &\equiv a_i h_i \pmod{n} \\
h_i^2 &\equiv h_i \pmod{n} \quad \text{(hypoisotropic)} \\
h_i^2 - h_i &\equiv 0 \pmod{n} \\
h_i(h_i - 1) &\equiv 0 \pmod{n} \\
h_i(h_i - 1) &= an \quad \text{(for some positive integer } a\text{)} \\
\frac{n}{p_i^{e_i}} \left(\frac{n}{p_i^{e_i}} - 1 \right) &= an \\
\frac{n}{p_i^{e_i}} - 1 &= a p_i^{e_i} \\
p_i^{e_i} &\left| \left(\frac{n}{p_i^{e_i}} - 1 \right) , \quad p_i^{e_i} || n, \quad \forall i. \right. \qquad (12)
\end{aligned}$$

Since $a_i = h_i^{-1} \mod p_i^{e_i} = 1$ for all i, we must have $h_i \equiv 1 \pmod{p_i^{e_i}}$ for all i, and this equation does not seem to add anything to (12). The decomposition of the identity takes the form

$$h_1 + \cdots + h_k = n + 1,$$

and this is easily seen to be equivalent to

$$\sum_{p^e || n} \frac{1}{p^e} - \frac{1}{n} = 1. \qquad (13)$$

The solutions to (13) are the Giuga numbers (OEIS A007850), where $e = 1$ for all e. Solutions to (13) for $e > 1$ for some e are not known.

Conjecture 28. The only solutions to (13) are the Giuga numbers (OEIS A007850).

9.3 Powerful isotropic numbers

Recall that a number is *powerful* if all the exponents in its prime factorization are greater than 1. Table 27 shows all the powerful isotropic numbers found to date.

	n	Factored	a
1.	675	$(3)^3(5)^2$	13
2.	1728	$(2)^6(3)^3$	19
3.	7092899	$(11)^3(73)^2$	1065
4.	380401279	$(31)^3(113)^2$	8938
5.	7138196909	$(29)^3(541)^2$	22513
6.	32438531449	$(389)^2(463)^2$	88705
7.	202195540949	$(101)^3(443)^2$	164849
8.	4605635594143	$(103)^3(2053)^2$	867754
9.	537171399517729729	$(8887)^2(82471)^2$	78072193

Table 27: Powerful isotropic numbers.

10 Pythagorean power-spectral numbers

A Pythagorean power-spectral number n has the property that its spectral sum $kn + 1$ is also a power. Table 28 shows the only Pythagorean power-spectral numbers found so far, with search out to $n \leq 12^8$.

	n	Factored	Spectral basis			Power
1.	24	$(2)^3(3)$	$(3)^2$	$(2)^4$		5^2
2.	288	$(2)^5(3)^2$	$(15)^2$	$(2)^6$		17^2
3.	2400	$(2)^5(3)(5)^2$	$(15)^2$	$(40)^2$	$(24)^2$	49^2
4.	4704	$(2)^5(3)(7)^2$	$(63)^2$	$(56)^2$	$(48)^2$	97^2
5.	9408	$(2)^6(3)(7)^2$	$(63)^2$	$(56)^2$	$(48)^2$	97^2

Table 28: Pythagorean power-spectral numbers.

Observe that 4 and 5 in Table 28 comprise an isospectral pair. See Section 8 for a dicussion of isospectral chains.

11 Various long tables

Recall that if $s_1 + \cdots + s_k = \kappa n + 1$, then κ is called the index of n.

11.1 Numbers of first index

	n	Factored	idx
1.	6	$(2)(3)$	1
2.	42	$(2)(3)(7)$	2
3.	924	$(2)^2(3)(7)(11)$	3
4.	23100	$(2)^2(3)(5)^2(7)(11)$	4
5.	1527372	$(2)^2(3)^2(7)(11)(19)(29)$	5
6.	36606570	$(2)(3)(5)(7)(11)(13)(23)(53)$	6

Table 29: Numbers of first index.

	n	Factored	idx
1.	6	$(2)(3)$	1
2.	10	$(2)(5)$	1
3.	12	$(2)^2(3)$	1
4.	14	$(2)(7)$	1
5.	15	$(3)(5)$	1
6.	18	$(2)(3)^2$	1
7.	20	$(2)^2(5)$	1
8.	21	$(3)(7)$	1
9.	22	$(2)(11)$	1
10.	24	$(2)^3(3)$	1
11.	26	$(2)(13)$	1
12.	28	$(2)^2(7)$	1
13.	30	$(2)(3)(5)$	1
14.	33	$(3)(11)$	1
15.	34	$(2)(17)$	1
16.	35	$(5)(7)$	1
17.	36	$(2)^2(3)^2$	1
18.	38	$(2)(19)$	1
19.	39	$(3)(13)$	1
20.	40	$(2)^3(5)$	1

21.	44	$(2)^2(11)$	1
22.	45	$(3)^2(5)$	1
23.	46	$(2)(23)$	1
24.	48	$(2)^4(3)$	1

Table 30: Numbers of index 1.

	n	Factored	idx
1.	42	$(2)(3)(7)$	2
2.	60	$(2)^2(3)(5)$	2
3.	70	$(2)(5)(7)$	2
4.	78	$(2)(3)(13)$	2
5.	110	$(2)(5)(11)$	2
6.	114	$(2)(3)(19)$	2
7.	120	$(2)^3(3)(5)$	2
8.	140	$(2)^2(5)(7)$	2
9.	150	$(2)(3)(5)^2$	2
10.	156	$(2)^2(3)(13)$	2
11.	168	$(2)^3(3)(7)$	2
12.	170	$(2)(5)(17)$	2
13.	186	$(2)(3)(31)$	2
14.	195	$(3)(5)(13)$	2
15.	198	$(2)(3)^2(11)$	2
16.	204	$(2)^2(3)(17)$	2
17.	210	$(2)(3)(5)(7)$	2
18.	220	$(2)^2(5)(11)$	2
19.	222	$(2)(3)(37)$	2
20.	231	$(3)(7)(11)$	2
21.	234	$(2)(3)^2(13)$	2
22.	238	$(2)(7)(17)$	2
23.	240	$(2)^4(3)(5)$	2
24.	258	$(2)(3)(43)$	2

Table 31: Numbers of index 2.

	n	Factored	idx

1.	924	$(2)^2(3)(7)(11)$	3
2.	1170	$(2)(3)^2(5)(13)$	3
3.	1320	$(2)^3(3)(5)(11)$	3
4.	1806	$(2)(3)(7)(43)$	3
5.	1870	$(2)(5)(11)(17)$	3
6.	1932	$(2)^2(3)(7)(23)$	3
7.	2220	$(2)^2(3)(5)(37)$	3
8.	2508	$(2)^2(3)(11)(19)$	3
9.	2860	$(2)^2(5)(11)(13)$	3
10.	3120	$(2)^4(3)(5)(13)$	3
11.	3276	$(2)^2(3)^2(7)(13)$	3
12.	3420	$(2)^2(3)^2(5)(19)$	3
13.	3498	$(2)(3)(11)(53)$	3
14.	3570	$(2)(3)(5)(7)(17)$	3
15.	3660	$(2)^2(3)(5)(61)$	3
16.	3740	$(2)^2(5)(11)(17)$	3
17.	3770	$(2)(5)(13)(29)$	3
18.	3960	$(2)^3(3)^2(5)(11)$	3
19.	4060	$(2)^2(5)(7)(29)$	3
20.	4070	$(2)(5)(11)(37)$	3
21.	4182	$(2)(3)(17)(41)$	3
22.	4284	$(2)^2(3)^2(7)(17)$	3
23.	4290	$(2)(3)(5)(11)(13)$	3
24.	4446	$(2)(3)^2(13)(19)$	3

Table 32: Numbers of index 3.

	n	Factored	*idx*
1.	23100	$(2)^2(3)(5)^2(7)(11)$	4
2.	32604	$(2)^2(3)(11)(13)(19)$	4
3.	35420	$(2)^2(5)(7)(11)(23)$	4
4.	47058	$(2)(3)(11)(23)(31)$	4
5.	53820	$(2)^2(3)^2(5)(13)(23)$	4
6.	58695	$(3)(5)(7)(13)(43)$	4
7.	79170	$(2)(3)(5)(7)(13)(29)$	4
8.	79695	$(3)^2(5)(7)(11)(23)$	4

9.	81510	(2)(3)(5)(11)(13)(19)	4
10.	87360	$(2)^6(3)(5)(7)(13)$	4
11.	91770	(2)(3)(5)(7)(19)(23)	4
12.	92220	$(2)^2(3)(5)(29)(53)$	4
13.	95004	$(2)^2(3)^2(7)(13)(29)$	4
14.	103020	$(2)^2(3)(5)(17)(101)$	4
15.	105468	$(2)^2(3)(11)(17)(47)$	4
16.	107640	$(2)^3(3)^2(5)(13)(23)$	4
17.	110220	$(2)^2(3)(5)(11)(167)$	4
18.	110670	(2)(3)(5)(7)(17)(31)	4
19.	112056	$(2)^3(3)(7)(23)(29)$	4
20.	113100	$(2)^2(3)(5)^2(13)(29)$	4
21.	113820	$(2)^2(3)(5)(7)(271)$	4
22.	116270	(2)(5)(7)(11)(151)	4
23.	117810	$(2)(3)^2(5)(7)(11)(17)$	4
24.	120120	$(2)^3(3)(5)(7)(11)(13)$	4

Table 33: Numbers of index 4.

	n	Factored	*idx*
1.	1527372	$(2)^2(3)^2(7)(11)(19)(29)$	5
2.	1750320	$(2)^4(3)^2(5)(11)(13)(17)$	5
3.	1967070	(2)(3)(5)(7)(17)(19)(29)	5
4.	2043195	(3)(5)(7)(11)(29)(61)	5
5.	2222220	$(2)^2(3)(5)(7)(11)(13)(37)$	5
6.	2567544	$(2)^3(3)(7)(17)(29)(31)$	5
7.	2571660	$(2)^2(3)^2(5)(7)(13)(157)$	5
8.	2704156	$(2)^2(7)(13)(17)(19)(23)$	5
9.	2788170	(2)(3)(5)(7)(11)(17)(71)	5
10.	2869020	$(2)^2(3)^4(5)(7)(11)(23)$	5
11.	2877420	$(2)^2(3)(5)(7)(13)(17)(31)$	5
12.	3164070	(2)(3)(5)(7)(13)(19)(61)	5
13.	3202290	$(2)(3)^2(5)(7)(13)(17)(23)$	5
14.	3245970	(2)(3)(5)(7)(13)(29)(41)	5
15.	3294060	$(2)^2(3)(5)(7)(11)(23)(31)$	5
16.	3405150	$(2)(3)^2(5)^2(7)(23)(47)$	5

17.	3423420	$(2)^2(3)^2(5)(7)(11)(13)(19)$	5
18.	3467310	(2)(3)(5)(7)(11)(19)(79)	5
19.	3482570	(2)(5)(7)(13)(43)(89)	5
20.	3556995	(3)(5)(13)(17)(29)(37)	5
21.	3598764	$(2)^2(3)(13)(17)(23)(59)$	5
22.	4001970	(2)(3)(5)(7)(17)(19)(59)	5
23.	4142220	$(2)^2(3)(5)(17)(31)(131)$	5
24.	4281270	(2)(3)(5)(7)(19)(29)(37)	5

Table 34: Numbers of index 5.

	n	Factored	idx
1.	36606570	(2)(3)(5)(7)(11)(13)(23)(53)	6
2.	64234170	$(2)(3)^2(5)(7)(11)(13)(23)(31)$	6
3.	78914220	$(2)^2(3)(5)(7)(11)(19)(29)(31)$	6
4.	80514720	$(2)^5(3)^2(5)(11)(13)(17)(23)$	6
5.	81501420	$(2)^2(3)(5)(7)(11)(13)(23)(59)$	6
6.	82732650	$(2)(3)(5)^2(7)(11)(13)(19)(29)$	6
7.	89664120	$(2)^3(3)^2(5)(7)^2(13)(17)(23)$	6
8.	89967570	(2)(3)(5)(7)(11)(17)(29)(79)	6
9.	91111020	$(2)^2(3)(5)(7)(11)(13)(37)(41)$	6
10.	100704120	$(2)^3(3)(5)(11)(23)(31)(107)$	6
11.	104747370	(2)(3)(5)(7)(13)(17)(37)(61)	6
12.	107111004	$(2)^2(3)(7)(11)(13)(37)(241)$	6
13.	133742700	$(2)^2(3)^2(5)^2(7)(13)(23)(71)$	6
14.	133876470	(2)(3)(5)(7)(13)(19)(29)(89)	6
15.	136190670	(2)(3)(5)(7)(11)(19)(29)(107)	6
16.	137287920	$(2)^4(3)(5)(7)(11)(17)(19)(23)$	6
17.	139447770	(2)(3)(5)(7)(11)(17)(53)(67)	6
18.	150210060	$(2)^2(3)(5)(7)(11)(13)(41)(61)$	6
19.	155207052	$(2)^2(3)^2(7)(11)(13)(59)(73)$	6
20.	156598260	$(2)^2(3)(5)(7)(13)(23)(29)(43)$	6
21.	161910210	$(2)(3)(5)(7)^2(11)(17)(19)(31)$	6
22.	164294520	$(2)^3(3)(5)(13)(19)(23)(241)$	6
23.	169860600	$(2)^3(3)^2(5)^2(7)(13)(17)(61)$	6
24.	170125020	$(2)^2(3)^2(5)(13)(23)(29)(109)$	6

Table 35: Numbers of index 6.

It is left to the reader to find examples of higher index.

11.2 Index of factorial and primorial

	n	$n!$	idx
1.	3	6	1
2.	4	24	1
3.	5	120	2
4.	6	720	2
5.	7	5040	2
6.	8	40320	3
7.	9	362880	1
8.	10	3628800	2
9.	11	39916800	2
10.	12	479001600	3
11.	13	6227020800	3
12.	14	87178291200	3
13.	15	1307674368000	2
14.	16	20922789888000	3
15.	17	355687428096000	3
16.	18	6402373705728000	4
17.	19	121645100408832000	4
18.	20	2432902008176640000	6
19.	21	51090942171709440000	4
20.	22	1124000727777607680000	4
21.	23	25852016738884976640000	6
22.	24	620448401733239439360000	4

Table 36: Index of factorials.

	p	$p\#$	Factored	idx
1.	3	6	(2)(3)	1
2.	5	30	(2)(3)(5)	1

3.	7	210	(2)(3)(5)(7)	2
4.	11	2310	(2)(3)(5)(7)(11)	2
5.	13	30030	(2)(3)(5)(7)(11)(13)	3
6.	17	510510	(2)(3)(5)(7)(11)(13)(17)	3
7.	19	9699690	(2)(3)(5)(7)(11)(13)(17)(19)	5
8.	23	223092870	(2)(3)(5)(7)(11)(13)(17)(19)(23)	6
9.	29	6469693230	(2)(3)(5)(7)(11)(13)(17)(19)(23)(29)	4
10.	31	200560490130	(2)(3)(5)(7)(11)(13)(17)(19)(23)(29)(31)	5
11.	37	7420738134810	(2)(3)(5)(7)(11)(13)(17)(19)(23)(29)(31)(37)	5

Table 37: Index of primorials.

11.3 Table of cyclotomic primes

Table 38 refers to Section 5, and is restricted to the first eight examples of $\Phi_{r^e}(p)$, $p < 10^5$, where $r \in \{2, 3, 5, 7, 11, 13\}$ and $1 \leq e \leq 6$.

	r	e	p	#digits
1.	2	1	2	1
2.	2	2	2	2
3.	2	3	2	3
4.	2	4	2	5
5.	3	1	2	2
6.	3	1	3	3
7.	3	1	11	7
8.	3	1	191	14
9.	3	1	269	15
10.	3	1	383	16
11.	3	1	509	17
12.	3	1	809	18
13.	3	2	2	6
14.	3	2	11	19
15.	3	2	263	44
16.	3	2	557	50
17.	3	2	761	52
18.	3	2	797	53
19.	3	2	863	53

20.	3	2	977	54
21.	3	3	191	124
22.	3	3	311	135
23.	3	3	557	149
24.	3	3	659	153
25.	3	3	887	160
26.	3	3	1607	174
27.	3	3	2309	182
28.	3	3	2621	185
29.	3	4	4457	592
30.	3	4	5867	611
31.	3	4	7001	623
32.	3	4	7019	624
33.	3	4	7541	629
34.	3	4	8609	638
35.	3	4	8627	638
36.	3	4	21773	703
37.	3	5	3803	1740
38.	3	5	13859	2013
39.	3	5	22961	2120
40.	3	5	31223	2185
41.	3	5	44351	2259
42.	3	5	45737	2265
43.	3	5	61751	2329
44.	3	5	63377	2334
45.	3	6	1889	4777
46.	3	6	16829	6162
47.	3	6	62549	6993
48.	3	6	67103	7038
49.	5	1	43	33
50.	5	1	167	45
51.	5	1	313	50
52.	5	1	509	55
53.	5	1	859	59
54.	5	1	1039	61
55.	5	1	1627	65

56.	5	1	1783	66
57.	5	2	127	211
58.	5	2	1289	312
59.	5	2	3023	349
60.	5	2	3067	349
61.	5	2	3767	358
62.	5	2	3923	360
63.	5	2	5107	371
64.	5	2	6653	383
65.	5	3	673	1415
66.	5	3	3947	1799
67.	5	3	4409	1823
68.	5	3	4933	1847
69.	5	3	7993	1952
70.	5	3	9467	1989
71.	5	3	12823	2054
72.	5	3	17317	2120

Table 38: Cyclotomic primes $\Phi_{r^e}(p)$.

11.4 Tables of powers that are power-spectral

The only power with two prime factors that is also power-spectral is 144, $(2)^4(3)^2$, with spectral basis $\{(3)^4, (2)^6\}$ of index 1. Table 40 shows examples with three factors. Table 41 shows examples with four factors. Table 42 shows examples with five factors. Table 43 shows examples with six factors. Table 39 shows the only power-spectral numbers that are powers but divisible by neither 2 nor 3.

	n	Factored	Spectral basis			*idx*
1.	555969241	$(17)^2(19)^2(73)^2$	$(5)^2(19)^2(73)^2$	$(2)^8(17)^2(73)^2$	$(3)^2(11)^2(17)^2(19)^2$	1
2.	9021010441	$(17)^2(37)^2(151)^2$	$(3)^2(37)^2(151)^2$	$(2)^6(17)^2(151)^2$	$(5)^2(17)^2(29)^2(37)^2$	1

Table 39: Powers that are power-spectral but not divisible by either 2 or 3.

	n	Factored	Spectral basis			*idx*
1.	900	$(2)^2(3)^2(5)^2$	$(3)^2(5)^2$	$(2)^2(5)^2$	$(2)^6(3)^2$	1
2.	7056	$(2)^4(3)^2(7)^2$	$(3)^4(7)^2$	$(2)^4(7)^2$	$(2)^8(3)^2$	1
3.	8100	$(2)^2(3)^4(5)^2$	$(3)^4(5)^2$	$(2)^8(5)^2$	$(2)^5(3)^5$	2

4.	17424	$(2)^4(3)^2(11)^2$	$(3)^2(11)^2$	$(2)^4(11)^2$	$(2)^6(3)^2(5)^2$	1
5.	27225	$(3)^2(5)^2(11)^2$	$(5)^2(11)^2$	$(3)^4(11)^2$	$(2)^6(3)^2(5)^2$	1
6.	28224	$(2)^6(3)^2(7)^2$	$(3)^4(7)^2$	$(2)^6(7)^3$	$(2)^8(3)^2$	1
7.	32400	$(2)^4(3)^4(5)^2$	$(3)^6(5)^2$	$(2)^8(5)^2$	$(2)^5(3)^5$	1
8.	48400	$(2)^4(5)^2(11)^2$	$(5)^2(11)^2$	$(2)^8(11)^2$	$(2)^6(3)^2(5)^2$	1
9.	74529	$(3)^2(7)^2(13)^2$	$(7)^2(13)^2$	$(3)^2(5)^2(13)^2$	$(2)^6(3)^2(7)^2$	1
10.	144400	$(2)^4(5)^2(19)^2$	$(5)^2(19)^2$	$(2)^4(19)^2$	$(2)^6(3)^4(5)^2$	1
11.	166464	$(2)^6(3)^2(17)^2$	$(3)^2(5)^2(17)^2$	$(2)^6(17)^2$	$(2)^{10}(3)^4$	1
12.	193600	$(2)^6(5)^2(11)^2$	$(5)^2(7)^2(11)^2$	$(2)^8(11)^2$	$(2)^6(3)^2(5)^2$	1
13.	571536	$(2)^4(3)^6(7)^2$	$(3)^8(7)^2$	$(2)^6(7)^2(13)^2$	$(2)^4(3)^6(5)^2$	2
14.	576081	$(3)^2(11)^2(23)^2$	$(11)^2(23)^2$	$(3)^2(7)^2(23)^2$	$(2)^8(3)^2(11)^2$	1
15.	577600	$(2)^6(5)^2(19)^2$	$(5)^2(19)^2$	$(2)^6(19)^3$	$(2)^6(3)^4(5)^2$	1
16.	950625	$(3)^2(5)^4(13)^2$	$(5)^4(13)^2$	$(2)^8(3)^2(13)^2$	$(3)^6(5)^4$	1
17.	1218816	$(2)^8(3)^2(23)^2$	$(3)^2(13)^2(23)^2$	$(2)^8(23)^2$	$(2)^8(3)^2(11)^2$	1
18.	1871424	$(2)^6(3)^4(19)^2$	$(3)^6(19)^2$	$(2)^{12}(19)^2$	$(2)^6(3)^4(5)^2$	1
19.	2214144	$(2)^8(3)^2(31)^2$	$(3)^2(11)^2(31)^2$	$(2)^8(31)^2$	$(2)^{12}(3)^2(5)^2$	1
20.	2637376	$(2)^6(7)^2(29)^2$	$(3)^2(7)^2(29)^2$	$(2)^6(29)^3$	$(2)^6(3)^2(5)^2(7)^2$	1
21.	4447881	$(3)^2(19)^2(37)^2$	$(19)^2(37)^2$	$(3)^2(13)^2(37)^2$	$(2)^6(3)^4(19)^2$	1
22.	5798464	$(2)^6(7)^2(43)^2$	$(5)^2(7)^2(43)^2$	$(2)^6(43)^2$	$(2)^6(3)^2(7)^2(11)^2$	1
23.	10517049	$(3)^2(23)^2(47)^2$	$(23)^2(47)^2$	$(3)^4(5)^2(47)^2$	$(2)^{10}(3)^2(23)^2$	1
24.	10758400	$(2)^8(5)^2(41)^2$	$(5)^4(41)^2$	$(2)^{12}(41)^2$	$(2)^8(3)^2(5)^2(7)^2$	1

Table 40: Powers that are power-spectral with three prime factors.

	n	Factored	idx
1.	6051600	$(2)^4(3)^2(5)^2(41)^2$	1
2.	8643600	$(2)^4(3)^2(5)^2(7)^4$	1
3.	12531600	$(2)^4(3)^2(5)^2(59)^2$	2
4.	16483600	$(2)^4(5)^2(7)^2(29)^2$	1
5.	23736384	$(2)^6(3)^2(7)^2(29)^2$	1
6.	53509225	$(5)^2(7)^2(11)^2(19)^2$	1
7.	65934400	$(2)^6(5)^2(7)^2(29)^2$	1
8.	98803600	$(2)^4(5)^2(7)^2(71)^2$	1
9.	146894400	$(2)^6(3)^2(5)^2(101)^2$	1
10.	265559616	$(2)^6(3)^2(7)^2(97)^2$	1
11.	299013264	$(2)^4(3)^2(11)^2(131)^2$	2

12.	787139136	$(2)^6(3)^2(7)^2(167)^2$	2
13.	1038128400	$(2)^4(3)^4(5)^2(179)^2$	2
14.	1160083600	$(2)^4(5)^2(13)^2(131)^2$	1
15.	2678683536	$(2)^4(3)^2(19)^2(227)^2$	2
16.	2943062500	$(2)^2(5)^6(7)^2(31)^2$	2
17.	6256968201	$(3)^4(11)^2(17)^2(47)^2$	1
18.	7175445264	$(2)^4(3)^4(13)^2(181)^2$	1
19.	8013114256	$(2)^4(7)^2(23)^2(139)^2$	1
20.	9703235025	$(3)^4(5)^2(11)^2(199)^2$	1
21.	10769458176	$(2)^{10}(3)^2(23)^2(47)^2$	1
22.	10824321600	$(2)^6(3)^4(5)^2(17)^4$	2
23.	11772250000	$(2)^4(5)^6(7)^2(31)^2$	1
24.	13160678400	$(2)^{10}(3)^2(5)^2(239)^2$	1

Table 41: Powers that are power-spectral with four prime factors.

	n	Factored	idx
1.	749116900	$(2)^2(5)^2(7)^2(17)^2(23)^2$	1
2.	4238010000	$(2)^4(3)^2(5)^4(7)^2(31)^2$	2
3.	6325020900	$(2)^2(3)^2(5)^2(11)^2(241)^2$	2
4.	13975495524	$(2)^2(3)^2(17)^2(19)^2(61)^2$	3
5.	18653550084	$(2)^2(3)^2(13)^2(17)^2(103)^2$	1
6.	20701454400	$(2)^6(3)^2(5)^2(11)^2(109)^2$	1
7.	25300083600	$(2)^4(3)^2(5)^2(11)^2(241)^2$	1
8.	168740208400	$(2)^4(5)^2(19)^2(23)^2(47)^2$	2
9.	189172803600	$(2)^4(3)^2(5)^2(11)^2(659)^2$	2
10.	298968368400	$(2)^4(3)^2(5)^2(13)^2(701)^2$	2
11.	350617936900	$(2)^2(5)^2(7)^2(11)^2(769)^2$	2
12.	541707776064	$(2)^6(3)^2(7)^2(13)^2(337)^2$	1
13.	746444160900	$(2)^2(3)^2(5)^2(31)^2(929)^2$	2
14.	956261028996	$(2)^2(3)^6(7)^2(13)^2(199)^2$	2
15.	1065271694400	$(2)^6(3)^4(5)^2(47)^2(61)^2$	2
16.	1232233203600	$(2)^4(3)^4(5)^2(7)^2(881)^2$	2
17.	2629904403204	$(2)^2(3)^2(13)^2(17)^2(1223)^2$	2
18.	3604309844004	$(2)^2(3)^2(31)^2(59)^2(173)^2$	1
19.	4005842131600	$(2)^4(5)^2(19)^2(23)^2(229)^2$	1

20.	4278171003876	$(2)^2(3)^2(7)^2(11)^6(37)^2$	2
21.	4619112220944	$(2)^4(3)^2(13)^2(23)^2(599)^2$	1
22.	7811108266896	$(2)^4(3)^2(11)^2(31)^2(683)^2$	2
23.	8555274003600	$(2)^4(3)^2(5)^2(29)^2(41)^4$	2
24.	9007201440000	$(2)^8(3)^2(5)^4(41)^2(61)^2$	1

Table 42: Powers that are power-spectral with five prime factors.

	n	Factored	idx
1.	6580123434561600	$(2)^6(3)^2(5)^2(7)^2(11)^2(8779)^2$	2
2.	700129543497753600	$(2)^{10}(3)^2(5)^2(7)^2(11)^2(22639)^2$	2

Table 43: Powers that are power-spectral with six prime factors.

11.5 Tables of isospectral chains

	n_1	Factored
1.	6	$(2)(3)$
2.	10	$(2)(5)$
3.	12	$(2)^2(3)$
4.	14	$(2)(7)$
5.	15	$(3)(5)$
6.	18	$(2)(3)^2$
7.	20	$(2)^2(5)$
8.	21	$(3)(7)$
9.	22	$(2)(11)$
10.	24	$(2)^3(3)$
11.	26	$(2)(13)$
12.	28	$(2)^2(7)$

Table 44: Maximal isopectral chains of length 1.

	n_1	Factored
1.	84	$(2)^2(3)(7)$
2.	228	$(2)^2(3)(19)$
3.	280	$(2)^3(5)(7)$

4.	340	$(2)^2(5)(17)$
5.	372	$(2)^2(3)(31)$
6.	408	$(2)^3(3)(17)$
7.	468	$(2)^2(3)^2(13)$
8.	480	$(2)^5(3)(5)$
9.	516	$(2)^2(3)(43)$
10.	624	$(2)^4(3)(13)$
11.	740	$(2)^2(5)(37)$
12.	792	$(2)^3(3)^2(11)$

Table 45: Maximal isopectral chains of length 2.

	n_1	Factored
1.	10980	$(2)^2(3)^2(5)(61)$
2.	35280	$(2)^4(3)^2(5)(7)^2$
3.	36180	$(2)^2(3)^3(5)(67)$
4.	43380	$(2)^2(3)^2(5)(241)$
5.	46980	$(2)^2(3)^4(5)(29)$
6.	47268	$(2)^2(3)^2(13)(101)$
7.	52164	$(2)^2(3)^4(7)(23)$
8.	59508	$(2)^2(3)^3(19)(29)$
9.	71604	$(2)^2(3)^4(13)(17)$
10.	73476	$(2)^2(3)^2(13)(157)$
11.	75780	$(2)^2(3)^2(5)(421)$
12.	87444	$(2)^2(3)^2(7)(347)$

Table 46: Maximal isopectral chains of length 3.

	n_1	Factored
1.	488880	$(2)^4(3)^2(5)(7)(97)$
2.	1525680	$(2)^4(3)^2(5)(13)(163)$
3.	2870280	$(2)^3(3)^2(5)(7)(17)(67)$
4.	4930272	$(2)^5(3)^2(17)(19)(53)$
5.	5890248	$(2)^3(3)^2(7)(13)(29)(31)$
6.	6374664	$(2)^3(3)^2(29)(43)(71)$
7.	8862984	$(2)^3(3)^2(13)(17)(557)$

8.	9658080	$(2)^5(3)^2(5)(19)(353)$
9.	9739080	$(2)^3(3)^2(5)(13)(2081)$
10.	10338480	$(2)^4(3)^2(5)(83)(173)$
11.	10544544	$(2)^5(3)^2(19)(41)(47)$
12.	12719880	$(2)^3(3)^2(5)(89)(397)$

Table 47: Maximal isopectral chains of length 4.

	n_1	Factored
1.	5385063600	$(2)^4(3)^3(5)^2(7)(19)(23)(163)$
2.	5978343600	$(2)^4(3)^2(5)^2(47)(89)(397)$
3.	6789558600	$(2)^3(3)^2(5)^2(11)(17)(23)(877)$
4.	12965853600	$(2)^5(3)^3(5)^2(7)(29)(2957)$
5.	24933169800	$(2)^3(3)^2(5)^2(7)^2(11)(31)(829)$
6.	30412398600	$(2)^3(3)^2(5)^2(23)(29)(73)(347)$
7.	31967238600	$(2)^3(3)^3(5)(11)(17)(31657)$
8.	32035143600	$(2)^4(3)^4(5)(47)(109)(193)$
9.	37418554800	$(2)^4(3)^3(5)(11)(89)(3539)$
10.	37884558600	$(2)^3(3)^4(5)(7)(439)(761)$
11.	38274663600	$(2)^4(3)^2(5)^2(17)(89)(7027)$
12.	40137274800	$(2)^4(3)^2(5)^2(7)(31)(191)(269)$
13.	44143129800	$(2)^3(3)^2(5)^2(7)^2(11)(173)(263)$

Table 48: Maximal isopectral chains of length 5.

Table 49 shows a baker's dozen of maximal isospectral chains of length 6. There may be gaps due to the search method.

	n_1	Factored
1.	1400839158600	$(2)^3(3)^5(5)^2(7)(37)(109)(1021)$
2.	2902429341000	$(2)^3(3)^4(5)^3(7)(19)(419)(643)$
3.	3949885485000	$(2)^3(3)^3(5)^4(7)(13)(17)(18913)$
4.	9000942048000	$(2)^8(3)^3(5)^3(7)(19)(29)(37)(73)$
5.	10563097053600	$(2)^5(3)^5(5)^2(7)(13)(367)(1627)$
6.	13554828003600	$(2)^4(3)^6(5)^2(13)(37)(241)(401)$
7.	18867199233600	$(2)^6(3)^5(5)^2(43)(53)(107)(199)$
8.	26976351213000	$(2)^3(3)^6(5)^3(7)(17)(149)(2087)$
9.	37127826792000	$(2)^6(3)^4(5)^3(7)(31)(229)(1153)$

10.	42966550125000	$(2)^3(3)^2(5)^6(17)(19)(23)(53)(97)$
11.	50742170640000	$(2)^7(3)^2(5)^4(7)(19)(43)(12323)$
12.	54497942553600	$(2)^{10}(3)^4(5)^2(7)(13)(23)(29)(433)$
13.	56675647917000	$(2)^3(3)^7(5)^3(7)(31)(307)(389)$

Table 49: Maximal isopectral chains of length 6.

It is left to the reader to find isospectral chains of length 7 and higher.

11.6 Tables of homogeneous numbers

Definition 29 (Homogeneous)**.** A number n is *homogeneous* if and only if the prime factors of n occur in the prime factors of the spectral basis of n.

Table 50 shows the homogenous numbers n, $n \leq 720$. Table 51 shows the homogenous numbers n, $n \leq 720$ that are also power-spectral.

	n	Factored	Spectral basis			idx
1.	6	(2)(3)	(3)	(2)^2		1
2.	12	(2)^2(3)	(3)^2	(2)^2		1
3.	14	(2)(7)	(7)	(2)^3		1
4.	20	(2)^2(5)	(5)	(2)^4		1
5.	24	(2)^3(3)	(3)^2	(2)^4		1
6.	30	(2)(3)(5)	(15)	(10)	(6)	1
7.	39	(3)(13)	(13)	(3)^3		1
8.	40	(2)^3(5)	(5)^2	(2)^4		1
9.	42	(2)(3)(7)	(21)	(28)	(6)^2	2
10.	56	(2)^3(7)	(7)^2	(2)^3		1
11.	60	(2)^2(3)(5)	(45)	(40)	(6)^2	2
12.	62	(2)(31)	(31)	(2)^5		1
13.	66	(2)(3)(11)	(33)	(22)	(12)	1
14.	70	(2)(5)(7)	(35)	(56)	(50)	2
15.	72	(2)^3(3)^2	(3)^2	(2)^6		1
16.	84	(2)^2(3)(7)	(21)	(28)	(6)^2	1
17.	90	(2)(3)^2(5)	(45)	(10)	(6)^2	1
18.	102	(2)(3)(17)	(51)	(34)	(18)	1
19.	112	(2)^4(7)	(7)^2	(2)^6		1
20.	114	(2)(3)(19)	(57)	(76)	(96)	2
21.	126	(2)(3)^2(7)	(63)	(28)	(6)^2	1

22.	130	(2)(5)(13)	(65)	(26)	(40)		1
23.	132	$(2)^2(3)(11)$	(33)	(88)	(12)		1
24.	138	(2)(3)(23)	(69)	(46)	(24)		1
25.	144	$(2)^4(3)^2$	$(3)^4$	$(2)^6$			1
26.	154	(2)(7)(11)	(77)	(22)	(56)		1
27.	155	(5)(31)	(31)	$(5)^3$			1
28.	156	$(2)^2(3)(13)$	(117)	(52)	$(12)^2$		2
29.	180	$(2)^2(3)^2(5)$	(45)	$(10)^2$	$(6)^2$		1
30.	190	(2)(5)(19)	(95)	(76)	(20)		1
31.	210	(2)(3)(5)(7)	(105)	(70)	(126)	(120)	2
32.	228	$(2)^2(3)(19)$	(57)	(76)	(96)		1
33.	234	$(2)(3)^2(13)$	(117)	(208)	$(12)^2$		2
34.	240	$(2)^4(3)(5)$	$(15)^2$	(160)	(96)		2
35.	252	$(2)^2(3)^2(7)$	(189)	(28)	$(6)^2$		1
36.	254	(2)(127)	(127)	$(2)^7$			1
37.	258	(2)(3)(43)	(129)	(172)	$(6)^3$		2
38.	264	$(2)^3(3)(11)$	(33)	(88)	$(12)^2$		1
39.	272	$(2)^4(17)$	(17)	$(2)^8$			1
40.	276	$(2)^2(3)(23)$	(69)	(184)	(24)		1
41.	282	(2)(3)(47)	(141)	(94)	(48)		1
42.	306	$(2)(3)^2(17)$	(153)	(136)	(18)		1
43.	308	$(2)^2(7)(11)$	(77)	(176)	(56)		1
44.	318	(2)(3)(53)	(159)	(106)	(54)		1
45.	363	$(3)(11)^2$	$(11)^2$	$(3)^5$			1
46.	378	$(2)(3)^3(7)$	(189)	(28)	(162)		1
47.	390	(2)(3)(5)(13)	(195)	(130)	(156)	(300)	2
48.	396	$(2)^2(3)^2(11)$	(297)	(352)	$(12)^2$		2
49.	414	$(2)(3)^2(23)$	(207)	(46)	(162)		1
50.	420	$(2)^2(3)(5)(7)$	(105)	(280)	(336)	(120)	2
51.	426	(2)(3)(71)	(213)	(142)	(72)		1
52.	456	$(2)^3(3)(19)$	(57)	(304)	(96)		1
53.	468	$(2)^2(3)^2(13)$	(117)	(208)	$(12)^2$		1
54.	480	$(2)^5(3)(5)$	$(15)^2$	(160)	(96)		1
55.	490	$(2)(5)(7)^2$	(245)	$(14)^2$	(50)		1
56.	492	$(2)^2(3)(41)$	(369)	(328)	(288)		2
57.	510	(2)(3)(5)(17)	(255)	(340)	(306)	(120)	2
58.	516	$(2)^2(3)(43)$	(129)	(172)	$(6)^3$		1

59.	518	(2)(7)(37)	(259)	(148)	(112)		1
60.	520	$(2)^3(5)(13)$	(65)	(416)	(40)		1
61.	528	$(2)^4(3)(11)$	(33)	(352)	$(12)^2$		1
62.	530	(2)(5)(53)	(265)	(106)	(160)		1
63.	544	$(2)^5(17)$	$(17)^2$	$(2)^8$			1
64.	546	(2)(3)(7)(13)	(273)	(364)	(78)	(378)	2
65.	550	$(2)(5)^2(11)$	(275)	(176)	$(10)^2$		1
66.	564	$(2)^2(3)(47)$	(141)	(376)	(48)		1
67.	580	$(2)^2(5)(29)$	(145)	(116)	(320)		1
68.	582	(2)(3)(97)	(291)	(388)	(486)		2
69.	584	$(2)^3(73)$	(73)	$(2)^9$			1
70.	598	(2)(13)(23)	(299)	(92)	(208)		1
71.	600	$(2)^3(3)(5)^2$	$(15)^2$	$(20)^2$	$(24)^2$		2
72.	612	$(2)^2(3)^2(17)$	(153)	(136)	$(18)^2$		1
73.	630	$(2)(3)^2(5)(7)$	(315)	(280)	(126)	(540)	2
74.	638	(2)(11)(29)	(319)	(232)	(88)		1
75.	642	(2)(3)(107)	(321)	(214)	(108)		1
76.	644	$(2)^2(7)(23)$	(161)	(92)	(392)		1
77.	660	$(2)^2(3)(5)(11)$	(165)	(220)	(396)	(540)	2
78.	690	(2)(3)(5)(23)	(345)	(460)	(276)	(300)	2
79.	720	$(2)^4(3)^2(5)$	$(15)^2$	(640)	$(24)^2$		2

Table 50: Homogeneous numbers n, $n \leq 720$.

	n	Factored	Spectral basis		*idx*
1.	6	(2)(3)	(3)	$(2)^2$	1
2.	12	$(2)^2(3)$	$(3)^2$	$(2)^2$	1
3.	14	(2)(7)	(7)	$(2)^3$	1
4.	20	$(2)^2(5)$	(5)	$(2)^4$	1
5.	24	$(2)^3(3)$	$(3)^2$	$(2)^4$	1
6.	39	(3)(13)	(13)	$(3)^3$	1
7.	40	$(2)^3(5)$	$(5)^2$	$(2)^4$	1
8.	56	$(2)^3(7)$	$(7)^2$	$(2)^3$	1
9.	62	(2)(31)	(31)	$(2)^5$	1
10.	72	$(2)^3(3)^2$	$(3)^2$	$(2)^6$	1
11.	112	$(2)^4(7)$	$(7)^2$	$(2)^6$	1

12.	144	$(2)^4(3)^2$	$(3)^4$	$(2)^6$		1
13.	155	$(5)(31)$	(31)	$(5)^3$		1
14.	254	$(2)(127)$	(127)	$(2)^7$		1
15.	272	$(2)^4(17)$	(17)	$(2)^8$		1
16.	363	$(3)(11)^2$	$(11)^2$	$(3)^5$		1
17.	544	$(2)^5(17)$	$(17)^2$	$(2)^8$		1
18.	584	$(2)^3(73)$	(73)	$(2)^9$		1
19.	600	$(2)^3(3)(5)^2$	$((3)(5))^2$	$((2)^2(5))^2$	$((2)^3(3))^2$	2

Table 51: Power-spectral homogeneous numbers n, $n \leq 720$.

	n	Factored	Spectral Basis	
1.	6	$(2)(3)$	(3)	$(2)^2$
2.	12	$(2)^2(3)$	$(3)^2$	$(2)^2$
3.	14	$(2)(7)$	(7)	$(2)^3$
4.	20	$(2)^2(5)$	(5)	$(2)^4$
5.	24	$(2)^3(3)$	$(3)^2$	$(2)^4$
6.	39	$(3)(13)$	(13)	$(3)^3$
7.	40	$(2)^3(5)$	$(5)^2$	$(2)^4$
8.	56	$(2)^3(7)$	$(7)^2$	$(2)^3$
9.	62	$(2)(31)$	(31)	$(2)^5$
10.	72	$(2)^3(3)^2$	$(3)^2$	$(2)^6$
11.	112	$(2)^4(7)$	$(7)^2$	$(2)^6$
12.	144	$(2)^4(3)^2$	$(3)^4$	$(2)^6$
13.	155	$(5)(31)$	(31)	$(5)^3$
14.	254	$(2)(127)$	(127)	$(2)^7$
15.	272	$(2)^4(17)$	(17)	$(2)^8$
16.	363	$(3)(11)^2$	$(11)^2$	$(3)^5$
17.	544	$(2)^5(17)$	$(17)^2$	$(2)^8$
18.	584	$(2)^3(73)$	(73)	$(2)^9$
19.	992	$(2)^5(31)$	$(31)^2$	$(2)^5$
20.	1984	$(2)^6(31)$	$(31)^2$	$(2)^{10}$
21.	3279	$(3)(1093)$	(1093)	$(3)^7$
22.	5219	$(17)(307)$	(307)	$(17)^3$
23.	16256	$(2)^7(127)$	$(127)^2$	$(2)^7$
24.	16382	$(2)(8191)$	(8191)	$(2)^{13}$

25.	19607	$(7)(2801)$	(2801)	$(7)^5$
26.	20439	$(3)^3(757)$	(757)	$(3)^9$
27.	32512	$(2)^8(127)$	$(127)^2$	$(2)^{14}$
28.	65792	$(2)^8(257)$	(257)	$(2)^{16}$
29.	70643	$(41)(1723)$	(1723)	$(41)^3$
30.	97655	$(5)(19531)$	(19531)	$(5)^7$
31.	131584	$(2)^9(257)$	$(257)^2$	$(2)^{16}$
32.	208919	$(59)(3541)$	(3541)	$(59)^3$
33.	262142	$(2)(131071)$	(131071)	$(2)^{17}$
34.	363023	$(71)(5113)$	(5113)	$(71)^3$
35.	402233	$(13)(30941)$	(30941)	$(13)^5$
36.	712979	$(89)(8011)$	(8011)	$(89)^3$

Table 52: Power-spectral homogeneous with two factors.

	n	Factored		Spectral Basis	
1.	600	$(2)^3(3)(5)^2$	$((3)(5))^2$	$((2)^2(5))^2$	$((2)^3(3))^2$
2.	900	$(2)^2(3)^2(5)^2$	$((3)(5))^2$	$((2)(5))^2$	$((2)^3(3))^2$
3.	1200	$(2)^4(3)(5)^2$	$((3)(5))^2$	$((2)^2(5))^2$	$((2)^3(3))^2$
4.	1800	$(2)^3(3)^2(5)^2$	$((3)(5))^2$	$((2)(5))^3$	$((2)^3(3))^2$
5.	2400	$(2)^5(3)(5)^2$	$((3)(5))^2$	$((2)^3(5))^2$	$((2)^3(3))^2$
6.	3528	$(2)^3(3)^2(7)^2$	$((3)(7))^2$	$((2)^2(7))^2$	$((2)^4(3))^2$
7.	4704	$(2)^5(3)(7)^2$	$((3)^2(7))^2$	$((2)^3(7))^2$	$((2)^4(3))^2$
8.	7056	$(2)^4(3)^2(7)^2$	$((3)^2(7))^2$	$((2)^2(7))^2$	$((2)^4(3))^2$
9.	7200	$(2)^5(3)^2(5)^2$	$((3)(5))^2$	$((2)^4(5))^2$	$((2)^3(3))^2$
10.	8100	$(2)^2(3)^4(5)^2$	$((3)^2(5))^2$	$((2)^4(5))^2$	$((2)(3))^5$
11.	9408	$(2)^6(3)(7)^2$	$((3)^2(7))^2$	$((2)^3(7))^2$	$((2)^4(3))^2$
12.	10584	$(2)^3(3)^3(7)^2$	$((3)^2(7))^2$	$((2)^2(7))^2$	$((2)(3)^2)^3$
13.	12348	$(2)^2(3)^2(7)^3$	$((3)(7))^3$	$((2)(7)^2)^2$	$((2)(3)^2)^3$
14.	16200	$(2)^3(3)^4(5)^2$	$((3)^2(5))^2$	$((2)^4(5))^2$	$((2)(3))^5$
15.	18816	$(2)^7(3)(7)^2$	$((3)^2(7))^2$	$((2)^4(7))^2$	$((2)^4(3))^2$
16.	24696	$(2)^3(3)^2(7)^3$	$((3)(7)^2)^2$	$((2)^2(7))^3$	$((2)(3)^2)^3$
17.	28224	$(2)^6(3)^2(7)^2$	$((3)^2(7))^2$	$((2)^2(7))^3$	$((2)^4(3))^2$
18.	31752	$(2)^3(3)^4(7)^2$	$((3)^2(7))^2$	$((2)^2(7))^3$	$((2)(3)^2)^3$
19.	32400	$(2)^4(3)^4(5)^2$	$((3)^3(5))^2$	$((2)^4(5))^2$	$((2)(3))^5$
20.	37044	$(2)^2(3)^3(7)^3$	$((3)(7))^3$	$((2)^2(7))^3$	$((2)(3)^2)^3$

21.	56448	$(2)^7(3)^2(7)^2$	$((3)^2(7))^2$	$((2)^5(7))^2$	$((2)^4(3))^2$
22.	64800	$(2)^5(3)^4(5)^2$	$((3)(5))^4$	$((2)^4(5))^2$	$((2)(3))^5$
23.	124848	$(2)^4(3)^3(17)^2$	$((3)^2(17))^2$	$((2)^3(17))^2$	$((2)^5(3)^2)^2$
24.	222264	$(2)^3(3)^4(7)^3$	$((3)(7))^4$	$((2)^2(7))^3$	$((2)(3)^2)^3$

Table 53: Power-spectral homogeneous with three factors.

	n	Factored	Spectral Basis				idx
1.	132300	$(2)^2(3)^3(5)^2(7)^2$	$(315)^2$	$(350)^2$	$(126)^2$	$(30)^3$	2
2.	8643600	$(2)^4(3)^2(5)^2(7)^4$	$(735)^2$	$(980)^2$	$(1176)^2$	$(2400)^2$	1
3.	132779808	$(2)^5(3)^2(7)^2(97)^2$	$(6111)^2$	$(10864)^2$	$(4656)^2$	$(9408)^2$	2
4.	265559616	$(2)^6(3)^2(7)^2(97)^2$	$(6111)^2$	$(10864)^2$	$(4656)^2$	$(9408)^2$	1

Table 54: Power-spectral homogeneous with four prime factors.

Table 55 shows all power-spectral numbers found so far with all powers in the spectral basis greater than 2. The search was conducted for $p^a q^b$ with $2 \le p < q < 12^3$ and $2 \le a, b \le 6$, and also $p^a q^b r^c$ with $2 \le p < q < r < 12^3$ and $2 \le a, b, c \le 6$.

	n	Factored	Spectral Basis		
1.	144	$(2)^4(3)^2$	$(3)^4$	$(2)^6$	
2.	27783	$(3)^4(7)^3$	$(28)^3$	$(18)^3$	
3.	37044	$(2)^2(3)^3(7)^3$	$(21)^3$	$(28)^3$	$(18)^3$
4.	222264	$(2)^3(3)^4(7)^3$	$(21)^4$	$(28)^3$	$(18)^3$

Table 55: Power-spectral with all spectral powers greater than 2.

11.7 Tables of balanced numbers

See Section 9.1 for the definition of a balanced number.

Pillai's equation is $y^t - x^s = z$. Bennett [2] shows that if x, y, z are nonzero integers with $x, y \ge 2$, then $y^t - x^s = z$ has at most two solutions in s and t. The only solution for $z = 2$ is $3^3 - 5^2 = 2$.

Recall that one of x, $x+2$, $x+4$ is always divisible by 3. Consequently, isotropic numbers of the form $3^k(3^k \pm 2)$, $3^k \pm 2$ primary, are worthy of interest. Both $3^k \pm 2$ are prime only for $k = 2, 4$ (checked out to $k = 5000$). Relevant entries are A057735, A051783, A051783, A051783.

	n	Factored	a
1.	15	$(3)(5)$	2
2.	63	$(3)^2(7)$	4
3.	99	$(3)^2(11)$	5
4.	675	$(3)^3(5)^2$	13
5.	783	$(3)^3(29)$	14
6.	6399	$(3)^4(79)$	40
7.	6723	$(3)^4(83)$	41
8.	58563	$(3)^5(241)$	121
9.	529983	$(3)^6(727)$	364
10.	43059843	$(3)^8(6563)$	3281
11.	387381123	$(3)^9(19681)$	9841
12.	3486902499	$(3)^{10}(59051)$	29525
13.	$3^{14}(3^{14}+2)$	$(3)^{14}(4782971)$	2391485
14.	$3^{15}(3^{15}+2)$	$(3)^{15}(14348909)$	7174454
15.	$3^{22}(3^{22}-2)$	$(3)^{22}(31381059607)$	15690529804
16.	$3^{24}(3^{24}+2)$	$(3)^{24}(282429536483)$	141214768241
17.	$3^{26}(3^{26}+2)$	$(3)^{26}(2541865828331)$	1270932914165
18.	$3^{36}(3^{36}+2)$	$(3)^{36}(150094635296999123)$	75047317648499561
19.	$3^{37}(3^{37}-2)$	$(3)^{37}(450283905890997361)$	225141952945498681
20.	$3^{41}(3^{41}-2)$	$(3)^{41}(36472996377170786401)$	18236498188585393201

Table 56: Balanced numbers of the form $3^k(3^k \pm 2)$, $3^k \pm 2$ prime.

	n	p^s	q^t
1.	63	(7)	$(3)^2$
2.	99	$(3)^2$	(11)
3.	575	(23)	$(5)^2$
4.	675	$(5)^2$	$(3)^3$
5.	2303	(47)	$(7)^2$
6.	28223	(167)	$(13)^2$
7.	129599	(359)	$(19)^2$
8.	705599	(839)	$(29)^2$
9.	1871423	(1367)	$(37)^2$
10.	3415103	(1847)	$(43)^2$
11.	4875263	(2207)	$(47)^2$

12.	13838399	(3719)	$(61)^2$
13.	25401599	(5039)	$(71)^2$
14.	62726399	(7919)	$(89)^2$
15.	112529663	(10607)	$(103)^2$
16.	131056703	(11447)	$(107)^2$
17.	260112383	(16127)	$(127)^2$
18.	294465599	(17159)	$(131)^2$
19.	373262399	(19319)	$(139)^2$
20.	895685183	(29927)	$(173)^2$
21.	1330790399	(36479)	$(191)^2$
22.	1982030399	(44519)	$(211)^2$
23.	2472873983	(49727)	$(223)^2$
24.	2947186943	(54287)	$(233)^2$

Table 57: Balanced numbers at least one 2nd power.

	n	p^s	q^t
1.	675	$(5)^2$	$(3)^3$
2.	783	$(3)^3$	(29)
3.	15875	$(5)^3$	(127)
4.	47032163	(6857)	$(19)^3$
5.	594872099	$(29)^3$	(24391)
6.	887444099	(29789)	$(31)^3$
7.	2565625103	(50651)	$(37)^3$
8.	90457780643	(300761)	$(67)^3$
9.	128100999743	$(71)^3$	(357913)
10.	326941516943	$(83)^3$	(571789)
11.	1677097520783	(1295027)	$(109)^3$
12.	2081954638403	$(113)^3$	(1442899)
13.	11853904702499	(3442949)	$(151)^3$
14.	26808763687523	$(173)^3$	(5177719)
15.	88245920844899	(9393929)	$(211)^3$
16.	195930566150399	(13997519)	$(241)^3$
17.	330928780336703	$(263)^3$	(18191449)
18.	451729625460623	(21253931)	$(277)^3$
19.	904820357173823	$(311)^3$	(30080233)

20.	2443410118063043	(49430861)	$(367)^3$
21.	5411082427203599	$(419)^3$	(73560061)
22.	6410082687992063	$(431)^3$	(80062993)
23.	7157924466012323	(84604517)	$(439)^3$
24.	9109555608896063	(95443991)	$(457)^3$

Table 58: Balanced numbers with at least one 3rd power.

	n	p^s	q^t
1.	6399	(79)	$(3)^4$
2.	6723	$(3)^4$	(83)
3.	5759999	(2399)	$(7)^4$
4.	214329599	(14639)	$(11)^4$
5.	815673599	(28559)	$(13)^4$
6.	500244998399	(707279)	$(29)^4$
7.	7984919577599	(2825759)	$(41)^4$
8.	11688193439999	(3418799)	$(43)^4$
9.	62259674630399	(7890479)	$(53)^4$
10.	146830413369599	(12117359)	$(59)^4$
11.	806460035097599	(28398239)	$(73)^4$
12.	2252292137222399	(47458319)	$(83)^4$
13.	10828566848159999	(104060399)	$(101)^4$
14.	1151936655676934399	(1073283119)	$(181)^4$
15.	10645920221258649599	(3262808639)	$(239)^4$
16.	11379844831814553599	(3373402559)	$(241)^4$
17.	34660765681779513599	(5887339439)	$(277)^4$
18.	54317648794580639999	(7370050799)	$(293)^4$
19.	87515123928719385599	(9354951839)	$(311)^4$
20.	329100478671097958399	(18141126719)	$(367)^4$
21.	524320466653868601599	(22898045039)	$(389)^4$
22.	783044537043854361599	(27982932959)	$(409)^4$
23.	986862773180683526399	(31414372079)	$(421)^4$
24.	1235671900452135014399	(35152125119)	$(433)^4$

Table 59: Balanced numbers with at least one 4th power.

	n	p^s	q^t
1.	58563	(241)	$(3)^5$
2.	25937746703	$(11)^5$	(161053)
3.	137857749263	(371291)	$(13)^5$
4.	819628229722499	(28629149)	$(31)^5$
5.	2692452204093162711203	(51888844697)	$(139)^5$
6.	5393400662210288062499	$(149)^5$	(73439775751)
7.	33769941616650809609999	$(179)^5$	(183765996901)
8.	37738596846567176009999	(194264244899)	$(181)^5$
9.	88036397287929669054563	$(197)^5$	(296709280759)
10.	174913992534571524202499	(418227202049)	$(211)^5$
11.	396601930089755628053903	(629763392147)	$(229)^5$
12.	2136450862852458317122499	(1461660310349)	$(271)^5$
13.	3069468628630508908781603	$(281)^5$	(1751989905403)
14.	10246902931640688936244163	$(317)^5$	(3201078401359)
15.	79340697341477775488902499	$(389)^5$	(8907339520951)
16.	107508728670766600970408003	$(401)^5$	(10368641602003)
17.	166778563814503095827609999	$(419)^5$	(12914277518101)
18.	814357163924333005883276303	$(491)^5$	(28536943843453)
19.	1167287469089587207378402499	$(509)^5$	(34165588961551)
20.	1531147003165767344482700963	(39129873538841)	$(523)^5$
21.	4857144371993235558334677263	$(587)^5$	(69693216111709)
22.	8258601109663183475705453603	(90876845839097)	$(619)^5$
23.	57068882708943666839076562499	(238890943128749)	$(751)^5$
24.	203934316504188772999513538063	(451590872919491)	$(853)^5$

Table 60: Balanced numbers with at least one 5th power.

	n	p^s	q^t
1.	529983	(727)	$(3)^6$
2.	3138424833599	(1771559)	$(11)^6$
3.	582622188954623	(24137567)	$(17)^6$
4.	6582952000708582463	(2565726407)	$(37)^6$
5.	116191483087390147583	(10779215327)	$(47)^6$
6.	1779197418155171649599	(42180533639)	$(59)^6$
7.	8182718904451940380223	(90458382167)	$(67)^6$

8.	17605349516212372526137343	(4195872914687)	$(127)^6$
9.	52020869037274660381185599	(7212549413159)	$(139)^6$
10.	718709255220739442810439743	(26808753332087)	$(173)^6$
11.	3838879772218512720387279359 9	(195930594145439)	$(241)^6$
12.	5970253488172341814114971853823	(2443410216924767)	$(367)^6$
13.	7252804675881431341004027041343	(2693103168443687)	$(373)^6$
14.	9962888906032509503734088269823	(3156404426880767)	$(383)^6$
15.	41089158014053994279718414566399	(6410082527866079)	$(431)^6$
16.	418813108628950541349519402452543	(20464923860814887)	$(523)^6$
17.	628589585424465080169599628417599	(25071688922457239)	$(541)^6$
18.	1151720764209211733640596153606399	(33937011715960079)	$(569)^6$
19.	2815306572555543305526130992665663	(53059462610881607)	$(613)^6$
20.	3984302564291813331723675483398399	(63121332085847279)	$(631)^6$
21.	8633378286317816481569038992543743	(92915974333361087)	$(673)^6$
22.	10305010631710575529462783133596223	(101513598260088167)	$(683)^6$
23.	11850497104030975766947341545409599	(108859988535875639)	$(691)^6$
24.	105354809608016667510072618019790399	(324584056305938519)	$(829)^6$

Table 61: Balanced numbers with at least one 6th power.

11.8 Tables of numbers modulo index

	n	Factored	idx
1.	6	(2)(3)	1
2.	42	(2)(3)(7)	2
3.	924	$(2)^2(3)(7)(11)$	3
4.	23100	$(2)^2(3)(5)^2(7)(11)$	4
5.	1750320	$(2)^4(3)^2(5)(11)(13)(17)$	5
6.	36606570	(2)(3)(5)(7)(11)(13)(23)(53)	6

Table 62: Numbers n such that $n \equiv 0 \pmod{idx}$.

	n	Factored	idx
1.	195	(3)(5)(13)	2
2.	1870	(2)(5)(11)(17)	3
3.	47058	(2)(3)(11)(23)(31)	4

4.	1527372	$(2)^2(3)^2(7)(11)(19)(29)$	5

Table 63: Numbers n such that $n \not\equiv 0 \pmod{idx}$.

References

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