Complete Proof of the Collatz Conjecture

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PROOF OF THE COLLATZ CONJECTURE

Abstract

The Collatz conjecture has remained unsolved for a long time. In this paper, a proof of this conjecture will be provided. We know that almost all numbers will reach one by following the steps of the Collatz algorithm. Terence Tao has previously proven this. This paper uses Tao's proof to demonstrate that all numbers will eventually reach one. If almost all numbers reach the number one, then the probability that a randomly selected number from an infinite set will reach one is one. Conversely, the probability of not reaching one is zero. The probability of selecting elements of a sequence associated with a number n that violates the conjecture from an infinite set is a non-zero number, such as c, but this contradicts the proof that almost all numbers will reach one. Therefore, there is no such number n that violates the conjecture, and the conjecture holds true for all numbers. To prove that the probability of selecting elements of a sequence associated with a number n that violates the conjecture from an infinite set is a non-zero number, such as c, we look at the sequences associated with the number one. If the probability of selecting elements of each sub-branch of these sequences from an infinite set is a non-zero number, then we reach the proof we are looking for.

Introduction

The Collatz conjecture simply states the following: Pick a positive integer. If this number is even, divide it by two; if it's odd, multiply it by three and add one. Repeat the same operation with the new number you get. Continue this way, and eventually, you'll reach the number one. The problem here is to prove that every positive integer will eventually reach one. This paper will demonstrate this proof. There could be two scenarios that violate the conjecture: either the numbers increase indefinitely, or there are some closed loops.

As a result, Tao has proven that almost all numbers will reach one according to the Collatz conjecture. Now, let's prove that all numbers reach one.

To do this, let's examine a number n . Assume that this number does not reach one, meaning it violates the conjecture. Undoubtedly, we can reach the number n from some other numbers as well. Here, let's reverse the Collatz algorithm and see from which numbers we can reach n . We can reach n by dividing 2n by two. Similarly, we can reach n by dividing 4n by four. These numbers form a sequence:

n-2n-4n-8n-16n-... This means that if a single number n violates the conjecture, an infinite number of other numbers also violate the conjecture. However, this sequence is not limited to just these numbers. The initial number n could be in the form of 3k , 3k+1 , or 3k+2 . If the number is in the form of 3k , we take the next odd number. That is, (3n+1)/2 or (3n+1)/4... Our goal is to start with a number in the form of 3k+1 or 3k+2 because numbers that are powers of three do not lead to new sequences in subsequent steps. Let me explain what I mean with an example. To reach the number three, we have the sequence 3-6-12-24-48-96-... This sequence does not lead to new odd numbers because we cannot reach these even numbers by multiplying an odd number by three and adding one, but let's look at the sequence of the number five: 5-10-20-40-...

This sequence leads to new elements. For instance, we can reach the number 10 from the number three, or we can reach the number 40 from the number 13. In the next step, the number 13 leads to a new sequence like 13-26-52-104-..., and these steps can be repeated infinitely. We started with a number n , and if n is a multiple of three, it can be proven that the next odd number will not be a power of three in the subsequent step. If our first number is n , our next odd number will be (3n+1)/2^k . This number is not a power of three. As a result, if we have a number n , and this number violates the conjecture, we are faced with a tree-like structure with infinite branches and each branch leading to an infinite number of new branches. Example:

```
                                 21-42-84-…
                                  ↑
1-  2-  4-  8-   16-   32-   64  -  128-…
                  ↓
                 5-10-20-40-…
                    ↓
                   3-6-12-24-48-…
```

Our proof claims that the probability of selecting the elements of such a tree, starting from a number n , from an infinite set is a probability c . In this case, it contradicts the claim that almost all numbers reach one. Therefore, if it is proven that this probability is a number like c , the conclusion would be that such elements are not found in Collatz sequences. In other words, there is no situation that violates the conjecture because if the probability of selecting the elements of sequences that violate the conjecture from an infinite set is a non-zero probability like c , this would contradict the proof that almost every number reaches one. Therefore, it is sufficient to prove that the probability of selecting the elements of sequences related to a number n that violates the conjecture from an infinite set is a number like c . For this, let's first look at the number one. We know that the branches of the sequence starting from the number one contain almost all the numbers. The probability of selecting the elements of the sequences starting from the number one from an infinite set is one. The first sequence starting from the number one has an infinite number of branches. Each of those branches also has an infinite number of branches. This continues indefinitely. Now, we must show that the probability of selecting the elements of any branch of the sequence starting from the number one from an infinite set is c . The total contributions of all branches are one. This can happen in two ways. Either each branch contributes c(1), c(2), c(3) , etc., or each branch contributes zero. In other words, 1 = c(1) + c(2) + ... or 1 = 0 + 0 + ... The second representation contains the indeterminacy of 0*∞ . There is a third possibility, which is 1 = 1 + 0 + 0 +... , meaning only the first branch can contribute to the probability. Let's refute this possibility. If the first branch contributes to the entire probability, the first sub-branch of this branch also contributes to the probability. Because if other branches contribute a non-zero amount to the probability, it would be the same for the upper branches as well, but only the first branch of the upper branches contributes to the probability. As a result, the first sub-branch's sub-branch also contributes to the probability. This continues, and the numbers grow. At some point, the first element of the sub-branch that contributes to the probability will be greater than the first element of the second branch of the upper branches. At this point, it would be illogical for this sub-branch to contribute more to the probability than the second upper branch because if we compare the number of elements of these branches, the number of elements in the second upper branch would be greater than the number of elements in the sub-branch related to the first branch. Therefore, it is illogical for the first upper branch to make such a contribution to the probability. Let's explain this visually:

```
                         85
                         ↑
1-2-4-8-16-32-64-128-256-…
         ↓
         5- 10-  20-  40-…
            ↓         ↓
            3-6-…    13-26-52-…
                            ↓
                            17-34-68-…
                                ↓
                                11-22-44-…
                                    ↓
                                    7-  14- 28-   56-  112
                                            ↓          ↓
                                            9-18-…    37-74-148-…
                                                             ↓
                                                             49-98-196-…
                                                                    ↓
                                                                    65-130-260-…
                                                                         ↓
                                                                         43-86-172-344-688...
                                                                                 ↓        ↓
                                                                                 57       229
```

In this sequence that branches out, the starting number of the last branch we reach is 229. 229 > 85 . Therefore, if the probability of selecting the elements of the sequence starting with 229 from an infinite set is c , then the probability for the sequence starting with 85 cannot be zero because the sequence starting with 85 has more elements than the sequence starting with 229. In the visual above, we progressed by taking the first sub-branches of the sequences, but we did not take into account the sequences starting with elements that are powers of three. This is because the probability of selecting the elements of sequences starting with elements that are powers of three from an infinite set is zero. As a result, the probability of selecting the elements of the sequence starting with the number one from an infinite set cannot be 1 = 1 + 0 + 0 +... because the probability of selecting the elements of the first sub-branch from an infinite set cannot be one. Now, let's prove that the sum 1 = 0 + 0 + 0 +... is also invalid.

Theorem

For each branch of the sequence starting with 1, the probability of selecting the elements in that branch from an infinite set cannot be zero.

Proof

Consider the sequence:
1-2-4-8-16-...

For each element, the probability of selecting it is $1/\infty = 0$. We know that the probability of selecting an element of this sequence from the set of natural numbers is zero. Therefore, $0 = 0 + 0 + 0... = 0 * \infty$. For branches starting with the number 1, $1 \neq 0 + 0 + 0... = 0 * \infty = 0$ because the number of zeros in the sum for the above sequence is greater than the number of zeros in the sequence starting with 1 and branching out infinitely. Therefore, if $0 * \infty = 0$ for one, then $0 * \infty = 0$ for the other as well. "Here, the important point to remember is that each zero in the sum represents each first sub-branch of the sequence starting with 1, in a sequence that begins with 1 and branches out. Let's explain this visually.

```
                              21-42-84-…
                              ↑
1-  2-  4-  8-   16-   32-   64  -  128-…
                 ↓
                 5-10-20-40-…
                   ↓
                   3-6-12-24-48-…
```

In this sequence, the first sub-branch starts with the number 16 and is represented by the first zero in the sum. The second sub-branch starts with the number 64 and is represented by the second zero in the sum. Since the probability of selecting all elements like 4, 8, 32, etc., from the second sequence out of an infinite set is zero, excluding them from the sum does not change the result. If we include these numbers in the sum, the number of zeros will be equal for both sequences, and $0 * \infty$ will still be zero for both sequences.

Conclusion

For a number n that violates the conjecture, there exist infinitely extending branches and sequences associated with n. The probability of selecting the elements of the set associated with this n from the set of natural numbers is equal to a non-zero number c. However, this is impossible because the probability of selecting numbers that reach 1 from the set of natural numbers is 1. Therefore, there cannot exist a number n that violates the conjecture, and all numbers must reach 1 according to the Collatz algorithm.To prove that the probability of selecting the elements of a set associated with a number n from an infinite set is a non-zero number c, we look at the sequence starting with the number 1. This sequence has sub-branches, and for each sub-branch, we proved that the probability of selecting the numbers in this branch from the set of natural numbers is greater than zero. Hence, the equality $1 = c(1) + c(2) + c(3) + ...$ holds true for the sub-branches of the sequence associated with 1. No branch can contribute zero to the probability, and in this case, the probability of selecting the elements of a total set associated with a sequence starting with a number t, which is greater than n and does not violate the conjecture, from the set of natural numbers would be a non-zero number c(t).We know that the number t does not violate the conjecture, meaning that t is the starting number of one of the sub-branches of the sequence beginning with 1. However, the number of elements in the set associated with the sequence starting with n is greater than the number of elements in the set associated with the sequence starting with t because $t > n$. Therefore, the probability of selecting the elements of the set associated with the sequence starting with n from the set of natural numbers is a number greater than zero, c(n).

As a result, since it is proven that nearly all numbers reach 1, the probability of selecting the elements of a block of numbers that violate the conjecture from the set of natural numbers is zero. Hence, if the probability of selecting the elements of sequences starting with n that violates the conjecture from an

infinite set is a number like c(n), that is, not zero, then ultimately, there is no number n that violates the conjecture.
The Collatz conjecture is true for all numbers. The conjecture has been proven.

References

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