Andrey V. Voron

The Magic squares of Khajuraho, Durer and the Golden Proportion

Introduction. Based on the theoretical analysis of Khajuraho, Durer squares and similar 4×4 squares, the features of their "structure" are revealed: the invariants of the structure of 4×4 pandiagonal squares are pairs of numbers equal in sum to one of the two Fibonacci numbers – 13 or 21.

A magic square is an n×n square table filled with n2 different numbers in such a way that the sum of the numbers in each row, each column and on both diagonals is the same. The earliest unique 4×4 magic square was discovered in an inscription of the XI century in the Indian city of Khajuraho. The 4×4 square depicted in Albrecht Durer's engraving "Melancholy" is considered the earliest in European art (1514). The sum of the numbers of the Durer square on any horizontal, vertical and diagonal is 34. This sum is also found in all 2×2 corner squares, in the central square, in the square of corner cells, in squares constructed by the "knight's move" (2+12+15+5 and 3+8+14+9), at the vertices of rectangles parallel to the diagonals (2+8+15+9 and 3+12+14+5), in rectangles formed by pairs of middle cells on opposite sides (3+2+15+14 and 5+8+9+12). Most of the additional symmetries are due to the fact that the sum of any two centrally symmetrically arranged numbers is 17.

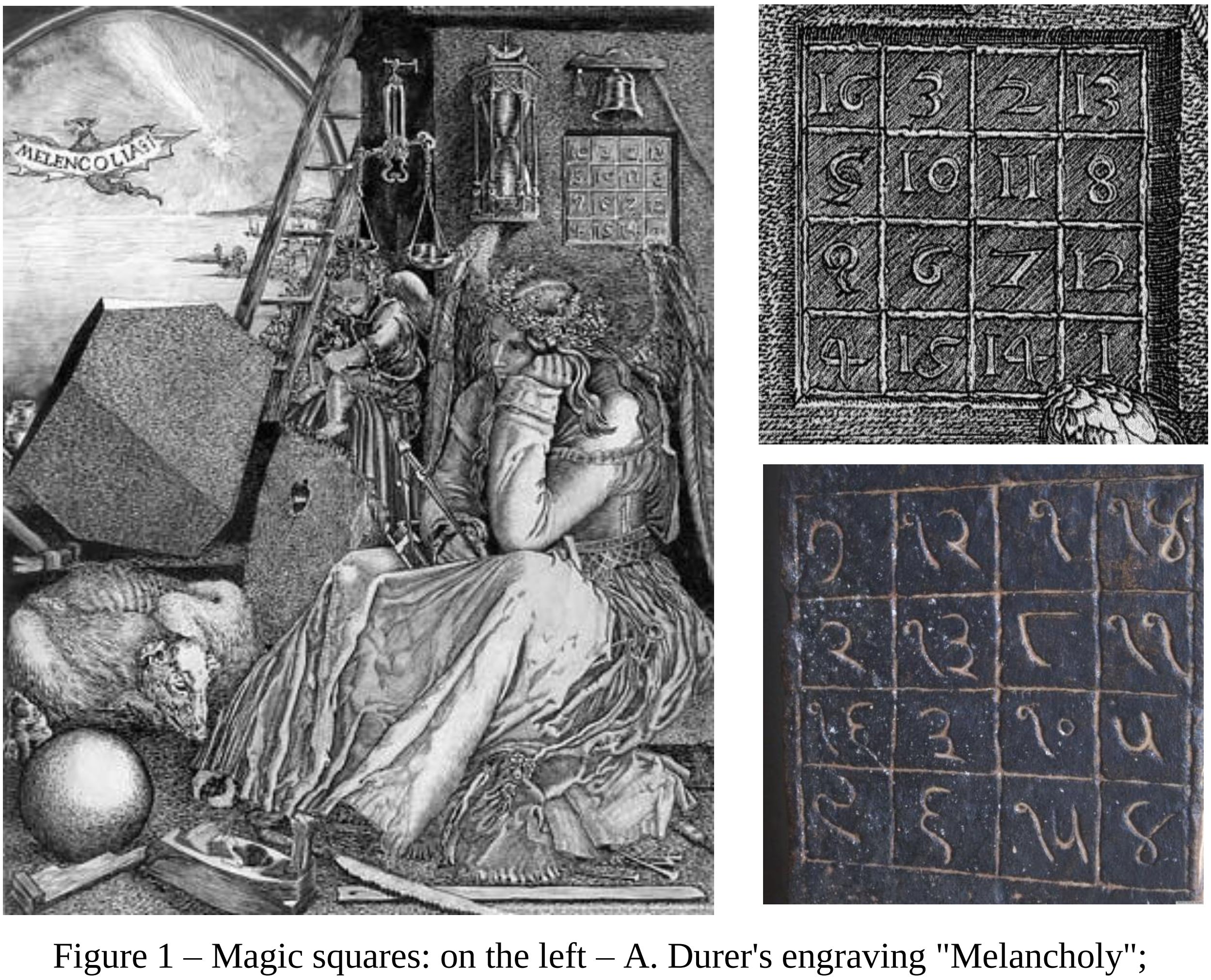

Figure 1 – Magic squares: on the left – A. Durer's engraving "Melancholy";
on the top right – Durer's square; on the bottom right – Khajuraho square

There are 48 4×4 pandiagonal squares with precision to rotations and reflections. If we also take into account the symmetry with respect to toric parallel transfers, then there are only 3 significantly different squares (Figure 2).

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Figure 2 – Magic squares. Three variants of squares (right universal)

The main part. We have analyzed the "structure" of 4×4 pandiagonal squares and identified invariant parts of their structure (Figure 3). The invariants of the structure of 4×4 pandiagonal squares are pairs of numbers equal in total to one of the two Fibonacci numbers – 13 or 21. Various variants of the symmetrical combination of these numerical pairs form a set of 4×4 pandiagonal squares.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

a)

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

b)

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

c)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

d)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

e)

Figure 3 – Magic squares: a, b, c – basic variants of 4×4 squares;
d – Khajuraho square; e – Durer square

The Durer square (and similar 4×4 pandiagonal squares) have the symmetry of the golden proportion. For example, Figure 4 shows variants of symmetries with red and blue squares, in which the arithmetic mean of the sum of the red components of the squares in possible positions (4 or 2, when rotating in different directions) is 51. Thus, the sum of all the numbers of the square is 136, of which 85 are blue, 51 are red. 136/85=1,6; 85/51=1,667.

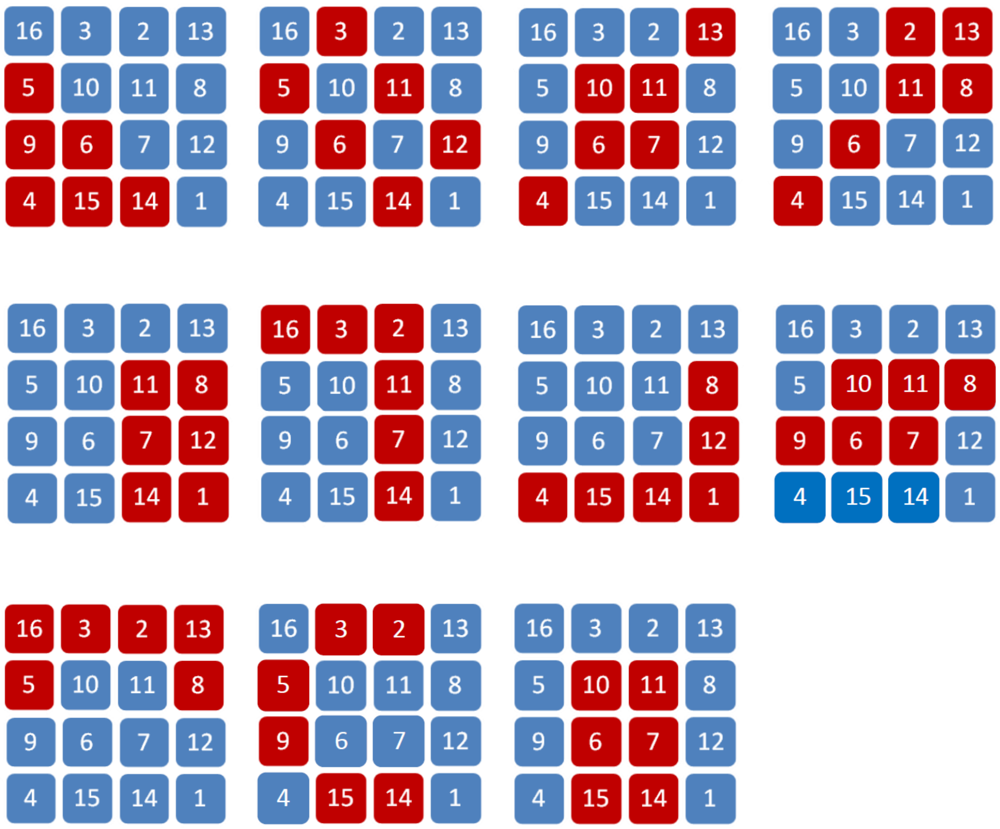

Figure 4 – Magic squares. Variants of symmetries based on the Durer square

On the basis of the Durer square, we have constructed a geometric figure "cube within a cube", which has the symmetry properties of 4×4 pandiagonal squares (Figure 5). Such a "transformation" became possible when the vertical columns of the Durer square numbers were positioned at a certain angle, thus forming a cube within a cube. At the same time, all the numbers of the diagonals of the cube have the properties of "golden symmetry" (two numbers form in one case the total number 13, in the other – 21), and all planes having 4 angles (numbers) Both the inner and outer squares of the constructed figure add up to the Fibonacci number – 34.

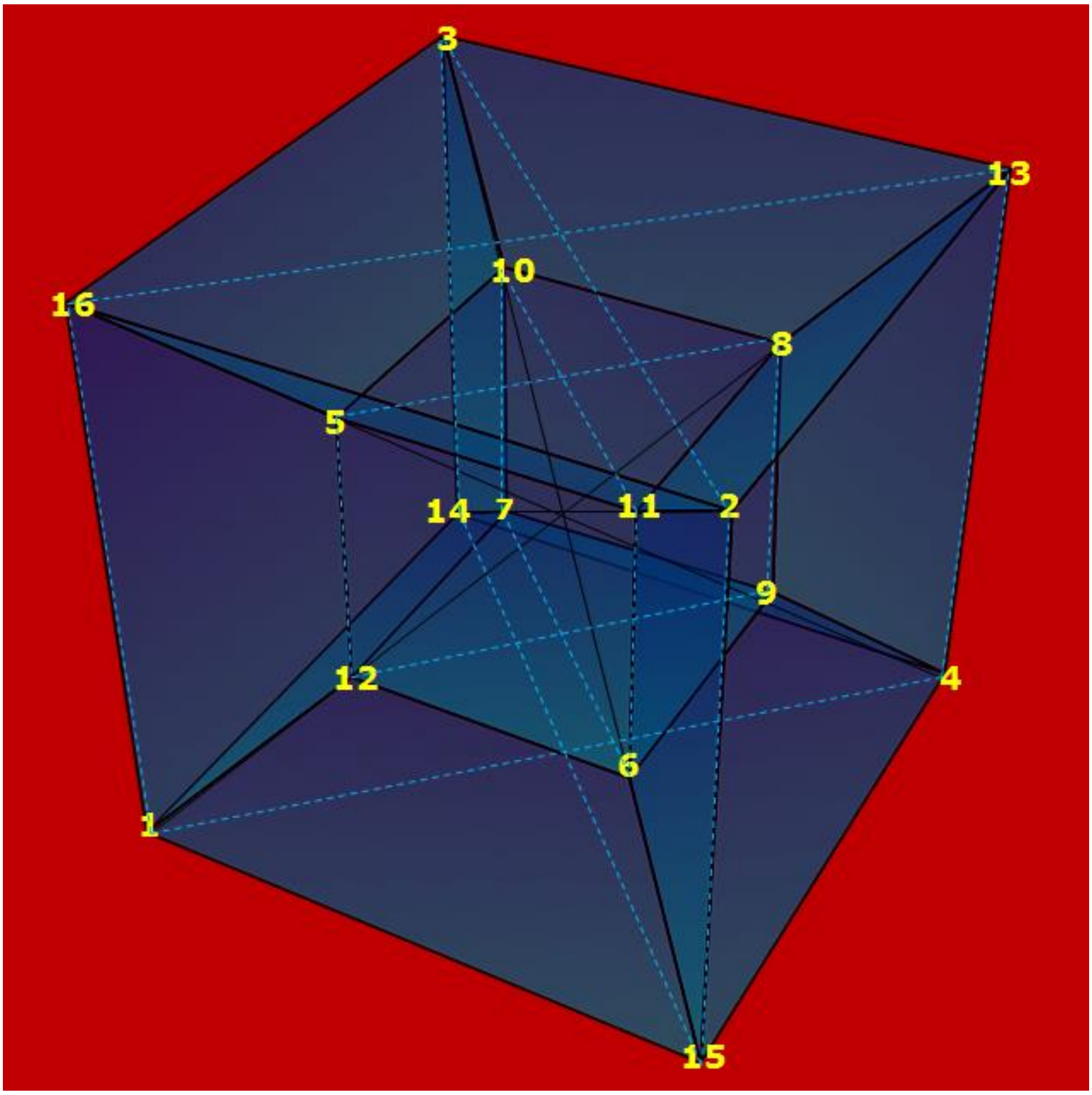

Figure 5 – Geometric shape "cube in a cube"

Conclusion.

1. Based on the theoretical analysis of 4×4 pandiagonal squares, their "structure" features are shown: the invariants of the structure of 4×4 pandiagonal squares are pairs of numbers equal in sum to one of the two Fibonacci numbers – 13 or 21.

2. It is revealed that any variant of the set of six digits of the Durer square and similar 4×4 pandiagonal squares, forming a continuous symmetric configuration, is equal in total to the integer 51.

3. A geometric figure "cube in a cube" is constructed, which has the properties of the "golden symmetry" of 4×4 pandiagonal squares. All the numbers of the diagonals of the cube have the properties of "golden symmetry" (two numbers form in one case the total number 13, in the other – 21), and all planes having 4 angles (numbers) both the inner and outer squares of the geometric figure form a total of the Fibonacci number – 34.