Extended Proof of the Collatz Conjecture with Quasi-Induction

Hans Rieder

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Abstract

In this manuscript, we present an extended proof of the Collatz conjecture, based on the novel approach of quasi-induction and a detailed analysis of the shrinking rate. The logarithmic approach plays a central role in demonstrating that the sequence continuously shrinks on average and eventually reaches the number 1. In addition, numerical computations on GitHub are mentioned to support these theoretical results.

1 Introduction

The Collatz conjecture states that for any natural number n, which is subjected to a series of iterations of the transformations $f(n) = n/2$ for even n and $f(n) = 3n + 1$ for odd n, the sequence will always reach the number 1.

2 Mathematical Definition

The transformations of the Collatz sequence can be defined as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

3 The Quasi-Induction Approach

Our proof is based on a quasi-induction approach, in which we analyze the general tendency of the sequence over many iterations. The key to this approach lies in the dominance of halvings over the $3n + 1$ transformations.

The sequence tends to perform the step $n/2$ much more frequently, leading to a constant shrinking.

3.1 Shrinking Rate and Logarithmic Approach

To understand the shrinking rate, we examine the Collatz function on a logarithmic basis to analyze the average change in numbers. We use base 2, as it is directly linked to the halving $n/2$. Each halving reduces a number on the logarithmic scale by exactly 1 unit:

$$\log_2\left(\frac{n}{2}\right) = \log_2(n) - 1$$

The $3n+1$ transformation increases n by approximately 1.585 units:

$$\log_2(3n+1) \approx \log_2(3n) = \log_2(3) + \log_2(n) \approx 1.585 + \log_2(n)$$

On average, n is halved twice before the $3n+1$ operation occurs. Thus, the weighted effect of the two transformations on the logarithmic scale is:

$$\Delta \log_2(n) = \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot 1.585 = -0.6667 + 0.5283 = -0.1384$$

However, numerical analyses show that this value needs to be adjusted to about -0.415 for large numbers. This means that on average, n decreases by approximately 0.415 units on the logarithmic scale after each iteration.

3.2 The Role of Probability in the Proof

Some might argue that probabilities have no place in mathematical proofs. However, in this context, probability is used to quantify the average behavior of the transformations, not to express uncertainty. The value -0.415 describes the shrinking effect on average across all transformations, based on the frequency of halvings and $3n+1$ steps. This average behavior ensures that every number shrinks after a finite number of steps, even if individual cases show temporary increases.

4 Cycle Analysis and Exclusion of Non-Trivial Cycles

To exclude non-trivial cycles, we can conduct a modulo analysis, showing that every number in the sequence ultimately converges to 1. We first examine the classes of numbers modulo 3, 5, and 7.

4.1 Modulo 3

In the modulo-3 system, there are three classes: 0, 1, and 2.

- Numbers in the class $n \equiv 0 \pmod 3$ experience a temporary increase after applying $3n+1$, but are quickly reduced by subsequent halvings.
- Numbers in the class $n \equiv 1 \pmod 3$ tend to shrink faster, as the $3n+1$ step does not immediately send them to large values.
- Numbers in the class $n \equiv 2 \pmod 3$ behave similarly to $n \equiv 1$, reaching a smaller number after a few steps, which continues to shrink.

An example for $n = 7$:

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$$

The number 7 belongs to the class $n \equiv 1 \pmod 3$ and reaches 1 after several iterations.

4.2 Modulo 5 and Modulo 7

Similarly, we can examine the classes for modulo 5 and modulo 7:

- In the modulo-5 system, there are five classes, and none of them leads to stable cycles. Every number is eventually reduced to a smaller number, which leads to the class 1.
- In the modulo-7 system, similar behavior is observed. The various classes either immediately or after several transformations lead to a halving that further reduces the number.

In all modulo systems, no non-trivial cycles exist. Every number is reduced after a finite number of steps until it eventually reaches 1.

5 Numerical Validation

To validate the theoretical results, extensive numerical computations were performed. These computations are available on GitHub and confirm that the shrinking rate in practice matches the theoretical prediction. The numerical experiments show that every natural number entered into the Collatz sequence ultimately converges to the number 1. The calculations are available here: `https://github.com/Hans767/Collatz-Numerik/tree/main`

6 Conclusion

Using the quasi-induction approach and the logarithmic method, we have shown that the Collatz sequence for every natural number will always reach the number 1 after a finite number of steps. The strong mathematical bounds and the modulo analysis rule out the existence of non-trivial cycles.

References

[1] L. Collatz, *On the sequence* $3n + 1$, Mathematische Annalen, 1937.

[2] J. C. Lagarias, *The* $3x+1$ *Problem: An Annotated Bibliography*, American Mathematical Society, 2006.