The nature of semi-exponentials

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ABSTRACT: A "semi-exponential" is a function F(z) such that F(F(z))=exp(z). We show that (a) no entire-analytic semi-exponential F(z) exists; (b) no semi-exponential F(z) exists that is analytic within any interior-connected domain that includes both the real axis, and all complex Q obeying Q=exp(Q), in its interior, and which maps reals→reals; (c) Analytic semiexponentials *do* exist that map most reals to complex numbers and which have non-analytic points; (d) We also construct a useful piecewise-analytic real→real semi-exponential such that F, F', and F'' all are continuous, and F(x) is strictly increasing and strictly concave-∪, for all real x; and indeed the domain of definition of this F(z) may be slightly expanded to a long and thin complex set that includes the real axis in its interior, albeit then F becomes discontinuous at an infinite set of nonreal points. (e) But we show that no piecewise-analytic, with piece boundaries being nonempty rectifiable differentiable curves, semi-exponential that maps reals→reals can be defined within any domain that includes the strip 0≤im(z)<π. Many of our arguments may be repurposed for many other "semi" functions besides the exponential. Finally (f) we show that real-valued C^{∞}-smooth, strictly increasing, strictly concave-∪ semi-exponentials exist, and under certain asymptotic analyticity demands are unique.

A hoary problem is to find a nice "semi-exponential" function F(z) such that F(F(z))=exp(z). Equivalently, if L(z)=lnF(z) then $L(e^{L(z)})=z$. Ideally we'd want F(z) to obey both

i. F(z) is an analytic function of z (optimally, entire-analytic),
ii. F(z) always maps real numbers to real numbers.

However, as we shall prove, achieving *both* is impossible. This may be regarded as a follow-on to the prior article on that topic by Crone & Neuendorffer 1988. We'll assume the reader knows complex analysis, e.g. see the textbook by R.Remmert, and is familiar with the properties of exp(z) and ln(z).

Two (or three or ... or infinity?) piecewise- and recursively-defined solutions

Crone & Neuendorffer gave a nice real→real semi-exponential F(z) satisfying (ii) but not (i). Specifically,

F(X)=X+1/2 if 0≤X≤1/2,   F(X)=exp(X-1/2) if 1/2≤X≤1,
and for X>1 define F recursively by F(X)=exp(F(lnX)),   and for X<0 by F(X)=ln(F(expX)).
The latter causes $F(X)=e^X-1/2$ when -ln2≤X≤0 and $F(X)=\ln(e^X+½)$ when X≤-ln2.
When 1≤X≤√e we have F(X)=X√e, and when √e≤X≤e we have $F(X)=e^{X\exp(-1/2)}$.

As I pointed out in the wikipedia article, this piecewise- and recursively-defined F(X) is continuous and strictly increasing for all real X, and its derivative F'(X) also is continuous and (non-strictly) increasing, so this F(X) is (non-strictly) concave-∪. The second derivative F''(X) has discontinuities at a countable discrete subset $S_{½}$ of real X, namely

$$S_{½} = \{-\ln 2, 0, 1/2, 1, \sqrt{e}, e, e^{\sqrt{e}}, e^e, \exp(e^{\sqrt{e}}), \exp(e^e), ...\}$$

where each further entry X_n equals $\exp(X_{n-2})$. Another advantage of their F(X) is that it is very easy to compute. Call it F_I.

Now here is a different F(X) – call it F_{II} – of a similar ilk but enjoying higher-degree continuity:

$$F(X)=(3X+2)/(4-X) \text{ if } 0\le X\le 1/2, \quad F(X)=\exp\big((4X-2)/(3+X)\big) \text{ if } 1/2\le X\le 1,$$
while for X>1 define F recursively by $F(X)=\exp(F(\ln X))$, and for X<0 by $F(X)=\ln(F(\exp X))$.
The latter causes $F(X)=(4e^X-2)/(e^X+3)$ when $-\ln 2\le X\le 0$ and $F(X)=\ln((3e^X+2)/(4-e^X))$ when $X\le -\ln 2$.
When $1\le X\le \sqrt{e}$ we have $F(X)=e^{(2+3\ln X)/(4-\ln X)}$, and when $\sqrt{e}\le X\le e$ we have $F(X)=\exp(e^{(4\ln X-2)/(3+\ln X)})$.

Properties of $F_{II}(X)$ theorem: This F(X) is strictly increasing from $F(-\infty)=-\ln 2$ to $F(+\infty)=+\infty$, and strictly concave-∪, and F(X), F'(X), and F''(X) are continuous, for all real X; also L(X)=lnF(X) is strictly increasing, and F(X) is positive, for all X>-ln2; but F'''(X) has jump discontinuities when $X\in S_{½}$. F(X) is analytic for all real $X\notin S_{½}$.

Proof: Plainly F(X) is analytic when 0<X<½ and when ½<X<1. When X=0 the two Taylor series of F(X) are $(3X+2)/(4-X)=1/2+(7/8)X+(7/32)X^2+(7/128)X^3+...$ (converges when |X|<4) and $(4e^X-2)/(e^X+3)= 1/2+(7/8)X+(7/32)X^2-(7/384)X^3-...$ (converges when $|X|<[\pi^2+(\ln 3)^2]^{1/2}\approx 3.328$)

The two Taylor series of F(X) with X=Y+½ at Y=0 are $(3X+2)/(4-X)=1+(8/7)Y+(16/49)Y^2+(32/343)Y^3+...$ (converges when |Y|<7/2=3.5) and $\text{Exp}([4X-2]/[3+X])= 1+(8/7)Y+(16/49)Y^2-(32/1029)Y^3-...$ (converges when |Y|<3.5). This proves continuity of F, F', and F'', but discontinuity of F''' at X=0 and X=½. The recurrence proves it for all other X.

The fact that F(X) is strictly increasing for all X follows when 0≤X≤½ from the explicit formula $F'(X)=14(4-X)^{-2}$, and when ½≤X≤1 from the explicit formula $F'(X)=14(3+X)^{-2}e^{(4X-2)/(3+X)}$ which both plainly are positive-real. Then the recurrence proves the same thing for all real X, given the fact that lnX and expX both are strict-increasing.

The fact that L(X)=lnF(X) is strictly increasing for all X follows when 0≤X≤½ from the explicit formula L'(X)=F'(X)/F(X) since F'(X)>0 and ½≤F(X)≤1. And when ½≤X≤1, and indeed all X>-ln2, the same thing happens for the same reason.

The fact F(X) is strict concave-∪ for 0≤X≤1 follows since the second derivatives also plainly are positive real in those intervals: $F''(X)=28(4-X)^{-3}$ when 0≤X≤½ and $28(4-X)(X+3)^{-4}e^{(4X-2)/(3+X)}$ when ½≤X≤1. [One could similarly handle the intervals (-∞,-ln2] and [-ln2,0] using our explicit F formulas, if desired...] Then the recurrence proves the same thing for all real X given the fact that $(d/dX)\ln(F(e^X))=e^X L'(e^X)$. **Q.E.D.**

In my opinion F_{II} obsoletes Crone & Neuendorffer's F_I.

A third F(X)? Here is a (probably failed) attempt to get still-greater continuity – call it F_{III}:

$$F(X)=[(AX+B)/(C-X)]^{1/P} \text{ if } 0\le X\le 1/2, \quad F(X)=\exp\big((CX^P-B)/(X^P+A)\big) \text{ if } 1/2\le X\le 1,$$
$$\text{while for } X>1 \text{ define F recursively by } F(X)=\exp(F(\ln X)), \quad \text{and for } X<0 \text{ by } F(X)=\ln(F(\exp X)).$$
$$\text{The latter causes } F(X)=(Ce^{XP}-B)/(e^{XP}+A) \text{ when } -\ln 2\le X\le 0.$$

The question is whether any real parameter-4-tuple (A,B,C,P) exists causing this F(X), and F', F'', and F''' all to be continuous. Unfortunately adding P to the mix made it too difficult to solve the continuity equations for A,B,C,P in closed form. Therefore I do not know the answer.

$$A=4.56212, \quad B=3.39375, \quad C=6.17481, \quad P=0.863515$$

come *close* to achieving those goals, but does not. This failure is unsurprising given that each of these numbers has been truncated after 6 significant figures – I would expect the true A,B,C,P (if they exist) would all be irrational. Can we keep doing better by providing more decimals, approaching perfection arbitrarily closely? I suspect the answer is "no" – some discontinuities in at least one of those three derivatives are unavoidable – but am not 100% sure.

A fourth F(X)? Here is an (also probably failed) attempt F_{IV} that is more likely to work (or even if it cannot be made to work, then it will be able to *approximately* achieve continuity to greater accuracy), since it has two extra parameters D and M (0<M<1):

$$F(X)=[(AX+B)/(C-X)]^{1/P}+D \text{ if } 0\le X\le M, \quad F(X)=\exp\big((C(X-D)^P-B)/((X-D)^P+A)\big) \text{ if } M\le X\le 1,$$
$$\text{while for } X>1 \text{ define F recursively by } F(X)=\exp(F(\ln X)), \quad \text{and for } X<0 \text{ by } F(X)=\ln(F(\exp X)).$$
$$\text{The latter causes } F(X)=(Ce^{(X-D)P}-B)/(e^{(X-D)P}+A) \text{ when } \ln M\le X\le 0.$$

The parameter choice

$$A=4.515046, \quad B=3.251113, \quad C=6.058416, \quad D=0.01025116, \quad P=0.87208825, \quad M=0.5000563$$

comes closer to making F, F', F'', and F''' all continuous at all real X, with F(0)=M and F(M)=1, but again fails. Again I do not know whether all these continuities can be achieved by F_{IV} to arbitrarily good accuracy, but doubt it.

A fifth F_V(z) trying to be a semi-exponential with everywhere continuous F, F', F'', F''', and F'''' – and this one almost certainly does work – is

$$F(X)=A(X)/B(X) \text{ if } 0\le X\le M, \quad F(X)=e^{C(X)} \text{ if } M\le X\le 1,$$
$$\text{while for } X>1 \text{ define F recursively by } F(X)=\exp(F(\ln X)), \quad \text{and for } X<0 \text{ by } F(X)=\ln(F(\exp X)).$$

Here A(X) and B(X) are **quartic polynomials** (one of them, without loss of generality, can be demanded to be monic) chosen so that F(X) is monotone-increasing for 0<X<M with F(0)=M and F(M)=1; here M obeys 0<M<1, and the algebraic function Y=C(X) obeys A(Y)=B(Y)X. Note: if the coefficients inside A(X) and B(X) are known, then an explicit, albeit complicated, formula for C(X) may be written with the aid of the quartic formula; and because F(X) is monotone-increasing for 0<X<M this formula will yield a real-valued C(X) which increases monotonically from C(M)=0 to C(1)=M, so that $F(X)=e^{C(X)}$ increases monotonically from F(M)=1 to $F(1)=e^M$. A and B contain, in all, 9 variable real coefficients, and M also is variable between 0 and 1, so we have 10 degrees of freedom in all. The demands for continuity of F', F'', F''', and F'''' at X=0 and X=M (the demands at X=1 are the same as at X=0 and those at $X=e^M$ the same as those at X=M, and etc thanks to the recursion, so only X=0 and X=M matter) then are eight equations, and the demands F(0)=M and F(M)=1 are two more, making 10 in all. Furthermore, the "monotone increasing" demand corresponds to certain inequalities. Presumably those 10 equations in 10 real variables (with some extra inequalities) have at least one solution; and if so, each will yield a piecewise- and recursively-defined semi-exponential with continuous 4th derivative, albeit some derivative of order≥5 should have discontinuities at $X\in S_M$ where

$$S_M = \{\ln M, 0, M, 1, e^M, e, \exp(e^M), \exp(e), ...\}.$$

Furthermore, I can prove there are only a *finite* set of solutions (since all the answer-numbers are algebraic numbers of bounded degree), which means this solution is *unique* up to a finite set of choices.

A sixth $F_{VI}(X)$: is exactly the same as $F_V(X)$ above, except that A(X) and B(X) now are **degree-d polynomials**. If d≥5, since there is no quintic formula, the algebraic function Y=C(X) will have no explicit formula, but nevertheless is implicitly defined by A(Y)=B(Y)X, and *uniquely* defined if $F_{VI}(X)$ is monotonically increasing for 0≤X≤M, and may be found numerically by e.g. the foolproof "bisection algorithm." The 2d+1 coefficients inside A and B, as well as the value of M with 0<M≤1, are 2d+2 variables which we demand satisfy the 2d+2 equations corresponding to continuity of F(X), F'(X), F''(X), ..., $F^{(d)}(X)$ across both X=0 and X=M. If for any d those 2d+2 simultaneous equations have at least one real solution, then $F_{VI}(X)$ is a semi-exponential with continuous derivatives of all orders≤d, but which should have some derivative of some order≥d+1 which is discontinuous when $X\in S_M$. Furthermore I can prove there will be at most a *finite* set of solutions, causing this order≤d-continuous $F_{VI}(X)$ to be *unique* up to a finite set of choices. And if we always select the one *closest* (in the sense of L^2 distance over the real interval 0<X<1), to the preceding (next smaller d) choice (starting with our F_{II} for the case d=2), then I would expect genuine uniqueness.

I then would conjecture that the **limit** as d→∞ along some sequence of integer d≥1, of those order≤d-continuous $F_{VI}(X)$ *exists*, which then necessarily would provide a construction of a C^∞-smooth real-valued monotonically-increasing semi-exponential. Our construction would (I conjecture) yield a unique such F(X), probably concave-∪.

Open questions: Prove (or disprove) that this limit, in fact, exists. Compute its Chebyshev series over the interval 0≤X≤1 accurate to $\pm 10^{-20}$. And: Is it the *unique* monotone-increasing concave-∪ C^∞-smooth real-valued semi-exponential F(X)?

I can in fact prove by a different method that a C^∞-smooth real-valued monotonically-increasing semi-exponential exists. It presumably equals the construction above, but I have not proven that.

C^∞-smooth real-valued monotonically-increasing semi-exponential Existence & Uniqueness Theorem: A C^∞-smooth real-valued monotonically-increasing F(X) exists such that $F(F(X))=e^X$. Furthermore, if F(X) is insisted to agree (and its first k derivatives also agree, for any fixed integer k≥0) with an *analytic* function of X to within additive errors that approach 0 when X→+∞ with k held fixed, then it is *unique*.

Proof: Instead of the function e^X consider the function $G(X)=e^X-1$ which has a (mildly repulsive) fixpoint at X=0. For integer N≥1, the "iterative N^{th} root" of G(X), call it $G^{[1/N]}(X)$, is a function

i. whose N-fold iteration equals G(X),
ii. whose formal power series $G^{[1/N]}(X)=X+\sum_{m\geq 2}a_m X^m$ with all-real coeffcients a_m (if N>1 this series will not converge for any X≠0 and is valid only in the sense of an "asymptotic series" when X→0+; see Baker 1958 for proof of nonconvergence) if iterated N times, agrees with $G(X)=x+\sum_{m\geq 2}X^m/m!$,
iii. is an analytic function of X.

Szekeres 1958 proved that such a $G^{[1/N]}(X)$ *exists*, is *unique*, and is monotonically increasing, for all X≥0. We now simply demand when X→+∞ that our semi-exponential F(X) must equal $G^{[1/2]}(X)$ to within additive error that approaches 0 (and the first k derivatives must also agree to within errors appraoching 0, for any fixed k≥0, when X→+∞). Then F(X) plainly exists and is uniquely determined when 0≤X≤1 by running its recurrence F(X)=exp(F(lnX)) "backwards" from X→∞. Furthermore, the resulting F(X) automatically will be C^∞ and strictly monotonically increasing, and automatically will obey F(X)>X for all X≥0. Therefore M=F(0) automatically will lie in (0,1). Hence our recursion then may be used to define F(X) uniquely for all real X. **Q.E.D.**

Unfortunately, both Crone & Neuendorffer's F_I, and my new F_{II}, F_{III}, F_{IV}, F_V, F_{VI} are only defined for real z, and as yet we have provided no clue how to extend any of their definitions to work for **complex** z. Because exp(z) is periodic with period=2πi, it suffices to define it when |im(z)|<π; and hence seems a plausibly-good idea to insist that F(z) also be thus-periodic. In any case let us attempt to extend F(z)'s domain at least to the infinite rectangular **strip** 0≤im(z)<π.

An obvious idea then to extend our piecewise-defined F(z) off the real axis into the complex plane, is to replace the countable set S_M of piece-boundary *points*, by some set of **curves** passing through those points vertically, and then use the analytic continuations of each function-piece to fill the resulting pieces of the strip. However, because an analytic function is fully and uniquely specified by its values on any curve-segment (however short) we see that it is *impossible* for the resulting F(z) to be continuous across the piece-boundary curves. There will be a jump-discontinuity almost everywhere we cross every curve.

But let's say you are willing to accept all that discontinuity. Then can we create suitable jigsaw-puzzle curves? The answer is "**no**":

"No jigsaw" theorem: There cannot exist any partitioning of the infinite strip $0 \le im(z) < \pi$ by a nonempty countable set of 1-dimensional curves, (intersecting the real axis at the countable point-set S_M for some M with $0<M<1$; and we assume these curves have 2-dimensional measure zero and have at least one-sided derivatives at least at the points where they cross the real axis) such that functions analytic within each "puzzle piece," but discontinuous almost everywhere across each curve, yield a piecewise-defined semi-exponential valid throughout that strip off the curves.

Proof: The first question is: *what* should the jigsaw-puzzle curves be? That is quite easy: For n=2,3,4,... the *n*th curve $\mathscr{C}_n$ must be just the exp(z) map applied to all $z \in \mathscr{C}_{n-2}$. [And the $(n+2)^{th}$ piece-interior is the exp-map applied to the n^{th}.] All these curves need to be *disjoint*. Unfortunately, the obvious choice to make $\mathscr{C}_0$ be "the vertical line with real part $-\infty$" would force $\mathscr{C}_2$ to be the single point (not a curve!) z=0. Therefore, we instead are forced to make $\mathscr{C}_0$ be a different curve e.g. *asymptoting* to $-\infty$ on the real axis, and then there would also need to be curves $\mathscr{C}_{-1}$, $\mathscr{C}_{-2}$, etc. Now for some integer k we must specify two particular $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$, whereupon the whole collection automatically becomes defined.

If any $\mathscr{C}_k$ contains any point z=x+iy with $x>(\ln 2)/2 \approx 0.3466$ and $0<y<\pi/4$ (and some $\mathscr{C}_k$ must) then, after we apply the z'=exp(z) map, the new z'=x'+iy' will have x'>x and y'>y. After enough iterations of the exp-map, this y-growth will necessarily eventually yield some $\mathscr{C}_n$ containing points z=x+iy for *every* y with $\pi/2 \le y \le \pi$. But if $z=x+i\pi$ then after one more exping we will find that $\mathscr{C}_{n+2}$ contains a point on the negative real-axis and hence must be (if we also demanded mirror symmetry across the real axis) a *closed* curve! And then $\mathscr{C}_{n+4}$ will topologically-necessarily *cross* it, a *contradiction*. **Q.E.D.**

However, it is possible to extend the domains of definition of both $F_I(z)$ and $F_{II}(z)$ to, e.g, the set of z=x+iy with $|y|<\text{sech}(x)/5$ by making, e.g, $\mathscr{C}_1$ and $\mathscr{C}_2$ be the vertical lines with real parts -ln2 and 0 respectively. This domain is so thin, and F(z)'s discontinuities across the curves $\mathscr{C}_n$ at all nonreal points so annoying, that this seems of almost no interest.

An infinitude of analytic solutions

Crone & Neuendorffer (and at least two others much earlier) also suggested a different F(z) now attempting to satisfy (i) but not (ii). Begin with a **fixpoint** Q of exp(z). There are a countable infinitude of such Q, all of which are complex with none real, one being

$$Q \approx 0.31813150520476413531265425158766451 + 1.33723570143068940890116214319371 06\, i.$$

This $Q=Q_1$ where Q_n with n>0 denote the fixpoints of exp(z) with $im(Q_n)>0$ sorted into increasing-$im(Q_n)$ order [indeed $im(Q_n)$ and $re(Q_n)$ and $|Q_n|$ for $n \ge 1$ form three strictly-increasing sequences of positive reals], and $Q_{-n}=\bar{Q}_n$. Asymptotically $reQ_n \sim \ln(2\pi|n|)$ and $imQ_n \sim 2\pi n i$ for large integer |n|. Any of the Q_n with $n \ne 0$ could have been used instead of our Q, thus yielding not one, but actually a countably infinite set, of analytic semi-exponential functions F(z). (It also might be natural to define $Q_0=+\infty$, but we shall not.)

Expand exp(z) in a Taylor series with basepoint Q, that is

$$\exp(z) = Q + \sum_{n\geq 1} (z-Q)^n Q/n!, \quad \text{convergent for all complex } z.$$

Even though Q=expQ and every summand are complex (and every series-truncation is complex nonreal for generic real z), this whole series of course yields a real-valued result for real z. Now let

$$F(z) = Q + \sum_{n\geq 1} (z-Q)^n c_n .$$

where F(Q)=exp(Q)=Q, and solve for the series coefficients

$$c_1 = R, \quad c_2 = (R/2!)\,(R+1)^{-1}, \quad c_3 = (R/3!)\,(Q-R+1)\,(R+1)^{-2}\,(Q+1)^{-1},$$
$$c_4 = (R/4!)\,(Q^2-4QR+7Q-5R+1)\,(R+1)^{-3}\,(Q+1)^{-1}\,(Q-R+1)^{-1}, \quad \ldots$$

one by one to cause F(F(z))=exp(z) for all small-enough |z-Q|. There are **exactly two solutions**, arising from R=±√Q. For z outside the circle of convergence of our series for F(z), define F(z) by analytic continuation.

Non-existence theorems

Finite-disk Theorem: Any Taylor series, based anywhere, for any semi-exponential F(z), has a *finite* convergence-disk. [For the specific F(z) defined in the preceding section one is interested in disk-center z=Q, but this theorem works for all F(z) and for any series-basepoint.]

Proof: Plainly F(0)≠0 since F(0)=0 would cause exp(0)=0, which is false since exp(0)=1. Either F(z)=0 has no solution z=P, or if it does, then F(z)=P has no solution z=Y. [Because: if Y existed, then P=F(Y) would too, and then F(P)=F(F(Y))=exp(Y)=0, which is impossible.] Therefore the range of F(·) omits either 0 or P or both. But if P exists, then F(F(lnP))=P, therefore either Y=F(lnP) exists, whereupon P=F(Y) exists (contradiction!); or the *domain* of F(·), even after maximally-extending it via "analytic continuation," omits (all values of) lnP. In the latter case, there is a "natural barrier" so plainly F must have a finite convergence disk. So if F is entire-analytic we conclude that P and Y *both* necessarily are missing from the range of F(·). And P=Y is impossible since if P=Y then F(P)=P hence P=0 (contradiction!). Now by Picard's theorem, F(z) *cannot* be an entire-analytic function because such functions can omit at most *one* value from their range! Therefore, every Taylor series for F(z) has *finite* convergence-disk. **Q.E.D.**

> And actually that "finite-disk" theorem works not only for the semi-exponential, but for any "semi" version of *any* real→real entire-analytic function whose range omits one complex value. [For the case of exp(z), the omitted value is 0.]

Unfortunately

Non-reality Theorem: Any semi-exponential F(z) defined using the fixpoint-and-series trick assumes nonreal values for generic real z.

Proof: Numerical computation and graphical argument sketched in fig.2 of Crone & Neuendorffer. **Q.E.D.**

And actually, this generic non-reality happens not only for semi-exponentials, but actually for essentially *any* "semi" version of *any* real→real analytic function got by the fixpoint-and-series trick based at *any* non-real fixpoint.

"No analytic real→real semi-exponential" theorem: No function F(z), analytic within any domain with a connected interior that includes both the real axis, and all fixpoints Q_n of exp(z), exists such that F(F(z))=exp(z) and F(z) maps reals to reals.

Proof: From the non-reality theorem, every semi-exponential F(z) that has any fixpoint anywhere, is nonreal for generic real z. Therefore, if there is any analytic semi-exponential F(z) that maps reals→reals, it must have *no fixpoints* anywhere – although it must have plenty of 2-orbit points, since every Q_n with $Q_n=\exp(Q_n)$ is a 2-orbit point, i.e. fixpoint of F(F(z)). [And more generally: by essentially the same argument applied to $\exp(\exp(...\exp(z)))=\exp^{[2n+1]}(z)$, F(z) cannot have any odd-cardinality orbits.]

Now F(F(z))=exp(z) has a countably-infinite set of fixpoints Q, which are exactly the 2-orbit points for F(z). Let Q'=F(Q). Then Q=F(Q'). If Q is a fixpoint for F(F(z))=exp(z), then Q'=F(Q) automatically also is. Therefore the action of F(z) on the fixpoints of exp(z) is to *permute* them in a self-inverse way, i.e. the permutation consists entirely of *2-cycles*. There are two ways to make those 2-cycles:

1. $F(Q_n)=\overline{Q}_n$ (complex conjugation),
2. $F(Q_a)=Q_b$ for various disjoint pairs (a,b) of nonzero integers with $a+b\neq 0$ and $a\neq b$.

If (1) then $F(z)=\overline{F}(\overline{z})$ holds for various $z=Q_n$ (which if it happened for an infinite set of Q_n would be an infinite set of z), which helps, or at least does not hurt, the prospect this holds for all z, which is the only way F maps reals→reals. But if (1) holds for every $z=Q_n$, then the degree-k Taylor coefficients c_k at $\overline{Q}$ must be the complex conjugates of those at Q, causing $|c_1|^2=Q=\exp Q=\exp' Q$, which is impossible because all the Q_n are nonreal.

If (2) then pick one of the integer pairs (a,b) and try to find the Taylor series coefficients of F(z) based at $z=Q_a$ and those based at $z=Q_b$ one by one. But then the product of the $|c_1|$'s must equal $|Q|=|\exp Q|=|\exp' Q|$ for both Q_a and Q_b simultaneously, which is impossible since $|Q_a|\neq|Q_b|$.

Q.E.D.

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