Geometric Symmetry of Non-Trivial Zeros of the Riemann Zeta Function in Polar Coordinates

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Abstract

This paper investigates the symmetry of non-trivial zeros of the Riemann zeta function $\zeta(s)$ through geometric analysis in polar coordinates. By transforming the complex number $s = \sigma + it$ into polar form, we demonstrate that the symmetry about the critical line $\sigma = \frac{1}{2}$ necessitates $\sigma = \frac{1}{2}$ for all non-trivial zeros. Numerical simulations further confirm the accuracy and consistency of this geometric approach. And we introduce a formula for the distribution pattern of all non-trivial zeros:

$$\zeta\left(\sqrt{\frac{1}{4} + t^2}\, e^{i\,\arctan(2t)}\right) = 0$$

where:

$$r = \sqrt{\frac{1}{4} + t^2} \quad \text{and} \quad \theta = \arctan(2t)$$

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1 Introduction

The Riemann Hypothesis, proposed by Bernhard Riemann in 1859, posits that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ have a real part equal to $\frac{1}{2}$ [1]. Formally, for any complex number $s = \sigma + it$ where $\zeta(s) = 0$, the hypothesis asserts that $\sigma = \frac{1}{2}$.

Understanding the distribution of these non-trivial zeros is crucial, as it has significant implications for number theory, particularly in the distribution of prime numbers [2]. Despite extensive numerical evidence supporting the hypothesis, a rigorous proof remains elusive. This paper aims to demonstrate that σ is indeed constant and must be $\frac{1}{2}$ using geometric analysis in a polar coordinate framework.

The Riemann zeta function $\zeta(s)$ for a complex number $s = \sigma + it$ is initially defined for $\sigma > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For the purposes of analytic continuation, we use the following key formulas:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)[3]$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

The functional equation for the zeta function is:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

The symmetry about the critical line $\sigma = \frac{1}{2}$ is maintained:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

To transform $s = \sigma + it$ to polar coordinates, we set:

$$s = re^{i\theta}$$

where

$$r = \sqrt{\sigma^2 + t^2} \quad \text{and} \quad \theta = \arctan\left(\frac{t}{\sigma}\right)$$

Assuming that $\sigma = \frac{1}{2}$, we obtain:

$$r = \sqrt{\frac{1}{4} + t^2} \quad \text{and} \quad \theta = \arctan(2t)$$

Thus, the zeta function for non-trivial zeros in polar coordinates becomes:

$$\zeta\left(\sqrt{\frac{1}{4} + t^2}\, e^{i\arctan(2t)}\right) = 0$$

This transformation simplifies the expression of the zeta function and provides a unified formula for all non-trivial zeros, assuming $\sigma = \frac{1}{2}$.

2 Polar Coordinate Representation of Complex Zeros

In this section, we explore the polar coordinate representation of complex numbers, specifically the zeros of the Riemann zeta function. This representation provides insight into the symmetry properties of these zeros.

2.1 Polar Coordinate Representation of the Complex Number $s = \sigma + it$

For the complex number $s = \sigma + it$, where σ is the real part and t is the imaginary part, the polar coordinate representation is as follows:

- **Modulus (Radius)** r:

$$r = |s| = \sqrt{\sigma^2 + t^2}$$

- **Argument (Angle)** θ:

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Thus, the complex number $s = \sigma + it$ can be represented in polar coordinates as:

$$s = r(\cos\theta + i\sin\theta)$$

or equivalently:

$$s = re^{i\theta}$$

where $r = \sqrt{\sigma^2 + t^2}$ is the modulus and $\theta = \arctan\left(\frac{t}{\sigma}\right)$ is the argument.

2.2 Polar Coordinate Representation of the Complex Number $s = \sigma - it$

Similarly, for the complex number $s = \sigma - it$:

- **Modulus (Radius)** r:

$$r = |s| = \sqrt{\sigma^2 + t^2}$$

- **Argument (Angle)** θ changes because the sign of the imaginary part t becomes negative. In this case:

$$\theta = \arctan\left(\frac{-t}{\sigma}\right)$$

  Note that:

$$\arctan\left(\frac{-t}{\sigma}\right) = -\arctan\left(\frac{t}{\sigma}\right)$$

Therefore, the complex number $s = \sigma - it$ in polar coordinates is represented as:

$$s = r(\cos(-\theta) + i\sin(-\theta)) = r(\cos\theta - i\sin\theta)$$

or equivalently:

$$s = re^{-i\theta}$$

where $r = \sqrt{\sigma^2 + t^2}$ is the modulus, and $-\theta = -\arctan\left(\frac{t}{\sigma}\right)$ is the argument.

2.3 Summary of the Symmetry

To summarize:

- The complex number $s = \sigma + it$ in polar coordinates is represented as $re^{i\theta}$, where $r = \sqrt{\sigma^2 + t^2}$ and $\theta = \arctan\left(\frac{t}{\sigma}\right)$.

- The complex number $s = \sigma - it$ in polar coordinates is represented as $re^{-i\theta}$, where $r = \sqrt{\sigma^2 + t^2}$ and $\theta = -\arctan\left(\frac{t}{\sigma}\right)$.

These two complex numbers $s = \sigma + it$ and $s = \sigma - it$ have the same modulus but opposite arguments. This is the geometric explanation of their symmetry about the real axis in the complex plane.

2.4 The Symmetry of Arguments

More precisely, what is symmetric is the argument (i.e., the angle in polar coordinates). In the complex plane, the two complex numbers $s = \sigma + it$ and $s = \sigma - it$ have the same modulus (the same distance from the origin), but their arguments are opposites:

- The argument of $s = \sigma + it$ is $\theta = \arctan\left(\frac{t}{\sigma}\right)$.
- The argument of $s = \sigma - it$ is $-\theta = \arctan\left(\frac{-t}{\sigma}\right)$.

This means that they are symmetric about the real axis (i.e., the line $\theta = 0$). These two complex numbers represent vectors of the same length but with angles opposite relative to the real axis. This is why they are symmetric about the real axis.

3 Geometric Symmetry and the Critical Line

In this chapter, we explore the relationship between the symmetry of the non-trivial zeros of the Riemann zeta function and the critical line $\sigma = \frac{1}{2}$. We provide a rigorous mathematical proof demonstrating that if the non-trivial zeros must exhibit symmetry about the critical line, then the real part σ of these zeros must equal $\frac{1}{2}$. This conclusion is essential in understanding why σ is conjectured to be a constant value of $\frac{1}{2}$, as posited by the Riemann Hypothesis.

3.1 Implications for Symmetry About $\sigma = \frac{1}{2}$

Now, consider the requirement that non-trivial zeros of the Riemann zeta function must be symmetric about the line $\sigma = \frac{1}{2}$ in the complex plane. If a zero $s = \sigma + it$ is symmetric about $\sigma = \frac{1}{2}$, there must exist a corresponding zero $s' = (1 - \sigma) + it$.

If we demand this symmetry, then $s = \sigma + it$ and $s' = (1 - \sigma) + it$ must have the same modulus r and arguments θ, which leads to the following equation:

$$\sqrt{\sigma^2 + t^2} = \sqrt{(1 - \sigma)^2 + t^2}$$

This equation simplifies to:

$$\sigma^2 + t^2 = (1 - \sigma)^2 + t^2$$

Cancelling t^2 from both sides, we get:

$$\sigma^2 = (1-\sigma)^2$$

Expanding and simplifying:

$$\sigma^2 = 1 - 2\sigma + \sigma^2$$

$$0 = 1 - 2\sigma$$

$$\sigma = \frac{1}{2}$$

4 Conclusion: $\sigma = \frac{1}{2}$

The above derivation shows that in order for non-trivial zeros $s = \sigma + it$ and their corresponding symmetric points $s' = (1-\sigma)+it$ to satisfy the symmetry condition about $\sigma = \frac{1}{2}$, the real part σ must indeed equal $\frac{1}{2}$. This conclusion supports the Riemann Hypothesis, which asserts that all non-trivial zeros of the Riemann zeta function lie on the critical line $\sigma = \frac{1}{2}$.

Therefore, the symmetry of the zeros of the Riemann zeta function, when expressed in polar coordinates, demands that $\sigma = \frac{1}{2}$ is not just a possibility but a necessity if the zeros are to be symmetric about the critical line.

Thus, this formula reveals the distribution pattern of all non-trivial zeros:

$$\zeta\left(\sqrt{\frac{1}{4} + t^2}\, e^{i \arctan(2t)}\right) = 0$$

And we have:

$$r = \sqrt{\frac{1}{4} + t^2} \quad \text{and} \quad \theta = \arctan(2t)$$

5 Acknowledgement

The researcher acknowledges that polar coordinates in exponential form may be an already-established method for analyzing the Riemann zeta function. The transformation into polar coordinates may not be novel; it has been used by some mathematicians, as can be seen in discussions on platforms like Math

Stack Exchange. Therefore, the validity and utility of polar coordinates in this context are assumed.

Nevertheless, the interchangeability between polar and Cartesian coordinate systems allows for the transformation of the Riemann zeta function and its non-trivial zeros into polar coordinates while preserving their validity. This transformation is feasible because polar coordinates emphasize positional orientation and avoid the complexities associated with negative numbers, thus simplifying calculations. More importantly, this conversion reframes the Riemann zeta function, its non-trivial zeros, and the Riemann Hypothesis as a geometric problem. Consequently, the application of trigonometric functions and the Pythagorean theorem could facilitate a straightforward, rapid, and possible proof of the Riemann Hypothesis.

The researcher initially proposed a perspective on the fundamental nature of numbers, suggesting that negative numbers and zero, while useful as abstract concepts, lack direct physical representations in reality. However, this viewpoint may be recognized as erroneous in the context of complex numbers. Complex numbers are neither positive nor negative and are not ordered in the same way real numbers are. Their components can be both positive and negative in both polar and rectangular coordinates. Furthermore, zero plays a crucial role in the Riemann Hypothesis, as it focuses on finding the zeros of the zeta function. Thus, it is argued that zero cannot be disregarded in this context.

Despite this, the perspective, which may be recognized as erroneous, sparked further investigation into the geometric properties and positional locations of non-trivial zeros. This led to the proposal of using polar coordinates to verify the Riemann Hypothesis. By representing complex numbers geometrically within this system, it was hypothesized that this approach could streamline the verification process of the hypothesis and yield new insights into the distribution of non-trivial zeros of the Riemann zeta function.

The proposed approach of employing a positive coordinate system aims to provide a fresh perspective on mathematical problems, potentially simplifying complex calculations and offering a clearer understanding of mathematical properties traditionally considered abstract. However, it is important to acknowledge that the initial arguments against negative numbers and zero were recognized as incorrect, yet they served as a catalyst for further exploration into the geometric analysis of the zeta function's non-trivial zeros.

Moreover, the author(researcher) extends heartfelt thanks to Kaylee Robert Tejeda, M.Sc., for his valuable review and insightful comments on this paper.

6 The Use of AI Statement

During the preparation of this work, the author used ChatGPT-4 to facilitate discussions on the nature of negative numbers, zero, and imaginary numbers, which helped refine the researcher's ideas. The perspective that negative numbers and zero are abstract without direct physical representations was provided by the researcher. The idea of a new positive coordinate system to replace the traditional system containing negative numbers and zero was proposed by the researcher.

The AI assisted in articulating and structuring the methodology for transforming the traditional complex plane into a positive coordinate system and utilizing polar coordinates to represent complex numbers. It provided support in defining the transformations needed to shift all values to positive and in creating a clear mathematical framework.

ChatGPT-4 helped implement and execute the mathematical calculations required to verify the Riemann zeta function in the new coordinate system and supported the verification of known non-trivial zeros of the zeta function using the new positive coordinate system.

The AI assisted in analyzing the results of the calculations, ensuring consistency and accuracy. It also helped draft the discussion and conclusion sections, articulating the significance of the findings and suggesting potential future research directions.

The AI contributed to the writing of the paper, including the abstract, introduction, methodology, results, discussion, and conclusion sections. It provided editing and formatting support, ensuring the paper met academic standards for clarity, coherence, and structure.

The researcher revised and corrected the mistakes in the paper.

Additionally, Claude 3 Opous was employed to critically evaluate this paper and offered suggestions for improvements.

Throughout the research and writing process, ChatGPT-4 adhered to ethical guidelines, providing support within its capabilities while ensuring the primary intellectual contribution remained with the human researcher.

After using these tools/services, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

This paper is a collaborative effort between the human researcher, ChatGPT-4, and Claude 3 Opous.

7 Declarations

- **Funding**: No Funding
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- **Materials availability**: Not Applicable
- **Code availability**: Not Applicable
- **Author contribution**: Bryce Petofi Towne had the original ideas and hypotheses. ChatGPT-4 and Claude 3 Opous, although not qualified as authors, assisted in articulating and structuring the methodology and provided mathematical validation and evaluations.

References

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[2] Edwards, H. M. (1974). *Riemann's Zeta Function.* Dover Publications.

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A Appendix A: Mathematical Transformations and Properties

A.1 Transformation to Polar Coordinates

The Riemann zeta function $\zeta(s)$ is defined for complex numbers $s = \sigma + it$ with $\sigma > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} = \sum_{n=1}^{\infty} \frac{1}{n^s} [3]$$

This series converges absolutely for $\sigma > 1$ and can be analytically continued to other values of s (except $s = 1$).

The analytic continuation of the zeta function extends its domain to the entire complex plane, excluding $s = 1$. This continuation is essential for defining $\zeta(s)$ beyond the region where the original series converges.

Two key formulas used in analytic continuation are:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)[3]$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$$

These integral representations converge for all s in the complex plane except $s = 1$, preserving the analytic nature of $\zeta(s)$ in polar coordinates as well. The functional equation of the Riemann zeta function implies a symmetry about the critical line $\sigma = \frac{1}{2}$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

The Ξ function, defined as:

$$\Xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the simpler functional equation:

$$\Xi(s) = \Xi(1-s)$$

To express s in polar coordinates, we write:

$$s = \sigma + it = re^{i\theta}$$

where:

$$r = \sqrt{\sigma^2 + t^2}$$
$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Given a complex number $s = \sigma + it$, we transform it into polar coordinates as follows:

$$s = r(\cos\theta + i\sin\theta)$$

where:

$$r = \sqrt{\sigma^2 + t^2}$$
$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

The magnitude r of the complex number s in the traditional system is:

$$|s| = \sqrt{\sigma^2 + t^2}$$

In the polar coordinate system, the magnitude r is defined as:

$$r = \sqrt{\sigma^2 + t^2}$$

Since the magnitude is preserved, we have:

$$|s| = r$$

The phase θ in the traditional system is:

$$\phi = \arctan\left(\frac{t}{\sigma}\right)$$

In the polar coordinate system, the phase θ is:

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Since the phase is preserved, we have:

$$\phi = \theta$$

To show that $s = \sigma + it$ is preserved in polar coordinates, we start with:

$$s = r(\cos\theta + i\sin\theta)$$

Substitute r and θ:

$$s = \sqrt{\sigma^2 + t^2}\left(\cos\left(\arctan\left(\frac{t}{\sigma}\right)\right) + i\sin\left(\arctan\left(\frac{t}{\sigma}\right)\right)\right)$$

Using the trigonometric identities:

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

Let $x = \frac{t}{\sigma}$, then:

$$\cos\left(\arctan\left(\frac{t}{\sigma}\right)\right) = \frac{\sigma}{\sqrt{\sigma^2+t^2}}$$

$$\sin\left(\arctan\left(\frac{t}{\sigma}\right)\right) = \frac{t}{\sqrt{\sigma^2+t^2}}$$

Substituting these back:

$$s = \sqrt{\sigma^2+t^2}\left(\frac{\sigma}{\sqrt{\sigma^2+t^2}} + i\frac{t}{\sqrt{\sigma^2+t^2}}\right)$$

Simplifying:

$$s = \sigma + it$$

This confirms that the transformation preserves the representation $s = \sigma + it$ and this also applies to $s = \sigma - it$.

A.2 Verification of Properties

To verify that the zeta function's properties are consistent in polar coordinates, we provide detailed steps:

1. **Series Representation**:

$$\zeta(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{1}{n^{re^{i\theta}}}$$

2. **Continuity and Differentiability**: The transformation from Cartesian to polar coordinates is smooth, and $\zeta(re^{i\theta})$ inherits the continuity and differentiability of $\zeta(s)$.

3. **Functional Equation in Polar Form**: The functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$ in polar coordinates becomes:

$$\zeta(re^{i\theta}) = 2^{re^{i\theta}}\pi^{re^{i\theta}-1}\sin\left(\frac{\pi re^{i\theta}}{2}\right)\Gamma(1-re^{i\theta})\zeta(1-re^{i\theta})$$

   Given that the gamma function $\Gamma(s)$ and the sine function $\sin(s)$ are well-defined and analytic in the complex plane, the symmetry and analytic continuation properties hold in the polar form.

4. **Symmetry**: Using the Ξ function, which satisfies $\Xi(s) = \Xi(1-s)$, we confirm that the symmetry about the critical line $\sigma = \frac{1}{2}$ is maintained:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

B Appendix B: Verification and Analysis of Non-Trivial Zeros

B.1 Verification of Formula for Non-Trivial Zeros

To verify the formula $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right) = 0$ for non-trivial zeros of the Riemann zeta function, we selected 30 known non-trivial zeros and computed the zeta function values using the given formula.

The results are summarized in the following table:

B.2 Analysis

The values of $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right)$ for the selected non-trivial zeros are extremely close to zero, with both real and imaginary parts being on the order of 10^{-13} or smaller. This strongly suggests that the given formula holds true for these zeros.

These results indicate that the formula $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right) = 0$ accurately represents the non-trivial zeros of the Riemann zeta function for the tested cases, providing further support to the hypothesis that all non-trivial zeros lie on the critical line $\sigma = \frac{1}{2}$.

t (Imaginary part of zero)	$\Re(\zeta)$	$\Im(\zeta)$
14.1347251417347	-1.61×10^{-16}	4.93×10^{-15}
21.0220396387716	1.41×10^{-14}	4.77×10^{-14}
25.0108575801457	-4.07×10^{-15}	1.50×10^{-14}
30.4248761258595	-2.80×10^{-15}	-1.03×10^{-14}
32.9350615877392	-5.02×10^{-15}	1.17×10^{-14}
37.5861781588256	-3.73×10^{-14}	-1.26×10^{-13}
40.9187190121475	7.62×10^{-15}	3.32×10^{-15}
43.327073280914	1.11×10^{-12}	-1.46×10^{-12}
48.0051508811672	3.72×10^{-14}	2.82×10^{-14}
49.7738324776723	-3.64×10^{-15}	6.71×10^{-15}
52.9703214777145	-6.67×10^{-15}	9.52×10^{-14}
56.4462476970634	1.33×10^{-14}	4.67×10^{-15}
59.3470440026026	1.57×10^{-13}	3.08×10^{-13}
60.8317785246098	1.17×10^{-14}	-3.83×10^{-15}
65.1125440480819	4.63×10^{-13}	4.89×10^{-13}
67.0798105294942	-5.28×10^{-15}	3.99×10^{-14}
69.5464017111739	6.91×10^{-14}	-1.62×10^{-13}
72.0671576744819	1.79×10^{-14}	-1.92×10^{-15}
75.7046906990839	-5.06×10^{-14}	-3.79×10^{-14}
77.1448400688748	8.12×10^{-15}	-6.14×10^{-15}
79.3373750202493	1.27×10^{-13}	-1.37×10^{-13}
82.910380854086	-5.32×10^{-14}	-5.85×10^{-14}
84.7354929805171	-5.97×10^{-15}	9.47×10^{-14}
87.4252746131252	-1.32×10^{-14}	-5.74×10^{-14}
88.8091112076345	-2.34×10^{-14}	5.16×10^{-14}
92.4918992705583	-3.63×10^{-13}	-3.92×10^{-13}
94.6513440405198	-6.37×10^{-14}	-1.07×10^{-13}
95.8706342282453	4.79×10^{-14}	-2.13×10^{-14}
98.8311942181937	-1.89×10^{-14}	8.13×10^{-14}
101.317851005731	-3.04×10^{-13}	-1.19×10^{-12}

Table 1: Verification of the formula for known non-trivial zeros of the Riemann zeta function