Sixth degree Diophantine polynomial equation

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Abstract

(abc)(ab+bc+ca)(a+b+c) = (pqr)(pq+pr+qr)(p+q+r) -------- (1)

Historicaly, we note that finding a prametrization of degree six has not been easy. In the below paper the author has followed in the footsteps of below mentioned paper, ref. no. (1). In ref. no.(1), Ajai Choudhry on page 356 has parametrized equation: m(abc)(ab+bc+ca)(a+b+c)=n(pqr)(pq+pr+qr)(p+q+r), where, (m,n)=(3,32) using one parameter. In the below paper the author has parametrized the said equation for (m,n)=(1,1), but with two parameters. The trick used is to split equation (1) into three parts, so as to balance the coefficents (m,n) & such that, (m,n)=[(uvw),(xyz)]. And [(uvw),(xyz)] =(72,72), & hence they cancel each other out.

Consider the below equation:

(abc)(ab+bc+ca)(a+b+c)=(pqr)(pq+pr+qr)(p+q+r) ---(1)

By the below method we arrive at a parametrization of equation (1) with two variables:

We split the above equation (1) as below:

u(a+b+c)=x(p+q+r) ------------ (2)

v(ab+bc+ca)=y(pq+qr+pr) ------- (3)

w(abc)=z(pqr) ------------------ (4)

We take: (u,v,w)=(8,1,9) & (x,y,z)=(9,4,2)

8(a+b+c)=9(p+q+r) ------------ (2)

(ab+bc+ca)=4(pq+qr+pr) ------- (3)

9abc=2pqr -------------------- (4)

We note that: (uvw)=(xyz)=72

We take:

(a,b,c)=[(pu),(qv),(2r/9uv)] ----- (5)

We notice that (4) is satisfied by (5) & Substituting (5) in (2) we get the value of variable 'r' as below:

$$r = \frac{9uv\left(p(8u - 9) + q(8v - 9)\right)}{(81uv - 16)} \quad ------(6)$$

We substitute value of 'r' from (6) in equation (3) & we simplify.

Hence we get a quadratic equation in (p,q) & after making the coefficient of (q^2) in it equal to zero at ($u = \frac{1}{18}$), we get the below values of variables (p, q).

$$p = (81v^2u^2 - 288vu^2 - 288uv^2 + 340uv - 18u - 18v + 64) \; ------- \; (7)$$

$$q = 2u(144uv - 8u - 162v + 9) \quad \text{------------------- (8)}$$

Substituting, ($u = \frac{1}{18}$), in (7) & (8) we get:

$$p = \frac{9(v^2 - 4)}{4} \; ---------- \; (9)$$

$$q = \frac{11(18v - 1)}{81} \; ---------- \; (10)$$

After substituting value of $\left(u = \frac{1}{18}\right)$ & (p, q) from (9) & (10) in (6) we get the value of variable (r) as below:

$$r = \frac{11v(v-72)}{324} \quad \text{------------- (11)}$$

Since: $(a,b,c) = \left[(pu),(qv),\left(\frac{2r}{9uv}\right)\right]$, we get after substituting the values of $(u = \frac{1}{18})$ & (p,q,r) $from$ $(9),(10)$ & (11) $we\ get$:

$$a = \frac{(v^2-4)}{8}$$

$$b = \frac{11v(18v-1)}{81}$$

$$c = \frac{11(v-72)}{81}$$

After removing common factors we get, (a,b,c,p,q,r), as below:

$$a = 81(v^2-4)$$

$$b = 88v(18v-1)$$

$$c = 88(v-72)$$

$$p = 1458(v^2-4)$$

$$q = 88(18v-1)$$

$$r = 22v(v-72)$$

We put, ($v = \frac{m}{n}$) & we get a parametrization in two variables & is shown below:

$$\boldsymbol{a = 81(m+2n)(m-2n)}$$

$$\boldsymbol{b = 88m(18m-n)}$$

$$\boldsymbol{c = 88n(m-72n)}$$

$$\boldsymbol{p = 1458(m+2n)(m-2n)}$$

$$\boldsymbol{q = 88n(18m-n)}$$

$$\boldsymbol{r = 22m(m-72n)}$$

For, $(m, n) = (3, 2)$, we get:

$$(a, b, c) = (567, -13728, 24816)$$

$$(p, q, r) = (10206, -9152, 9306)$$

I thank mathematician Dr. Ajai Choudhry, who readily sent (by email) the below parameterization (in three variables) of equation (1):

$$a = mu(m^2 + 1)(m^4 - m^2 + 1)(u + v)$$

$$b = mv(m^2 + 1)(m^4 - m^2 + 1)(u + v)$$

$$c = -m(m^6uv + u^2 + v^2 + uv)$$

$$p = u(m^2 + 1)(m^4 - m^2 + 1)(u + v)$$

$$q = v(m^2 + 1)(m^4 - m^2 + 1)(u + v)$$

$$r = -(m^6uv + m^6u^2 + m^6v^2 + uv)$$

Small numerical soln for above at, (m,u,v)=(2,2,1):

$$(a, b, c, p, q, r) = (52, 26, -18, 26, 13, -30)$$

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