Sliding Discrete Fourier Transform of Windowed Samples

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Abstract

In general the sliding discrete Fourier transform (SDFT) for analyzing the frequency characteristics of shift data is not applied to the windowed shift data and so in this case a lot of calculations are needed. But in practical applications calculating the spectrum of the windowed shift data efficiently is needed.

This paper proposes a SDFT of the windowed shift data using the window functions. Window functions have the symmetric property. Using this symmetric property, discrete Fourier transform (DFT) of the windowed shift data can be calculated by recursive algorithm.

And this paper verifies its correctness and analyzes the number of calculations through the simulation using MATLAB. Several window functions, i.e. Bartlett, hanning, hamming, Blackman window functions are used to prove the proposed algorithm. These algorithms can be efficiently used in the analysis of frequency characteristics of the shift data and implemented using dsPIC, ARM and FPGA.

Keywords: sliding discrete Fourier transform (SDFT), DFT of shift data, windowed DFT, fast algorithm of DFT

1. Introduction

The discrete Fourier transform (DFT) is widely used in the frequency analysis and the fast algorithms of DFT are proposed and used[1-5]. One way is just SDFT for the shift data and in this case it is the very effective algorithm[1,2].

This algorithm is as follows:

In the first step, find the DFT of N samples $\{x(0), x(1), \ldots, x(N-1)\}$.

At the arbitrary n^{th} moment, find the DFT of N samples $\{x(n-N+1), x(n-N+2), \ldots, x(n)\}$.

And then at the n+1$^{\text{th}}$ moment, find the DFT of N samples $\{x(n-N+2), x(n-N+3), \ldots, x(n+1)\}$.

Comparing the n^{th} samples with the n+1$^{\text{th}}$ samples, only the first sample $x(n-N+1)$ and the last sample $x(n+1)$ differ, but the others remain.

Using the above relation, the SDFT is introduced and used in the case of spectrum analysis of shift data.

Let's denote the DFT of n^{th} samples as $X_n(k)$.

$$X_n(k) = x(n-N+1) + x(n-N+2)W_N^{1k} + x(n-N+3)W_N^{2k} + \cdots + x(n)W_N^{(N-1)k} \tag{1}$$

where $W_N^k = e^{-j2\pi k/N}$

And then the DFT of n^{th} samples as $X_{n+1}(k)$.

$$X_n(k) = x(n-N+2) + x(n-N+3)W_N^{1k} + \cdots + x(n+1)W_N^{(N-1)k} \tag{2}$$

Comparing the Eq. (2) with Eq. (1),

$$X_{n+1}(k) = X_n(k)W_N^{-k} - x(n-N+1)W_N^{-k} + x(n+1)W_N^{k(N-1)}$$

Simplifying the above equation,

$$X_{n+1}(k) = W_N^{-k}\left[X_n(k) - x(n-N+1) + x(n+1)\right], \qquad k = 0,1,\ldots,N-1 \tag{3}$$

The Fig. 1 shows the realization of sliding discrete Fourier transform.

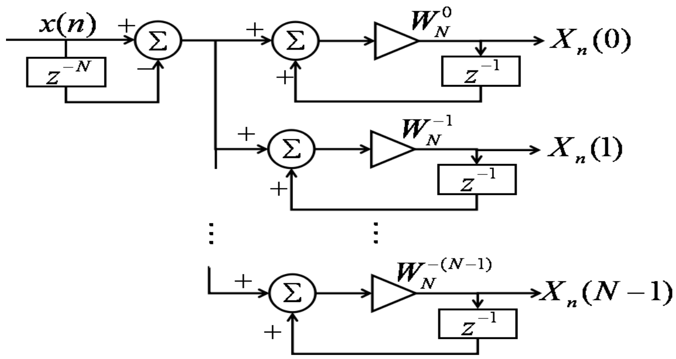

Fig. 1 The realization of SDFT

In SDFT, the number of calculation of complex multiplications is N and the number of calculation of complex additions is $2N$.

In general the DFT of windowed data is widely used to eliminate the parasitic sidelobe of signal[6-10].

But from the above result the SDFT cannot be used in the windowed samples and in the case of windowed data there are many calculations for finding the spectrum characteristics.

This paper proposes the algorithm to use the SDFT in not only non-windowed shift data but also windowed shift data for eliminating the parasitic sidelobe.

2. SDFT algorithm of windowed data

This paper proposes the algorithm of SDFT of signals windowed using Bartlett, Hanning, Hamming and Blackman windows.

2.1 SDFT of Bartlett windowed signals

The Bartlett window is defined as follows[5,11].

$$w(n) = \begin{cases} \dfrac{2n}{N-1} & 0 \le n \le \dfrac{N}{2} - 1 \\ 2 - \dfrac{2n}{N-1} & \dfrac{N}{2} \le n \le N-1 \end{cases} \tag{4}$$

If the windowed data is expressed as $x_w(n) = x(n)w(n)$, DFT of the n^{th} windowed samples can be expressed as Eq. (5).

$$X_{nw}(k) = x(n-N+1)w(0) + x(n-N+2)w(1)W_N^k + \cdots + x(n)w(N-1)W_N^{(N-1)k} \tag{5}$$

And then DFT of the $n+1^{th}$ windowed samples is Eq. (6).

$$X_{(n+1)w}(k) = x(n-N+2)w(0) + x(n-N+3)w(1)W_N^k + \cdots + x(n+1)w(N-1)W_N^{(N-1)k} \tag{6}$$

From the above equations, the relation between the DFTs of n^{th} and $n+1^{th}$ samples can be written as Eq. (7)

$$\begin{aligned} X_{(n+1)w}(k) &= W_N^{-k} X_{nw}(k) + x(n-N+2)[w(0)-w(1)] + x(n-N+3)W_N^k[w(1)-w(2)] + \\ &+ \cdots + x(n)W_N^{(N-2)k}[w(N-2)-w(N-1)] + x(n+1)W_N^{(N-1)k}w(N-1) \end{aligned} \tag{7}$$

From the definition of the Bartlett window, if $n < \frac{N}{2}$, $w(n-1) - w(n) = -\frac{2}{N-1}$ and if $n > \frac{N}{2}$, $w(n-1) - w(n) = \frac{2}{N-1}$. And then if $n = \frac{N}{2}$, $w(n-1) - w(n) = 0$.

If the first half samples are reversed, the Eq. (7) can be calculated by DFT of new samples which are obtained by reversing the first $\frac{N}{2}$ samples.

The DFT of the new samples $\left\{-x(n-N+2), -x(n-N+3), \cdots, -x\left(n-\frac{N}{2}+1\right), x\left(n-\frac{N}{2}+2\right), \cdots, x(n), x(n+1)\right\}$ is expressed as $X'_{n+1}(k)$, and then $X'_{n+1}(k)$ can be found by recursive algorithm.

The recursive form is as follows.

$$\begin{aligned} X'_{n+1}(k) &= W_N^{-k}\left[X'_n(k) + x(n-N+1) + 2x(n-N/2+1) + x(n+1)\right] \quad k: odd \\ X'_{n+1}(k) &= W_N^{-k}\left[X'_n(k) + x(n-N+1) - 2x(n-N/2+1) + x(n+1)\right] \quad k: even \end{aligned} \tag{8}$$

Using the Eq. (8), the DFT of windowed data $x_w(n)$ can be found by recursive algorithm.

$$\begin{aligned} X_{(n+1)w}(k) &= W_N^{-k}\left[X_{nw}(k) + \frac{2}{N-1}\left(X'_n(k) + x(n-N+1) + x(n-N/2+1)\right)\right] \quad k: odd \\ X_{(n+1)w}(k) &= W_N^{-k}\left[X_{nw}(k) + \frac{2}{N-1}\left(X'_n(k) + x(n-N+1) - x(n-N/2+1)\right)\right] \quad k: \text{even} \end{aligned} \tag{9}$$

2.2 SDFT of signals windowed using Hanning, Hamming and Blackman windows

The DFT of signals windowed using Hanning, Hamming and Blackman windows can be calculated more easily than the DFT using Blackman window.

The Hanning, Hamming and Blackman windows are defined as follows[3, 12].

$$\begin{aligned} w(n) &= 0.5[1 - \cos(2\pi n/(N-1))] \\ w(n) &= 0.54 - 0.46\cos[2\pi n/(N-1)] \\ w(n) &= 0.42 - 0.5\cos[2\pi n/(N-1)] + 0.08\cos[4\pi n/(N-1)] \end{aligned} \tag{10}$$

It is difficult to find the SDFT using above definition.

In this paper the Hanning, Hamming and Blackman windows are redefined and they are called modified Hanning, Hamming and Blackman windows.

The modified windows are as follows.

$$\begin{aligned} w(n) &= 0.5[1 - \cos(2\pi n/N)] \\ w(n) &= 0.54 - 0.46\cos(2\pi n/N) \\ w(n) &= 0.42 - 0.5\cos(2\pi n/N) + 0.08\cos(4\pi n/N) \end{aligned} \tag{11}$$

The DFT of signals windowed using above windows can be more easily calculated using the DFT property.

The windowed signal $x_w(n) = x(n)w(n)$ can be expressed as follows by Euler formula in the case of modified Hanning window.

$$x_w(n) = x(n)w(n) = x(n)\left[0.5 - 0.25\left(e^{j2\pi n/N} + e^{-j2\pi n/N}\right)\right]$$

Using the DFT property, the DFT of the above windowed samples can be obtained simply.

$$X_w(k) = 0.5X(k) - 0.25[X(k-1) + X(k+1)] \tag{12}$$

In Eq. (12) the DFT $X(k)$ of input samples can be calculated by applying Eq. (3).

That is, the DFT of modified Hanning windowed signal is found easily using recursive algorithm.

In such a way, the DFT of modified Hamming and Blackman windowed signal respectively can be calculated.

$$X_w(k) = 0.54X(k) - 0.23[X(k-1) + X(k+1)] \tag{13}$$

$$X_w(k) = 0.42X(k) - 0.25[X(k-1) + X(k+1)] + 0.04[X(k-2) + X(k+2)] \tag{14}$$

As a result, the SDFT of input samples is calculated by applying the recursive algorithm and then by using that result the DFT of windowed signal is found.

Eqs. (12), (13) and (14) are similar to each other and so Fig. 2 shows the recursive realization of Hamming windowed samples.

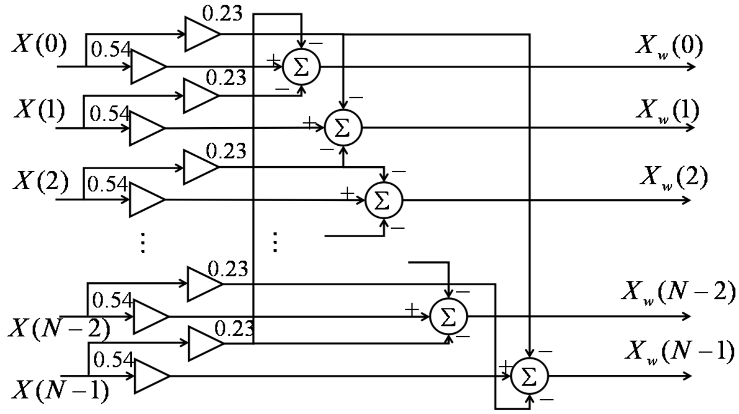

Fig. 2 The recursive realization of samples windowed using Hamming window

3. Result analysis

This section analyzes the number of calculations and correctness applying the proposed algorithms.

3.1 Analysis of the number of calculations.

This part analyzes the number of complex multiplications and additions for FFT and SDFT.

Using FFT the number of multiplications for windowing the shift samples is N, the number of complex multiplications is $\frac{\log_2 N}{2}N$ and the number of total complex multiplications is $\left(\frac{\log_2 N}{2}+1\right)N$. On the other hand the number of complex additions is $\log_2 N \cdot N$.

But using SDFT of Bartlett windowed data, the number of complex multiplications and additions is $2.5N$ and $6N$ from Eqs. (8) and (9).

Using SDFT of Hanning and Hamming windowed data, the number of complex multiplications and additions is $2N$ and $4N$ from Eqs. (12) and (13).

Also using SDFT of Blackman windowed data the number of complex multiplications and additions is $2.5N$ and $6N$ from Eq. (14).

Figs. 3 and 4 illustrate the ratios of multiplications and additions for FFT and SDFT.

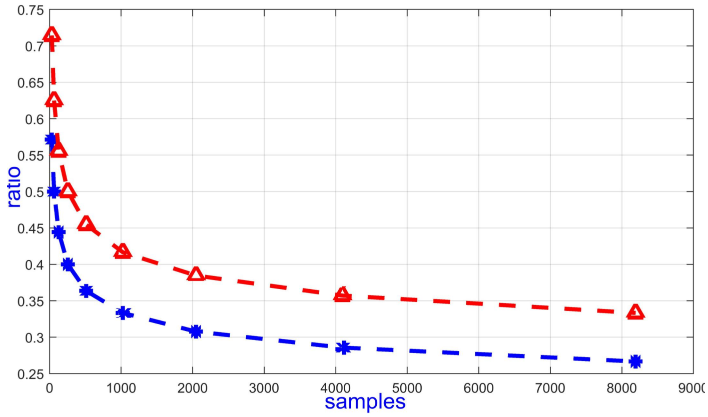

Fig. 3 The ratio of complex multiplications for FFT and SDFT

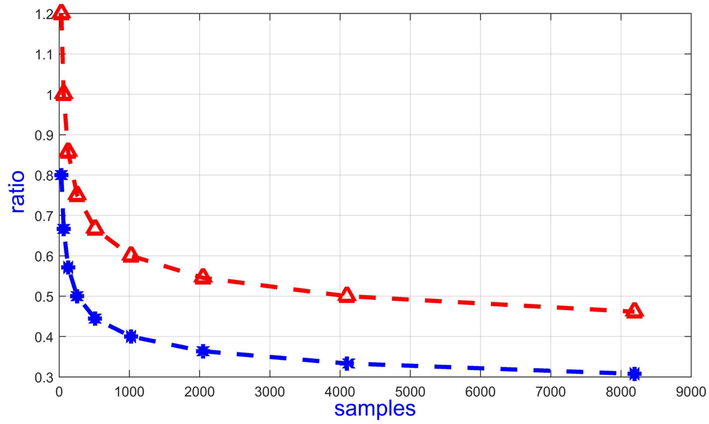

Fig. 4 The ratio of complex additions for FFT and SDFT

Figs 3 and 4 show that the higher the number of samples is, the lower the ratio gets. Especially using Hamming window if the number of samples is 64 or 1024, the ratio of calculations gets 1/ 2 or 1/3 respectively.

3.2 Correctness analysis

This paper proposed the SDFT of Bartlett, modified Hanning, Hamming and Blackman windowed data.

This part involves the analysis of the error between the SDFT using modified windows and FFT using original window.

The following figures show the relative errors according to time when using modified windows separately.

In this simulation, we use the harmonic signal with $f = 100$ Hz and $SNR = 10$ dB. And the length of the window is 256.

From the figures, the relative errors are about 1% and the results of the SDFT and FFT are almost same. So using the above algorithms the number of calculations can be reduced while its results are almost correct.

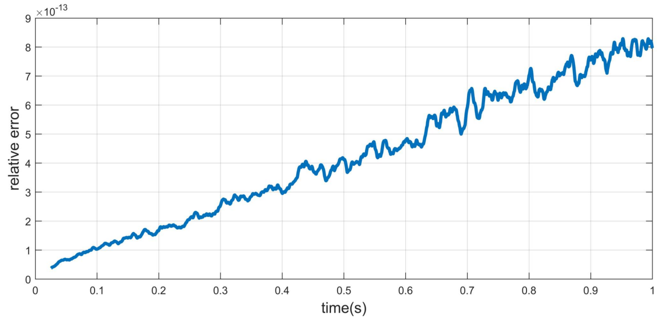

Fig. 5 A relative error when applying Bartlett window

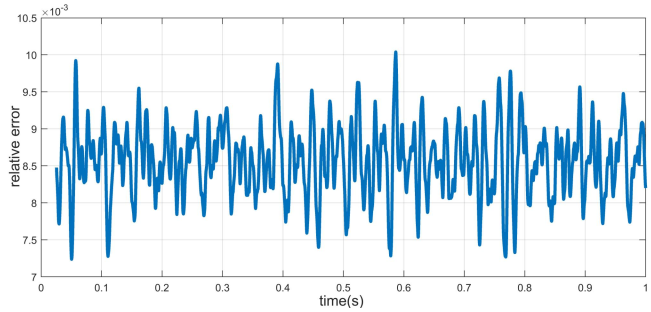

Fig. 6 A relative error when applying Hanning window

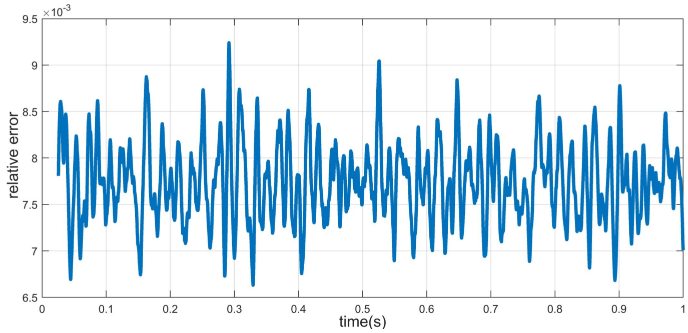

Fig. 7 A relative error when applying Hamming window

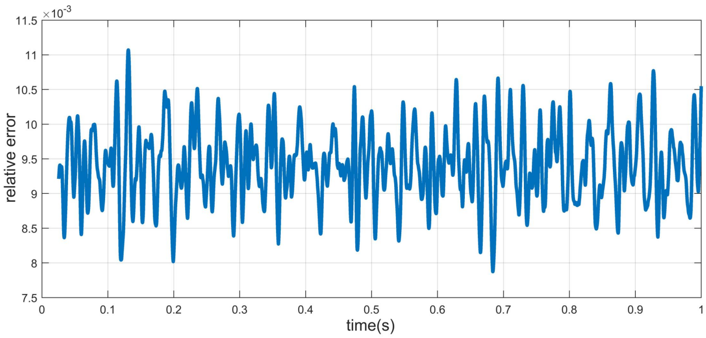

Fig. 8 A relative error when applying Blackman window

4. Conclusions

This paper proposes the SDFT of the windowed shift samples. Based on analysis of the windowed shift signal and by using the property of DFT, the recursive algorithms have been developed for DFT so that the number of calculations is lower and the DFT can be obtained in real-time. For example, Using the proposed algorithms if the number of samples is more 64, the ratio of calculations gets value less than 1/ 2 respectively. And the processing time can also be reduced to the value less than 1/2.

And then this paper proves the correctness of the suggested algorithms and analyzes a ratio of calculations using MATLAB.

Declarations

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Availability of data and material: The data that supports the findings of this study are available within the article and its supplementary material.

Code availability: The authors send their custom code in MATLAB to you. The code's name is "ProposedMethod.m".

Authors' contributions: Conceptualization: Jon Un Song; Methodology: Jon Un Song, Kwon Ryong Il; Formal analysis and investigation: Jon Un Song; Resources: Jon Un Song; Writing: Kwon Ryong Il; Editing: Kwon Ryong Il

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