Generalisation of $\lim_{x\to 0}\frac{x}{\sin x}=1$

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July 24, 2024

Abstract

It is known that most of the formulae that hold for ordinary trigonometric functions hold for generalised trigonometric functions. In this study, we succeeded in generalizing $\lim_{x\to 0}\frac{x}{\sin x}=1$. This makes it possible to discuss the generalised case in unsolved problems involving trigonometric functions, such as the generalisation of the Flint Hills series.

1 Introduction

For $p, q > 1$, we define the function

$$F_{p,q}(x)=\int_0^x (1-t^q)^{-\frac{1}{p}}\,dt \quad (x\in[0,1]) .$$

Since this function is strictly increasing it has an inverse, which we denote by $\sin_{p,q} x$

$$\sin_{p,q} x = F_{p,q}^{-1}(x) \quad \left(x\in\left[0,\frac{\pi_{p,q}}{2}\right]\right),$$

where

$$\pi_{p,q}=2\int_0^1 (1-t^q)^{-\frac{1}{p}}\,dt .$$

Note that $\sin_{p,q} x$ is strictly increasing on $\left[0,\frac{\pi_{p,q}}{2}\right]$, we observe that $\sin_{p,q} x\in[0,\ 1]$. We can extend $\sin_{p,q} x$ to $[0,\pi_{p,q}]$ by defining

$$\sin_{p,q} x=\sin_{p,q}(\pi_{p,q}-x) \quad \left(x\in\left[\frac{\pi_{p,q}}{2},\pi_{p,q}\right]\right).$$

Furthermore we can extend to $[-\pi_{p,q},\pi_{p,q}]$ by defining

$$\sin_{p,q}(-x)=-\sin_{p,q} x \quad (x\in[0,\pi_{p,q}]) .$$

Finally $\sin_{p,q} x$ is extended to whole of $\mathbb{R}$.

On the other hand, we define $\cos_{p,q} x$ by

$$\cos_{p,q} x = \frac{d}{dx}\left(\sin_{p,q} x\right).$$

Generalising trigonometric function makes it possible to generalise various open problems. For example, Flint Hills series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 |\sin n|^2}.$$

Meiburg [2] studied the convergence of the Flint Hills series by extending the problem by defining a new function called sine-like function.

In this study, the aim was to extend the $\lim_{x\to 0} \frac{x}{\sin x} = 1$ to a generalised form as shown in Theorem 1.

2 The value of $\lim_{x\to 0} \frac{x}{\sin_{p,q} x}$

Theorem 1. $\lim_{x\to 0} \frac{x}{\sin_{p,q} x} = 1$.

Lemma 2. If $x \in \left[0, \frac{\pi_{p,q}}{2}\right]$, then $\sin_{p,q} x \le x \le \frac{\sin_{p,q} x}{\cos_{p,q} x}$.

Proof. We defined the $f(x)$ and $g(x)$ as $f(x) = x - \sin_{p,q} x$, $g(x) = \frac{\sin_{p,q} x}{\cos_{p,q} x} - x$. The value of $f(0)$ and $g(0)$ is zero. Furthermore

$$\frac{d}{dx} f(x) = 1 - \cos_{p,q} x = 1 - \left(1 - \left(\sin_{p,q} x\right)^q\right)^{\frac{1}{p}} \ge 1 - 1 = 0 \tag{1}$$

$$\frac{d}{dt} g(x) = \frac{q}{p} \cdot \frac{\left(\sin_{p,q} x\right)^q}{\left(\cos_{p,q} x\right)^p} \ge 0. \tag{2}$$

In (2) we used the fact that

$$\left(\sin_{p,q} x\right)^q + \left(\cos_{p,q} x\right)^p = 1 \tag{3}$$

holds. According to Edmunds et.al [1] (3) holds when $x > 0$ is close enough to zero. So (1) and (2), both $f(x)$ and $g(x)$ are found to be monotonically increasing functions. Therefore, since $f(x)$, $g(x) > 0$ whenever $x > 0$, so

$$\sin_{p,q} x \le x \le \frac{\sin_{p,q} x}{\cos_{p,q} x}$$

holds.

□

Theorem 1. $\lim_{x \to 0} \frac{x}{\sin_{p,q} x} = 1$.

Proof. Since if $x \in \left[0, \frac{\pi_{p,q}}{2}\right]$, then $\sin_{p,q} x > 0$, the inequality in Lemma 2 can be transformed as follows that

$$1 \leq \frac{x}{\sin_{p,q} x} \leq \frac{1}{\cos_{p,q} x} . \tag{4}$$

Since (5) holds, the squeeze theorem can be used in conjunction with (4) .

$$\lim_{x \to +0} \frac{1}{\cos_{p,q} x} = \lim_{x \to +0} \left(1 - \left(\sin_{p,q} x\right)^q\right)^{-\frac{1}{p}} = 1 \tag{5}$$

Therefore

$$\lim_{x \to +0} \frac{x}{\sin_{p,q} x} = 1 .$$

Next, we want to prove

$$\lim_{x \to -0} \frac{x}{\sin_{p,q} x} = 1 .$$

Let $x = -t$, then

$$1 = \lim_{t \to -0} \frac{-t}{\sin_{p,q}(-t)} = \lim_{t \to -0} \frac{t}{\sin_{p,q} t}$$

holds. Therefore

$$\lim_{x \to 0} \frac{x}{\sin_{p,q} x} = 1 .$$

□

3 Conclusion

In this study, it was shown that $\lim_{x \to 0} \frac{x}{\sin x} = 1$ is also valid for generalised trigonometric functions. Edmunds et.al [1] also succeeded in generalising well-known formulas such as $\sin^2 x + \cos^2 x = 1$, so it is expected that many of the formulas that hold for ordinary trigonometric functions will hold in the generalised case.

References

[1] David E. Edmundsa, Petr Gurkab, and Jan Langc. *Journal of Approximation Theory*, 164:47-56, 2012.

[2] Alex Meiburg. Bounds on Irrationality Measures and the Flint-Hills Series. arXiv:2208.13356, 2022.