# What does the Central Limit Theorem have to say about General Relativity?

Réjean Plamondon

Département de génie électrique
Polytechnique Montréal
Montréal, QC, Canada H3C 3A7
rejean.plamondon@polymtl.ca

**Abstract.** In this paper, we speculate on a possible connection between the Bayes's law and the Einstein's general relativity equation to support the use of a metric based on an *erfc* gravitational potential that has been recently proposed to provide some cues to open problems in the solar systems. Starting from a basic interdependence premise, an analogy between Einstein's equation and Bayes's law is used to analyze the linear case of a weak field static symmetric massive object, providing a probabilistic context that takes into account the probability of presence of a given energy density in its corresponding 4D curved space-time manifold. Using the Central Limit Theorem to model globally the very slow process of star formation and mathematically express the corresponding probability density, the new framework provides a rationale for the emergence of a weighted Newton's law of gravitation. One key feature of this modified gravity model is that it relies on the existence of an intrinsic emergent physical constant $\sigma$ , a star-specific proper length that scales all its surroundings.

**Keywords:** Modified gravity, weighted Newton's law, Central limit theorem, Bayes's law, star proper length, *erfc* potential, emergence, self-organization.

In its most general configuration, a complex system is a network of heterogeneous and usually simpler subsystems that interact among each other to give rise to emergent features that guides its self-organization into a more complex system. The description of the whole process simplifies at a given level of representation, leading to some emergent properties [1]. These emergent systems are omnipresent in physics, chemistry and biology [2].

Among the tools that can be used to study such systems and their convergence is the Central Limit Theorem [3]. This theorem has been developed over four centuries in the context of searching for the asymptotic probabilistic behavior of a sum of independent or quasi-independent random variables. The key feature of this theorem, which makes it practical for the study of complex systems, is that although the details of the individual sub-processes are unknown, the behavior of the whole system can be predicted, under some non-restrictive conditions, to converge towards multivariate Gaussian functions.

In this paper, we use this modeling approach to conjecture about emergent gravity, speculating from a possible connection between the Bayes's law and the Einstein's general relativity equation. The idea of modifying gravity to come up with new relativistic field descriptions has been proposed time and again in the last decades to provide among other things alternative explanations to some open problems in astronomy and astrophysics [4,5,6]. These extensions aimed at correcting and enlarging Einstein's theory to encompass several shortcomings when cosmological, astrophysical, mathematical and quantum mechanical observations and objections are taken into account [7,8,9]. In this mindset, in a recent paper [10], the static non-empty symmetric geometry described by a metric based on an *erfc* gravitational potential have been proposed and studied in detail. This new metric provides a consistent set of predictions and interpretations regarding some open problems in the solar system, like the fly-by anomalies, the secular increase of the astronomical unit, the residual Pioneers' delays [11].

In the present manuscript, a fundamental question is addressed: can we lay the foundations for an emergent model that predicts the existence of an *erfc* potential using the central limit theorem? In the next section, starting from a complementarity that has been pointed out by Wheeler, we propose a comparison between Einstein's equation and Bayes's law of conditional probabilities and use it to support our analogical and speculative argumentation. The whole framework relies on a global probabilistic description of a star formation from which a fundamental law of gravitation comes out as a consequence of an asymptotic convergence predicted by the Central Limit Theorem. In section 2, we put the general relativity in a probabilistic context and in Section 3, we present the conditions under which a weighted Newton's law automatically emerges from this new scheme and then conclude.

**2. Introducing a probabilistic context in general relativity**

Einstein's gravitation equation, which links the space-time curvature tensor *G* to the energy-momentum tensor *T,*

$$G = KT \tag{1}$$

has been encapsulated by Wheeler as, "Space-time tells matter how to move; matter tells space-time how to curve." [12]. This points out an interesting interdependence that can be used to put general relativity into a probabilistic context if this assertion is converted into a general and fundamental premise:

"Space-time curvature (S) and energy momentum (E) are two inextricable descriptive approaches to define the physically observable probabilistic universe (U); they must be mutually exploited to describe any subset $U_i$ of this universe. The probability of observing and describing a given subset of the universe $P(U_i)$, i.e. the joint probability $P(S_i,E_i)$, can be studied from two equivalent methods: either by analyzing the curvature of space-time $S_i$ after hypothesizing a given energy momentum $E_i$ or by analyzing the energy momentum $E_i$ under the hypothesis of a given space-time curvature $S_i$. In terms of conditional probabilities, this leads to two equivalent descriptions:

$$\mathrm{P}(U_i) = \mathrm{P}(S_i,E_i) = \mathrm{P}(S_i/E_i)\mathrm{P}(E_i) = \mathrm{P}(E_i/S_i)\mathrm{P}(S_i) \tag{2}$$

Using the corresponding probability density function $f(\ )$ of these conditional probabilities $P(\ )$ and rewriting (2) in a 4D Bayesian format, we get:

$$\begin{gathered} f(S_{\mu\nu}/E_{\mu\nu})f(E_{\mu\nu}) = f(E_{\mu\nu}\ /\ S_{\mu\nu})f(S_{\mu\nu}) \\ f(S_{\mu\nu}/E_{\mu\nu}) = \frac{f(S_{\mu\nu})}{f(E_{\mu\nu})} \times f(E_{\mu\nu}\ /\ S_{\mu\nu}) \end{gathered} \tag{3}$$

In other words, we consider the space-time curvature and the energy momentum tensors as continuous 4D random variables and the values of their probability density functions define the probability that these random variables have a particular range of values within an infinitesimal space-time interval, providing an estimate of the relative likelihood that these random variables have these values in this interval.

This latter equation can be linked to Einstein's equation (1) through the following analogy:

$$G_{\mu\nu} \Leftrightarrow f(S_{\mu\nu}/E_{\mu\nu}) \tag{4}$$

and

$$KT_{\mu\nu} \Leftrightarrow \frac{f(S_{\mu\nu})}{f(E_{\mu\nu})} \times f(E_{\mu\nu}\ /\ S_{\mu\nu}) \tag{5}$$

In other words, $f(S_{\mu\nu}/E_{\mu\nu})$ can be interpreted as describing the probability of space-time to be curved under the conditional probability of observing a given energy momentum density $\left[\frac{f(S_{\mu\nu})}{f(E_{\mu\nu})} \times f(E_{\mu\nu}\ /\ S_{\mu\nu})\right]$, which can be linked to $G_{\mu\nu}$ and $T_{\mu\nu}$ respectively.

**3. Emergence of a weighted Newton's law of gravitation**

Under weak field, low speed, classical conditions, only the 00-component of (1) is significant:

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = K T_{00} \tag{6}$$

Applying the previous analogy to such a system and using dimensional analysis, one can associate $f\left(S_{00} / E_{00}\right)$, in (E/L$^{-4}$), to the probability density of the space-time subset $S_{00}$ to be curved given a matter-energy $E_{00}$, to $G_{00}$, a curvature component in (L$^{-2}$):

$$f(S_{00} / E_{00}) = k_1 G_{00} \tag{7}$$

Similarly,

$$\frac{f(S_{00})}{f(E_{00})} f(E_{00} / S_{00}) = k_2 K T_{00} \tag{8}$$

where the coefficients $k_1$ and $k_2$ are unit-balancing constants

A way to perform estimate $f(E_{00} / S_{00})$ is to analyze the very slow process of star formation using a simple stochastic model. Assuming that in a remote and isolated part of the Universe, a star is slowly building up from the gradual agglomeration of chunks of matter-energy. Considering these chunks as random variables described by their own density functions, this process, which involves hydrodynamics, thermodynamics, radiation transport, etc, is equivalent, from a global probabilistic point of view, to adding up random variables, i.e. making the convolution of their corresponding probability density functions. Since these densities respect the Lindeberg conditions [13], in the sense that they are real, normalized, non-negative functions with a finite third moment and a scaled dispersion, then the Central Limit Theorem applies and predicts that in a flat Euclidean space-time, when the number of random chunks is very large (N→∞),

$$f(E_{00} / S_{00}) \propto f(\mathbf{x}) = \lim_{N\to\infty}\left[f_1(\mathbf{x}) * f_2(\mathbf{x}) * \ldots * f_N(\mathbf{x})\right] \tag{9}$$

and the ideal form of the global probability density $f(\mathbf{x})$ will be a normal multivariate and will tend to the following general form:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right], \tag{10}$$

where $\mathbf{x}$ is an $n$ dimension random vector measuring the distance from the mean vector $\boldsymbol{\mu}$ of the distribution, and $\Sigma = \mathrm{E}[(\mathbf{x}-\boldsymbol{\mu}),(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}]$ is the statistical covariance matrix measuring the expected (E) dispersion of this distribution.

For a 4D pseudo-Euclidean static system $n = 4$, with the quadri-vector $\mathbf{x} = (c\overline{t}, x, y, z)$, centered at $\boldsymbol{\mu} = (c\overline{t}, 0, 0, 0)$, Equation (10) can be rewritten as:

$$f(\overline{r}) = \frac{1}{4\pi^2\sigma^4} \exp\left(-\frac{\overline{r}^2}{2\sigma^2}\right) \tag{11}$$

where $\overline{r}$ is the Euclidian distance $\left(\overline{r}^2 = x^2 + y^2 + z^2\right)$ from the zero-centered mean of the $f(\overline{r})$ density. The diagonal covariance matrix $(\Sigma)$ reduces to $\sigma^2$, a weighting parameter that scales the norm of the quadri-vector $\mathbf{x}$, a Lorentz invariant. This scalar $\sigma^2$ is de facto a Lorentz invariant, an intrinsic and emergent feature of the central limit process. It reflects the system intrinsic proper length. This specific scale is the basic feature that can be used to get a curved space description $f(\overline{r})$ of the star's probability density and to point out some of its specific inherent properties.

Indeed, Equation (11) is not practically useful in its present form, since it is only valid in a flat space-time that is, when an observer is at infinity from the star or locally, on a geodesic.

In other words, $\bar{r}$ defines the distance from the apparent star centre as seen from infinity in a hypothetical flat space while $\hat{r}$ defines the physical curvilinear distance from the star centre in the curved space-time.

In their simplest algebraic form, the relationship between $\bar{r}$ and $\hat{r}$ can be summarized as follows:

$$\begin{bmatrix} \bar{r} = 0 \text{ at } \hat{r} = \infty \\ \bar{r} = \infty \text{ at } \hat{r} = 0 \end{bmatrix} \Rightarrow \bar{r} = \frac{s}{\hat{r}} \tag{12}$$

where $s$ is a scale parameter that can be determined from the invariance of $\sigma$ :

$$\bar{r} = \sigma = \hat{r} \Rightarrow s = \sigma^2 \tag{13}$$

This leads to making the following change of coordinates:

$$\frac{\bar{r}}{\sigma} = \frac{\sigma}{\hat{r}} \tag{14}$$

to get $f(\hat{r})$ , a projection of $f(\bar{r})$ on a manifold of variable curvature described locally by the coordinate $\hat{r}$. Making this change of coordinates making sure that the normalization of the probability densities in both representation spaces is maintained, this leads to:

$$f(\hat{E}_{00} \ / \ \hat{S}_{00}) = k_3 f \ \hat{r} \ = \frac{k_3}{4\pi^2\sigma^2\hat{r}^2}\exp\left(\frac{-\sigma^2}{2\hat{r}^2}\right) \tag{15}$$

Equation (15) expresses the probability density of finding the star within an equivalent 3-ball of radius $\hat{r}$ , in a curved manifold under static, symmetric, weak field and low speed conditions.

Pursing on this analogy, one can define the energy priori probability density by:

$$f(E_{00}) = \frac{1}{M_{tot}c^2} \tag{16}$$

Taking into account the mapping defined by equation (14) $f(S_{00})$ can be estimated in two steps. First, an invariant reference surface, valid both in the flat and curved descriptions, must be established. This is done assuming that the total energy of the star is distributed on the reference 2-spheres of constant curvature $1/\sigma^2$ in both representations:

$$f(S_{00\ \bar{r}=\sigma}) = \frac{1}{4\pi\sigma^2} = f(S_{00\ \hat{r}=\sigma}), \tag{17}$$

Second, a value of $f(S_{00(\bar{r})})$ valid all over the flat space is obtained by weighting the by the corresponding 3-ball volumes defined at $\bar{r}$ and $\sigma$ respectively:

$$f(S_{00\ \bar{r}}) = \frac{1}{4\pi\sigma^2}\frac{V_{3b.\bar{r}}}{V_{3b.\sigma}} = \frac{1}{4\pi\sigma^2}\frac{\frac{4\pi}{3}\bar{r}^3}{\frac{4\pi}{3}\sigma^3} = 4\pi\sigma^2\frac{\bar{r}^3}{\sigma^3} \tag{18}$$

which, using Equation (14), leads to an expression for $f(S_{00(\hat{r})})$ valid at any corresponding $\bar{r}$ position:

$$f(S_{00\ \hat{r}}) = \frac{1}{4\pi\sigma^2}\frac{\sigma^3}{\hat{r}^3} = \frac{\sigma}{4\pi\hat{r}^3} \tag{19}$$

In other words, the mapping resulting from Equation (14) guarantees that the energy-momentum tensor $T_{00\ \hat{r}}$ affecting the curvature at a radial distance $\hat{r}$ in the curved space is consistent with the energy-momentum tensor component $T_{00\,\bar{r}}$ that would be measured in an ideal flat space at the corresponding distance $\bar{r}$ from the Gaussian density center.

Substituting equations (15) and (19) in equation (6) leads to the description of a central force field as a function of the curvilinear distance from the star center:

$$\nabla^2\Phi(\mathrm{r}) = \frac{2Kc^2\sigma^2Mc^2}{4\pi\sigma^3 r^5}\exp\left(-\frac{\sigma^2}{2r^2}\right), \tag{20}$$

where $K = k_3/k_1$ (with $k_1$ and $k_2$ in $L^{-2}$ and $k_3$ in $L^4\,E^{-1}$) and where, from now on, the curved hat over the coordinate $r$ is omitted $(\hat{r} \rightarrow r)$.

This Laplacian can be solved to get an expression for the magnitude of the linear radial gravitational field:

$$g(r) = -\left|\vec{\nabla}\Phi(\mathrm{r})\right| = -\frac{2Kc^2Mc^2}{4\pi\sigma^3 r^2}\exp\left(-\frac{\sigma^2}{2r^2}\right), \tag{21}$$

which reduces to the Newton description for large $r$ values.
The gravitational potential can be obtained, if Equation (21) is integrated:

$$\Phi(r) = \int_0^r g(r)dr = \frac{2Kc^2Mc^2}{(4\pi\sigma)^3}\left(\frac{\sqrt{\pi}}{\sqrt{2}\sigma}\right)erfc\left(\frac{\sigma}{\sqrt{2}r}\right) = \Phi_{erfc}(r) \tag{22}$$

where the integration limits, from 0 to $r$, are consistent, according to Equation (14), with integration from $\bar{r} = \infty$ to 0 in the corresponding flat space representation, which leads to an *erfc* potential. In other words, in a flat space, the gravitational potential is fixed to zero at $\bar{r} = \infty$ and the Minkowskian metric is recovered under this condition. But when the flat model is projected into a curved space, according to the inverse relationship (14) between $\bar{r}$ and $\hat{r}$, equation (22) predicts a constant potential at r=infinity. This is the particular feature of an *erfc* potential which leads to an original description of the space-time surrounding a massive object as previously published [7,8]. Equation (22) also converges towards the Newton limit, if the constant term included in the *erfc* function is arbitrarily subtracted, which leads to an *erf* potential that tends towards a $1/r$ behavior at large $r$ values.

**4 Conclusion**

Under the paradigm of a self-organizing universe, the laws of physics should emerge from the space time and matter energy distribution. From a global perspective, the analogical and speculative approach presented in this paper can be seen as a heuristic strategy to mathematically take into account Mach's principle [9]. On top of Einstein's arguments [14], Equation (3) provides a rationale based on a fundamental law of probabilities, the Bayes's law [15]. Applying this interdependence principle and the Central Limit Theorem, we have pinpointed a possible explanation for the emergence of a weighted Newton's law of gravitation in such a system. One key feature of the present theory [16] is that it is based on the existence of an intrinsic star specific physical constant, the parameter $\sigma^2$, which automatically emerges from a convergence process described by the Central Limit Theorem. As reported in [7] and [8], the new *erfc* potential once incorporated into a spherically symmetric metric, describes various features of the resulting modified Schwarzchild geometry. For examples, computing the systematic errors that emerge when the effect of $\sigma$ is neglected, the Hubble constant $H_0$ can be linked to $\sigma_{Sun}$ and the secular increase of the astronomical unit $V_{AU}$ to $\sigma_{Earth}$, which leads to accurate numerical predictions: $H_0 = 74.42(.02)(\mathrm{km/s})/\mathrm{Mpc}$ and $V_{\mathrm{AU}} \cong 7.8\mathrm{cm\text{-}yr}^{-1}$. Moreover, investigating the expected impacts of the *erfc* potential on the flybys anomalies and the residual Pioneers' delay lead to corrections for the osculating asymptotic velocity of a flyby at the perigee of the order of 10 mm/s and an inward radial acceleration of $8.34\times10^{-10}\,\mathrm{m/s^2}$ affecting the Pioneer space crafts.

To the best of our knowledge, the Bayesian paradigm proposed in this paper has never been investigated. Bayesian approaches have extensively been used to explore complex problems from a probabilistic point of view in numerous fields of science [17,18], including quantum physics [19], astronomy [20], artificial intelligence [21] and neuroscience [22],to name a few examples. The present model adds up to this exhaustive list.

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