New Formulas containing Bessel numbers

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August 27, 2024

Abstract : We present a new formula that generates Bessel numbers.

Keywords : Bessel numbers, Bessel polynomials, recurrent sum, falling factorial, rising factorial, primitive function.

1-First results

Lemma 1. Let $n \in \mathbb{N}^*$. For all $1 \leq k \leq n$

$$\forall\, 1 \leq k \leq n, \quad b_n^k = \sum_{i=0}^{k} (n-k+1)^{(k-i)} b_{n-1}^i \quad (1)$$

where b_n^k is Bessel number (see https://oeis.org/A001498), with $b_n^0 = 1$ for all $n \in \mathbb{N}$, and $x^{(m)} = x(x+1)\ldots(x+m-1)$ is the rising factorial function.

Proof.

In Grosswald's book (see [1]), a recurrence relation for b_n^k is given by :

$$\forall\, 1 \leq k \leq n, \quad b_n^k = b_{n-1}^k + (n-k+1) b_n^{k-1}$$

The repetitive substitution gives :

$$
\begin{aligned}
b_n^k &= b_{n-1}^k + (n-k+1)b_n^{k-1} \\
&= b_{n-1}^k + (n-k+1)\{b_{n-1}^{k-1} + (n-k+2)b_n^{k-2}\} \\
&= b_{n-1}^k + (n-k+1)b_{n-1}^{k-1} + (n-k+1)^{(2)}b_n^{k-2} \\
&= b_{n-1}^k + (n-k+1)b_{n-1}^{k-1} + (n-k+1)^{(2)}\{b_{n-1}^{k-2} + (n-k+3)b_n^{k-3}\} \\
&= b_{n-1}^k + (n-k+1)b_{n-1}^{k-1} + (n-k+1)^{(2)}b_{n-1}^{k-2} + (n-k+1)^{(3)}b_n^{k-3} \\
&= \dots \\
&= b_{n-1}^k + (n-k+1)b_{n-1}^{k-1} + (n-k+1)^{(2)}b_{n-1}^{k-2} + \dots + (n-k+1)^{(k)}b_n^0 \\
&= b_{n-1}^k + (n-k+1)b_{n-1}^{k-1} + (n-k+1)^{(2)}b_{n-1}^{k-2} + \dots + (n-k+1)^{(k)}b_{n-1}^0 \quad (b_n^0 = b_{n-1}^0 = 1) \\
&= \sum_{i=0}^{k} (n-k+1)^{(k-i)} b_{n-1}^i
\end{aligned}
$$

Remark 1. Formula (1) impliess that b_n^k is a recurrent sum of order n, we have :

$$
b_n^k = \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0} \quad (1')
$$

where δ_m is Kronecker delta.

Corollary 1.

$$
\forall\, 1 \le n, \quad b_n^n = \sum_{i=0}^{n} (n-i)!\, b_{n-1}^i \quad (1'')
$$

Proof.

Set $k = n$ in (1). Notice that $1^{(n-i)} = (n-i)!$. ■

Lemma 2. Let $(\alpha, \beta) \in \mathbb{N}^2$. We have :

$$
\int_0^A x^\alpha F_\beta(x)\, dx = \sum_{k=0}^{\alpha} (\alpha)_k (-1)^k A^{\alpha-k} F_{\beta+k+1}(A) + \alpha!\, (-1)^{\alpha+1} F_{\beta+\alpha+1}(0) \quad (2)
$$

where $(x)_m = x(x-1)\dots(x-m+1)$ is the falling factorial function.

Proof. 1. Base case : verify true for $\alpha = 0$.

$$\int_0^A x^0 F_\beta(x)\,dx = F_{\beta+1}(A) - F_{\beta+1}(0) = \sum_{k=0}^{0} (0)_k\,(-1)^k A^{0-k} F_{\beta+k+1}(A) + 0!\,(-1)^1 F_{\beta+1}(0)$$

2. Induction hypothesis : assume the statement is true until α.

$$\int_0^A x^\alpha F_\beta(x)\,dx = \sum_{k=0}^{\alpha} (\alpha)_k\,(-1)^k A^{\alpha-k} F_{\beta+k+1}(A) + \alpha!\,(-1)^{\alpha+1} F_{\beta+\alpha+1}(0)$$

3. Induction step : we will show that this statement is true for $(\alpha+1)$.

$$\int_0^A x^{\alpha+1} F_\beta(x)\,dx = \sum_{k=0}^{\alpha+1} (\alpha+1)_k\,(-1)^k A^{\alpha+1-k} F_{\beta+k+1}(A) + (\alpha+1)!\,(-1)^{\alpha+2} F_{\beta+\alpha+2}(0)$$

By integrating by parts we see that :

$$\begin{aligned}
\int_0^A x^{\alpha+1} F_\beta(x)\,dx &= \left[x^{\alpha+1} F_{\beta+1}(x)\right]_0^A - (\alpha+1)\int_0^A x^\alpha F_{\beta+1}(x)\,dx \\
&= A^{\alpha+1} F_{\beta+1}(A) - (\alpha+1)\left(\sum_{k=0}^{\alpha} (\alpha)_k\,(-1)^k A^{\alpha-k} F_{\beta+k+2}(A) + \alpha!\,(-1)^{\alpha+1} F_{\beta+\alpha+2}(0)\right) \\
&= A^{\alpha+1} F_{\beta+1}(A) - \left(\sum_{k=0}^{\alpha} (\alpha+1)_{k+1}\,(-1)^k A^{\alpha-k} F_{\beta+k+2}(A) + (\alpha+1)!\,(-1)^{\alpha+1} F_{\beta+\alpha+2}(0)\right) \\
&= A^{\alpha+1} F_{\beta+1}(A) - \left(\sum_{k=1}^{\alpha+1} (\alpha+1)_k\,(-1)^{k-1} A^{\alpha-k+1} F_{\beta+k+1}(A) + (\alpha+1)!\,(-1)^{\alpha+1} F_{\beta+\alpha+2}(0)\right) \\
&= \left(A^{\alpha+1} F_{\beta+1}(A) + \sum_{k=1}^{\alpha+1} (\alpha+1)_k\,(-1)^k A^{\alpha-k+1} F_{\beta+k+1}(A)\right) + (\alpha+1)!\,(-1)^{\alpha+2} F_{\beta+\alpha+2}(0) \\
&= \sum_{k=0}^{\alpha+1} (\alpha+1)_k\,(-1)^k A^{\alpha-k+1} F_{\beta+k+1}(A) + (\alpha+1)!\,(-1)^{\alpha+2} F_{\beta+\alpha+2}(0)
\end{aligned}$$

The proposition is proven by induction. ■

2- The formula

Theorem. (*Benmoussa's formula*) Let $n \in \mathbb{N}$ and f a function of x. We have :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x f(x)\, dx\, dx_1\, dx_2 \dots dx_{n-1} = \sum_{k=0}^{n} b_n^k\, (-1)^k {x_n}^{n-k} F_{n+k}(x_n) + \sum_{k=0}^{n-1} b_{k+1}^{k+1}\, (-1)^k F_{2k+2}(0) \frac{{x_n}^{2(n-1)-2k}}{(2(n-1)-2k)!!} \quad (3)$$

where F_m is the m-th primitive function of f (*meaning the expression of the indefinite integral of f excluding the constants of integration*).

Proof.

For simplicity, set :

$$\vartheta^n = \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x f(x)\, dx\, dx_1\, dx_2 \dots dx_{n-1}$$

The aim is to prove that :

$$\vartheta^n = \sum_{k=0}^{n} b_n^k\, (-1)^k {x_n}^{n-k} F_{n+k}(x_n) + \sum_{k=0}^{n-1} b_{k+1}^{k+1}\, (-1)^k F_{2k+2}(0) \frac{{x_n}^{2(n-1)-2k}}{(2(n-1)-2k)!!}$$

1. Base case : verify true for $n = 0$.

$$\vartheta^0 = f(x) = \textstyle\sum_{k=0}^{0} b_0^k\, (-1)^k {x_0}^{0-k} F_{0+k}(x_0) + \sum_{k=0}^{-1} b_{k+1}^{k+1}\, (-1)^k F_{2k+2}(0) \frac{{x_0}^{-2k}}{(-2k)!!}$$

2. Induction hypothesis : assume the statement is true until n.

3. Induction step : we will show that it is true for $(n+1)$.

$$\vartheta^{n+1} = \int_0^{x_{n+1}} x_n \vartheta^n dx_n$$

$$= \int_0^{x_{n+1}} x_n \left(\sum_{k=0}^{n} b_n^k (-1)^k x_n{}^{n-k} F_{n+k}(x_n) + \sum_{k=0}^{n-1} b_{k+1}^{k+1} (-1)^k F_{2k+2}(0) \frac{x_n{}^{2(n-1)-2k}}{(2(n-1)-2k)!!} \right) dx_n$$

$$= \int_0^{x_{n+1}} \left(\sum_{k=0}^{n} b_n^k (-1)^k x_n{}^{n+1-k} F_{n+k}(x_n) + \sum_{k=0}^{n-1} b_{k+1}^{k+1} (-1)^k F_{2k+2}(0) \frac{x_n{}^{2(n-1)-2k+1}}{(2(n-1)-2k)!!} \right) dx_n$$

$$= \int_0^{x_{n+1}} \sum_{k=0}^{n} b_n^k (-1)^k x_n{}^{n+1-k} F_{n+k}(x_n)\, dx_n + \int_0^{x_{n+1}} \sum_{k=0}^{n-1} b_{k+1}^{k+1} (-1)^k F_{2k+2}(0) \frac{x_n{}^{2(n-1)-2k+1}}{(2(n-1)-2k)!!}\, dx_n$$

$$= \sum_{k=0}^{n} b_n^k (-1)^k \int_0^{x_{n+1}} x_n{}^{n+1-k} F_{n+k}(x_n)\, dx_n + \sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1} (-1)^k F_{2k+2}(0)}{(2(n-1)-2k)!!} \int_0^{x_{n+1}} x_n{}^{2(n-1)-2k+1}\, dx_n$$

$$= S_1 + S_2$$

We must deal with each sum individually.

$$S_1 = \sum_{k=0}^{n} b_n^k (-1)^k \int_0^{x_{n+1}} x_n{}^{n-k+1} F_{n+k}(x_n)\, dx_n$$

$$= \sum_{k=0}^{n} b_n^k (-1)^k \left(\sum_{i=0}^{n-k+1} (n-k+1)_i (-1)^i x_{n+1}{}^{n-k+1-i} F_{n+k+i+1}(x_{n+1}) + (n-k+1)!\,(-1)^{n-k+2} F_{2n+2}(0) \right)$$

Now we use the following property which enables us to transform falling factorial to rising factorial.

$$\forall (x,m) \in \mathbb{N}^2, x \geq m, \quad (x)_m = (x-m+1)^{(m)}$$

So :

$$S_1 = \sum_{k=0}^{n} b_n^k (-1)^k \left(\sum_{i=0}^{n-k+1} ((n+1)-(k+i)+1)^{(i)} (-1)^i x_{n+1}{}^{n-k+1-i} F_{n+k+i+1}(x_{n+1}) + (n-k+1)!\,(-1)^{n-k+2} F_{2n+2}(0) \right)$$

$$\sum_{k=0}^{n} b_n^k (-1)^k \sum_{i=0}^{n-k+1} ((n+1)-(k+i)+1)^{(i)} (-1)^i x_{n+1}{}^{n-k+1-i} F_{n+k+i+1}(x_{n+1}) + \sum_{k=0}^{n} b_n^k\, (n-k+1)!\,(-1)^n F_{2n+2}(0)$$

$$= \sum_{k=0}^{n} \sum_{j=k}^{n+1} b_n^k (-1)^k ((n+1)-j+1)^{(j-k)} (-1)^{j-k} x_{n+1}{}^{n-j+1} F_{n+j+1}(x_{n+1}) + \sum_{k=0}^{n} b_n^k\, (n-k+1)!\,(-1)^n F_{2n+2}(0)$$

Now we use the following summation interchange formula (see [2] for a proof):

$$\sum_{i=0}^{n}\sum_{j=i}^{m} a_{i,j} = \sum_{j=0}^{m}\sum_{i=0}^{\min(j,n)} a_{i,j}$$

Valid for all $(m,n) \in \mathbb{N}^2$ with $m \geq n$ and $(a_{i,j})_{0\leq i\leq n, 0\leq j\leq m}$ a doubly indexed sequence of real numbers.

So :

$$S_1 \quad = \quad \sum_{j=0}^{n+1}\sum_{k=0}^{\min(j,n)} ((n+1)-j+1)^{(j-k)}\, b_n^k (-1)^j {x_{n+1}}^{n-j+1} F_{n+j+1}(x_{n+1}) + \sum_{k=0}^{n} b_n^k\, (n-k+1)!\, (-1)^n F_{2n+2}(0)$$

By cases on j and using the property that $b_n^j = 0$ for all $j > n$, we can prove that :

$$\sum_{k=0}^{\min(j,n)} ((n+1)-j+1)^{(j-k)} b_n^k \quad = \quad \sum_{k=0}^{j} ((n+1)-j+1)^{(j-k)} b_n^k$$

So :

$$S_1 \quad = \quad \sum_{j=0}^{n+1}\left(\sum_{k=0}^{j}((n+1)-j+1)^{(j-k)} b_n^k\right)(-1)^j {x_{n+1}}^{n-j+1} F_{n+j+1}(x_{n+1}) + \sum_{k=0}^{n} b_n^k\, (n-k+1)!\, (-1)^n F_{2n+2}(0)$$

using (1) we deduce that :

$$S_1 \quad = \quad \sum_{j=0}^{n+1} b_{n+1}^j (-1)^j {x_{n+1}}^{n+1-j} F_{n+1+j}(x_{n+1}) + \sum_{k=0}^{n} b_n^k\, (n-k+1)!\, (-1)^n F_{2n+2}(0)$$

$$= \quad T_1 + T_2$$

To finish, we must prove that :

$$T_2 + S_2 = \sum_{k=0}^{n} b_{k+1}^{k+1}\, (-1)^k F_{2k+2}(0)\, \frac{{x_{n+1}}^{2n-2k}}{(2n-2k)!!}$$

But we already have that:

$$
\begin{aligned}
S_2 &= \sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1}(-1)^k F_{2k+2}(0)}{(2(n-1)-2k)!!} \int_0^{x_{n+1}} x_n{}^{2(n-1)-2k+1}\, dx_n \\
&= \sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1}(-1)^k F_{2k+2}(0)}{(2(n-1)-2k)!!} \left[\frac{x_n{}^{2(n-1)-2k+2}}{2(n-1)-2k+2}\right]_0^{x_{n+1}} \\
&= \sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1}(-1)^k F_{2k+2}(0)}{(2(n-1)-2k)!!} \left(\frac{x_{n+1}{}^{2n-2k}}{2n-2k}\right) \\
&= \sum_{k=0}^{n-1} b_{k+1}^{k+1}(-1)^k F_{2(k+1)}(0) \frac{x_{n+1}{}^{2n-2k}}{(2n-2k)!!}
\end{aligned}
$$

and :

$$
\begin{aligned}
T_2 &= (-1)^n F_{2n+2}(0) \sum_{k=0}^{n} b_n^k\,(n-k+1)! \\
&= (-1)^n F_{2(n+1)}(0) b_{n+1}^{n+1} \qquad \text{using } (1'')
\end{aligned}
$$

So the proposition is proven by induction. ∎

3-Some applications

We present some applications of formula (3)

Proposition 1. Let $n \in \mathbb{N}$, we have

$$\frac{1}{2^n} = \sum_{k=0}^{n} b_n^k (-1)^k \frac{1}{(n+1)^{(k)}} \quad (4)$$

Proof.

Setting $f(x) = 1$ in (3), knowing that $F_m(x) = \frac{x^m}{m!}$, we find that :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x\, dx\, dx_1\, dx_2 \dots dx_{n-1} = \frac{x_n{}^{2n}}{n!} \sum_{k=0}^{n} b_n^k\,(-1)^k \frac{1}{(n+1)^{(k)}}$$

But we have also :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x\, dx\, dx_1\, dx_2 \dots dx_{n-1} = \frac{{x_n}^{2n}}{(2n)!!}$$

Equating we arrive to the result.

In general, if we set $f = x^t$ $(x > 0)(t \in \mathbb{Q}^+)$, we get the following formula,

$$\frac{(t+1)^{(n)}}{\prod_{i=1}^{n}(t+2i)} = \sum_{k=0}^{n} b_n^k (-1)^k \frac{1}{(t+n+1)^{(k)}} \quad (5)$$

Proposition 2. Let

$$\frac{e^x}{x^n} y_n\left(-\frac{1}{x}\right) = \left\{\sum_{k=0}^{\infty} \frac{x^k}{k! \prod_{i=1}^{n}(k+2i)}\right\} - \frac{1}{x^2}\left\{\sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1}}{(2(n-1)-2k)!!}(-1)^k \frac{1}{x^{2k}}\right\} \quad (6)$$

Where $y_n(x) = \sum_{k=0}^{n} b_n^k x^k$ is Bessel polynomial.

Proof.

Set $f(x) = \exp x$ in (3), you get :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x e^x\, dx\, dx_1\, dx_2 \dots dx_{n-1} = e^{x_n} \sum_{k=0}^{n} b_n^k (-1)^k {x_n}^{n-k} + \sum_{k=0}^{n-1} b_{k+1}^{k+1} (-1)^k \frac{{x_n}^{2(n-1)-2k}}{(2(n-1)-2k)!!}$$

We have from the other hand :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x e^x\, dx\, dx_1\, dx_2 \dots dx_{n-1} = \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \left\{\sum_{k=0}^{\infty} \frac{x^k}{k!}\right\} dx\, dx_1\, dx_2 \dots dx_{n-1}$$

$$= \sum_{k=0}^{\infty} \frac{{x_n}^{k+2n}}{k! \prod_{i=1}^{n}(k+2i)}$$

Equating we reach the desired formula.

Proposition 3. Let We have for all $x \in [-1,0[\cup]0,1]$:

$$x^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\prod_{i=0}^{n}(2k+2i+1)} = \arctan x \sum_{k=0}^{n} b_n^k (-1)^k \frac{\phi_{n+k}(x)}{x^k} + \ln(1+x^2) \sum_{k=0}^{n} b_n^k (-1)^k \frac{\psi_{n+k}(x)}{x^k} \quad (7)$$

where :

$$
\begin{cases}
\phi_{2t}(x) = \sum_{j=0}^{t} (-1)^{t-j} \frac{1}{(2j)!\,(2t-2j)!} x^{2j} \\
\phi_{2t+1}(x) = \sum_{j=0}^{t} (-1)^{t-j} \frac{1}{(2j+1)!\,(2t-2j)!} x^{2j+1}
\end{cases}
$$

$$
\begin{cases}
\psi_{2t}(x) = \sum_{j=1}^{t} (-1)^{t-j+1} \frac{1}{2(2j-1)!\,(2t+1-2j)!} x^{2j-1} \\
\psi_{2t+1}(x) = \sum_{j=0}^{t} (-1)^{t-j+1} \frac{1}{2(2j)!\,(2t+1-2j)!} x^{2j}
\end{cases}
$$

See [3] for more properties about ϕ_t and ψ_t.

Proof.

If we set in (3) $f(x) = \arctan x$, and knowing from reference [3] that :

$$
F_m(x) = \phi_m(x) \arctan x + \psi_m(x) \ln(1 + x^2)
$$

We get that :

$$
\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \arctan x \, dx \, dx_1 \, dx_2 \dots dx_{n-1}
$$

$$
= {x_n}^n \left(\arctan x_n \sum_{k=0}^{n} b_n^k (-1)^k \frac{\phi_{n+k}(x_n)}{{x_n}^k} + \ln(1 + {x_n}^2) \sum_{k=0}^{n} b_n^k (-1)^k \frac{\psi_{n+k}(x_n)}{{x_n}^k} \right)
$$

also :

$$
\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \arctan x \, dx \, dx_1 \, dx_2 \dots dx_{n-1} = \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \right\} dx \, dx_1 \, dx_2 \dots dx_{n-1}
$$

$$
= \sum_{k=0}^{\infty} \frac{(-1)^k {x_n}^{2n+2k+1}}{\prod_{i=0}^{n} (2k + 2i + 1)}
$$

Equating we reach our formula.

Proposition 4. Let for all x

$$
(-1)^n x^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)! \prod_{i=1}^{n} (2k+2i+1)} = \sin\left(x + \frac{n\pi}{2}\right) \sum_{k=0}^{n} b_n^k \frac{\cos\left(\frac{k\pi}{2}\right)}{x^k} + \cos\left(x + \frac{n\pi}{2}\right) \sum_{k=0}^{n} b_n^k \frac{\sin\left(\frac{k\pi}{2}\right)}{x^k} \quad (8)
$$

Proof.

If we set $f = \sin x$, and knowing that $F_m(x) = (-1)^m \sin\left(x + \frac{m\pi}{2}\right)$ we get that :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \sin x \, dx \, dx_1 \, dx_2 \dots dx_{n-1} = \sum_{k=0}^{n} b_n^k (-1)^k {x_n}^{n-k} (-1)^{n+k} \sin\left(x_n + \frac{(n+k)\pi}{2}\right) + \sum_{k=0}^{n-1} \frac{b_{k+1}^{k+1} (-1)^k \sin((k+1)\pi) \, {x_n}^{2(n-1)-2k}}{(2(n-1)-2k)!!}$$

$$= (-1)^n {x_n}^n \left\{ \sin\left(x_n + \frac{n\pi}{2}\right) \sum_{k=0}^{n} b_n^k \frac{\cos\left(\frac{k\pi}{2}\right)}{{x_n}^k} + \cos\left(x_n + \frac{n\pi}{2}\right) \sum_{k=0}^{n} b_n^k \frac{\sin\left(\frac{k\pi}{2}\right)}{{x_n}^k} \right\}$$

On the other hand we have :

$$\int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \sin x \, dx \, dx_1 \, dx_2 \dots dx_{n-1} = \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1 \int_0^{x_1} x \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right\} dx \, dx_1 \, dx_2 \dots dx_{n-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k {x_n}^{2n+2k+1}}{(2k+1)! \prod_{i=1}^{n} (2k+2i+1)}$$

Equating we reach formula (7).

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