On a formula similar to the formula of Grunert

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Abstract :

We believe that recurrent sums are a way to express mathematical functions just like continued fractions are, so we present a theorem wich enable us to write some sequences as recurrent sums of a special type [1].

1. Introduction

Let $n \in \mathbb{N}$ and f an arbitrary function of x, Grunert's formula is the following remarkable formula [2]:

$$\underbrace{x(\dots x(x(x}_{n} f \overbrace{)')'\dots)'}^{n} = \sum_{k=0}^{n} S_n^k x^k f^{(k)}$$

Where $'$ stands for the derivation operation, $f^{(k)}$ is the k-th derivative of f and S_n^k are Stirling numbers of the second kind.

A natural question that might come to mind is : what happens when we replace the operation of derivation in the above formula with integration, what kind of formula will we get and what kind of coefficients will appear in this formula ?

The answer to this question will enable us to get some beautiful formulas like the following one :

$$\frac{1}{n!} = \sum_{k=0}^{n(n-1)} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n^2-n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n^2-3n+3-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (1-i_1)^{(i_1-i_0)}\, \delta_{i_0}$$

Where $x^{(m)} = x(x+1)\dots(x+m-1)$ is the rising factorial of order m of x and δ_m is the Kronecker delta.

Another remarkable formula is :

$$\frac{1}{1.2.4\dots 2n} + \frac{1}{1.3.5\dots(2n+1)} + \frac{1}{2.4.6\dots(2n+2)} + \frac{1}{6.5.7\dots(2n+3)} + \cdots$$
$$= e\left(\sum_{k=0}^{n} (-1)^k \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)}\, \delta_{i_0}\right)$$

Wich combines the number e with a recurrent sum to give an infinite series.

2. the operator I & some lemmas.

Let I be the linear map that assigns to each function f its primitive function F_1, *by primitive function we mean the expression of the indefinite integral without the constant of integration*.

We have :

$$I(f) = F_1$$

In general :

$$I^n(f) = \underbrace{I(\dots(I}_{n}\, f)\dots) = F_n$$

where F_n is the n-th primitive function of f, *meaning the expression of the n-th indefinite integral of* f *excluding the constants of integration*.

It is also obvious that (integration by parts) :

$$I(fg') = fg - I(f'g)$$

Lemma 1.

Let $\alpha, \beta \in \mathbb{N}$. We have :

$$I\left(x^\alpha I^\beta f\right) = \sum_{k=0}^{\alpha} (\alpha)_k \,(-1)^k x^{\alpha-k} I^{\beta+k+1} f$$

where $(x)_n = x(x-1)\dots(x-n+1)$ is the falling factorial function.

Proof. 1. Base case : verify true for $\alpha = 0$.

$I\left(x^0 I^\beta f\right) = I\left(I^\beta f\right) = I^{\beta+1} f$ and $\sum_{k=0}^{0} (0)_k \,(-1)^k x^{0-k} I^{\beta+k+1} f = I^{\beta+1} f.$

2. Induction hypothesis : assume the statement is true until α.

$$I\left(x^\alpha I^\beta f\right) = \sum_{k=0}^{\alpha} (\alpha)_k \,(-1)^k x^{\alpha-k} I^{\beta+k+1} f$$

3. Induction step : we will show that this statement is true for $(\alpha + 1)$.

$$I\left(x^{\alpha+1} I^\beta f\right) = \sum_{k=0}^{\alpha+1} (\alpha+1)_k \,(-1)^k x^{\alpha+1-k} I^{\beta+k+1} f$$

By integrating by parts we see that :

$$
\begin{aligned}
I\left(x^{\alpha+1}I^{\beta}f\right) &= x^{\alpha+1}I^{\beta+1}f-(\alpha+1)I(x^{\alpha}I^{\beta+1}f)\\
&= x^{\alpha+1}I^{\beta+1}f-(\alpha+1)\sum_{k=0}^{\alpha}(\alpha)_k\,(-1)^k x^{\alpha-k}I^{\beta+k+2}f\\
&= x^{\alpha+1}I^{\beta+1}f-\sum_{k=0}^{\alpha}(\alpha+1)_{k+1}\,(-1)^k x^{\alpha-k}I^{\beta+k+2}f\\
&= x^{\alpha+1}I^{\beta+1}f-\sum_{k=1}^{\alpha+1}(\alpha+1)_{k}\,(-1)^{k-1} x^{\alpha+1-k}I^{\beta+k+1}f\\
&= \sum_{k=0}^{\alpha+1}(\alpha+1)_{k}\,(-1)^{k} x^{\alpha+1-k}I^{\beta+1+k}f
\end{aligned}
$$

The proposition is proven by induction. ■

The following lemma gives a transformation formula.

Lemma 2.

Let $x, i \in \mathbb{N}$.

$$(x)_i = (x-i+1)^{(i)}$$

Proof. The proof is obvious.

Lemma 3.

Let $n, m \in \mathbb{N}$ with $m \geq n$, and $(a_{i,j})_{0\leq i\leq n, 0\leq j\leq m}$ a doubly indexed sequence of real numbers. We have :

$$\sum_{i=0}^{n}\sum_{j=i}^{m} a_{i,j} = \sum_{j=0}^{m}\sum_{i=0}^{\min(j,n)} a_{i,j}$$

Proof. See [2].

3. The formula

We begin by studying the following three structures :

$$
\begin{cases}
f_n = \underbrace{(Ix)\ldots(Ix)}_{n} f\\
\tilde{f}_n = \underbrace{(Ix^2)\ldots(Ix^2)}_{n} f\\
\acute{f}_n = \underbrace{(Ix^{p_n-p_{n-1}-1})\ldots(Ix)}_{n} f
\end{cases}
$$

Where p_n is the n-th prime number with $p_0 = 0$.

1-f_n

The first four expansions of f_n are :

$$f_0 = f$$

$$\begin{aligned} f_1 &= (Ix)f \\ &= I(xf) \\ &= x.If - I^2 f \end{aligned}$$

$$\begin{aligned} f_2 &= (Ix)(Ix)f \\ &= (Ix)(x.If - I^2 f) \\ &= I(x^2.If - x.I^2 f) \\ &= I(x^2.If) - I(x.I^2 f) \\ &= \begin{cases} x^2.I^2 f - 2x.I^3 f + 2I^4 f \\ \qquad - (x.I^3 f - I^4 f) \end{cases} \text{Using lemma 1} \\ &= x^2.I^2 f - 3x.I^3 f + 3I^4 f \end{aligned}$$

$$f_3 = x^3 I^3 f - 6x^2 I^4 f + 15x I^5 f - 15 I^6 f$$

$$f_4 = x^4 I^4 f - 10x^3.I^5 f + 45x^2.I^6 f - 105x.I^7 f + 105 I^8 f$$

…

We conjecture that :

$$f_n = \sum_{k=0}^{n} A_n^k \,(-1)^k x^{n-k}.I^{n+k} f$$

Where the numbers A_n^k can be arranged in an arithmetical triangle as follows :

$n\,/\,k$	0	1	2	3	4	…
0	1					
1	1	1				
2	1	3	3			
3	1	6	15	15		
4	1	10	45	105	105	
…	…	…	…	…	…	…

Table 1 : Table of values for the numbers A_n^k, with $0 \le k \le n$ and $0 \le n \le 4$

Some of the properties of the numbers A_n^k that can be observed from the triangle are :

$$\begin{cases} \forall n \geq 0, \quad A_n^0 = 1 \\ \forall k > n, \quad A_n^k = 0 \\ \forall n \geq 1, \quad A_n^{n-1} = A_n^n \end{cases}$$

These numbers can be calculated by the following recursive relation:

$$A_n^k = \sum_{i=0}^{k} (n+1-k)^{(k-i)} A_{n-1}^i$$

Substituting repeatidly we get that :

$$A_n^k = \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} A_0^{i_0}$$

We observe that :

$$\forall i_0 \in \mathbb{N}, \quad A_0^{i_0} = \delta_{i_0}$$

Where δ_{i_0} is the Kronecker delta.

So :

$$A_n^k = \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

2- $\tilde{f}_n$

We have :

$$\tilde{f}_0 = f$$

$$\begin{aligned} \tilde{f}_1 &= (Ix^2)f \\ &= x^2.If - 2x.I^2f + 2I^3f \end{aligned}$$

$$\tilde{f}_2 = x^4.I^2f - 6x^3.I^3f + 20x^2.I^4f - 40x.I^5f + 40I^6f$$

Hence :

$$\tilde{f}_n = \sum_{k=0}^{2n} B_{2n}^k \, (-1)^k x^{2n-k}.I^{n+k}f$$

Where the numbers B_{2n}^k are given by :

$$B_{2n}^k = \sum_{i=0}^{k} (2n+1-k)^{(k-i)} B_{2(n-1)}^i$$

Substituting repeatidly we get :

$$B_{2n}^{k} = \sum_{i_{n-1}=0}^{k} (2n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (2n-1-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

3- $\acute{f}_n$

We have :

$$\acute{f}_0 = f$$

$$\acute{f}_1 = x.If - I^2 f$$

$$\acute{f}_2 = x.I^2 f - 2I^3 f$$

$$\acute{f}_3 = x^2.I^3 f - 4x.I^4 f + 4I^5 f$$

$$\acute{f}_4 = x^3.I^4 f - 7x^2.I^5 f + 18x.I^6 f - 18I^7 f$$

...

$$\acute{f}_n = \sum_{k=0}^{p_n-n} C_{p_n-n}^{k} (-1)^k x^{p_n-n-k}.I^{n+k} f$$

Where the numbers $C_{p_n-n}^{k}$ are given by :

$$C_{p_n-n}^{k} = \sum_{i=0}^{k} (p_n-n+1-k)^{(k-i)} D_{p_{n-1}-(n-1)}^{i}$$

Substituting we get :

$$C_{p_n-n}^{k} = \sum_{i_{n-1}=0}^{k} (p_n-n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (p_{n-1}-n+2-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

4- The general case

Theorem.

Let f be a function of x, and $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of natural numbers. If :

$$f_n = \underbrace{(Ix^{a_n-a_{n-1}-1}) \dots (Ix^{a_1-a_0-1})}_{n} f$$

Then :

$$f_n = \sum_{k=0}^{a_n - a_0 - n} U_{a_n-a_0-n}^{k} (-1)^k x^{a_n-a_0-n-k}. I^{n+k} f$$

Where the numbers $U_{a_n-a_0-n}^{k}$ are given by:

$$\begin{cases} U_0^0 & = & 1 \\ U_{a_n-a_0-n}^{k} & = & \sum_{i=0}^{k} (a_n - a_0 - n + 1 - k)^{(k-i)} U_{a_{n-1}-a_0-(n-1)}^{i} \end{cases}$$

Proof. 1. Base case :verify true for $n = 0$.

$$f_0 = f \quad \text{and} \quad \sum_{k=0}^{0} U_0^k (-1)^k x^{0-k}. I^k f = U_0^0 f = f$$

2. Induction hypothesis : assume the statement is true until n.

3. Induction step : we will show that this statement is true for $(n+1)$.

$$\begin{aligned}
f_{n+1} &= (I x^{a_{n+1}-a_n-1}) f_n \\
&= (I x^{a_{n+1}-a_n-1}) \left(\sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^{k} (-1)^k x^{a_n-a_0-n-k}. I^{n+k} f \right) \\
&= \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^{k} (-1)^k I (x^{a_{n+1}-a_0-(n+1)-k}. I^{n+k} f) \\
&= \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^{k} (-1)^k. \sum_{i=0}^{a_{n+1}-a_0-(n+1)-k} (a_{n+1} - a_0 - (n+1) - k)_i (-1)^i x^{a_{n+1}-a_0-(n+1)-k-i}. I^{n+k+i+1} f \\
&= \sum_{k=0}^{(a_n-a_0-n)} \sum_{i=0}^{(a_{n+1}-a_0-(n+1)-k)} (a_{n+1} - a_0 - (n+1) + 1 - k - i)^{(i)} U_{a_n-a_0-n}^{k} (-1)^{k+i} x^{a_{n+1}-a_0-(n+1)-(k+i)}. I^{n+1+(k+i)} f \\
&= \sum_{k=0}^{(a_n-a_0-n)} \sum_{j=k}^{(a_{n+1}-a_0-(n+1))} (a_{n+1} - a_0 - (n+1) + 1 - j)^{(j-k)} U_{a_n-a_0-n}^{k} (-1)^j x^{a_{n+1}-a_0-(n+1)-j}. I^{n+1+j} f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} \sum_{k=0}^{\min(j, a_n-a_0-n)} (a_{n+1} - a_0 - (n+1) + 1 - j)^{(j-k)} U_{a_n-a_0-n}^{k} (-1)^j x^{a_{n+1}-a_0-(n+1)-j}. I^{(n+1)+j} f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} \left(\sum_{k=0}^{j} (a_{n+1} - a_0 - (n+1) + 1 - j)^{(j-k)} U_{a_n-a_0-n}^{k} \right) (-1)^j x^{a_{n+1}-a_0-(n+1)-j}. I^{(n+1)+j} f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} U_{a_{n+1}-a_0-(n+1)}^{j} (-1)^j x^{a_{n+1}-a_0-(n+1)-j}. I^{(n+1)+j} f
\end{aligned}$$

We must prove that :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k} = \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k}$$

If $j \leq a_n - a_0 - n$ we get :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k} = \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k}$$

If $j = a_{n+1} - a_0 - (n+1)$ we get :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k} = \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^{k}$$

And :

$$\sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k} = \sum_{k=0}^{a_{n+1}-a_0-(n+1)} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^{k}$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^{k} + 1^{(0)} U_{a_n-a_0-n}^{a_{n+1}-a_0-(n+1)}$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^{k} + 0 = \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^{k}$$

So in all cases :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k} = \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^{k}$$

The theorem is proven by induction. ∎

4. Applications.

Firstly, we must answer the question of the introduction. The formula obtained when substituting derivation by integration in Grunert's formula is :

$$\underbrace{(xI)\dots(xI)}_{n}f=\sum_{k=0}^{n-1}U_{n-1}^{k}\,(-1)^{k}x^{n-k}.I^{n+k}f$$

Where :

$$U_{n-1}^{k}=\sum_{i_{n-1}=0}^{k}(n-k)^{(k-i_{n-1})}\sum_{i_{n-2}=0}^{i_{n-1}}(n-1-i_{n-1})^{(i_{n-1}-i_{n-2})}\dots\sum_{i_0=0}^{i_1}(1-i_1)^{(i_1-i_0)}\,\delta_{i_0}$$

Corollary 1.

Let $m\in\mathbb{N}$, $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of natural numbers. We have :

$$\frac{(m+1)^{(n)}}{(m+a_1-a_0)(m+a_2-a_0)\dots(m+a_n-a_0)}=\sum_{k=0}^{a_n-n-a_0}U_{a_n-a_0-n}^{k}\,(-1)^{k}.\frac{1}{(m+n+1)^{(k)}}$$

Proof. By setting $f=x^m$ $(m\in\mathbb{N})$ in the theorem, we get on the one hand :

$$(x^m)_n\quad=\quad\frac{x^{m+a_n-a_0}}{(m+1)^{(n)}}\sum_{k=0}^{a_n-a_0-n}U_{a_n-a_0-n}^{k}(-1)^{k}.\frac{1}{(m+n+1)^{(k)}}$$

On the other hand we have :

$$\begin{aligned}(x^m)_n &= \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n}x^m\\ &= \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_2-a_1-1})}_{n-1}\left(\frac{x^{m+a_1-a_0}}{m+a_1-a_0}\right)\\ &= \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_3-a_2-1})}_{n-2}\left(\frac{x^{m+a_2-a_0}}{(m+a_1-a_0)(m+a_2-a_0)}\right)\\ &= \dots\\ &= \frac{x^{m+a_n-a_0}}{(m+a_1-a_0)(m+a_2-a_0)\dots(m+a_n-a_0)}\end{aligned}$$

Equating we get the desired identity. ■

EXAMPLE :

By setting $m=0, a_n=2n, a_0=1$ we get the formula :

$$\frac{n!}{(2n-1)!!}=\sum_{k=0}^{n-1}\frac{(-1)^k}{(n+1)^{(k)}}\sum_{i_{n-1}=0}^{k}(n-k)^{(k-i_{n-1})}\sum_{i_{n-2}=0}^{i_{n-1}}(n-1-i_{n-1})^{(i_{n-1}-i_{n-2})}\dots\sum_{i_0=0}^{i_1}(1-i_1)^{(i_1-i_0)}\,\delta_{i_0}$$

Corollary 2.

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. We have :

$$\frac{1}{b_1 b_2 \dots b_n} = \sum_{k=0}^{n(b_n-1)} U_{n(b_n-1)}^{k} (-1)^k . \frac{1}{(n+1)^{(k)}}$$

Proof. By setting $m = 0, a_n = nb_n$ in the previous corollary we get the desired identity.

EXAMPLES :

By setting in corollary 2 :

$b_n = A$

Where $A \in \mathbb{N}^*$.

We get :

$$\frac{1}{A^n} = \sum_{k=0}^{n(A-1)} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n(A-1)+1-k)^{(k-i_{n-1})} \dots \sum_{i_0=0}^{i_1} (A-i_1)^{(i_1-i_0)}\, \delta_{i_0}$$

For example, if $A = 2$ we get the following formula:

$$\frac{1}{2^n} = \sum_{k=0}^{n} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)}\, \delta_{i_0}$$

Setting :

$b_n = n$

We get :

$$\frac{1}{n!} = \sum_{k=0}^{n^2-n} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n^2-n+1-k)^{(k-i_{n-1})} \dots \sum_{i_0=0}^{i_1} (1-i_1)^{(i_1-i_0)}\, \delta_{i_0}$$

Remark.

We remark that :

$$\frac{(n+1)^{(n(b_n-1))}}{\sum_{k=0}^{n(b_n-1)} (-1)^k (n+k+1)^{(n(b_n-1)-k)} U_{n(b_n-1)}^{k}} \in \mathbb{N}^*$$

Proof.

$$\frac{1}{b_1 b_2 \dots b_n} = \sum_{k=0}^{n(b_n-1)} U_{n(b_n-1)}^k (-1)^k . \frac{1}{(n+1)^{(k)}}$$

$$= \frac{\sum_{k=0}^{n(b_n-1)} (-1)^k U_{n(b_n-1)}^k (n+k+1)^{(n(b_n-1)-k)}}{(n+1)^{(n(b_n-1))}}$$

Inverting both sides we get the desired result. ∎

Corollary 3.

Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. We have for all $x \in \mathbb{R}$:

$$x^n \sum_{k=0}^{\infty} \frac{x^k}{k!\,(k+a_1-a_0)(k+a_2-a_0)\dots(k+a_n-a_0)} = e^x \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k \frac{1}{x^k}$$

Proof. If we set $f = e^x$ in the theorem we get :

$$(e^x)_n = e^x \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k x^{a_n-n-a_0-k}$$

On the other hand we have :

$$(e^x)_n = \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} e^x$$

$$= \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} x^k \right)$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+a_n-a_0}}{k!\,(k+a_1-a_0)(k+a_2-a_0)\dots(k+a_n-a_0)}$$

Equating we get desired formula. ∎

Setting $x = 1, a_n = 2n$ and substituting U_n^k we obtain :

$$e\left(\sum_{k=0}^{n}(-1)^k \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}\right) = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+2)(k+4)\dots(k+2n)}$$

Corollary 4.

Let $n \in \mathbb{N}$. We have for all $x \in [-1,1]$:

$$x^n \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{4^k (2k+1)(2k+1+a_1-a_0)(2k+1+a_2-a_0)\dots(2k+1+a_n-a_0)}$$
$$= \arcsin(x) \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k \frac{g_{n+k}(x)}{x^k}$$
$$+ \sqrt{1-x^2} \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k \frac{f_{n+k}(x)}{x^k}$$

Where $g_m(x)$ and $f_m(x)$ are polynomials explicited in [4].

Proof. If we set $f = \arcsin(x)$ in the theorem we get :

$$(\arcsin(x))_n = \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} \arcsin(x)$$

$$= \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} \left(\sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{4^k(2k+1)} \right)$$

$$= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} \left(\underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} x^{2k+1} \right)$$

$$= \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1+a_n-a_0}}{4^k(2k+1)(2k+1+a_1-a_0)(2k+1+a_2-a_0)\dots(2k+1+a_n-a_0)}$$

On the other hand we have :

$$(\arcsin(x))_n = \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k x^{a_n-n-a_0-k} . I^{n+k} \arcsin(x)$$
$$= \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k x^{a_n-n-a_0-k} \left(g_{n+k}(x) \arcsin(x) + f_{n+k}(x)\sqrt{1-x^2} \right)$$
$$= \arcsin(x) \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k x^{a_n-n-a_0-k} g_{n+k}(x)$$
$$+ \sqrt{1-x^2} \sum_{k=0}^{a_n-n-a_0} U_{a_n-n-a_0}^k (-1)^k x^{a_n-n-a_0-k} f_{n+k}(x)$$

For more details about $g_n(x)$ and $f_n(x)$ see [4].

Equating we reach the desired formula. ■

Setting $x = 1, a_n = 2n$ and substituting U_n^k we obtain :

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)(2k+3)(2k+5)\dots(2k+2n+1)}$$
$$= \frac{\pi}{2}\left(\sum_{k=0}^{n}(-1)^k g_{n+k}(1) \sum_{i_{n-1}=0}^{k}(n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}}(n-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1}(2-i_1)^{(i_1-i_0)}\,\delta_{i_0}\right)$$

We finish this research with the observation that in Table 1, if we sum the coefficients A_n^k horizontally for each $1 \leq n \leq 3$ we find that :

$$\sum_{k=0}^{1} A_1^k = 2, \quad \sum_{k=0}^{2} A_2^k = 7, \quad \sum_{k=0}^{3} A_3^k = 37$$

We have also:

$$\sum_{k=0}^{2} B_2^k = 5, \quad \sum_{k=0}^{4} B_4^k = 107, \quad \sum_{k=0}^{1} C_1^k = 3$$

So we state the following conjecture.

Conjecture.

There is an infinite number of triples (n, a_n, p) such that :

$$p = \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^k$$

Where $n \in \mathbb{N}$, p a prime number and $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of natural numbers.

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