ON A FORMULA SIMILAR TO THE FORMULA OF GRUNERT

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Abstract

We begin by defining the operation of integration without constants, then we present some lemmas important to prove our main theorem and finally we give some applications of the theorem.

1. INTRODUCTION

Let $n \in \mathbb{N}$ and f an arbitrary function of x, Grunert's formula is the following remarkable formula [1]:

$$\underbrace{x(\dots x(x(x}_{n} f \overbrace{')')' \dots)'}^{n} = \sum_{k=0}^{n} S_n^k . x^k . f^{(k)}$$

Where $'$ stands for the derivation operation, $f^{(k)}$ is the k-th derivative of f and S_n^k are Stirling numbers of the second kind.

But what happens when we replace the operation of derivation in the above formula with integration, what kind of formulas we get and what kind of coefficients appear in these formulas ?

That is the question that answers this paper, one of the results of this research is that we got new formulas such as, for example, the following one:

$$\frac{1}{n!} = \sum_{k=0}^{n(n-1)} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n^2-n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n^2-3n+3-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (1-i_1)^{(i_1-i_0)} \delta_{i_0}$$

Where $x^{(n)} = x(x+1)\dots(x+n-1)$ is the rising factorial function and δ_n is the Kronecker delta.

2. THE OPERATION OF "INTEGRATION WITHOUT CONSTANTS".

Definition.

The opertaion of "integration without constants" is a linear map $I\colon \quad f \quad \mapsto \quad F_1$, where F_1 is the primitive function of f, *by primitive we mean the expression of the indefinite integral without the constant of integration*.

One can easily show that $\underbrace{I(\dots(I}_{n} f)\dots) = F_n$ where F_n is the n-th primitive function of f, *meaning the expression of the n-th indefinite integral of f excluding the constants of integration*. For simplicity we will denote $\underbrace{I(\dots(I}_{n} f)\dots)$ as $I^n f$. It is also obvious that the operator I verifies the rule of integration by parts, that is $I(fg') = fg - I(f'g)$.

Lemma 1.

Let $\alpha, \beta \in \mathbb{N}$. We have :

$$I\left(x^\alpha I^\beta f\right) = \sum_{k=0}^{\alpha} (\alpha)_k \, (-1)^k x^{\alpha-k} I^{\beta+k+1} f$$

where $(x)_n = x(x-1)\dots(x-n+1)$ is the falling factorial function.

Proof. 1. Base case : verify true for $\alpha = 0$.

$I\left(x^0 I^\beta f\right) = I\left(I^\beta f\right) = I^{\beta+1} f$ and $\sum_{k=0}^{0} (0)_k \, (-1)^k x^{0-k} I^{\beta+k+1} f = I^{\beta+1} f$.

2. Induction hypothesis : assume the statement is true until α.

$$I\left(x^\alpha I^\beta f\right) = \sum_{k=0}^{\alpha} (\alpha)_k \, (-1)^k x^{\alpha-k} I^{\beta+k+1} f$$

3. Induction step : we will show that this statement is true for $(\alpha + 1)$.

$$I\left(x^{\alpha+1} I^\beta f\right) = \sum_{k=0}^{\alpha+1} (\alpha+1)_k \, (-1)^k x^{\alpha+1-k} I^{\beta+k+1} f$$

By integrating by parts we see that :

$$
\begin{aligned}
I\left(x^{\alpha+1}I^{\beta}f\right) &= x^{\alpha+1}I^{\beta+1}f-(\alpha+1)I(x^{\alpha}I^{\beta+1}f)\\
&= x^{\alpha+1}I^{\beta+1}f-(\alpha+1)\sum_{k=0}^{\alpha}(\alpha)_k\,(-1)^k x^{\alpha-k}I^{\beta+k+2}f\\
&= x^{\alpha+1}I^{\beta+1}f-\sum_{k=0}^{\alpha}(\alpha+1)_{k+1}\,(-1)^k x^{\alpha-k}I^{\beta+k+2}f\\
&= x^{\alpha+1}I^{\beta+1}f-\sum_{k=1}^{\alpha+1}(\alpha+1)_{k}\,(-1)^{k-1} x^{\alpha+1-k}I^{\beta+k+1}f\\
&= \sum_{k=0}^{\alpha+1}(\alpha+1)_{k}\,(-1)^{k} x^{\alpha+1-k}I^{\beta+1+k}f
\end{aligned}
$$

The proposition is proven by induction. ■

The following lemma gives a formula to trasform falling factorial to rising factorial.

Lemma 2.

Let $m, i \in \mathbb{N}$.

$$(m)_i = (m-i+1)^{(i)}$$

Proof. The proof is obvious.

Lemma 3.

Let $n, m \in \mathbb{N}$ with $m \geq n$, and $(a_{i,j})_{0\leq i\leq n, 0\leq j\leq m}$ a doubly indexed sequence of real numbers. We have :

$$\sum_{i=0}^{n}\sum_{j=i}^{m} a_{i,j} = \sum_{j=0}^{m}\sum_{i=0}^{\min(j,n)} a_{i,j}$$

Proof. See [2].

3. THE NEW FORMULA.

We begin by studying the following three structures :

$$
\begin{cases}
f_n = \underbrace{(Ix)\dots(Ix)}_{n} f\\
\bar{f}_n = \underbrace{(Ix^2)\dots(Ix^2)}_{n} f\\
\tilde{f}_n = \underbrace{(Ix^{p_n-p_{n-1}-1})\dots(Ix)}_{n} f
\end{cases}
$$

Where p_n is the n th prime number with $p_0 = 0$.

1-f_n

The first four expansions of f_n are :

$$f_0 = f$$

$$\begin{aligned} f_1 &= (Ix)f \\ &= I(xf) \\ &= x.If - I^2 f \end{aligned}$$

$$\begin{aligned} f_2 &= (Ix)(Ix)f \\ &= (Ix)(x.If - I^2 f) \\ &= I(x^2.If - x.I^2 f) \\ &= I(x^2.If) - I(x.I^2 f) \\ &= \begin{cases} x^2.I^2 f - 2x.I^3 f + 2I^4 f \\ \qquad - (x.I^3 f - I^4 f) \end{cases} \quad \text{Using lemma 1} \\ &= x^2.I^2 f - 3x.I^3 f + 3I^4 f \end{aligned}$$

$$f_3 = x^3 I^3 f - 6x^2 I^4 f + 15xI^5 f - 15I^6 f$$

$$f_4 = x^4 I^4 f - 10x^3.I^5 f + 45x^2.I^6 f - 105x.I^7 f + 105I^8 f$$

…

We conjecture that :

$$f_n = \sum_{k=0}^{n} A_n^k (-1)^k x^{n-k}.I^{n+k} f$$

Where the numbers A_n^k can be arranged in an arithmetical triangle as follows :

n / k	0	1	2	3	4	…
0	1					
1	1	1				
2	1	3	3			
3	1	6	15	15		
4	1	10	45	105	105	
…	…	…	…	…	…	…

Table 1 : Table of values for the numbers A_n^k, with $0 \leq k \leq n$ and $0 \leq n \leq 4$

Some of the properties of the numbers A_n^k that can be observed from the triangle are :

$$\begin{cases} \forall n \geq 0, \quad A_n^0 = 1 \\ \forall k > n, \quad A_n^k = 0 \\ \forall n \geq 1, \quad A_n^{n-1} = A_n^n \end{cases}$$

These numbers can be calculated by the following recursive relation:

$$A_n^k = \sum_{i=0}^{k} (n+1-k)^{(k-i)} A_{n-1}^i$$

Substituting repeatidly we get that :

$$A_n^k = \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \ldots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} A_0^{i_0}$$

We observe that :

$$\forall i_0 \in \mathbb{N}, \quad A_0^{i_0} = \delta_{i_0}$$

Where δ_{i_0} is the Kronecker delta.

So :

$$A_n^k = \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n-i_{n-1})^{(i_{n-1}-i_{n-2})} \ldots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

Thus the numbers A_n^k are a special type of recurrent sums [3].

2- $\tilde{f}_n$

We have :

$$\tilde{f}_0 = f$$

$$\begin{aligned} \tilde{f}_1 &= (Ix^2)f \\ &= x^2.If - 2x.I^2 f + 2I^3 f \end{aligned}$$

$$\tilde{f}_2 = x^4.I^2 f - 6x^3.I^3 f + 20x^2.I^4 f - 40x.I^5 f + 40I^6 f$$

Hence :

$$\tilde{f}_n = \sum_{k=0}^{2n} B_{2n}^k (-1)^k x^{2n-k}. I^{n+k} f$$

Where the numbers B_{2n}^k are given by :

$$B_{2n}^k = \sum_{i=0}^{k} (2n+1-k)^{(k-i)} B_{2(n-1)}^i$$

Substituting repeatidly we get :

$$B_{2n}^k = \sum_{i_{n-1}=0}^{k} (2n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (2n-1-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

3- $\acute{f}_n$

We have :

$\acute{f}_0 = f$

$\acute{f}_1 = x. If - I^2 f$

$\acute{f}_2 = x. I^2 f - 2I^3 f$

$\acute{f}_3 = x^2. I^3 f - 4x. I^4 f + 4I^5 f$

$\acute{f}_4 = x^3. I^4 f - 7x^2. I^5 f + 18x. I^6 f - 18I^7 f$

...

$$\acute{f}_n = \sum_{k=0}^{p_n-n} C_{p_n-n}^k (-1)^k x^{p_n-n-k}. I^{n+k} f$$

Where the numbers $C_{p_n-n}^k$ are given by :

$$C_{p_n-n}^k = \sum_{i=0}^{k} (p_n-n+1-k)^{(k-i)} D_{p_{n-1}-(n-1)}^i$$

Substituting we get :

$$C_{p_n-n}^k = \sum_{i_{n-1}=0}^{k} (p_n-n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (p_{n-1}-n+2-i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2-i_1)^{(i_1-i_0)} \delta_{i_0}$$

4- The general case

In this section we prove a theorem that generalizes the previous three structures.

Theorem.

Let f be a function of x, and $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of natural numbers. If :

$$f_n = \underbrace{(Ix^{a_n-a_{n-1}-1})\dots(Ix^{a_1-a_0-1})}_{n} f$$

Then :

$$f_n = \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^{k}\,(-1)^k x^{a_n-a_0-n-k}.I^{n+k} f$$

Where the numbers $U_{a_n-a_0-n}^{k}$ are given by:

$$\begin{cases} U_0^0 = 1 \\ U_{a_n-a_0-n}^{k} = \sum_{i=0}^{k} (a_n-a_0-n+1-k)^{(k-i)} U_{a_{n-1}-a_0-(n-1)}^{i} \end{cases}$$

Proof. 1. Base case :verify true for $n = 0$.

$$f_0 = f \quad \text{and} \quad \sum_{k=0}^{0} U_0^k\,(-1)^k x^{0-k}.I^k f = U_0^0 f = f$$

2. Induction hypothesis : assume the statement is true until n.

3. Induction step : we will show that this statement is true for $(n+1)$.

$$
\begin{aligned}
f_{n+1} &= (Ix^{a_{n+1}-a_n-1})f_n \\
&= (Ix^{a_{n+1}-a_n-1})\left(\sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^k (-1)^k x^{a_n-a_0-n-k}.I^{n+k}f\right) \\
&= \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^k (-1)^k I(x^{a_{n+1}-a_0-(n+1)-k}.I^{n+k}f) \\
&= \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^k (-1)^k. \sum_{i=0}^{a_{n+1}-a_0-(n+1)-k} (a_{n+1}-a_0-(n+1)-k)_i (-1)^i x^{a_{n+1}-a_0-(n+1)-k-i}.I^{n+k+i+1}f \\
&= \sum_{k=0}^{(a_n-a_0-n)} \sum_{i=0}^{(a_{n+1}-a_0-(n+1)-k)} (a_{n+1}-a_0-(n+1)+1-k-i)^{(i)} U_{a_n-a_0-n}^k (-1)^{k+i} x^{a_{n+1}-a_0-(n+1)-(k+i)}.I^{n+1+(k+i)}f \\
&= \sum_{k=0}^{(a_n-a_0-n)} \sum_{j=k}^{(a_{n+1}-a_0-(n+1))} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k (-1)^j x^{a_{n+1}-a_0-(n+1)-j}.I^{n+1+j}f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} \sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k (-1)^j x^{a_{n+1}-a_0-(n+1)-j}.I^{(n+1)+j}f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} \left(\sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k\right) (-1)^j x^{a_{n+1}-a_0-(n+1)-j}.I^{(n+1)+j}f \\
&= \sum_{j=0}^{a_{n+1}-a_0-(n+1)} U_{a_{n+1}-a_0-(n+1)}^j (-1)^j x^{a_{n+1}-a_0-(n+1)-j}.I^{(n+1)+j}f
\end{aligned}
$$

We must prove that :

$$
\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k = \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k
$$

If $j \leq a_n - a_0 - n$ we get :

$$
\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k = \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k
$$

If $j = a_{n+1} - a_0 - (n+1)$ we get :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^k$$

And :

$$\sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k$$
$$= \sum_{k=0}^{a_{n+1}-a_0-(n+1)} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^k$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^k + 1^{(0)} U_{a_n-a_0-n}^{a_{n+1}-a_0-(n+1)}$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^k + 0$$
$$= \sum_{k=0}^{a_n-a_0-n} 1^{(a_{n+1}-a_0-(n+1)-k)} U_{a_n-a_0-n}^k$$

So in all cases :

$$\sum_{k=0}^{\min(j,a_n-a_0-n)} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k$$
$$= \sum_{k=0}^{j} (a_{n+1}-a_0-(n+1)+1-j)^{(j-k)} U_{a_n-a_0-n}^k$$

And the theorem is proven by induction. ∎

4. APPLICATIONS OF THE THEOREM.

Corollary 1.

Let $m \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of natural numbers. We have :

$$\frac{(m+1)^{(n)}}{(m+a_1-a_0)(m+a_2-a_0)\dots(m+a_n-a_0)} = \sum_{k=0}^{a_n-n-a_0} U_{a_n-a_0-n}^k (-1)^k . \frac{1}{(m+n+1)^{(k)}}$$

Proof. By setting $f = x^m$ $(m \in \mathbb{N})$ in the theorem, we get on the one hand :

$$(x^m)_n = \frac{x^{m+a_n-a_0}}{(m+1)^{(n)}} \sum_{k=0}^{a_n-a_0-n} U^k_{a_n-a_0-n}(-1)^k.\frac{1}{(m+n+1)^{(k)}}$$

On the other hand we have :

$$(x^m)_n = \underbrace{(Ix^{a_n-a_{n-1}-1}) \dots (Ix^{a_1-a_0-1})}_{n} x^m$$

$$= \underbrace{(Ix^{a_n-a_{n-1}-1}) \dots (Ix^{a_2-a_1-1})}_{n-1} \left(\frac{x^{m+a_1-a_0}}{m+a_1-a_0}\right)$$

$$= \underbrace{(Ix^{a_n-a_{n-1}-1}) \dots (Ix^{a_3-a_2-1})}_{n-2} \left(\frac{x^{m+a_2-a_0}}{(m+a_1-a_0)(m+a_2-a_0)}\right)$$

$$= \dots$$

$$= \frac{x^{m+a_n-a_0}}{(m+a_1-a_0)(m+a_2-a_0)\dots(m+a_n-a_0)}$$

Equating we get the desired identity. ■

Corollary 2.

Let $(b_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. We have :

$$\frac{1}{b_1 b_2 \dots b_n} = \sum_{k=0}^{n(b_n-1)} U^k_{n(b_n-1)} (-1)^k.\frac{1}{(n+1)^{(k)}}$$

Proof. By setting $a_n = nb_n$ in the previous corollary we get the desired identity.

Remark.

We remark that :

$$\frac{(n+1)^{(n(b_n-1))}}{\sum_{k=0}^{n(b_n-1)} (-1)^k U^k_{n(b_n-1)} (n+k+1)^{(n(b_n-1)-k)}} \in \mathbb{N}^*$$

Proof.

$$\frac{1}{b_1 b_2 \dots b_n} = \sum_{k=0}^{n(b_n-1)} U^k_{n(b_n-1)} (-1)^k.\frac{1}{(n+1)^{(k)}}$$

$$= \frac{\sum_{k=0}^{n(b_n-1)} (-1)^k U^k_{n(b_n-1)} (n+k+1)^{(n(b_n-1)-k)}}{(n+1)^{(n(b_n-1))}}$$

Inverting both sides we get the desired result.

EXAMPLES :

By setting in corollary 2 :

$b_n = nA + 1$

Where $A \in \mathbb{N}$.

We get :

$$\frac{1}{(A+1)\dots(nA+1)} = \sum_{k=0}^{An^2} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (An^2 - k)^{(k-i_{n-1})} \dots \sum_{i_0=0}^{i_1} (A - i_1)^{(i_1-i_0)} \, \delta_{i_0}$$

Setting :

$b_n = q$

Where $q \in \mathbb{N}^*$.

We get :

$$\frac{1}{q^n} = \sum_{k=0}^{n(q-1)} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n(q-1) + 1 - k)^{(k-i_{n-1})} \dots \sum_{i_0=0}^{i_1} (q - i_1)^{(i_1-i_0)} \, \delta_{i_0}$$

Setting :

$b_n = n$

We get :

$$\frac{1}{n!} = \sum_{k=0}^{n^2-n} \frac{(-1)^k}{(n+1)^{(k)}} \sum_{i_{n-1}=0}^{k} (n^2 - n + 1 - k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n^2 - 3n + 3 - i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (1 - i_1)^{(i_1-i_0)} \, \delta_{i_0}$$

Result.

Let $n \in \mathbb{N}$. We have :

$$e\left(\sum_{k=0}^{n} (-1)^k \sum_{i_{n-1}=0}^{k} (n+1-k)^{(k-i_{n-1})} \sum_{i_{n-2}=0}^{i_{n-1}} (n - i_{n-1})^{(i_{n-1}-i_{n-2})} \dots \sum_{i_0=0}^{i_1} (2 - i_1)^{(i_1-i_0)} \, \delta_{i_0}\right) = \sum_{k=0}^{\infty} \frac{1}{1.2\dots k(k+2)(k+4)\dots(k+2n)}$$

Proof. If we set $f = e^x$, $a_n = 2n$ in the theorem we get :

$$(e^x)_n = e^x \sum_{k=0}^{n} U_n^k \, (-1)^k x^{n-k}$$

On the other hand we have :

$$\begin{aligned}(e^x)_n &= \underbrace{(Ix)\dots(Ix)}_{n} e^x \\ &= \underbrace{(Ix)\dots(Ix)}_{n}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\underbrace{(Ix)\dots(Ix)}_{n} x^k\right) \\ &= \sum_{k=0}^{\infty}\frac{x^{k+2n}}{k!\,(k+2)(k+4)\dots(k+2n)}\end{aligned}$$

Equating we get :

$$\sum_{k=0}^{\infty}\frac{x^{k+2n}}{k!\,(k+2)(k+4)\dots(k+2n)} = e^x \sum_{k=0}^{n} U_n^k \, (-1)^k x^{n-k}$$

Putting $x = 1$ and substituting U_n^k by its explicit expression we obtain the desired formula. ■

Result.

Let $n \in \mathbb{N}$. We have :

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k(2k+1)(2k+3)(2k+5)\dots(2k+2n+1)} = \frac{\pi}{2}\left(\sum_{k=0}^{n}(-1)^k g_{n+k}(1)\sum_{i_{n-1}=0}^{k}(n+1-k)^{(k-i_{n-1})}\sum_{i_{n-2}=0}^{i_{n-1}}(n-i_{n-1})^{(i_{n-1}-i_{n-2})}\dots\sum_{i_0=0}^{i_1}(2-i_1)^{(i_1-i_0)}\,\delta_{i_0}\right)$$

Where $g_m(x)$ is a polynomial of degree m with rational coefficients.

Proof. If we set $f = \arcsin(x)$, $a_n = 2n$ in the theorem we get :

$$
\begin{aligned}
(\arcsin(x))_n &= \underbrace{(Ix)\dots(Ix)}_{n}\arcsin(x)\\
&= \underbrace{(Ix)\dots(Ix)}_{n}\left(\sum_{k=0}^{\infty}\frac{\binom{2k}{k}x^{2k+1}}{4^k(2k+1)}\right)\\
&= \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k(2k+1)}\left(\underbrace{(Ix)\dots(Ix)}_{n}x^{2k+1}\right)\\
&= \sum_{k=0}^{\infty}\frac{\binom{2k}{k}x^{2k+1+2n}}{4^k(2k+1)(2k+3)(2k+5)\dots(2k+2n+1)}
\end{aligned}
$$

On the other hand we have :

$$
\begin{aligned}
(\arcsin(x)\,)_n &= \sum_{k=0}^{n} U_n^k\,(-1)^k x^{n-k}.\,I^{n+k}\arcsin(x)\\
&= \sum_{k=0}^{n} U_n^k\,(-1)^k x^{n-k}\left(g_{n+k}(x)\arcsin(x)+f_{n+k}(x)\sqrt{1-x^2}\right)
\end{aligned}
$$

Properties of the polynomials $g_m(x)$ and $f_m(x)$ are given in [4].

Equating we get :

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\binom{2k}{k}x^{2k+1+2n}}{4^k(2k+1)(2k+3)(2k+5)\dots(2k+2n+1)}\\
&\qquad= \sum_{k=0}^{n} U_n^k\,(-1)^k x^{n-k}\left(g_{n+k}(x)\arcsin(x)+f_{n+k}(x)\sqrt{1-x^2}\right)
\end{aligned}
$$

Putting $x = 1$ and substituting U_n^k by its explicit expression we obtain the desired formula. ■

If we go back to Table 1 and sum the coefficients A_n^k horizontally for each $1 \le n \le 3$ we find that :

$$
\sum_{k=0}^{1} A_1^k = 2,\quad \sum_{k=0}^{2} A_2^k = 7,\quad \sum_{k=0}^{3} A_3^k = 37
$$

We have also:

$$
\sum_{k=0}^{2} B_2^k = 5,\quad \sum_{k=0}^{4} B_4^k = 107,\quad \sum_{k=0}^{1} C_1^k = 3
$$

So is there an infinite number of couples (n,p) with $n \in \mathbb{N}$, $p \in \mathbb{P}$ and $p = \sum_{k=0}^{a_n-a_0-n} U_{a_n-a_0-n}^k$, where $(a_n)_{n\in\mathbb{N}}$ is some strictly increasing sequence of natural numbers ?

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