Doppler effect approximation: the source of Hubble tension

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Abstract:

The significant difference of the Hubble constant value estimated from the local distance ladder and from the cosmic microwave background radiation attracts substantial attention in the recent decade and has been dubbed as Hubble tension. Many researchers set out to find the source of Hubble tension, e.g. identifying the possible errors in distance estimation and exploring new theories or physical causes to fix the possible problems in the ΛCDM model. However, the tension is still unresolved. This paper examines the process of converting redshift to recessional velocity and reveals that the converting formula for local estimations is inappropriately approximated. By using the relativistic Doppler effect formula, the paper reduces the estimated Hubble constant from Cepheid method by 6%, agreeing with the estimates from the CMB method at about 1σ level. It is expected that the right formula can bring the estimates from the TRGB method to the same level of the CMB estimates, so the Hubble tension should disappear.

Key words: Hubble constant, Doppler effect, local distance ladder, proper distance, recessional velocity

1. Introduction

The Hubble constant at the current epoch is an extremely important constant in Astronomy. Since the time of Hubble, astronomers have kept improving the ways to obtain more accurate Hubble constant (Hodge, 1982[i], Tammann et al, 2008[ii], Freedman and Madore, 2010[iii]). So far, there are many methods to measure Hubble constant, but the main methods can be classified into two approaches. The first approach is a direct, model-free approach. This approach aims at measuring the distance and recessional velocity of galaxies in the local universe and then obtaining the Hubble constant through Hubble's law directly. The recessional velocity can be obtained through the measurement of cosmological redshift while the distance of galaxies in local universe can be measured by local distance ladder such as type Ia supernovae (SNe Ia), Cepheid variables, tips of the red giant branch (TRGB), maser galaxies, etc. Cepheid variables are widely used as the local distance ladder.

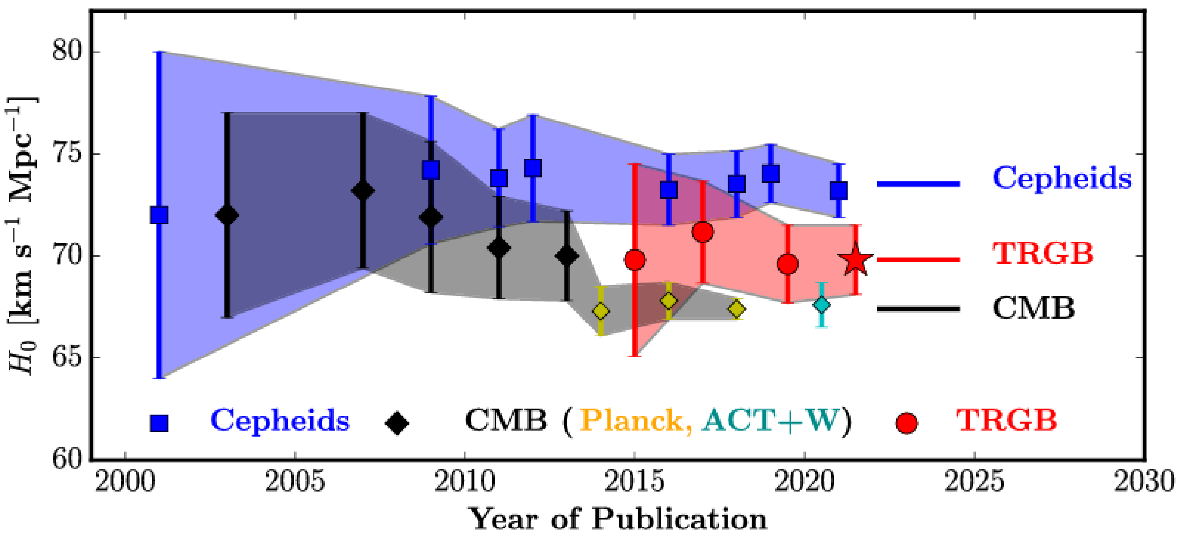
The second approach is an indirect one, which utilizes other physical phenomena that are related to Hubble constant. One method measures the relics after the Big Bang and use a cosmological model – the ΛCDM model – to determine the path of expansion of universe and thus the Hubble constant at current epoch. Two such relics are cosmic microwave background (CMB) radiation (Peebles and Yu, 1970[iv], Komatsu et al, 2009[v]) and galaxy matter power spectrum (Eisenstein, et al, 2005[vi]; Percival et al, 2010[vii]). The particle creation during the inflation epoch generated a small density perturbation. With the gravity as driving force and the photon radiation pressure as the restoring force, the density perturbation caused the anisotropies of acoustic oscillations in CMB radiation. Measuring these anisotropies can calibrate the parameters in the cosmological model and thus determine the Hubble constant through the model. Similarly, the competing force of gravity and radiation pressure in the hot plasma also created the sound waves and leave an imprint of baryon acoustic oscillations (BAO) in galaxy mater power spectrum. The measurement of BAO can break degeneracies in the CMB measurements and thus help the calibration of parameters in ΛCDM model and derivation of Hubble constant.

Another indirect method is gravitational lensing (e.g. Blandford and Narayan, 1992[viii]). The gravitational lens method measures the difference in arrival time and angular separation of the images of a time-variable object after passing through a strong gravitational field. In principle, the time difference between images is inversely related to Hubble constant, so the measurement of time delay can infer Hubble constant. However, this method involves many technical difficulties and uncertainties (Myers, 1999[ix]; Schechter, 2005[x]).

The measured Hubble constants from two approaches before 2010 were largely agree with each other, but with large uncertainty. With the refinement in both approaches, the uncertainty in both measurements improved significantly, but the measured values of Hubble constant do not agree with each other. The results from the first approach (the model-independent local calibration) fall in a range of 70-76 km s^{-1} Mpc^{-1}(Freedman et al, 2012[xi], 2019[xii], Riess et al, 2016, 2019; Pesce et al, 2020[xiii], Huang, 2020[xiv], Khetan et al, 2021[xv], Blakeslee et al, 2021[xvi]), but the results from the CMB and BAO studies derived a much lower Hubble constant with low uncertainty, e.g. 67.4±0.5 km s^{-1} Mpc^{-1} in Plank

Collaboration et al (2020[xvii]), 67.6±1.1 in Aiola et al (2020)[xviii], 67.3±1.1 in Aubourg et al (2015)[xix]; 67.8±1.3 in Macaulay et al (2019)[xx]. Recent gravitational lensing studies also estimated a low value of Hubble constant but with a high uncertainty range 67.4±4.1 (Birrer et al, 2020[xxi]; Birrer and Treu, 2021[xxii]). The difference of Hubble constant estimated from the Cepheid method and the from the CMB data amount to 8% or at about 5σ-6σ level. Some researchers (e.g. Riess et al, 2019[xxiii], 2021[xxiv], Di Valentino et al, 2021) think that the discrepancy of Hubble constant values from two approaches is too high and label it Hubble constant tension or Hubble tension. To resolve this tension, astronomers focused on a new measurement using the tip of the red giant branch (TRGB) as local distance ladder. The results of this measurement (e,g. Krisciunas et al, 2017[xxv], Freedman et al, 2020[xxvi]) are about 69.6±2.5 km s^{-1} Mpc^{-1}, which are between the those from Cepheids method and CMB method. Freeman (2021)[xxvii] combined 4 independent calibrations of the TRGB and led to a Hubble constant of 69.8±2.2 km s^{-1} Mpc^{-1}. To prove that the TRGB method gives better estimation, Freeman (2021) also provided a graphic summary of the measurement of Hubble constant since year 2000, which is shown in Fig.1. However, later TRGB calibration found bigger Huble constant. Anand et al (2022)[xxviii] estimated a value of 71.5±1.8 km s^{-1} Mpc^{-1}. In a comparative analysis of the TRGB, Scolnic et al (2023)[xxix] estimated a value of 73.22 ± 2.06 km s^{-1} Mpc^{-1} for the Pantheon+ SN Ia sample.

Fig.1 Estimated Hubble constant from year 2000



Source: Freedman (2021). Cepheids estimates shown as blue squares, TRGB estimates as red circles and star, CMB estimates including WMAP (black diamonds), Plank (yellow diamonds), and ACT+WMAP (cyan diamond).

The Hubble tension has triggered substantial interests. While some (e.g. Rameez and Sarkar, 2021[xxx]) consider the discrepancies in estimated Hubble constant is reasonable given the discrepancies in observation data and the distribution of estimated results, most researchers think the discrepancies show that either there are systematic errors in local estimations or the ΛCDM model may have some issues. To address the possible errors in local estimations, Riess et al (2023)[xxxi] re-estimated Hubble constant using the data from James Webb Space Telescope, which has much sharp vision than the Hubble telescope. They found that Webb's data have dramatically reduced the noise in the Cepheid measurements, but the earlier Hubble space telescope measurement is accurate. As a result, their estimation results confirmed the previous Cepheid estimation with higher certainty.

If we can trust the direct local measurements of Hubble constant from Cepheids, we must conclude that the universe is expanding about 8% faster than what predicted from the ΛCDM model. This would cast doubt on the celebrated ΛCDM model that is supported by a wide range of cosmological experiments and observations. Many researchers set out to fix possible issues in the ΛCDM model by modifying old theories or proposing new ones, including early dark energy (Herold and Ferreira, 2022 [xxxii]), late dark energy (Zhao et al, 2017[xxxiii]), models with extra relativistic degree of freedom (Carneiro et al 2019[xxxiv]), modified gravity (Quiros, 2019[xxxv]), models with extra interactions (Yao and Meng, 2022[xxxvi]), etc. Di Valentino et al (2021)[xxxvii] gives a detailed review and classification of the proposed solutions to the Hubble tension.

While researchers paid much effort to fix the possible errors in measuring the distance and to find a remedy to the ΛCDM model, so far no one has checked the possibility of errors from converting the measured redshift to recessional velocity. This may result from the common wisdom that the measurement of velocity from redshift is easy because the conversion is based on the well-known Doppler effect which is unlikely has a room for error. However, since this conversion process is critical for Hubble constant estimation and has never been examined by researchers, this paper takes a close look at the process, aiming at finding a possible cause of Hubble tension.

2. A close examination of the Doppler effect

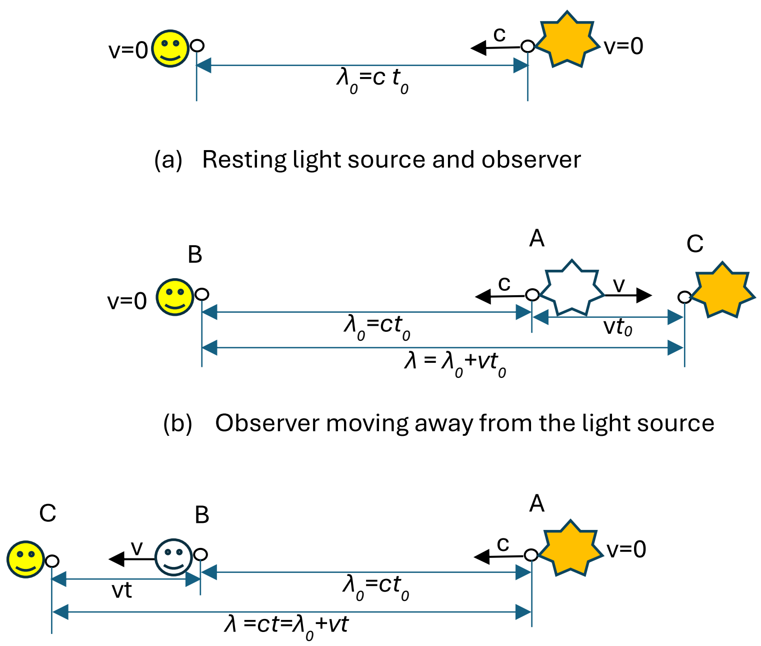
Although most of introductory cosmology textbooks differentiate the redshift due to the expansion of the universe from the redshift due to Doppler effect. However, through parallel transportation, it has been proved that that the cosmological redshift is equivalent to redshift caused by the Doppler effect (Synge, 1960[xxxviii], Narlikar, 1994[xxxix], Bunn and Hogg, 2009[xl]). Because the Doppler effect is crucial in explaining redshift and obtaining recession velocity for calculating Hubble constant, here we revisit it in detail.

Fig. 2 shows the Doppler effect in different scenarios. In panel (a), the light source and observer are both stationary. The observer is so positioned as to see the first photon (or wavefront) when the source start to emit the second photon, so the distance between the observer and the source is the wavelength of the light λ_0. Given the speed of light c and emission period t_0 (or frequency f_0), the wavelength can be expressed as:

$$\lambda_0 = ct_0 = c/f_0 \qquad (1)$$

Since the wavelength viewed from the source is the same as that perceived by the observer, the Doppler effect in this case is zero.

Fig.2 Doppler effect of moving light source or observer



(a) Resting light source and observer

(b) Observer moving away from the light source

(c) Light source moving away from the observer

In panel (b), the observer at point B is stationary while the source recedes at speed v from the observer. After the source emits the first photon at point A, the observer waits for a period of t_0 and perceives the photon. In the period of t_0, the light source moves from A to C and starts to emit the second photon. The distance between the first and second photon BC is the wavelength λ perceived by the observer, which can be calculated as:

$$\lambda = \lambda_0 + vt_0 = \lambda_0(1 + v/c) \qquad (2)$$

The redshift z is defined as

$$z = \frac{\lambda-\lambda_0}{\lambda_0}\ or,\ z + 1 = \frac{\lambda}{\lambda_0} \qquad (3)$$

Substituting eq. (3) into eq. (2), We have:

$$z + 1 = \frac{\lambda}{\lambda_0} = \frac{c+v}{c}, or\ z = \frac{v}{c} \qquad (4)$$

This is the conversion equation commonly used for low redshifts (z < 1) to obtain recessional speed to calculate the Hubble constant. It seems that the practice has a sound foundation; however, this is only one scenario of Doppler effect.

Panel (c) shows the case where the source is stationary at position A but the observer moves away at the speed v. When the observer at position B perceives the first photon, the source starts emit the second photon. Then, the observer moves to left at speed v. In the period of time t, the observer moves to position C and observes the second photon. The time t can be calculated based on the following equation:

$$\lambda_0 + vt = ct \qquad (5)$$

Solving for t, we have:

$$t = \lambda_0/(c - v) \qquad (6)$$

Here t is the perceived emission period, so the perceived wavelength by the observer is the distance the photon travels in time t:

$$\lambda = ct = c\lambda_0/(c - v) \qquad (7)$$

In terms of redshift z, we can rewrite eq. (7) as:

$$z + 1 = \frac{\lambda}{\lambda_0} = \frac{c}{c-v}\quad or\ \ z = \frac{v}{c-v} \qquad (8)$$

Comparing eq. (4) and (8), we find that the redshifts are different for scenarios (b) and (c). This is the well-known asymmetry property of the Doppler effect in a general case. This asymmetry proves to be problematic for any work in astronomy: when we observe the light from a cosmological object, it is impossible to determine with confidence if the object is moving or we are moving, or perhaps both are moving. Fortunately, Einstein's special relativity indicates that the Doppler effect of light is symmetric and that the redshift caused by the speed in the radial direction in any cases (moving source or moving observer) can be calculated as:

$$z + 1 = \frac{\lambda}{\lambda_0} = \sqrt{\frac{c+v}{c-v}} \quad or \quad v = \frac{(1+z)^2-1}{(1+z)^2+1}c \qquad (9)$$

Comparing eqs. (4), (8) and (9), we can conclude that eq.(9) is in fact the geometric average of eqs. (4) and (8).

3. The impact of Doppler effect approximation

The current process of converting redshift to recessional velocity in astronomy depends on the size of redshift z. If $z<1$, eq. (4) is used. If $z\geq 1$, eq. (9) is used. Ther reasoning behind this practice is that, for small z, the relativistic effect is tiny and thus can be ignored, so relativistic Doppler effect in eq. (9) can be approximated by the classical Doppler effect in eq.(4); for large z, the relativistic effect needs to be taken into account, so eq. (9) has to be used.

We argue in this paper that even for small z, we cannot use eq. (4) to calculate recessional speed. Based on discussions in the previous section, the condition for eq.(4) to be valid is that the observer is stationary while the light source is moving away. Since it is a known fact that the detector (a telescope on the earth or in earth's orbits) is not stationary, the precondition for eq. (4) is not satisfied and thus it is in valid to use eq. (4) even when the relativistic effect can be ignored. In other words, eq. (9) should be used for all cases.

What would be the bias caused by using eq. (4)? We can estimate it based on a simple calculation first and then by a re-estimation exercise. Since the SNe la data with redshift less than 0.15 is commonly used with the local Cepheid-calibrated distance ladder to measure the Hubble constant to increasingly high precision (Riess et al, 2016[xli]), we use z=0.15 as the upper limit to examine the bias in the resultant recessional speed caused by approximation.

Using eq. (4), we have the recessional speed v1:

$$v1 = cz = 0.15c$$

Using eq. (9), we have the recessional speed v2:

$$v2 = \frac{(1+z)^2 - 1}{(1+z)^2 + 1}c = 0.1388c$$

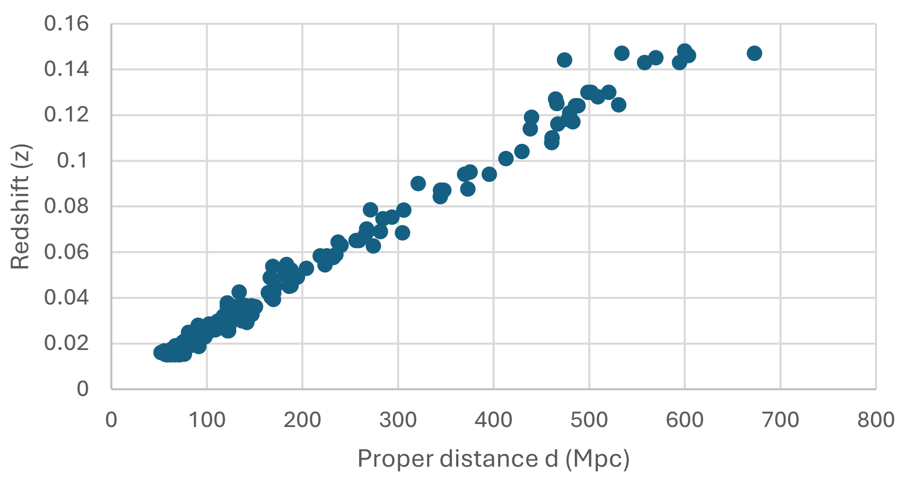
The difference between v1 and v2 is:

$$\frac{\Delta v}{v2} = \frac{v1 - v2}{v2} = 8.8\%$$

This difference is just slightly greater than the current 8% difference between the estimates of Hubble constant from the Cepeid method and the CMB method, indicating that the use of eq. (4) is likely the source of Hubble tension.

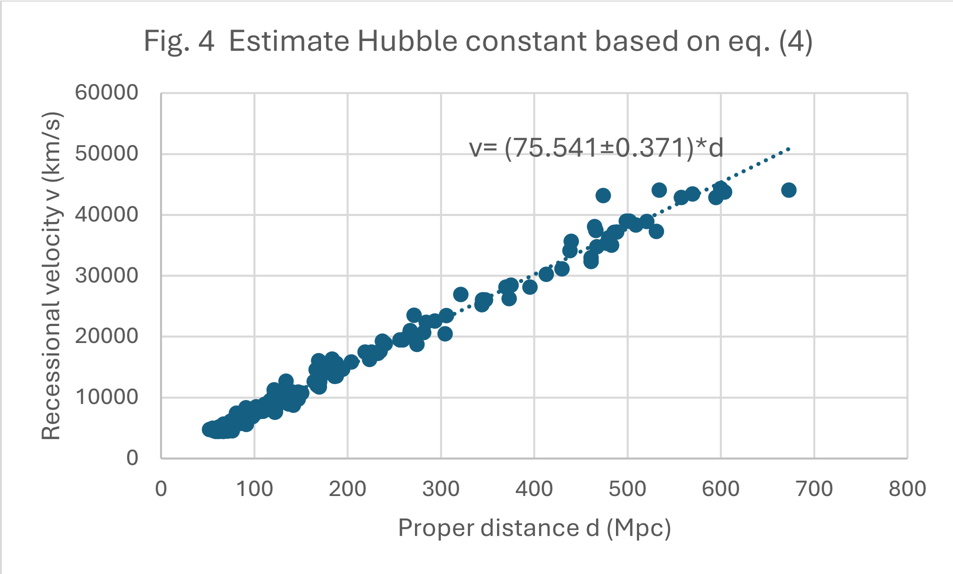
Next, we use the Union2 data from Amanullah et al. (2010)[xlii] to perform an exercise of Hubble constant estimation. Using the criterium of z<0.15, we select 192 samples. Their redshift and luminosity distance are plotted in Fig. 3.

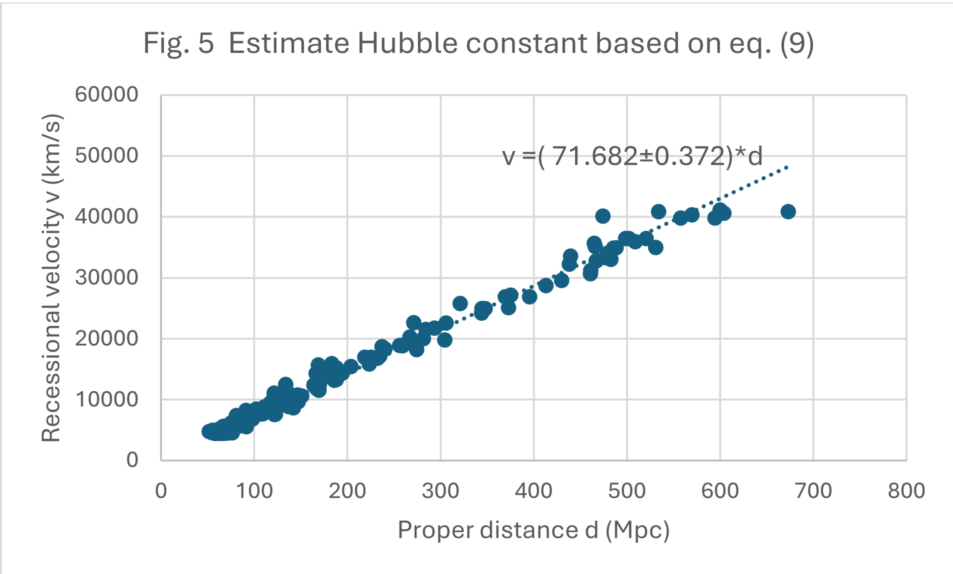
Fig. 3 Samples of z<0.15 in Amanullah et al (2010)



The samples in Fig.3 show clear linear relationship, despite the more noise for samples of z>0.11. Using the eq. (4) to convert redshifts to recessional velocity, we can estimate the Hubble constant through least squares method. The estimation results shown in Fig.4 suggest that H_0=75.541±0.371 km $s^{-1}pc^{-1}$.

Using the same approach, we can convert the redshifts to recessional velocity by eq. (9) and estimate the Hubble constant $H_0' = 71.682 \pm 0.372$, shown in Fig.5.





Comparing with the value of H_0 and H_0', we find that using eq.(9) we can reduce the estimation value by about 6%. As expected, this difference is less than the upper limit of 8.8% but reasonably close to the Hubble tension of 8%. This is just a simple estimation exercise. Estimation by taking into account various errors in distance estimation may produce more refined results. If the use of eq.(9) instead of eq. (4) can reduce the refined Hubble constant value by 6%, we only have 2% or about 1σ difference with the CMB estimation, so the Hubble tension disappears.

Since the TRGB estimation also uses eq.(4) to calculate recessional velocity, its estimation result can also be improved by adopting eq. (9). Since the current TRGB estimations of Hubble constant are significantly lower than those from Cepheid method, using eq. (9) can potentially bring the estimates very close to the CMB estimates. If we can trust that the TRGB method can measure the distance more accurately due to the much less interstellar extinction for the light of long wavelengths, the close results from both the TRGB and CMB methods tend to lend support for both the TRGB method and the ΛCDM model.

4. Summary

The paper has examined the formula for converting the redshift to recessional velocity in local estimation of Hubble constant. It is found that the formula for local estimation is an inappropriate approximation from the relativistic Doppler effect formula. With the correct formula for converting the redshift to recessional velocity, the paper shows that the Hubble constant estimated from local distance ladder can be reduced by 6%, so the estimates from Cepheid method will be consistent with the CMB estimates at about 1σ level. Due to the lower estimates from the TRGB method, the right formula could bring the estimates at the same level as the CMB estimates. As a result, the Hubble tension should vanish.

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