Generalized Hodge conjecture on 3-folds

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July 17, 2024

Abstract

The generalized Hodge conjecture can be viewed as a prediction on the coniveau filtration at all levels

$$degree - 2coniveau,$$

an integer associated to each cohomological subgroup in the filtration (see (1.3) below). At the level 0, the conjecture coincides with the millennium Hodge conjecture on Hodge classes, which remains outstanding. At the non negative levels, Voisin has worked out the detail of Grothendieck's vision: the conjecture at the level 1 is implied by the conjecture at the level 0. In this paper, we would like to introduce a method with a new interpretation of the coniveau filtration in general. In case of 3-folds, the new method shows that Voisin's proof for the level 1 could work in the same way without the millennium Hodge conjecture.

1 Introduction

Let X be a complex projective manifold of dimension n. Let p, i be two whole numbers. In the singular cohomology $H^i(X;\mathbb{Q})$ with rational coefficients, we define $N^pH^i(X;\mathbb{Q})$ to be the subspace spanned by all the subspaces

$$Ker\bigg(H^i(X;\mathbb{Q}) \to H^i(X-V;\mathbb{Q})\bigg) \tag{1.1}$$

with the subvariety V of codimension $\geq p$. They form a decreasing filtration known as the coniveau filtration,

$$N^nH^i(X;\mathbb{Q}) \subset \cdots \subset N^pH^i(X;\mathbb{Q}) \subset \cdots \subset N^0H^i(X;\mathbb{Q}) = H^i(X;\mathbb{Q}), \tag{1.2}$$

where p is the geometric coniveau, and

$$i - 2p \tag{1.3}$$

Key words: generalized Hodge conjecture
2020 Mathematics subject classification: 14C30

is defined to be the geometric level: the level in the abstract above. On the other hand, the cohomology has another subgroup: the maximal sub-Hodge structure $M^pH^i(X;\mathbb{Q})$ with the Hodge coniveau p. It is defined to be the union

$$\bigcup_{all\ with\ p} \mathbf{L}^p$$

where $\mathbf{L}^p \subset H^i(X;\mathbb{Q})$ is a sub-Hodge structure with Hodge coniveau p. Correspondingly, they also form a decreasing filtration

$$M^nH^i(X;\mathbb{Q}) \subset \cdots \subset M^pH^i(X;\mathbb{Q}) \subset \cdots \subset M^0H^i(X;\mathbb{Q}) = H^i(X;\mathbb{Q}).$$

The generalized Hodge conjecture in [2] predicts that

Conjecture 1.1. *These two filtrations agree, i.e.*

$$N^pH^i(X;\mathbb{Q}) = M^pH^i(X;\mathbb{Q}). \tag{1.4}$$

In the past, there has been no substantial progress in the conjecture due to the same difficulty in its origin. This paper tries another possibility. In the case of 3-folds, it leads to the following main theorem.

Main theorem 1.2. *If a supportive intersection exists on 4-dimensional complex projective manifolds, the conjecture 1.1 is correct on 3-dimensional complex projective manifolds.*

A supportive intersection is a bilinear map that preserves the support. Let's define it. Let Y be a differentiable manifold. Denote the space of currents by $\mathscr{D}'(Y)$. The group $S^\infty(Y)$ is defined to be the Abelian group freely generated by regular cells. We should note that a regular cell is a pair of an oriented polyhedron Π and a differential embedding of a neighbourhood of Π to Y. A chain in $S^\infty(Y)\otimes\mathbb{Q}$ is called a regular chain, furthermore a closed regular chain a regular cycle. It is well-known that a singular cohomology class with rational coefficients is represented by a regular cycle.

Definition 1.3. *A supportive intersection denoted by* $[\bullet \wedge \bullet]$ *is a bilinear map*

$$\begin{array}{ccc} S^\infty(Y)\times S^\infty(Y) & \rightarrow & \mathscr{D}'(Y) \\ (c_1, c_2) & \rightarrow & [c_1\wedge c_2] \end{array}$$

such that

1) *(Cohomologicality)* $[c_1\wedge c_2]$ *descends to the cup-product on cohomology, i.e. if* c_1, c_2 *are closed, so is* $[c_1\wedge c_2]$ *and it satisfies*

$$\big\langle [c_1\wedge c_2]\big\rangle = \langle c_1\rangle \smile \langle c_2\rangle \tag{1.5}$$

where $\langle\bullet\rangle$ *denotes the cohomology class of a singular cycle, and* $\smile$ *the* *cup-product.*

2) (Supportivity) for any pair of chains c_1, c_2 in $S^\infty(Y)$,

$$supp([c_1 \wedge c_2]) \subset supp(c_1) \cap supp(c_2). \tag{1.6}$$

where supp stands for support.

So, a supportive intersection is a set-theoretical version of the cup-product. The definition is for real manifolds in differential topology. To apply it, we come back to the complex projective manifold X in algebraic geometry where it is known that Conjecture 1.1 boils down to the containment

$$M^pH^i(X;\mathbb{Q}) \subset N^pH^i(X;\mathbb{Q}) \tag{1.7}$$

(see Conjecture 4, [4]). At the level 1, Voisin's work in [4] shows that the Hodge structure on the left hand side of (1.7) can be realized through the geometry of an Abelian variety. This leads to her construction: for a sub-Hodge structure $\mathbf{L}^1 \subset H^i(X;\mathbb{Q})$, there is a complex projective curve C in the Abelian variety such that $\mathbf{L}^1$ is the image of $H^1(C;\mathbb{Q})$ through the correspondence of a Hodge class Ψ of degree $i+1$ on $C \times X$. To go further in geometry, Voisin

(1) first assumed the millennium Hodge conjecture to obtain an algebraic cycle Γ_Ψ with rational coefficients that represents the class Ψ,

(2) then projected Γ_Ψ to X to obtain another algebraic cycle A_Ψ that supports a singular-cycle-representative of each class in $\mathbf{L}^1$.

This proves the containment (1.7).

We'll follow this procedure but with a new interpretation of the coniveau filtration. Let $T^pH^i(X;\mathbb{Q})$ be the linear span of all classes in $H^i(X;\mathbb{Q})$ that have representative currents supported on an algebraic subset of codimension $\geq p$. Then we claim

$$T^pH^i(X;\mathbb{Q}) = N^pH^i(X;\mathbb{Q}), for\ i-2p \geq 0 \tag{1.8}$$

(non-trivial, but will be proved in Proposition 2.1 below). The new interpretation leads to the alternative approach in which the cup-product on the right hand side of (1.8) is replaced by the supportive intersection on the left. Thus currents for the purpose of the support could play the same role as algebraic cycles. In this paper, we'll show how this idea can be used for 3-folds in Voisin's method to avoid the millennium Hodge conjecture. First, we step back from the general X to focus on the 3-fold X, where the conjecture 1.1 only has one remaining non-trivial case with $i=3, p=1$. So, it suffices to prove

$$T^1H^3(X;\mathbb{Q}) \supset M^1H^3(X;\mathbb{Q}).$$

Then in Voisin's method we replace the algebraic cycle Γ_Ψ by a regular cycle c_Ψ that represents the class Ψ. It follows that the projection of the singular cycle c_Ψ to the 3-fold X, by Lefschetz $(1,1)$ class theorem, also yields an algebraic cycle A_Ψ. Then the two properties of the supportive intersection directly imply that each class in $\mathbf{L}^1$ is represented by a current supported on the algebraic cycle A_Ψ. In terms of cohomology, this means that $\mathbf{L}^1 \subset T^1 H^3(X;\mathbb{Q})$.

Historical note. Hodge's original conjecture, after the adjustment of coefficients, was

$$N^p H^i(X;\mathbb{Q}) = F^p H^i(X;\mathbb{C}) \cap H^i(X;\mathbb{Q}), \tag{1.9}$$

where $F^\bullet$ denotes the Hodge filtration. Grothendieck in [2] pointed out that (1.9) is false due to the Hodge structure on the left hand side. He substantiated it with a concrete example of the Hodge structure $N^1 H^3(X;\mathbb{Q})$ on a triple product of an elliptic curve. Thus 3-folds provided the first non-trivial example where the problem we are facing is not about Hodge classes, rather about the coniveau filtration.

2 The new interpretation of the coniveau filtration

Let X be a complex projective manifold of an arbitrary dimension. Let $H^i(X;\mathbb{G})$ be the singular cohomology of degree i with coefficients in the Abelian group

$$\mathbb{G} = \mathbb{Q} \text{ } or \text{ } \mathbb{R}.$$

The coniveau filtration is defined on cohomology classes through homological algebra (1.1). However, Hodge's original expression had the focus on a particular type of representatives ([3]). So, we'll work with an approach on the representatives, but not Hodge's. Let $\mathcal{Z}(X)$ be the space of closed currents and $\mathcal{E}(X)$ be the space of exact currents. Then de Rham theory gives the equality

$$\frac{\mathcal{Z}(X)}{\mathcal{E}(X)} \simeq \sum_i H^i(X;\mathbb{R}) \tag{2.1}$$

Let $T^p H^i(X;\mathbb{Q})$ be the subgroup of $H^i(X;\mathbb{Q})$, whose elements are represented by some closed currents supported on some subvarieties of codimension at least p, i.e.

$$T^p H^i(X;\mathbb{Q}) := \frac{\bigcup_{cod(V)\geq p} Ker\Big(\mathcal{Z}^i(X) \to \mathcal{Z}^i(X-V)\Big) + \mathcal{E}(X)}{\mathcal{E}(X)} \cap H^i(X;\mathbb{Q}),$$

where the superscript i is the degree of the currents, and V is a subvariety. We call $T^p H^i(X;\mathbb{Q})$ the current-supported subgroup. In the following, we claim that the filtration of the current-supported subgroups is the same as the coniveau filtration.

Proposition 2.1. *Let X be a complex projective manifold. Then*

$$T^pH^i(X;\mathbb{Q}) = N^pH^i(X;\mathbb{Q}), \quad for\ i-2p\geq 0. \tag{2.2}$$

Proof. We recall the coniveau filtration's subgroup $N^pH^i(X;\mathbb{Q})$ is defined as the subgroup

$$\bigcup_{cod(V)\geq p} ker\Big\{H^i(X;\mathbb{Q})\to H^i(X-V;\mathbb{Q})\Big\}.$$

For each $\alpha\in T^pH^i(X;\mathbb{Q})$, let t_α be a current that represents α and is supported on an algebraic subvariety V of codimension $\geq p$. Since V is a closed set, the restricted current $t_\alpha|_{X-V}$ is well-defined and is equal to zero. Hence

$$t_\alpha \in Ker\Big(\mathcal{Z}^i(X)\to\mathcal{Z}^i(X-V)\Big).$$

Then the de Rham theory (2.1) implies

$$\alpha\in Ker\Big(H^i(X;\mathbb{Q})\to H^i(X-V;\mathbb{Q})\Big).$$

Hence

$$T^pH^i(X;\mathbb{Q})\subset N^pH^i(X;\mathbb{Q}). \tag{2.3}$$

Conversely, let $\alpha\in N^pH^i(X;\mathbb{Q})$. Since $i-2p\geq 0$, the cohomological degree's requirement for the Poincaré duality in Cor. 8.2.8, [1] * is met. Then the corollary holds and it directly implies that α has a singular representative c_α lying on some algebraic subvariety V of $codim(V)\geq p$. Since c_α can be regarded as the integration current on X, then the class α lies in $T^pH^i(X;\mathbb{Q})$. We complete the proof.

□

3 Proof

The Hodge coniveau satisfies $i-p\geq p$. Hence $i-2p\geq 0$. Thus the maximal sub-Hodge structure $M^pH^i(X;\mathbb{Q})$ in (1.4) is only well-defined at non-negative levels. This implies that Conjecture 1.1 is actually a prediction at non-negative levels. Notice that at the non-negative levels, all the other cases of Conjecture 1.1 for 3-folds have been proved previously except the case with $i=3, p=1$. Therefore the following Theorem 3.1 which proves this remaining case will imply Main theorem.

*The requirement $i-2p\geq 0$ for the corollary is quite implicit in the original paper.

Theorem 3.1. *If there is a supportive intersection on any 4-dimensional complex projective manifolds, then on a 3-dimensional complex projective manifold X,*

$$N^1H^3(X;\mathbb{Q}) = M^1H^3(X;\mathbb{Q}). \tag{3.1}$$

Proof. First, Deligne's corollary 8.2.8, [1] implies that

$$N^1H^3(X;\mathbb{Q}) \subset M^1H^3(X;\mathbb{Q}).$$

Thus it is sufficient to prove

$$M^1H^3(X;\mathbb{Q}) \subset N^1H^3(X;\mathbb{Q}). \tag{3.2}$$

Voisin's result 3.2. [†]

Let $\mathbf{L}^1 \subset M^1H^3(X;\mathbb{Q})$ be a sub-Hodge structure with Hodge coniveau 1. Then there exist a smooth projetive curve C, and a Hodge class

$$\Psi \in Hdg^4(C \times X) \tag{3.3}$$

such that

$$\Psi_*(H^1(C;\mathbb{Q})) = \mathbf{L}^1, \tag{3.4}$$

where $\Psi_* : H^1(C;\mathbb{Q}) \to H^3(X;\mathbb{Q})$ is the homomorphism in the correspondence induced from Ψ (see Section 3.2 in [4]).

While Voisin continued it with an algebraic cycle, we'll continue it with a singular cycle as follows. Let

$$\pi : C \times X \to X$$

be the projection. We denote its fibre integration map

$$H^\bullet(C \times X;\mathbb{Q}) \to H^{\bullet-2}(X;\mathbb{Q})$$

on cohomology by $\langle\pi\rangle_*$. Since the projection is a complex analytically smooth map, $\langle\pi\rangle_*$ is a morphism of the Hodge structure. Thus $\langle\pi\rangle_*(\Psi)$ is a Hodge class of degree 2 on the 3-fold X. By the Lefschetz (1,1) class theorem, it is represented by an algebraic cycle. So in the singular-cycle-representation, there is a singular cycle c (with rational coefficients) as a regular chain on $C \times X$ representing the class Ψ such that the projection in singular cycles satisfies

$$\pi_\#(c) = A_\Psi - dJ \tag{3.5}$$

[†] We apologize if there is any misquote on the origin of the result.

where A_Ψ is the singular cycle of a triangulated algebraic cycle, denoted also by A_Ψ (a divisor of X with rational coefficients), J is a regular chain of dimension 5, and $\pi_\#$ is the map on the singular chains. We express the sets in (3.5) as

$$\pi\left(\Big|c + d(\{a_0\} \times J)\Big|\right) = |A_\Psi| \tag{3.6}$$

where $a_0 \in C$ is a point, and $|\bullet|$ denotes the support of a singular chain. Then we define the singular cycle $c_\Psi := c + d(\{a_0\} \times J)$. By the assumption, there is a supportive intersection on the smooth variety $C \times X$. Hence there is a map, denoted by $(c_\Psi)_⋇$, referred to as the current-correspondence and defined as a homomorphism,

$$\begin{array}{ccc} S^\infty(C) \otimes \mathbb{Q} & \to & \mathscr{D}'(X) \\ \delta & \to & \pi_*\Big([c_\Psi \wedge (\delta \times X)]\Big) \end{array} \tag{3.7}$$

where π_* denotes the *push-forward* on currents, and the intersection $[\bullet \wedge \bullet]$ is the linear extension of Definition 1.3 to those with rational coefficients. By the part 1) of Definition 1.3, the map $(c_\Psi)_⋇$ descends to a cohomological homomorphism which coincides with the usual topological correspondence $\langle c_\Psi \rangle_* = \Psi_*$:

$$\begin{array}{ccc} \langle c_\Psi \rangle_* : H^1(C; \mathbb{Q}) & \to & H^3(X; \mathbb{Q}) \\ \bullet & \to & \langle \pi \rangle_* \Big(\Psi \smile \big((\bullet) \otimes 1\big)\Big). \end{array} \tag{3.8}$$

Suppose now the regular cycle $\delta \in S^\infty(C) \otimes \mathbb{Q}$ represents a class in $H^1(C; \mathbb{Q})$. Then Voisin's assertion (3.4) implies that the current

$$(c_\Psi)_⋇(\delta) \tag{3.9}$$

represents a cohomology class in $\mathbf{L}^1$, and such a representation covers $\mathbf{L}^1$ when δ runs through all regular cycles. On the other hand, the supportivity in the part 2) of Definition 1.3 implies that the current

$$[c_\Psi \wedge (\delta \times X)] \tag{3.10}$$

is supported on the set *supp*(c_Ψ), and whose projection to X is the algebraic set $|A_\Psi|$. Hence the current $(c_\Psi)_⋇(\delta)$ is supported on $|A_\Psi|$. So

$$\mathbf{L}^1 \subset T^1 H^3(X; \mathbb{Q}).$$

Proposition 2.1 implies $\mathbf{L}^1 \subset N^1 H^3(X; \mathbb{Q})$. We complete the proof. □

Remark Main theorem assumes the existence of a supportive intersection on complex projective manifolds of dimension 4. It is proved in [5] that a supportive intersection actually exists on any differential manifolds, and the existence is an invariant of the differential structures.

References

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