Some missed opportunities for Archimedes and early pi-computors

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ABSTRACT. We point out some simple improvements to Archimedes' "regular polygon methods" for computing and bounding π, which all the workers before 1650 could have used, but did not. All methods employed before the 1970s to compute the first D decimals of π required order D or more arithmetic operations (±, ×, ÷, $x^{1/2}$, $x^{-1/2}$). But we shall show that if Archimedes or his followers had been a bit smarter, they could have sped that up to $O(D^{2/3})$.

Early history of π-computing methods

To begin, let me briefly summarize computations of π≈3.1415926535897932384626433832795028841971693993751...

All important workers from Archimedes (ca. 287-212 BC) up to Ludolph van Ceulen (1540-1610) and apparently Christoph Grienberger (1561-1636) used some variant of the "regular polygon method." That is, for each n=3,4,5,... the area of the regular n-gons with circumradius=1 and inradius=1 provide lower and upper bounds on π. As n→∞ these bounds become arbitrarily tight because both n-gons approach the unit circle arbitrarily closely. These areas are, respectively, $n\cdot\sin(\pi/n)\cos(\pi/n)=(n/2)\sin(2\pi/n)=\pi-2\pi^3/(3n^2)+O(n^{-4})$ and $n\cdot\tan(\pi/n)=\pi+\pi^3/(3n^2)+O(n^{-4})$. We'll discuss how to compute these areas next section. Using this idea, Archimedes showed 3.14084≈223/71<π<22/7≈3.14286. Liu Hui (ca.225-295) showed 3.141024<π<3.142704 using a 96-gon and the fact that $96=6\times2^4$. Zu Chongzhi (429-500) used Liu Hui's technique to show 3.14159261864<π<3.141592706934 using a 12288-gon and the fact that $12288=6\times2^{11}$, and also estimated π≈355/113. Van Ceulen and his student Willebrord Snell (1580-1626) computed π to 35 decimal places, while Grienberger gave 3.14159 26535 89793 23846 26433 83279 50288 4196<π< 3.14159 26535 89793 23846 26433 83279 50288 4199 (38 correct decimals) in his 1630 book Elementa Trigonometrica. This already seems precise enough for every physical purpose.

After 1630, Archimedes' polygon method was supplanted by methods arising from Newton & Leibniz's calculus. E.g. John Machin calculated 100 digits in 1706 by combining his identity π=4arctan(1/5)-arctan(1/239) with Gregory's series $\arctan(x)=x-x^3/3+x^5/5-x^7/7-x^9/9-...$ Methods of Machin's ilk continued to hold the #decimals record until the 1980s when fancier series by Ramanujan, and various fancy algorithms, including "bianry splitting" hypergoemetric series summation methods, and Brent & Salamin's AGM-based π-algorithm, took over. I shall not discuss them, but they are asymptotically superior to, albeit more complicated to understand than, the methods we shall discuss.

How Archimedes and his followers computed their areas

Archimedes' simple idea was to use angle-doubling formulas for trig functions and hence angle-halving formulas. For the tan(x) function we have

$$\tan(2x) = 2\tan(x) / (1-\tan(x)^2)$$

from which we deduce

$$\tan(x) = \tan(2x) / (1 + [\tan(2x)^2+1]^{1/2}).$$

This allows us to start from the known values $\tan(\pi/4)=1$ or $\tan(\pi/6)=3^{-1/2}$ and repeatedly halve the angle to compute $\tan(2^{-n}\pi)$ for n=2,3,4,5,... or $\tan(2^{-n}\pi/3)$ for n=1,2,3,4,... using only division, addition, squaring, and square-rooting operations. In this way, the **upper bounds** $2^m\tan(2^{-m}\pi)$ and $2^m 3\tan(2^{-m}\pi/3)$ on π arising from a regular 2^m-gon and 2^m3-gon may be computed after $[4+o(1)]m$ such operations and should be accurate to additive errors at most 3.5×4^{-m} and 1.3×4^{-m} respectively.

For the sin(x) function we have

$$\sin(2x) = 2 \sin(x) (1-\sin(x)^2)^{1/2}$$

from which we deduce

$$\sin(x) = \sin(2x) (2 + 2[1-\sin(2x)^2]^{1/2})^{-1/2}.$$

This allows us to start from the known values $\sin(\pi/4)=2^{-1/2}$ or $\sin(\pi/6)=1/2$ and repeatedly halve the angle to compute $\sin(2^{-n}\pi)$ for n=2,3,4,5,... or $\sin(2^{-n}\pi/3)$ for n=1,2,3,4,... using only division, addition, subtraction, squaring, and square-rooting operations. In this way, the **lower bounds** $2^{m-1}\sin(2^{1-m}\pi)$ and $2^{m-1}3\sin(2^{1-m}\pi/3)$ on π arising from a regular 2^m-gon and 2^m3-gon may be computed after $[7+o(1)]m$ such operations and should be accurate to additive errors at most 4.6×4^{-m} and 0.6×4^{-m} respectively.

Tighter upper bound still accessible to Archimedes

Archimedes knew that the area under a parabolic arc equals (2/3) times the base times the height. For example, the area of the region $0<y<1-x^2$ equals $(2/3)\times 2\times 1=4/3$. Archimedes should also have been able to realize that if we replaced each side of the regular n-gon with inradius=1 by a parabolic arc osculatory to the circle at its midpoint, then we still get something strictly containing the circle, but smaller than the original n-gon, and hence whose area provides a tighter upper bound on π. Specifically,

$$\pi < n\cdot[\tan(\pi/n) - (2\tan(\pi/n) / ([2\tan(\pi/n)^2+1]^{1/2} + 1)])^3/3] = \pi - 3\pi^5/(10 n^4) + O(n^{-6}).$$

Tighter lower bound still accessible to Archimedes

Archimedes should have been able to realize that if we replaced the side of a regular n-gon inscribed in the unit circle, by a parabolic arc with the same endpoints, and tangent to the circle at its midpoint, then we still get something strictly contained inside the circle, but larger than the original n-gon, and hence whose area provides a tighter lower bound on π. Specifically,

$$\pi > n\cdot[\sin(2\pi/n)/2 + (4/3)\sin(\pi/n)\,[1-\cos(\pi/n)]] = n\cdot[4\sin(\pi/n)/3 - \sin(2\pi/n)/6] = \pi - \pi^5/(30\,n^4) + O(n^{-6}).$$

You still can use angle-halving to compute these when n is a power of 2 (or three times a power of 2). These tighter lower and upper bounds evidently would have enabled attaining roughly *twice* as many decimals of accuracy in the same number of arithmetic operations.

Much better approximations with same-order arithmetic-op count

We can *extrapolate* the π-approximations A_m arising from 2^m-gon areas (or B_m arising from $2^m 3$-gon areas) to m=∞ using Wynn's epsilon-algorithm. This simple modern extrapolation algorithm unfortunately was not known to the ancients.

Without extrapolation, A_m and B_m are each accurate to order m decimal places and computable via order m arithmetic operations. While our "parabola improvements" improve the constant factors, they do not alter the fundamental nature of that situation.

But if we Wynn-extrapolate the $1+\sqrt{m}$ values $A_m, A_{m+1}, \ldots, A_{m+\sqrt{m}}$ (or $B_m, B_{m+1}, \ldots, B_{m+\sqrt{m}}$) to m=∞, then we should null out the first √m nonzero terms in the error series in ascending powers of 2^{-m}, thus obtaining approximations to π accurate to order $m^{3/2}$ decimal places, while still only using O(m) arithmetic operations!

This "extrapolated Archimedes" method is an unboundedly huge improvement in computational efficiency, superior in terms of arithmetic-op-count to any method used by pi-computors until the advent of the quadratically-convergent Brent-Salamin algorithm in the 1970s. Extrapolated Archimedes should take $O(D^{2/3})$ arithmetic operations, each O(DlogD) compute-time using "fast arithmetic," to compute the first D decimals of π in $O(D^{4/3}\log D)$ bit-operations.

By contrast: Machin takes order D operations, each order D time, for $O(D^2)$ total single-precision ops (albeit somewhat more if D gets so huge it cannot fit in one machine word anymore). The iteration x←x+sin(x), which converges quadratically to x=π, takes order logD evaluations of the Maclaurin series for $\sin(x)=x-x^3/3!+x^5/5!-x^7/7!+\ldots$ out to, ultimately, order D/logD terms, although early iterations can use fewer series terms. The net arithmetic-op count then is O(D/logD). **Brent-Salamin** with fast arithmetic takes O(logD) arithmetic ops, which can be done via $O((\log D)^2 D)$ bit-ops.

References

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