Revised Attempt to Prove the Collatz Conjecture

Krishna Paliwal

July 8, 2024

Abstract

TThis paper tries to prove the Collatz Conjecture using a rigorous and logical approach trying to break down 80+ year old and proving that all sequences will eventually always reach to 1.

1 Introduction

The Collatz conjecture, proposed by Lothar Collatz in 1937, states that for any positive integer n, the sequence defined by the following rule will eventually reach 1:

- If n is even, the next term is $n/2$
- If n is odd, the next term is $3n + 1$

2 Definitions and Notation

Let $C(n)$ denote the Collatz function:

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We denote k applications of C as $C^k(n)$.

Let $S(n) = \{n, C(n), C^2(n), \ldots\}$ be the Collatz sequence starting from n.

3 Key Arguments

3.1 Decreasing Property

For any $n > 1$, there exists a positive integer k such that $C^k(n) < n$.

3.1.1 Proof for Even Numbers

For even n: $C(n) = \frac{n}{2} < n$ for all $n > 1$, so $k = 1$ suffices.

3.1.2 Proof for Odd Numbers

For odd n:

1. Let m be the smallest positive integer such that $C^m(n)$ is even.
2. We can express $C^m(n)$ as:

$$C^m(n) = \left\lfloor \frac{3^m n + b}{2^m} \right\rfloor$$

   where b is defined as:

$$b = \sum_{i=0}^{m-1} 3^i \cdot 2^{m-i-1}$$

3. Properties of b:
   - b is always a positive integer
   - b is always odd

   Proof:
   - Each term in the sum is a positive integer, so b is positive.
   - The last term ($i = m - 1$) is always 3^{m-1}, which is odd.
   - All other terms are even (they include a factor of 2^{m-i-1} where $m - i - 1 > 0$).
   - The sum of an odd number and any number of even numbers is always odd.

4. We can bound b:

$$b < \sum_{i=0}^{m-1} 3^i \cdot 2^{m-1} = 2^{m-1} \cdot \frac{3^m - 1}{2} = \frac{2^m \cdot 3^m - 2^m}{4}$$

5. Using this bound, we can refine our inequality:

$$\frac{3^m n + \frac{2^m \cdot 3^m - 2^m}{4}}{2^m} < n$$

6. Simplifying:

$$\frac{3^m n}{2^m} + \frac{3^m - 1}{4} < n$$

$$3^m n < 2^m n - 2^m \cdot \frac{3^m - 1}{4}$$

$$n \cdot (2^m - 3^m) > -2^m \cdot \frac{3^m - 1}{4}$$

$$n > \frac{2^m \cdot (3^m - 1)}{4 \cdot (2^m - 3^m)}$$

7. Therefore, we have proven that for any odd positive integer n and for the smallest m such that $C^m(n)$ is even:

$$C^m(n) < n \text{ for all } n > \frac{2^m \cdot (3^m - 1)}{4 \cdot (2^m - 3^m)}$$

3.2 Cycle Property

The only cycle in the Collatz sequence containing the numbers 1, 2, or 4 is $4 \to 2 \to 1 \to 4$.

3.2.1 Proof

This can be verified by direct computation of the Collatz function for these values.

3.3 No Other Cycles

There are no cycles containing only numbers greater than 4.

3.3.1 Proof

1. Assume, for contradiction, that such a cycle exists.
2. Let m be the smallest number in this cycle.
3. By (i), there exists k such that $C^k(m) < m$.
4. However, in a cycle, all numbers should return to themselves after some number of applications of C.
5. The fact that $C^k(m) < m$ contradicts the definition of a cycle.
6. Therefore, our assumption must be false.
7. We conclude that no such cycle can exist.

3.4 Boundedness

The Collatz sequence is bounded for any starting number n.

3.4.1 Proof

1. Let $M(n)$ be the maximum value in $S(n)$.
2. Assume, for contradiction, that $M(n)$ is unbounded.
3. This implies that for any K, there exists j such that $C^j(n) > K$.
4. Choose $K = 2n$.

5. Then there exists j such that $C^j(n) > 2n$.
6. But by (i), there exists k such that $C^k(C^j(n)) < C^j(n)$.
7. This process can be repeated indefinitely, creating an infinite decreasing sequence of integers greater than n.
8. However, this is impossible in the set of positive integers.
9. Our assumption must therefore be false.
10. We conclude that $M(n)$ must be bounded.

4 Main Theorem

Theorem: For any positive integer n, the Collatz sequence starting from n will eventually reach 1.
Proof:
Given:

- $S(n)$ is bounded (by the Boundedness property)
- $S(n)$ contains infinitely many terms (unless it reaches 1)
- $S(n)$ contains no cycles other than $4 \to 2 \to 1 \to 4$ (by the Cycle Property and No Other Cycles property)

By the pigeonhole principle, $S(n)$ must eventually repeat a value. The only possible repeat is the cycle $4 \to 2 \to 1 \to 4$.

Therefore, $S(n)$ must eventually reach 4, 2, or 1. If it reaches 4 or 2, it will subsequently reach 1.

Thus, for any starting number n, the sequence $S(n)$ will eventually reach 1.

5 Conclusion

In this paper, I have tried to constitute a rigorous proof of the Collatz sequence, a simple yet intriguing and intuitive problem and that it will alway end to 1 if we start with a positive integer (natural number).

Written by:
Krishna Paliwal
krishna.plwl264@gmail.com