Pi's Irrationality Using Maclaurin Polynomials

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Abstract

After reviewing Maclaurin series and the Alternating Series Estimation Theorem (ASET), we show how these can be combined with some algebraic observations to prove that π is irrational.

Introduction

There are many proofs of the irrationality of π [2, 4], but beginning calculus books tend not to use them [5, 8]. They are too hard. Even analysis books tend not to mention π's irrationality [7] and, if they do, they don't prove it in the text proper. In Apostol's *Mathematical Analysis* [1] it's relegated to a exercise. Here is a new proof that is a relatively easy way to prove this result. It is at the level of e's irrationality proofs that are generally in beginning calculus and analysis books [1, 5, 7, 8].

Review

We use the Maclaurin series

$$\sin(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}. \tag{1}$$

This is easily derived using the formula for a Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

The Maclaurin series is just a Taylor series with $c = 0$. The derivatives of $f(x) = \sin(x)$ at 0 are $\sin(0) = 0$, $\cos(0) = 1$, $-\sin(0) = 0$ and $-\cos(0) = -1$. With a little reflection this becomes (1).

To calculate the value of $\sin(x)$ at a particular point, approximations must be used and these give rise to Taylor and Maclaurin polynomials. When a value of x is substituted into (1) it becomes an alternating series and these polynomials become partial sums of this series. Alternating series have a key property we will use.

ASET has three parts. They are all implied by oscillations in partial sums; first too much, then too little, but the distance between the two goes to zero. Thus part 1 is $s_n < L < s_{n+1}$ where L is the limit of the series and s_k's are partial sums; part 2 is the absolute value of the error is less than the absolute value of a_{n+1}, the first omitted term of the series approximating partial; and part 3 is the sign of the tail, $L - s_n$ is the same as this first omitted term. There are many youtube animations that show all three parts.

We'll give a quick proof of part 3; we'll need it later. Consider

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n} a_n + (a_{n+1} + a_{n+2}) + (a_{n+3} + a_{n+4}) + \dots.$$

If the first omitted term, a_{n+1} is negative then, as $|a_n|$ is a descending sequence, $(a_{n+1} + a_{n+2}) < 0$, note a_{n+2} has to be positive; they're alternating. This pattern is maintained for all such pairings, so the tail is negative, thus the same sign as a_{n+1}. Likewise, if a_{n+1} is positive then a_{n+2} is negative and $(a_{n+1} + a_{n+2}) > 0$ and this pattern holds for subsequent pairs; the tail is positive, the same sign as a_{n+1}.

It follows that if r is a root of $\sin(x)$, then all Maclaurin polynomials can't be 0 at r: head(r) + tail(r) = 0; by way of ASET, tail(r) $\neq 0$; implies head(r) $\neq 0$ and head(r) is the partial. We'll need this implication as our particular interest is in the roots of Maclaurin polynomials.

First, let's get a picture. A TI84-CE calculator can be used to graph Maclaurin polynomials. The first few for our $\sin(x)$ series are given in Figure 1 and graphed in Figure 2. The $\sin(x)$ curve is slowly being formed. As the degree of

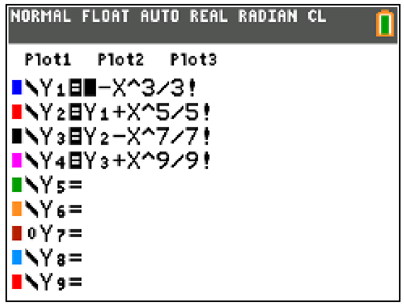

Figure 1: The first few Maclaurin polynomials for $\sin(x)$.

the polynomial grows the number of turning points [3] in the curve increases and the accuracy of the zero estimates of $\sin(x)$ get better; the non-zero root estimates are never perfect, per ASET as previously stated. In Figure 3 we can see that $Y_4(\pi) = 0.006$, almost zero.

Per the periodicity of $\sin(x)$, the roots of $\sin(x)$ are of the form $n\pi$ for integer n. The series (1) converges to $\sin(x)$ for all of the reals; an infinite circle of convergence.

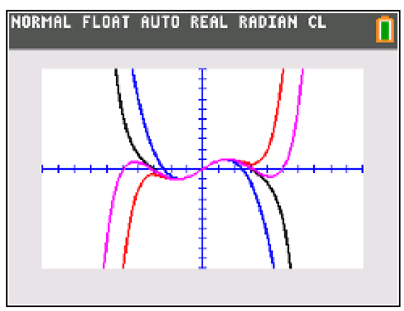

Figure 2: A few Maclaurin polynomials.

Algebraic Observations

Consider the zeros for the first few Maclaurin polynomials for $\sin(x)$ [5, 8]:

$$T_3(x) = x - \frac{x^3}{3!},$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

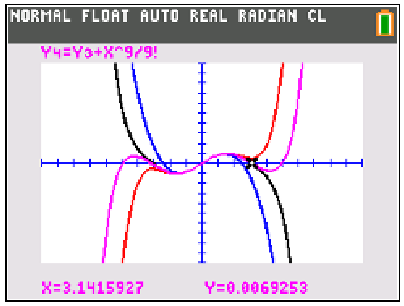

Figure 3: The calc feature of this TI84 calculator gives the value of $T_4(\pi)$.

and
$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$
As stated above, Maclaurin polynomials when evaluated at a point define partial sums of the alternating series (1). ASET [9] indicates that the sign of the remainder terms, the tail is the same as the first omitted term. As the terms are never zero at non-zero points, if the infinite series sums to 0, the partial can't be zero. They must be equal to the negative of the non-zero tail. This translates, as we showed, into the Maclaurin polynomials don't share roots with $\sin(x)$.

We can also observe that the roots of these $T_j(x)$ will have to be the same as

$$3!T_3(x) = -x(x^2 - 3!),$$

$$5!T_5(x) = x(x^4 - 5 \cdot 4x^2 + 5!),$$

and
$$7!T_7(x) = -x(x^6 - 7 \cdot 6x^4 + 7 \cdot 6 \cdot 5 \cdot 4x^2 - 7!).$$
Zero times even a large factorial number is still 0.

The non-zero roots of these will have to be the same as those of

$$\hat{T}_3(x) = x^2 - 3!, \tag{2}$$

$$\hat{T}_5(x) = x^4 - 5 \cdot 4x^2 + 5!, \tag{3}$$

and
$$\hat{T}_7(x) = x^6 - 7 \cdot 6x^4 + 7 \cdot 6 \cdot 5 \cdot 4x^2 - 7!. \tag{4}$$
We are now ready to prove π is irrational.

Proof

Theorem 1. *π is irrational.*

Proof. Define the partial series of the Maclaurin expansion of $\sin(x)$ as

$$T_j(x) = \sum_{k=1}^{j} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}$$

and consider

$$\hat{T}_j(x) = \frac{j! T_j(x)}{x} \quad \text{where } x \neq 0 \, .$$

Then $\hat{T}_j(x)$ is an integer polynomial that shares non-zero roots with $T_j(x)$. The sequence of these roots converges to the roots of $\sin(x)$ as

$$\lim_{j \to \infty} T_j(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!} = \sin(x).$$

Next, assume for a contradiction that $\pi = p/q$ then $q\pi = p$ and $\sin(p) = 0$. This means

$$\lim_{j \to \infty} \hat{T}_j(p) = 0.$$

But this implies that given an ϵ such that $0 < \epsilon < 1$, there exists N such that for all $j > N$,

$$0 < |\hat{T}_j(p)| < \epsilon, \tag{5}$$

but all $\hat{T}_j(x)$ are integer polynomials (see (2), (3), and (4)) and when evaluated at the integer p have to be a non-zero integer. A contradiction. Note, the left hand inequality in (5) is implied by ASET. □

Remarks

One can come to an understanding of the nature of this proof and of irrational numbers by considering what

$$\lim_{j \to \infty} \hat{T}_j(x) \tag{6}$$

must be. This is a power series with coefficients consisting of sequences that go to infinity. Hard to write down! With an integer x value and a finite j value it must evaluate to an integer. But if x is irrational, say π then

$$\lim_{k \to \infty} A_k\pi - B_k\pi = 0$$

is a possibility, where A_k and B_k are integer sequences going to infinity. The terms $A_k\pi$ and $B_k\pi$ always have infinite decimals and the difference can shrink to 0.

It is likely that (6) can define a function, but it must have a complicated nature. We just need the roots of $T_j(x)$ and $\hat{T}_j(x)$ are the same and as the former converges to roots of $\sin(x)$, so too will the latter.

Conclusion

This proof seems to be easier and as short as Niven's classic integral based 1947 proof [6]. It does require knowledge of infinite series, a topic later than integration in calculus textbooks. But the steps are simpler and not too removed from the level of beginning calculus. It almost seems to be simple algebra in nature. It might make a good application within a section on alternating series in calculus textbooks.

References

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