Is it really that difficult to prove the Goldbach conjecture?

Mary Anne Ji You & *Óscar E. Chamizo Sánchez.*

Abstract:

The Goldbach conjecture [1][2], that is to say, every even number greater than 4 can be represented by the sum of two primes, is a simple and intractable statement that has been torturing mathematicians for more than 250 years. We wondered if the *divide et impera* method, so useful in programming and algorithmics, could provide some service here. The goal is simplify and separate the whole problem into three independent and fairly manegeable subproblems. An approach that, as far as I know, has not been tested before.*

Definitions:

From now on, *m* and *n* are positive integer numbers, *p* and *q* are prime numbers. Note that all prime p $\geq$ 5 is of the form 6m$\pm$1; primes of the form 6m+1 are called **right primes**; primes of the form 6m-1 are called **left primes**.

Goldbach conjecture states that for all n and all prime p such that 3≤p≤n, at least some 2n-p is prime, i.e., not every 2n-p is composite. Let´s say: disprove, for all n and all 3 ≤ p ≤ n, that 2n-p is composite.

We shall divide the conjecture into 3 independent subproblems realizing that in three consecutive odd numbers, 2n-3, 2n-5 and 2n-7, one and only one of them must be multiple of 3. So we face:

Case A: 3|2n-7 or 2n ≡ 1 mod 3.
Case B: 3|2n-5 or 2n ≡ 2 mod 3.
Case C: 3|2n-3 or 2n ≡ 0 mod 3

Case A: 3|2n-7:

3|2n-7⇒ 3|2n-13, 3|2n-19 and so on. Hence 3|2n-(6m+1) for all m. If q is a right prime, 2n-q is a multiple of 3 and if q is a left prime, 2n-q is not a multiple of 3. Our goal is to prove that there are not enough factors p_i (i=1,2,3, …, k) from p_1=5 to p_k, where p_k is the largest prime $p_k \leq \sqrt{2n-3}$ to factorize every 2n-q with $\sqrt{2n-3} < q < n$. For example, to prove that 1000 (1000 ≡ 1 mod 3) must satisfy the Goldbach conjecture we shall prove that from 1000-41, (41 is the least left prime greater than or equal $\sqrt{1000-5}$), to 1000-491 necessarily there are numbers 1000-q that can not be factorized, i.e., prime numbers.

Sufficient and necessary conditions for all these q to be primes is:

$$6m-1 \not\equiv 0 \bmod p_i$$

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a…, let be $2n-a_i$ the odd number containing the first occurrence of prime factor p_i in that sequence.

Note that:

For each p_i, a_i is unique.

$3 \le a_i \le 2p_i+1$.

For some i, $a_i = 3$; for some i, $a_i=5$; for some i, $a_i=11$ MOD p_i; for some i, $a_i=17$ MOD p_i; for some i, $a_i=23$ MOD p_i and so on.

2n-q, i.e., 2n-(6m-1), is composite if and only if exists i such that $6m-1 \equiv a_1 \bmod p_i$. Indeed. Given the succession 2n-(6m-1), the answer to how often occurs the factor p_i in it is $6p_i$.

Now, let´s state conditions in order to find some 2n-q with q=6m-1 and q inside the interval $\sqrt{2n-3} < q < n$ that can not be factorized:

1) q is a left prime, i.e., q is not multiple of any p_i, so $6m-1 \not\equiv 0 \bmod p_i$ for all i.
2) There is no p_i factor available for 2n-q, so $6m-1 \not\equiv a_1 \bmod p_i$ for all i.

Prime condition for 6m-1	No factor available condition for 2n-(6m-1)
$6m \not\equiv 1 \bmod 5$	$6m \not\equiv (a_1+1) \bmod 5$
$6m \not\equiv 1 \bmod 7$	$6m \not\equiv (a_2+1) \bmod 7$
$6m \not\equiv 1 \bmod 11$	$6m \not\equiv (a_3+1) \bmod 11$
$6m \not\equiv 1 \bmod 13$	$6m \not\equiv (a_4+1) \bmod 13$
…………..	……………….
$6m \not\equiv 1 \bmod p_k$	$6m \not\equiv (a_k+1) \bmod p_k$

Hence for each p_i there are *at least* p_i-2 remainders moduli p_i that fullfill the conditions. That amounts up to a minimum of $(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $3.5.9.11.\ldots(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-3} < 6m < n$$

So let´s prove that at least one in 3.5.9.11…(p_k-2) solutions from 5.7.11.13…p_k possible systems lies inside the aformentioned interval.

Let be M the highest number of consecutive occurrences of 6m that do not fullfill the conditions.[1] Is not easy to figure out the value of M, given the unpredictable nature of prime number distribution. But we can prove that exists an upper bound S for M such that for sufficient large n:

$$S < \left\lceil \frac{n-\sqrt{2n-3}}{6} \right\rceil \qquad (1)$$

Given p_k, an upper bound for the total number of occurrences of each one of the two remainders moduli p are $2\left\lceil \frac{p_k}{p} \right\rceil$. So

$$S = 2\left(\left\lceil \frac{p_k}{5} \right\rceil + \left\lceil \frac{p_k}{7} \right\rceil + \left\lceil \frac{p_k}{11} \right\rceil + \left\lceil \frac{p_k}{13} \right\rceil + \ldots + \left\lceil \frac{p_k}{p_{k-1}} \right\rceil + 1\right)$$

is an upper bound for M:

k	p_k	M	S
1	5	2	2
2	7	4	6
3	11	8	11
4	13	13	16
5	17	19	24
6	19	22	28

In turn:

$$\left\lceil \frac{p_k}{5} \right\rceil + \left\lceil \frac{p_k}{7} \right\rceil + \left\lceil \frac{p_k}{11} \right\rceil + \left\lceil \frac{p_k}{13} \right\rceil + \ldots + \left\lceil \frac{p_k}{p_{k-1}} \right\rceil + 1 <$$

$$\frac{p_k}{2} + \frac{p_k}{3} + \frac{p_k}{5} + \frac{p_k}{7} + \frac{p_k}{11} + \ldots + \frac{p_k}{p_{k-1}} + 1 =$$

[1] For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for **M** is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in p_k days. What is the maximum number, as a function of p_k, of consecutive days off?

$$p_k\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}\ldots+\frac{1}{p_{k-1}}+\frac{1}{p_k}\right\}$$

The series between brackets is the well known partial summation of the reciprocal of the primes whose divergence was proved by Euler in 1737 together with the relationship:

$$\sum_{p\leq x}\frac{1}{p} \approx \mathrm{loglog(x)} \qquad (2)$$

Taking $x=p_k$ and given that an upper bound for all $x>e^4$ in (2) is loglogx+6 [3] allows us to state:

$$S < 2p_k(\mathrm{loglog}p_k+6)$$

Now it`s inmediate to conclude, since $p_k \leq \sqrt{2n-3}$, that (1) holds for, let´s say, every $2n \geq 10^6$.

For every $2n<10^6$ the verification of the conjecture have alredy been settled.

That completes the demonstration.

Hence, for all 2n such that 3|2n-7, i.e., for all 2n ≡ 1 mod 3, exists some 2n-q that can not be factorized, so 2n-q is prime and the conjecture holds for all 2n ≡ 1 mod 3.

Case B: 3|2n-5:

3|2n-5⇒ 3|2n-11, 3|2n-17 and so on. Hence 3|2n-(6m-1) for all m. If q is a left prime, 2n-q is a multiple of 3 and if q is a right prime, 2n-q is not a multiple of 3.

Following the same thought process than before, with q a right prime of the form 6m+1, it´s straightforward to conclude that the conjecture holds for all 2n such that 3|2n-5, i.e., for all 2n ≡ 2 mod 3.

Case C: 3|2n-3:

3|2n-3⇒ 3∤2n-(6m±1) for all m. All elements of the sequence:

2n-5, 2n-7, 2n-11, 2n-13, 2n-17, 2n-19, 2n-23, 2n-29, 2n-31, 2n-37, ... 2n-q

where q ≥ 5 is a prime, must be factorized. There are k consecutive primes p_i (i=1,2,3, …, k) from $p_1=5$ to p_k, where p_k is the largest prime $p_k \leq\sqrt{2n-3}$, available for that factorization.

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a…, let be $2n-a_i$ the number containing the first occurrence of prime factor p_i in that sequence.

Notice that:

For each p_i, a_i is unique.

$3 \le a_i \le 2p_i+1$.

For some i, $a_i = 3$; for some i, $a_i=5$; for some i, $a_i=11$ MOD p_i; for some i, $a_i=17$ MOD p_i; for some i, $a_i=23$ MOD p_i and so on.

2n-q, i.e., $2n-(6m\pm1)$, is composite if and only if exists i such that $6m\pm1 \equiv a_1 \mod p_i$.

Conditions in order to find some 2n-q with $q=6m\pm1$ and q inside the interval $\sqrt{2n-3} < q < n$ that can not be factorized:

Prime condition for $6m\pm1$	No factor available condition for $2n-(6m\pm1)$
$6m \not\equiv \pm1 \mod 5$	$6m \not\equiv (a_1\pm1) \mod 5$
$6m \not\equiv \pm1 \mod 7$	$6m \not\equiv (a_2\pm1) \mod 7$
$6m \not\equiv \pm1 \mod 11$	$6m \not\equiv (a_3\pm1) \mod 11$
$6m \not\equiv \pm1 \mod 13$	$6m \not\equiv (a_4\pm1) \mod 13$
…………..	……………….
$6m \not\equiv \pm1 \mod p_k$	$6m \not\equiv (a_k\pm1) \mod p_k$

Hence for each p_i there are *at least* $2(p_i-2)$ remainders moduli p_i that fullfill the conditions. That amounts up to a minimum of $2(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $2.3.5.9.11.\ldots.(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

Interesting to note here that this result is fully consistent with the fact that there are now twice as many composite numbers to factorize with the same number of factors than before (Cases **A** and **B**)

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-3} < 6m < n$$

The same considerations apply as in relation to the previous point, as to conclude that:

$$S < p_k(\log\log p_k+6)$$

is an upper bound for the highest number of consecutive occurrences of 6m that do not fullfill the previous conditions. Hence, as before, the conjecture also holds for every 2n ≡ 0 mod 3.

*The foundations and conclusions of this paper are an adaptation in a more intuitive way of a previous paper available here [4].

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roundkuniversity@gmail.com
PA^3.

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[3] Pollack, Paul. *Euler and the partial sums of the prime harmonic series*. University of Georgia. Athens. Georgia.

[4] https://vixra.org/pdf/2312.0021v1.pdf