Statistical foundation of black-body radiation

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Abstract

The conventional derivation of *Planck's distribution of black-body radiation* is based on the quantization in 1-D of the energy $\varepsilon = n\kappa\varepsilon_0$ of each descrete standing mode κ of the field into n photons of energy $\kappa\varepsilon_0$ where $\varepsilon_0 = hc/2\ell$; $\ell \sim$length of system. Considering interactions between radiation and matter in equilibrium T, it is accepted that the energy ε of each mode is governed by *Boltzmann's law* so that the conditional probability that there are n photons in the mode κ is

$$g(n/\kappa) = A\, e^{-n\frac{\kappa}{\theta}} \quad ; \quad \theta = T/\varepsilon_0 \quad ; \quad A = 1 - e^{-\frac{\kappa}{\theta}}$$

In the present article we derive alternatively g(n/κ) by quantizing the total energy $E = s\varepsilon_0$ of a closed 1-D system into s quanta that in turn form a random total number N=1, 2,…, s of photons of various energies. The number of *states* describing this photon gas is equal to the number p_s of integer solutions of the equation $n_1 + 2n_2 + 3n_3 + \ldots + sn_s = s$ where $n_1 \geq 0$; $n_2 \geq 0$; …. ; $n_s \geq 0$ are the numbers of photons occupying the energy levels κ=1,2,…,s respectively. Since photons are *indistinguishable particles* obeying *Bose statistics*, we assume that all above *quantum states occur with equal probabilities* $1/p_s$. As p_s represents also the *number of partitions of the integer s*, we express *exactly* the conditional probability g(n/κ) of each level κ in terms of partitions by introducing diagrams and then we study its behaviour for large s using the *Hardy-Ramanujan* formula. Thus, Planck's distribution is derived without resorting to Boltzmann's law and to interactions between radiation and matter.

1. Introduction

In the title of his important review article [1], Peter Enders is asking the question:

Why Boltzmann did not arrive at Planck's distribution. Apart from its historical perspective, this question shows how essential is Boltzmann's law in its quantized form for the derivation of Planck's distribution of black-body radiation. The conventional theory in this case is well known. The allowed standing wave spatial variation of the electromagnetic field in the cavity - identical in both classical and quantum theories [2] - creates in 1-D a descrete spectrum of frequencies: ν_0, $2\nu_0$, $3\nu_0$, ...where $\nu_0 = c/2\ell$; ℓ ~ length of system. Then, *Planck's hypothesis of* $\varepsilon = h\nu$ *and energy quantization* implies the occurrence of descrete energy levels ε_0, $2\varepsilon_0$, $3\varepsilon_0$, ... corresponding respectively to these frequencies, where $\varepsilon_0 = hc/2\ell$ is the energy of each quantum constituting the total energy $E = s\varepsilon_0$ of the system. Further, each level κ=1,2,3,... may be occupied by a mode which has frequency $\nu = \kappa\nu_0$ and quantized energy $\varepsilon = nh\nu = n\kappa\varepsilon_0$ where n is the number of *photons* of energy $\kappa\varepsilon_0$ contained in the mode.

The essence of today's quantum theory of radiation of the black-body system is based on the following two assumptions [2]:

I. For every mode at the energy level κ we introduce a separate sub-system containing a random number n of photons forming the mode, where each photon has energy $\kappa\varepsilon_0$.

II. Thermal equilibrium in each sub-system occurs through interactions between matter and radiation by the association of a quantum harmonic oscillator with each mode of the field.

Accordingly, in 1-D the probability density of the energy ε of a mode is given classically by *Boltzmann's law*:

$$P(\varepsilon) = \frac{1}{T} e^{-\frac{\varepsilon}{T}} \tag{1}$$

where the average energy of all modes is $\langle\varepsilon\rangle = \int_0^\infty \varepsilon P(\varepsilon) d\varepsilon = T$

By quantizing the energy $\varepsilon = n\kappa\varepsilon_0$ of a mode located at level κ , Eq.(1) transforms into the *conditional probability that there are* n *photons in energy level* κ i.e. that there are n photons in the system, each having κ quanta:

$$g(n/\kappa) = A\, e^{-n\frac{\kappa}{\theta}} \qquad ; \; n = 0,1,2,\ldots \tag{2}$$

where $\theta = T/\varepsilon_0$ and $A = 1 - e^{-\frac{\kappa}{\theta}}$ so that $g(n/\kappa)$ is normalized:

$$\sum_{n=0}^{\infty} g(n/\kappa) = \left(1 - e^{-\frac{\kappa}{\theta}}\right) \sum_{n=0}^{\infty} e^{-n\frac{\kappa}{\theta}} = 1 \tag{3}$$

The derivation of Planck's distribution from Eq.(2) is straightforward. The average number of photons in 1-D occupying level κ is given according to Eq.(2) by

$$\langle n_\kappa \rangle = \sum_{n=0}^{\infty} n\, g(n/\kappa) = \left(1 - e^{-\frac{\kappa}{\theta}}\right) \sum_{n=0}^{\infty} n\, e^{-n\frac{\kappa}{\theta}} = \frac{1}{e^{\frac{\kappa}{\theta}} - 1} \tag{4}$$

and the average number of quanta existing in level κ in 1-D reads

$$\langle q_\kappa \rangle = \kappa \, \langle n_\kappa \rangle = \frac{\kappa}{e^{\frac{\kappa}{\theta}} - 1} \tag{5}$$

where $\theta = T/\varepsilon_0$; $\varepsilon_0 = hc/2\ell$. Transforming Eq.(5) by using $\varepsilon = \varepsilon_0 \kappa$; $d\varepsilon = \varepsilon_0 \, d\kappa$,
we obtain the energy/cm within dε :

$$\rho(\varepsilon) d\varepsilon = \frac{1}{\ell} \, [\langle q_\kappa \rangle]_{\kappa = \varepsilon/\varepsilon_0} \, \varepsilon_0 \, d\kappa \tag{6}$$

so that

$$\rho(\varepsilon) = \frac{2}{hc} \frac{\varepsilon}{e^{\frac{\varepsilon}{T}} - 1} \tag{7}$$

and the 1-D energy/cm is given by

$$u = \int_0^\infty \rho(\varepsilon) d\varepsilon = \frac{2}{hc} \, T^2 \int_0^\infty \frac{x}{e^x - 1} \, dx = \frac{\pi^2}{3} \, \frac{T^2}{hc} \tag{8}$$

Introducing $u = E/\ell$; $E = s\varepsilon_0$; $\varepsilon_0 = hc/2\ell$ into Eq.(8) we also get the important 1-D relation

$$\theta \equiv \frac{T}{\varepsilon_0} = \frac{\sqrt{6s}}{\pi} \tag{9}$$

In 3-D, $\langle q_\kappa \rangle$ given by Eq.(5) should be multiplied [3] by the *Rayleigh-Jeans* coefficient $\pi\kappa^2$ containing the 1/8 positive shell correction of the frequency grid and the factor 2 regarding polarization:

$$\langle q_\kappa \rangle = \frac{\pi \kappa^3}{e^{\frac{\kappa}{\theta}} - 1} \tag{10}$$

where $\theta = T/\varepsilon_0$, $\varepsilon_0 = hc/2\ell$. Transforming Eq.(10) by using $\varepsilon = \varepsilon_0\kappa$; $d\varepsilon = \varepsilon_0 d\kappa$, the energy/cm^3 within dε reads:

$$\rho(\varepsilon) d\varepsilon = \frac{1}{\ell^3} \, [\langle q_\kappa \rangle]_{\kappa = \varepsilon/\varepsilon_0} \, \varepsilon_0 \, d\kappa \tag{11}$$

so that we obtain *Planck's distribution* for the energy/cm^3 per unit energy within the cavity:

$$\rho(\varepsilon) = \frac{8\pi}{h^3 c^3} \frac{\varepsilon^3}{e^{\frac{\varepsilon}{T}} - 1} \tag{12}$$

and the energy/cm^3 given by *Stephan's law*:

$$u = \int_0^\infty \rho(\varepsilon) d\varepsilon = \frac{8\pi}{h^3 c^3} T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{8\pi^5}{15} \frac{T^4}{h^3 c^3} \tag{13}$$

We observe that the derivation of the conditional probability $g(n/\kappa)$ [Eq.(2)] from Boltzmann's law [Eq.(1)], presents some conceptual difficulties:

The first problem is that separation into different sub-systems corresponding to modes of various frequencies (assumption I) is not a global approach to the construction of statistics for a closed system. However, in the conventional theory this separation into sub-systems is necessary because, although the number s of quanta of the closed system is *fixed*, the total number N of photons formed from these quanta is *random*. Therefore, Boltzmann's method of distributing s quanta into N fixed particles [4] cannot be used globally in the present case.

The second problem is that photons are *indistinguishable particles* obeying *Bose statistics*. Therefore, even separation of the system into sub-systems, cannot justify the use of Boltzmann's law for each sub-system since the derivation of this law [4] is based on *distinguishable* particles. This is why, additional consideration of interactions between radiation and matter becomes necessary in the conventional theory (assumption II).

Note that apart from the above problems, a photon differs from a classical particle because the latter exists as a statistical entity at zero energy whereas the former does not.

In the present paper, the conditional probability $g(n/\kappa)$ [Eq.(2)] will be derived without resorting to Boltzmann's law. Instead, the statistical foundations of black-body radiation will be based on the *number theory of partitions*. In particular, in section 2 a global approach is considered for a closed system containing s quanta of energy ε_0 and described by a single conservation equation. Also, *the principle of equal probabilities of quantum states* [4] is introduced and its relation to the number of partitions p_s of s is outlined. In section 3 the *states* of the system are represented by *diagrams* containing *all* statistical information of the problem. Also it is shown that the conditional probabilities $g(n/\kappa)$ calculated from the above diagrams as well as the average number of photons $\langle n_\kappa \rangle$ and quanta $\langle q_\kappa \rangle$ occupying the energy level κ, can be expressed *exactly* in terms of partitions. In section 4 a general theory valid for arbitrary s is developed and the behaviour of $g(n/\kappa)$ is studied for large s using the Hardy-Ramanujan formula. Hence, Eq.(2) leading to Planck's distribution is obtained explicitly without resorting to Boltzmann's law.

2. The principle of equal probabilities of quantum states

We consider a 1-D system of total energy $E = s\varepsilon_0$ containing s quanta of energy $\varepsilon_0 = hc/2\ell$ where $\ell \sim$length of system. These quanta imply the existence of energy levels $\kappa=1, 2, \ldots, s$ and form N photons where the number $N=1, 2, \ldots, s$ is random. † If a photon consists of κ quanta, then it occupies the energy level κ and has energy $\kappa\varepsilon_0$.

The number of *states* describing the photons is equal to the number p_s of integer solutions of the equation

$$n_1+2n_2+3n_3+\ldots+sn_s = s \qquad (14)$$

where $n_1 \geq 0$; $n_2 \geq 0$; … ; $n_s \geq 0$ are the numbers of photons occupying the energy levels $\kappa=1, 2, \ldots, s$ respectively. Assuming photons are *indistinguishable particles* obeying *Bose statistics,* it is the *states* $(n_1, n_2, \ldots, n_s)$ defined by Eq.(14) that consist the statistical basis of the 1-D photon gas. Thus, we introduce the main principle [4] of the present theory:

All quantum states defined by Eq.(14) occur with equal probability $1/p_s$

Now since p_s as defined by Eq.(14), represents also the number of partitions of the integer s, we will further study the problem of black-body radiation using the *number theory of partitions*. In particular, p_s is given exactly [5] by

$$p_s = \frac{2}{\pi}\int_0^{\pi/2} \prod_{\kappa=1}^{s} \frac{sin[(s+\kappa)x]}{sin(\kappa x)}\ cos[(s^2-2s)x]dx \qquad (15)$$

† Note here the difference between the present theory where the total number of photons $N=n_1+n_2+\ldots+n_s$ is random, and Boltzmann's method [4] where the total number of particles $N=n_1+n_2+\ldots+n_s$ is fixed and together with Eq.(14) defines the states and the configurations of the system.

and its asymptotic behaviour for large s can be expressed by the Hardy-Ramanujan formula [6]:

$$p_s = \frac{a}{s}\, e^{\,b\sqrt{s}} \quad ; \quad a = \frac{1}{4\sqrt{3}} \quad ; \quad b = \pi\sqrt{2/3} \tag{16}$$

Also, a very important expansion of p_s that relates to the present problem is the following

$$p_s = \sum_{N=1}^{s} \gamma_s(N) \tag{17}$$

where $\gamma_s(N)$; N=1, 2, ..., s is *the number of partitions of s that have N terms in their sum.* Up to s=10, Eq.(17) is represented by the following table:

s	p_s	$\gamma_s(1)$	$\gamma_s(2)$	$\gamma_s(3)$	$\gamma_s(4)$	$\gamma_s(5)$	$\gamma_s(6)$	$\gamma_s(7)$	$\gamma_s(8)$	$\gamma_s(9)$	$\gamma_s(10)$
1	1	1									
2	2	1	1								
3	3	1	1	1							
4	5	1	2	1	1						
5	7	1	2	2	1	1					
6	11	1	3	3	2	1	1				
7	15	1	3	4	3	2	1	1			
8	22	1	4	5	5	3	2	1	1		
9	30	1	4	7	6	5	3	2	1	1	
10	42	1	5	8	9	7	5	3	2	1	1

. . . .

Table 1: Apart from the terms $\gamma_s(1)=1$, s=1, 2, 3, ... all other terms can be derived from the symplectic relation:

$$\gamma_s(N) = \gamma_{s-1}(N-1) + \gamma_{s-N}(N) \quad ; \quad 2 \le N \le s \;\; ; \;\; s = 2, 3, \ldots$$

Table 1 has been previously introduced in number theory by H. Griffin [7] and was later extended by J. Leach [8].

In the present work we observe that according to the principle of equal probabilities of quantum states introduced above, *the probability $\Psi_s(N)$; N= 1, 2, ..., s that N photons are formed in a 1-D system consisting of s quanta*, is given by

$$\Psi_s(N) = \frac{\gamma_s(N)}{p_s} \tag{18}$$

where $\gamma_s(N)$ are defined in Eq.(17) and given in Table 1 so that $\Psi_s(N)$ is normalized. According to distribution (18), the *average number of photons* formed in a 1-D system consisting of s quanta reads

$$\langle N\rangle_s = \sum_{N=1}^{s} N\, \Psi_s(N) \tag{19}$$

and the *average quanta/photon* in a 1-D system consisting of s quanta can be expressed as

$$\langle \kappa\rangle_s = s \sum_{N=1}^{s} \frac{\Psi_s(N)}{N} \tag{20}$$

Note that the *conditional* average quanta / photon given that the system has N photons is $\langle \kappa/N\rangle_s = s/N$. The above averages will be calculated explicitly for various values of s together with the conditional probabilities $g(n/\kappa)$ in the next section.

3. Diagrammatic representation of quantum states

The states defined by Eq.(14) describing a 1-D closed system containing s quanta of energy ε_0 and a random number N=1, 2, …, s of photons formed by these quanta, can be represented by diagrams.

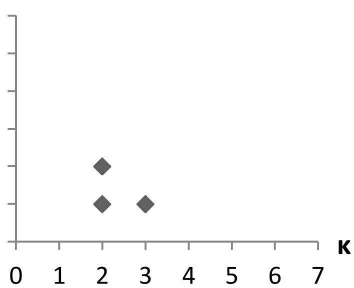


Fig. 1

For example, the diagram of Fig.1 represents a state of a system containing s=7 quanta forming N=3 photons. There are n_2=2 photons located at the energy level 2 (i.e. each photon has 2 quanta) and n_3=1 photon located at the energy level 3 (i.e. this photon has 3 quanta).

In this section the conditional probability $g(n/\kappa)$ that there are n photons in the energy level κ:

I. will be calculated directly from the diagrams according to the *principle of equal probabilities of quantum states* i.e.

$g(n/\kappa)$ = number of states where level κ has n photons / total number of states p_s

II. will be expressed *exactly* in terms of partitions (new idea of present article).

Also, the average number of photons occupying level κ :

$$\langle n_\kappa \rangle = \sum_{\kappa=1}^{s} n \, g(n/\kappa) \tag{21}$$

as well as the average number of quanta existing in level κ :

$$\langle q_\kappa \rangle = \kappa \, \langle n_\kappa \rangle \tag{22}$$

will be also expressed exactly in terms of partitions.

It becomes clear that always we have

$$\sum_{\kappa=1}^{s} \langle n_\kappa \rangle = \langle N \rangle_s \tag{23}$$

where the average number of photons $\langle N \rangle_s$ existing in a 1-D system of s quanta is defined by Eq.(19). Also, as expected

$$\sum_{\kappa=1}^{s} \langle q_\kappa \rangle = s \tag{24}$$

is always valid.

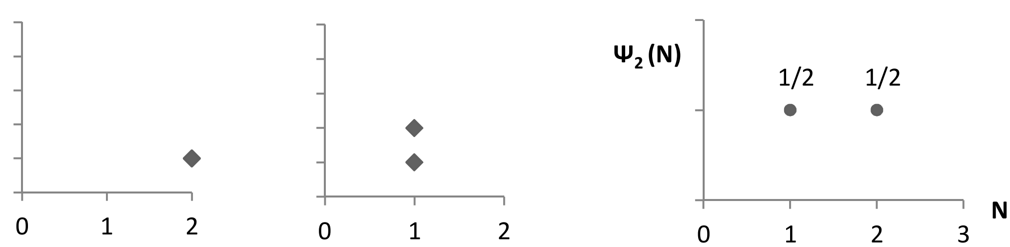
Let us consider next in detail the cases $s = 2, 3, \ldots, 9$.

$s = 2$ quanta ; $p_2 = 2$ states

Eq.(14) reads

$$n_1 + 2n_2 = 2 \tag{25}$$

Diagrams



N=1 N=2

From Eqs (19, 20) we have

$$\langle N \rangle_2 = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2} \; (ph) \tag{26}$$

$$\langle \kappa \rangle_2 = 2 \left\{ \frac{1}{1} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \right\} = \frac{3}{2} \; (qu/ph) \tag{27}$$

Conditional probabilities

Using the diagrams we calculate $g(n/\kappa)$; $\kappa=1,2$ and express the results in terms of partitions:

$\kappa=1$

$$g\,(0/1) = \frac{p_2 - p_1}{p_2} = \frac{1}{2} \qquad g\,(1/1) = \frac{p_1 - p_0}{p_2} = 0 \qquad g\,(2/1) = \frac{p_0}{p_2} = \frac{1}{2}$$

$$\text{Norm. } g(0/1) + g(1/1) + g(2/1) = 1 \tag{28}$$

$$\langle n_1 \rangle = 0 \cdot g(0/1) + 1 \cdot g(1/1) + 2 \cdot g(2/1) = \frac{1}{p_2}(p_1 + p_0) = 1$$

$$\langle q_1 \rangle = 1 \cdot \langle n_1 \rangle = \frac{1}{p_2}(p_1 + p_0) = 1$$

$\kappa=2$

$$g\,(0/2) = \frac{p_2 - p_0}{p_2} = \frac{1}{2} \qquad g\,(1/2) = \frac{p_0}{p_2} = \frac{1}{2} \qquad \text{Norm. } g(0/2) + g(1/2) = 1$$

$$\langle n_2 \rangle = 0 \cdot g(0/2) + 1 \cdot g(1/2) = \frac{p_0}{p_2} = \frac{1}{2} \qquad \langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2p_0}{p_2} = 1 \tag{29}$$

Also, we obtain the average number of photons existing in a system of $s=2$ quanta, given also by Eq.(26), as

$$\langle N \rangle_2 = \langle n_1 \rangle + \langle n_2 \rangle = \frac{1}{p_2}(p_1 + 2p_0) = \frac{3}{2}(ph) \tag{30}$$

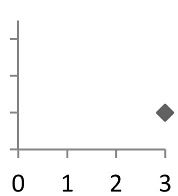
and as expected, the sum of the average number of quanta existing in energy levels 1,2 according to Eq.(24) reads

$$s = \langle q_1 \rangle + \langle q_2 \rangle = \frac{1}{p_2}(p_1 + 3p_0) = 2(qu) \tag{31}$$

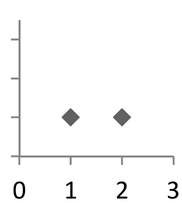
$s = 3$ quanta ; $p_3 = 3$ states

Eq.(14) reads

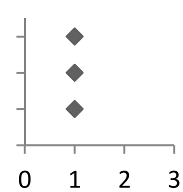
$$n_1 + 2n_2 + 3n_3 = 3 \tag{32}$$



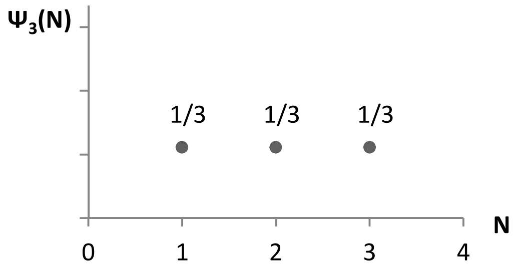
N=1



N=2



N=3



From Eqs (19,20) we have

$$\langle N\rangle_3 = 1\cdot\frac{1}{3}+2\cdot\frac{1}{3}+3\cdot\frac{1}{3}=2\ (ph) \tag{33}$$

$$\langle \kappa\rangle_3 = 3\left\{\frac{1}{1}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}\right\}=\frac{11}{6}\ (qu/ph) \tag{34}$$

Conditional probabilities

Using the diagrams we calculate g(n/κ); κ=1,2,3 and express the results in terms of partitions:

κ=1

$g(0/1) = \dfrac{p_3-p_2}{p_3}=\dfrac{1}{3}$ $\quad g(1/1)=\dfrac{p_2-p_1}{p_3}=\dfrac{1}{3}$ $\quad g(2/1)=\dfrac{p_1-p_0}{p_3}=0$

$g(3/1) = \dfrac{p_0}{p_3}=\dfrac{1}{3}$ $\quad$ Norm. g(0/1) + g(1/1) + g(2/1) + g(3/1) = 1

$$\langle n_1\rangle = 0\cdot g(0/1)+1\cdot g(1/1)+2\cdot g(2/1)+3\cdot g(3/1)=\frac{1}{p_3}(p_2+p_1+p_0)=\frac{4}{3} \tag{35}$$

$$\langle q_1\rangle = 1\cdot\langle n_1\rangle=\frac{1}{p_3}(p_2+p_1+p_0)=\frac{4}{3}$$

κ=2

$g(0/2) = \dfrac{p_3-p_1}{p_3}=\dfrac{2}{3}$ $\quad g(1/2)=\dfrac{p_1}{p_3}=\dfrac{1}{3}$ $\quad$ Norm. g(0/2) + g(1/2) = 1

$$\langle n_2\rangle = 0\cdot g(0/2)+1\cdot g(1/2)=\frac{p_1}{p_3}=\frac{1}{3} \qquad \langle q_2\rangle = 2\cdot\langle n_2\rangle=\frac{2p_1}{p_3}=\frac{2}{3} \tag{36}$$

κ=3

$$g(0/3) = \frac{p_3 - p_0}{p_3} = \frac{2}{3} \qquad g(1/3) = \frac{p_0}{p_3} = \frac{1}{3} \qquad \text{Norm. } g(0/3) + g(1/3) = 1$$

$$\langle n_3 \rangle = 0 \cdot g(0/3) + 1 \cdot g(1/3) = \frac{p_0}{p_3} = \frac{1}{3} \qquad \langle q_3 \rangle = 3 \cdot \langle n_3 \rangle = \frac{3p_0}{p_3} = 1 \qquad (37)$$

Also, we obtain the average number of photons existing in a system of s=3 quanta, given also by Eq.(33), as

$$\langle N \rangle_3 = \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle = \frac{1}{p_3}(p_2 + 2p_1 + 2p_0) = 2\ (ph) \qquad (38)$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3 according to Eq.(24) reads

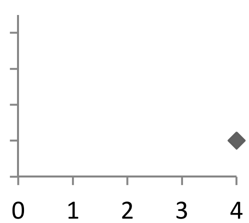
$$s = \langle q_1 \rangle + \langle q_2 \rangle + \langle q_3 \rangle = \frac{1}{p_3}(p_2 + 3p_1 + 4p_0) = 3\ (qu) \qquad (39)$$

s = 4 quanta ; $p_4 = 5$ states

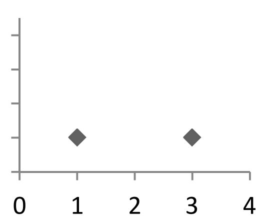
Eq.(14) reads

$$n_1 + 2n_2 + 3n_3 + 4n_4 = 4 \qquad (40)$$

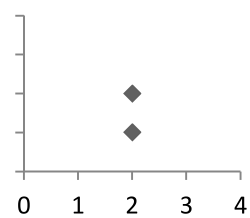
Diagrams



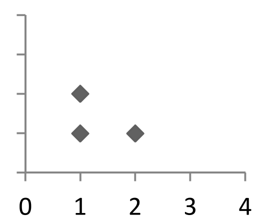
N=1



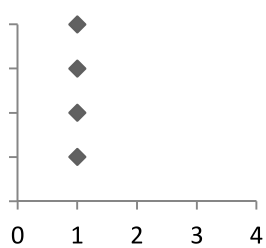
N=2



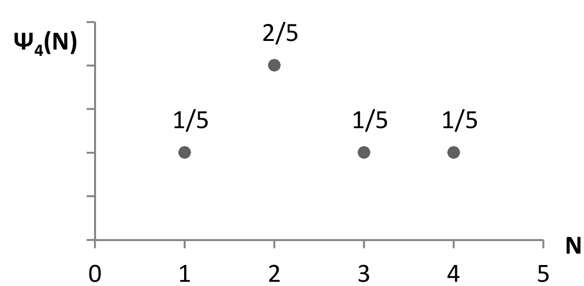
N=2



N=3



N=4



From Eqs (19,20) we have

$$\langle N\rangle_4 = 1\cdot\frac{1}{5} + 2\cdot\frac{2}{5} + 3\cdot\frac{1}{5} + 4\cdot\frac{1}{5} = \frac{12}{5}\ (ph) \tag{41}$$

$$\langle \kappa\rangle_4 = 4\left\{\frac{1}{1}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{1}{5}\right\} = \frac{31}{15}\ (qu/ph) \tag{42}$$

Conditional probabilities

Using the diagrams we calculate g(n/κ); κ=1,2,3,4 and express the results in terms of partitions:

κ=1

$$g(0/1) = \frac{p_4 - p_3}{p_4} = \frac{2}{5} \qquad g(1/1) = \frac{p_3 - p_2}{p_4} = \frac{1}{5} \qquad g(2/1) = \frac{p_2 - p_1}{p_4} = \frac{1}{5}$$

$$g(3/1) = \frac{p_1 - p_0}{p_4} = 0 \qquad g(4/1) = \frac{p_0}{p_4} = \frac{1}{5} \qquad \text{Norm.} \sum_{n=0}^{4} g(n/1) = 1 \tag{43}$$

$$\langle n_1\rangle = \sum_{n=0}^{4} n g(n/1) = \frac{1}{p_4}(p_3 + p_2 + p_1 + p_0) = \frac{7}{5}$$

$$\langle q_1\rangle = 1\cdot\langle n_1\rangle = \frac{1}{p_4}(p_3 + p_2 + p_1 + p_0) = \frac{7}{5}$$

κ=2

$$g(0/2) = \frac{p_4 - p_2}{p_4} = \frac{3}{5} \qquad g(1/2) = \frac{p_2 - p_0}{p_4} = \frac{1}{5} \qquad g(2/2) = \frac{p_0}{p_4} = \frac{1}{5}$$

$$\text{Norm.} \sum_{n=0}^{2} g(n/2) = 1 \tag{44}$$

$$\langle n_2\rangle = \sum_{n=0}^{2} n g(n/2) = \frac{1}{p_4}(p_2 + p_0) = \frac{3}{5} \qquad \langle q_2\rangle = 2\cdot\langle n_2\rangle = \frac{2}{p_4}(p_2 + p_0) = \frac{6}{5}$$

κ=3

$$g(0/3) = \frac{p_4 - p_1}{p_4} = \frac{4}{5} \qquad g(1/3) = \frac{p_1}{p_4} = \frac{1}{5} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/3) = 1 \tag{45}$$

$$\langle n_3\rangle = \sum_{n=0}^{1} n g(n/3) = \frac{p_1}{p_4} = \frac{1}{5} \qquad \langle q_3\rangle = 3\cdot\langle n_3\rangle = \frac{3p_1}{p_4} = \frac{3}{5}$$

κ=4

$$g(0/4)=\frac{p_4-p_0}{p_4}=\frac{4}{5}\qquad g(1/4)=\frac{p_0}{p_4}=\frac{1}{5}\qquad \text{Norm.}\sum_{n=0}^{1}g(n/4)=1 \tag{46}$$

$$\langle n_4\rangle = \sum_{n=0}^{1}ng(n/4)=\frac{p_0}{p_4}=\frac{1}{5}\qquad \langle q_4\rangle = 4\cdot\langle n_4\rangle=\frac{4p_0}{p_4}=\frac{4}{5}$$

Also, we obtain the average number of photons existing in a system of s=4 quanta, given also by Eq.(41), as

$$\langle N\rangle_4=\langle n_1\rangle+\langle n_2\rangle+\langle n_3\rangle+\langle n_4\rangle=\frac{1}{p_4}(p_3+2p_2+2p_1+3p_0)=\frac{12}{5}\ (ph) \tag{47}$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4 according to Eq.(24) reads

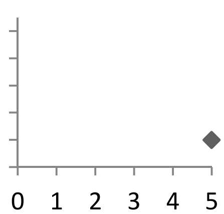
$$s=\langle q_1\rangle+\langle q_2\rangle+\langle q_3\rangle+\langle q_4\rangle=\frac{1}{p_4}(p_3+3p_2+4p_1+7p_0)=4\ (qu) \tag{48}$$

s = 5 quanta ; $p_5 = 7$ states

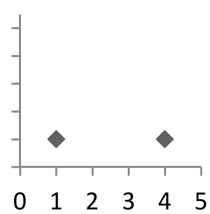
Eq.(14) reads

$$n_1+2n_2+3n_3+4n_4+5n_5=5 \tag{49}$$

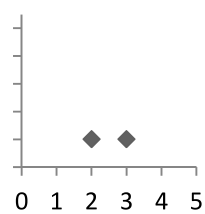
Diagrams



N=1



N=2



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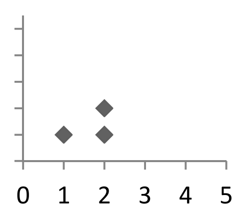

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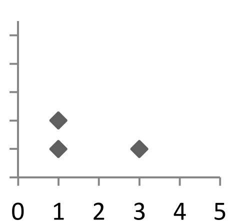
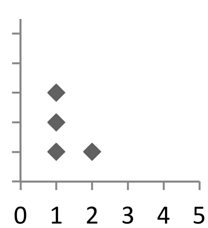
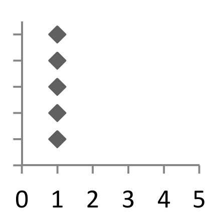


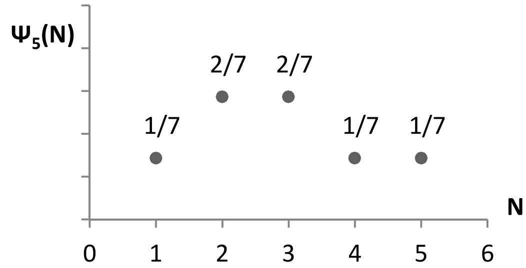
N=3



N=4



N=5



From Eqs (19,20) we have

$$\langle N\rangle_5 = 1\cdot\frac{1}{7} + 2\cdot\frac{2}{7} + 3\cdot\frac{2}{7} + 4\cdot\frac{1}{7} + 5\cdot\frac{1}{7} = \frac{20}{7}\ (ph) \tag{50}$$

$$\langle \kappa\rangle_5 = 5\left\{\frac{1}{1}\cdot\frac{1}{7} + \frac{1}{2}\cdot\frac{2}{7} + \frac{1}{3}\cdot\frac{2}{7} + \frac{1}{4}\cdot\frac{1}{7} + \frac{1}{5}\cdot\frac{1}{7}\right\} = \frac{187}{84}\ (qu/ph) \tag{51}$$

Conditional probabilities

Using the diagrams we calculate g(n/κ); κ=1,2,3,4,5 and express the results in terms of partitions:

κ=1

$g\,(0/1) = \dfrac{p_5 - p_4}{p_5} = \dfrac{2}{7}$ $\quad g\,(1/1) = \dfrac{p_4 - p_3}{p_5} = \dfrac{2}{7}$ $\quad g\,(2/1) = \dfrac{p_3 - p_2}{p_5} = \dfrac{1}{7}$

$g\,(3/1) = \dfrac{p_2 - p_1}{p_5} = \dfrac{1}{7}$ $\quad g\,(4/1) = \dfrac{p_1 - p_0}{p_5} = 0$ $\quad g\,(5/1) = \dfrac{p_0}{p_5} = \dfrac{1}{7}$

$$\text{Norm.} \sum_{n=0}^{5} g(n/1) = 1 \tag{52}$$

$$\langle n_1\rangle = \sum_{n=0}^{5} n\, g(n/1) = \frac{1}{p_5}(p_4 + p_3 + p_2 + p_1 + p_0) = \frac{12}{7}$$

$$\langle q_1\rangle = 1\cdot\langle n_1\rangle = \frac{1}{p_5}(p_4 + p_3 + p_2 + p_1 + p_0) = \frac{12}{7}$$

κ=2

$$g\,(0/2) = \frac{p_5 - p_3}{p_5} = \frac{4}{7} \qquad g\,(1/2) = \frac{p_3 - p_1}{p_5} = \frac{2}{7} \qquad g\,(2/2) = \frac{p_1}{p_5} = \frac{1}{7}$$

$$Norm. \sum_{n=0}^{2} g(n/2) = 1 \tag{53}$$

$$\langle n_2 \rangle = \sum_{n=0}^{2} n\, g(n/2) = \frac{1}{p_5}(p_3 + p_1) = \frac{4}{7} \qquad \langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2}{p_5}(p_3 + p_1) = \frac{8}{7}$$

κ=3

$$g\,(0/3) = \frac{p_5 - p_2}{p_5} = \frac{5}{7} \qquad g\,(1/3) = \frac{p_2}{p_5} = \frac{2}{7} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/3) = 1$$

$$\langle n_3 \rangle = \sum_{n=0}^{1} n\, g(n/3) = \frac{p_2}{p_5} = \frac{2}{7} \qquad \langle q_3 \rangle = 3 \cdot \langle n_3 \rangle = \frac{3p_2}{p_5} = \frac{6}{7} \tag{54}$$

κ=4

$$g\,(0/4) = \frac{p_5 - p_1}{p_5} = \frac{6}{7} \qquad g\,(1/4) = \frac{p_1}{p_5} = \frac{1}{7} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/4) = 1$$

$$\langle n_4 \rangle = \sum_{n=0}^{1} n\, g(n/4) = \frac{p_1}{p_5} = \frac{1}{7} \qquad \langle q_4 \rangle = 4 \cdot \langle n_4 \rangle = \frac{4p_1}{p_5} = \frac{4}{7} \tag{55}$$

κ=5

$$g\,(0/5) = \frac{p_5 - p_0}{p_5} = \frac{6}{7} \qquad g\,(1/5) = \frac{p_0}{p_5} = \frac{1}{7} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/5) = 1$$

$$\langle n_5 \rangle = \sum_{n=0}^{1} n\, g(n/5) = \frac{p_0}{p_5} = \frac{1}{7} \qquad \langle q_5 \rangle = 5 \cdot \langle n_5 \rangle = \frac{5p_0}{p_5} = \frac{5}{7} \tag{56}$$

Also, we obtain the average number of photons existing in a system of s=5 quanta, given also by Eq.(50), as

$$\langle N \rangle_5 = \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle + \langle n_4 \rangle + \langle n_5 \rangle = \frac{1}{p_5}(p_4 + 2p_3 + 2p_2 + 3p_1 + 2p_0) = \frac{20}{7}\ (ph) \tag{57}$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4,5 according to Eq.(24) reads

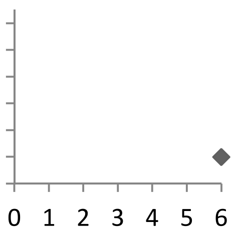
$$s = \langle q_1 \rangle + \langle q_2 \rangle + \langle q_3 \rangle + \langle q_4 \rangle + \langle q_5 \rangle = \frac{1}{p_5}(p_4 + 3p_3 + 4p_2 + 7p_1 + 6p_0) = 5\ (qu) \qquad (58)$$

s = 6 quanta ; $p_6 = 11$ states

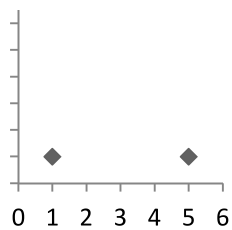
Eq.(14) reads

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 = 6 \qquad (59)$$

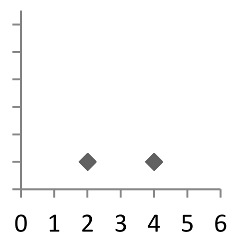
Diagrams



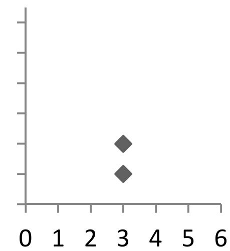
N=1



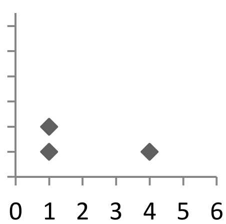
N=2



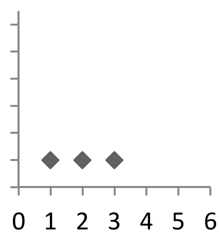
N=2



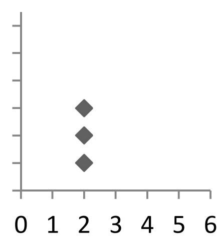
N=2



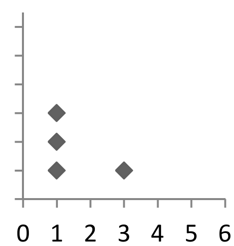
N=3



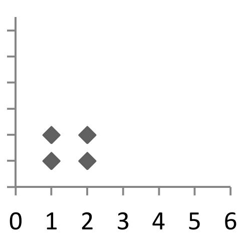
N=3



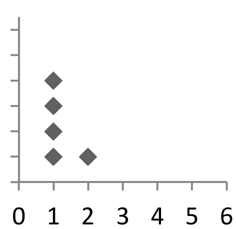
N=3



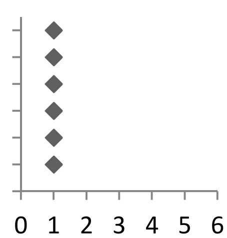
N=4



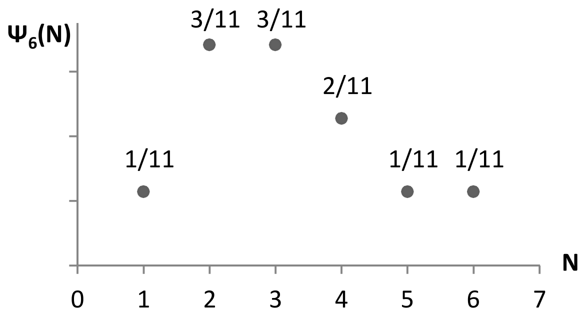
N=4



N=5



N=6



From Eqs (19,20) we have

$$\langle N\rangle_6 = 1\cdot\frac{1}{11}+2\cdot\frac{3}{11}+3\cdot\frac{3}{11}+4\cdot\frac{2}{11}+5\cdot\frac{1}{11}+6\cdot\frac{1}{11}=\frac{35}{11}(ph) \tag{60}$$

$$\langle\kappa\rangle_6 = 6\left\{\frac{1}{1}\cdot\frac{1}{11}+\frac{1}{2}\cdot\frac{3}{11}+\frac{1}{3}\cdot\frac{3}{11}+\frac{1}{4}\cdot\frac{2}{11}+\frac{1}{5}\cdot\frac{1}{11}+\frac{1}{6}\cdot\frac{1}{11}\right\}=\frac{131}{55}\ (qu/ph) \tag{61}$$

Conditional probabilities

Using the diagrams we calculate g(n/κ); κ=1,2,3,4,5,6 and express the results in terms of partitions:

κ=1

$$\mathrm{g}\,(0/1) = \frac{p_6-p_5}{p_6}=\frac{4}{11} \qquad \mathrm{g}\,(1/1) = \frac{p_5-p_4}{p_6}=\frac{2}{11} \qquad \mathrm{g}\,(2/1) = \frac{p_4-p_3}{p_6}=\frac{2}{11}$$

$$\mathrm{g}\,(3/1) = \frac{p_3-p_2}{p_6}=\frac{1}{11} \qquad \mathrm{g}\,(4/1) = \frac{p_2-p_1}{p_6}=\frac{1}{11} \qquad \mathrm{g}\,(5/1) = \frac{p_1-p_0}{p_6}=0$$

$$\mathrm{g}\,(6/1) = \frac{p_0}{p_6}=\frac{1}{11} \qquad \text{Norm.}\sum_{n=0}^{6} g(n/1) = 1 \tag{62}$$

$$\langle n_1\rangle = \sum_{n=0}^{6} n\, g(n/1) = \frac{1}{p_6}(p_5+p_4+p_3+p_2+p_1+p_0)=\frac{19}{11}$$

$$\langle q_1\rangle = 1\cdot\langle n_1\rangle = \frac{1}{p_6}(p_5+p_4+p_3+p_2+p_1+p_0)=\frac{19}{11}$$

κ=2

$$\mathrm{g}\,(0/2) = \frac{p_6-p_4}{p_6}=\frac{6}{11} \qquad \mathrm{g}\,(1/2) = \frac{p_4-p_2}{p_6}=\frac{3}{11} \qquad \mathrm{g}\,(2/2) = \frac{p_2-p_0}{p_6}=\frac{1}{11}$$

$$\mathrm{g}\,(3/2) = \frac{p_0}{p_6} = \frac{1}{11} \qquad Norm. \sum_{n=0}^{3} g(n/2) = 1 \tag{63}$$

$$\langle n_2 \rangle = \sum_{n=0}^{3} n\, g(n/2) = \frac{1}{p_6}(p_4 + p_2 + p_0) = \frac{8}{11}$$

$$\langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2}{p_6}(p_4 + p_2 + p_0) = \frac{16}{11}$$

κ=3

$$\mathrm{g}\,(0/3) = \frac{p_6 - p_3}{p_6} = \frac{8}{11} \qquad \mathrm{g}\,(1/3) = \frac{p_3 - p_0}{p_6} = \frac{2}{11} \qquad \mathrm{g}\,(2/3) = \frac{p_0}{p_6} = \frac{1}{11}$$

$$Norm. \sum_{n=0}^{2} g(n/3) = 1 \tag{64}$$

$$\langle n_3 \rangle = \sum_{n=0}^{2} n\, g(n/3) = \frac{1}{p_6}(p_3 + p_0) = \frac{4}{11} \qquad \langle q_3 \rangle = 3 \cdot \langle n_3 \rangle = \frac{3}{p_6}(p_3 + p_0) = \frac{12}{11}$$

κ=4

$$\mathrm{g}\,(0/4) = \frac{p_6 - p_2}{p_6} = \frac{9}{11} \qquad \mathrm{g}\,(1/4) = \frac{p_2}{p_6} = \frac{2}{11} \qquad \mathrm{Norm.} \sum_{n=0}^{1} g(n/4) = 1$$

$$\langle n_4 \rangle = \sum_{n=0}^{1} n\, g(n/4) = \frac{p_2}{p_6} = \frac{2}{11} \qquad \langle q_4 \rangle = 4 \cdot \langle n_4 \rangle = \frac{4p_2}{p_6} = \frac{8}{11} \tag{65}$$

κ=5

$$\mathrm{g}\,(0/5) = \frac{p_6 - p_1}{p_6} = \frac{10}{11} \qquad \mathrm{g}\,(1/5) = \frac{p_1}{p_6} = \frac{1}{11} \qquad \mathrm{Norm.} \sum_{n=0}^{1} g(n/5) = 1$$

$$\langle n_5 \rangle = \sum_{n=0}^{1} n\, g(n/5) = \frac{p_1}{p_6} = \frac{1}{11} \qquad \langle q_5 \rangle = 5 \cdot \langle n_5 \rangle = \frac{5p_1}{p_6} = \frac{5}{11} \tag{66}$$

κ=6

$$\mathrm{g}\,(0/6) = \frac{p_6 - p_0}{p_6} = \frac{10}{11} \qquad \mathrm{g}\,(1/6) = \frac{p_0}{p_6} = \frac{1}{11} \qquad \mathrm{Norm.} \sum_{n=0}^{1} g(n/6) = 1$$

$$\langle n_6 \rangle = \sum_{n=0}^{1} n\, g(n/6) = \frac{p_0}{p_6} = \frac{1}{11} \qquad \langle q_6 \rangle = 6 \cdot \langle n_6 \rangle = \frac{6p_0}{p_6} = \frac{6}{11} \tag{67}$$

Also, we obtain the average number of photons existing in a system of s=6 quanta, given also by Eq.(60), as

$$\langle N\rangle_6 = \langle n_1\rangle + \langle n_2\rangle + \langle n_3\rangle + \langle n_4\rangle + \langle n_5\rangle + \langle n_6\rangle$$
$$= \frac{1}{p_6}(p_5 + 2p_4 + 2p_3 + 3p_2 + 2p_1 + 4p_0) = \frac{35}{11}\ (ph) \tag{68}$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4,5,6 according to Eq.(24) reads

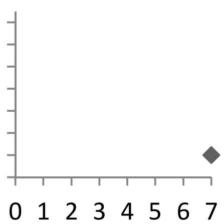
$$\mathrm{s} = \langle q_1\rangle + \langle q_2\rangle + \langle q_3\rangle + \langle q_4\rangle + \langle q_5\rangle + \langle q_6\rangle$$
$$= \frac{1}{p_6}(p_5 + 3p_4 + 4p_3 + 7p_2 + 6p_1 + 12p_0) = 6\ (qu) \tag{69}$$

s = 7 quanta ; $p_7 = 15$ states

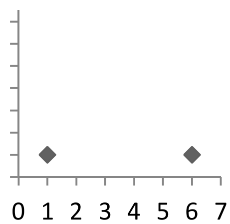
Eq.(14) reads

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 = 7 \tag{70}$$

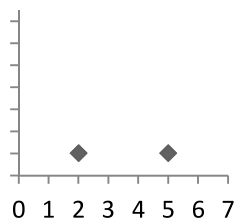
Diagrams



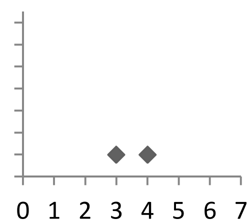
N=1



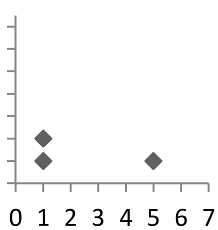
N=2



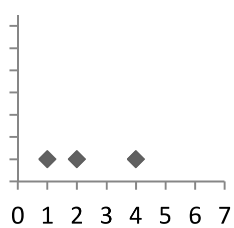
N=2



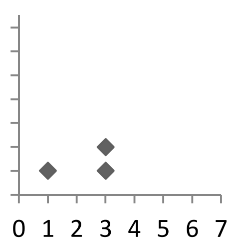
N=2



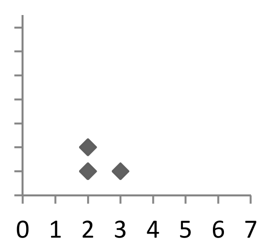
N=3



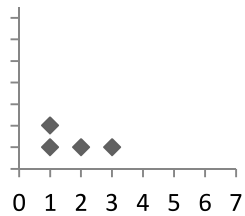
N=3



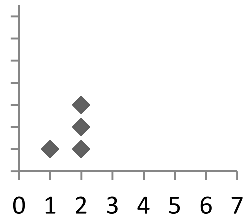
N=3



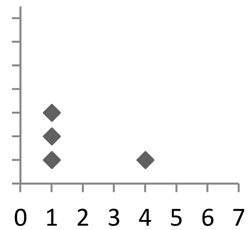
N=3



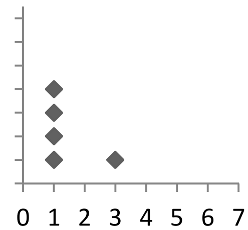
N=4



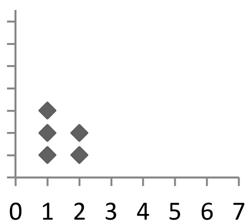
N=4



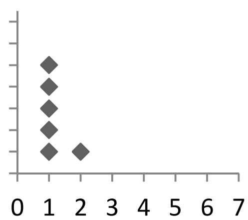
N=4



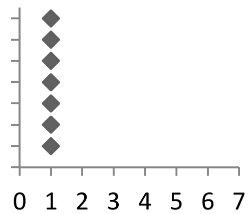
N=5



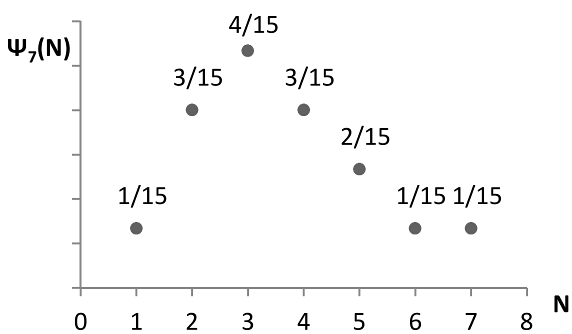
N=5



N=6



N=7



From Eqs (19,20) we have

$$\langle N\rangle_7 = 1\cdot\frac{1}{15} + 2\cdot\frac{3}{15} + 3\cdot\frac{4}{15} + 4\cdot\frac{3}{15} + 5\cdot\frac{2}{15} + 6\,\frac{1}{15} + 7\,\frac{1}{15} = \frac{54}{15}\,(ph) \tag{71}$$

$$\langle\kappa\rangle_7 = 7\left\{\frac{1}{1}\cdot\frac{1}{15} + \frac{1}{2}\cdot\frac{3}{15} + \frac{1}{3}\cdot\frac{4}{15} + \frac{1}{4}\cdot\frac{3}{15} + \frac{1}{5}\cdot\frac{2}{15} + \frac{1}{6}\cdot\frac{1}{15} + \frac{1}{7}\cdot\frac{1}{15}\right\} = \frac{247}{100}\ (qu/ph) \tag{72}$$

Conditional probabilities

Using the diagrams we calculate g(n/κ); κ=1,2,3,4,5,6,7 and express the results in terms of partitions:

κ=1

$$g\,(0/1) = \frac{p_7 - p_6}{p_7} = \frac{4}{15} \qquad g\,(1/1) = \frac{p_6 - p_5}{p_7} = \frac{4}{15} \qquad g\,(2/1) = \frac{p_5 - p_4}{p_7} = \frac{2}{15}$$

$$g\,(3/1) = \frac{p_4 - p_3}{p_7} = \frac{2}{15} \qquad g\,(4/1) = \frac{p_3 - p_2}{p_7} = \frac{1}{15} \qquad g\,(5/1) = \frac{p_2 - p_1}{p_7} = \frac{1}{15}$$

$$g\,(6/1) = \frac{p_1 - p_0}{p_7} = 0 \qquad g(7/1) = \frac{p_0}{p_7} = \frac{1}{15} \qquad \text{Norm.} \sum_{n=0}^{7} g(n/1) = 1 \tag{73}$$

$$\langle n_1 \rangle = \sum_{n=0}^{7} n\, g(n/1) = \frac{1}{p_7}(p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{30}{15}$$

$$\langle q_1 \rangle = 1 \cdot \langle n_1 \rangle = \frac{1}{p_7}(p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{30}{15}$$

κ=2

$$g\,(0/2) = \frac{p_7 - p_5}{p_7} = \frac{8}{15} \qquad g\,(1/2) = \frac{p_5 - p_3}{p_7} = \frac{4}{15} \qquad g\,(2/2) = \frac{p_3 - p_1}{p_7} = \frac{2}{15}$$

$$g\,(3/2) = \frac{p_1}{p_7} = \frac{1}{15} \qquad \text{Norm.} \sum_{n=0}^{3} g(n/2) = 1 \tag{74}$$

$$\langle n_2 \rangle = \sum_{n=0}^{3} n\, g(n/2) = \frac{1}{p_7}(p_5 + p_3 + p_1) = \frac{11}{15}$$

$$\langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2}{p_7}(p_5 + p_3 + p_1) = \frac{22}{15}$$

κ=3

$$g\,(0/3) = \frac{p_7 - p_4}{p_7} = \frac{10}{15} \qquad g\,(1/3) = \frac{p_4 - p_1}{p_7} = \frac{4}{15} \qquad g\,(2/3) = \frac{p_1}{p_7} = \frac{1}{15}$$

$$\text{Norm.} \sum_{n=0}^{2} g(n/3) = 1 \tag{75}$$

$$\langle n_3 \rangle = \sum_{n=0}^{2} n\, g(n/3) = \frac{1}{p_7}(p_4 + p_1) = \frac{6}{15} \qquad \langle q_3 \rangle = 3 \cdot \langle n_3 \rangle = \frac{3}{p_7}(p_4 + p_1) = \frac{18}{15}$$

κ=4

$$g\,(0/4) = \frac{p_7 - p_3}{p_7} = \frac{12}{15} \qquad g\,(1/4) = \frac{p_3}{p_7} = \frac{3}{15} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/4) = 1$$

$$\langle n_4 \rangle = \sum_{n=0}^{1} n\, g(n/4) = \frac{p_3}{p_7} = \frac{3}{15} \qquad \langle q_4 \rangle = 4 \cdot \langle n_4 \rangle = \frac{4p_3}{p_7} = \frac{12}{15} \tag{76}$$

κ=5

$$\text{g}\,(0/5) = \frac{p_7 - p_2}{p_7} = \frac{13}{15} \qquad \text{g}\,(1/5) = \frac{p_2}{p_7} = \frac{2}{15} \qquad Norm. \sum_{n=0}^{1} g(n/5) = 1$$

$$\langle n_5 \rangle = \sum_{n=0}^{1} n\, g(n/5) = \frac{p_2}{p_7} = \frac{2}{15} \qquad \langle q_5 \rangle = 5 \cdot \langle n_5 \rangle = \frac{5p_2}{p_7} = \frac{10}{15} \tag{77}$$

κ=6

$$\text{g}\,(0/6) = \frac{p_7 - p_1}{p_7} = \frac{14}{15} \qquad \text{g}\,(1/6) = \frac{p_1}{p_7} = \frac{1}{15} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/6) = 1$$

$$\langle n_6 \rangle = \sum_{n=0}^{1} n\, g(n/6) = \frac{p_1}{p_7} = \frac{1}{15} \qquad \langle q_6 \rangle = 6 \cdot \langle n_6 \rangle = \frac{6p_1}{p_7} = \frac{6}{15} \tag{78}$$

κ=7

$$\text{g}\,(0/7) = \frac{p_7 - p_0}{p_7} = \frac{14}{15} \qquad \text{g}\,(1/7) = \frac{p_0}{p_7} = \frac{1}{15} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/7) = 1$$

$$\langle n_7 \rangle = \sum_{n=0}^{1} n\, g(n/7) = \frac{p_0}{p_7} = \frac{1}{15} \qquad \langle q_7 \rangle = 7 \cdot \langle n_7 \rangle = \frac{7p_0}{p_7} = \frac{7}{15} \tag{79}$$

Also, we obtain the average number of photons existing in a system of s=7 quanta, given also by Eq.(71), as

$$\begin{aligned}\langle N \rangle_7 &= \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle + \langle n_4 \rangle + \langle n_5 \rangle + \langle n_6 \rangle + \langle n_7 \rangle \\ &= \frac{1}{p_7}(p_6 + 2p_5 + 2p_4 + 3p_3 + 2p_2 + 4p_1 + 2p_0) = \frac{54}{15}\ (ph)\end{aligned} \tag{80}$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4,5,6,7 according to Eq. (24) reads

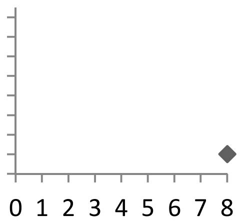
$$\begin{aligned}\text{s} &= \langle q_1 \rangle + \langle q_2 \rangle + \langle q_3 \rangle + \langle q_4 \rangle + \langle q_5 \rangle + \langle q_6 \rangle + \langle q_7 \rangle \\ &= \frac{1}{p_7}(p_6 + 3p_5 + 4p_4 + 7p_3 + 6p_2 + 12p_1 + 8p_0) = 7\ (qu)\end{aligned} \tag{81}$$

$s = 8$ quanta ; $p_8 = 22$ states

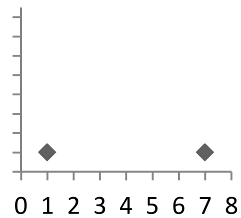
Eq.(14) reads

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 + 8n_8 = 8 \tag{82}$$

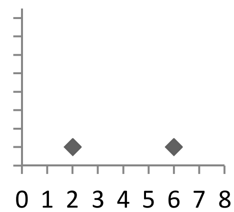
Diagrams



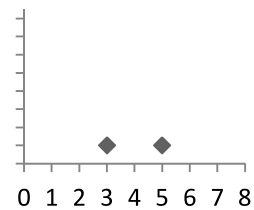
N=1



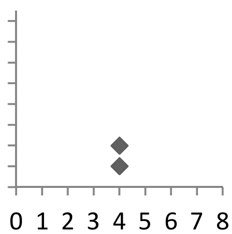
N=2



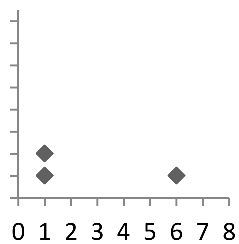
N=2



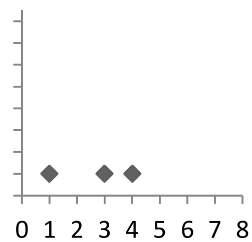
N=2



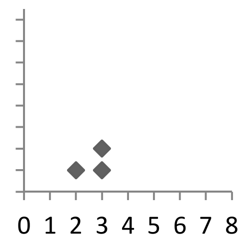
N=2



N=3



N=3



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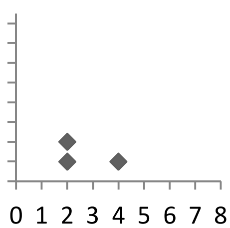

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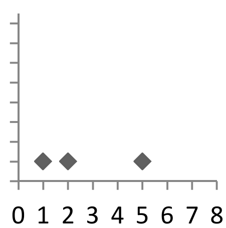
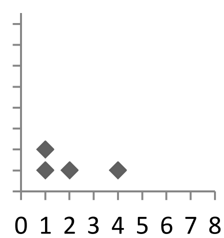
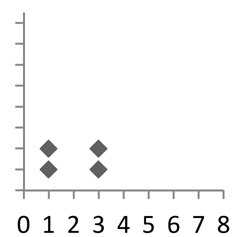


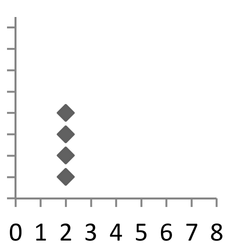
N=3



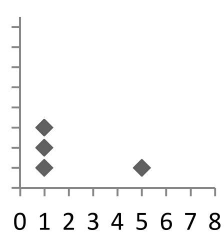
N=4



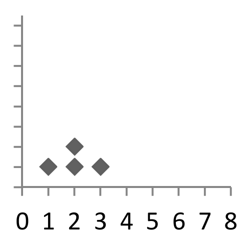
N=4



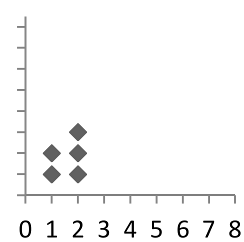
N=4



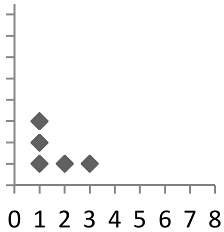
N=4



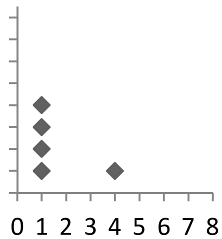
N=4



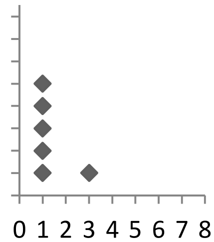
N=5



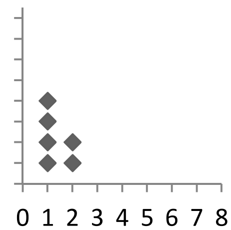
N=5



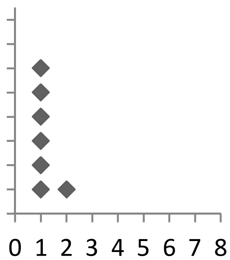
N=5



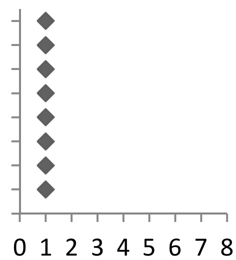
N=6



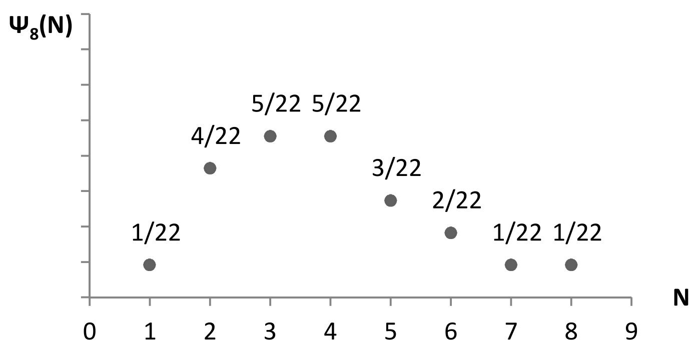
N=6



N=7



N=8



From Eqs (19,20) we have

$$\langle N\rangle_8 = 1\cdot\frac{1}{22}+2\cdot\frac{4}{22}+3\cdot\frac{5}{22}+4\cdot\frac{5}{22}+5\cdot\frac{3}{22}+6\cdot\frac{2}{22}+7\cdot\frac{1}{22}+8\cdot\frac{1}{22}=\frac{86}{22}(ph) \qquad (83)$$

$$\langle \kappa\rangle_8 = 8\left\{\frac{1}{1}\cdot\frac{1}{22}+\frac{1}{2}\cdot\frac{4}{22}+\frac{1}{3}\cdot\frac{5}{22}+\frac{1}{4}\cdot\frac{5}{22}+\frac{1}{5}\cdot\frac{3}{22}+\frac{1}{6}\cdot\frac{2}{22}+\frac{1}{7}\cdot\frac{1}{22}+\frac{1}{8}\cdot\frac{1}{22}\right\}=\frac{1993}{770}\ (qu/ph) \qquad (84)$$

Conditional probabilities

Using the diagrams we calculate $g(n/\kappa)$; $\kappa=1,2,3,4,5,6,7,8$ and express the results in terms of partitions:

κ=1

$$g\,(0/1) = \frac{p_8 - p_7}{p_8} = \frac{7}{22} \qquad g\,(1/1) = \frac{p_7 - p_6}{p_8} = \frac{4}{22} \qquad g\,(2/1) = \frac{p_6 - p_5}{p_8} = \frac{4}{22}$$

$$g\,(3/1) = \frac{p_5 - p_4}{p_8} = \frac{2}{22} \qquad g\,(4/1) = \frac{p_4 - p_3}{p_8} = \frac{2}{22} \qquad g\,(5/1) = \frac{p_3 - p_2}{p_8} = \frac{1}{22}$$

$$g\,(6/1) = \frac{p_2 - p_1}{p_8} = \frac{1}{22} \qquad g(7/1) = \frac{p_1 - p_0}{p_8} = 0 \qquad g(8/1) = \frac{p_0}{p_8} = \frac{1}{22}$$

$$\text{Norm.} \sum_{n=0}^{8} g(n/1) = 1 \tag{85}$$

$$\langle n_1 \rangle = \sum_{n=0}^{8} n\, g(n/1) = \frac{1}{p_8}(p_7 + p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{45}{22}$$

$$\langle q_1 \rangle = 1 \cdot \langle n_1 \rangle = \frac{1}{p_8}(p_7 + p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{45}{22}$$

κ=2

$$g\,(0/2) = \frac{p_8 - p_6}{p_8} = \frac{11}{22} \qquad g\,(1/2) = \frac{p_6 - p_4}{p_8} = \frac{6}{22} \qquad g\,(2/2) = \frac{p_4 - p_2}{p_8} = \frac{3}{22}$$

$$g\,(3/2) = \frac{p_2 - p_0}{p_8} = \frac{1}{22} \qquad g\,(4/2) = \frac{p_0}{p_8} = \frac{1}{22} \qquad \text{Norm.} \sum_{n=0}^{4} g(n/2) = 1 \tag{86}$$

$$\langle n_2 \rangle = \sum_{n=0}^{4} n\, g(n/2) = \frac{1}{p_8}(p_6 + p_4 + p_2 + p_0) = \frac{19}{22}$$

$$\langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2}{p_8}(p_6 + p_4 + p_2 + p_0) = \frac{38}{22}$$

κ=3

$$g\,(0/3) = \frac{p_8 - p_5}{p_8} = \frac{15}{22} \qquad g\,(1/3) = \frac{p_5 - p_2}{p_8} = \frac{5}{22} \qquad g\,(2/3) = \frac{p_2}{p_8} = \frac{2}{22}$$

$$\text{Norm.} \sum_{n=0}^{2} g(n/3) = 1 \tag{87}$$

$$\langle n_3 \rangle = \sum_{n=0}^{2} n\, g(n/3) = \frac{1}{p_8}(p_5 + p_2) = \frac{9}{22} \qquad \langle q_3 \rangle = 3 \cdot \langle n_3 \rangle = \frac{3}{p_8}(p_5 + p_2) = \frac{27}{22}$$

κ=4

$$g\,(0/4) = \frac{p_8 - p_4}{p_8} = \frac{17}{22} \qquad g\,(1/4) = \frac{p_4 - p_0}{p_8} = \frac{4}{22} \qquad g\,(2/4) = \frac{p_0}{p_8} = \frac{1}{22}$$

$$\text{Norm.} \sum_{n=0}^{2} g(n/4) = 1 \tag{88}$$

$$\langle n_4 \rangle = \sum_{n=0}^{2} n\, g(n/4) = \frac{1}{p_8}(p_4 + p_0) = \frac{6}{22} \qquad \langle q_4 \rangle = 4 \cdot \langle n_4 \rangle = \frac{4}{p_8}(p_4 + p_0) = \frac{24}{22}$$

κ=5

$$g\,(0/5) = \frac{p_8 - p_3}{p_8} = \frac{19}{22} \qquad g\,(1/5) = \frac{p_3}{p_8} = \frac{3}{22} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/5) = 1 \tag{89}$$

$$\langle n_5 \rangle = \sum_{n=0}^{1} n\, g(n/5) = \frac{p_3}{p_8} = \frac{3}{22} \qquad \langle q_5 \rangle = 5 \cdot \langle n_5 \rangle = \frac{5p_3}{p_8} = \frac{15}{22}$$

κ=6

$$g\,(0/6) = \frac{p_8 - p_2}{p_8} = \frac{20}{22} \qquad g\,(1/6) = \frac{p_2}{p_8} = \frac{2}{22} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/6) = 1 \tag{90}$$

$$\langle n_6 \rangle = \sum_{n=0}^{1} n\, g(n/6) = \frac{p_2}{p_8} = \frac{2}{22} \qquad \langle q_6 \rangle = 6 \cdot \langle n_6 \rangle = \frac{6p_2}{p_8} = \frac{12}{22}$$

κ=7

$$g\,(0/7) = \frac{p_8 - p_1}{p_8} = \frac{21}{22} \qquad g\,(1/7) = \frac{p_1}{p_8} = \frac{1}{22} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/7) = 1 \tag{91}$$

$$\langle n_7 \rangle = \sum_{n=0}^{1} n\, g(n/7) = \frac{p_1}{p_8} = \frac{1}{22} \qquad \langle q_7 \rangle = 7 \cdot \langle n_7 \rangle = \frac{7p_1}{p_8} = \frac{7}{22}$$

κ=8

$$g\,(0/8) = \frac{p_8 - p_0}{p_8} = \frac{21}{22} \qquad g\,(1/8) = \frac{p_0}{p_8} = \frac{1}{22} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/8) = 1 \tag{92}$$

$$\langle n_8 \rangle = \sum_{n=0}^{1} n\, g(n/8) = \frac{p_0}{p_8} = \frac{1}{22} \qquad \langle q_8 \rangle = 8 \cdot \langle n_8 \rangle = \frac{8p_0}{p_8} = \frac{8}{22}$$

Also, we obtain the average number of photons existing in a system of s=8 quanta, given also by Eq.(83), as

$$\langle N\rangle_8 = \langle n_1\rangle + \langle n_2\rangle + \langle n_3\rangle + \langle n_4\rangle + \langle n_5\rangle + \langle n_6\rangle + \langle n_7\rangle + \langle n_8\rangle$$
$$= \frac{1}{p_8}(p_7 + 2p_6 + 2p_5 + 3p_4 + 2p_3 + 4p_2 + 2p_1 + 4p_0) = \frac{86}{22}\ (ph) \qquad (93)$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4,5,6,7,8 according to Eq.(24) reads

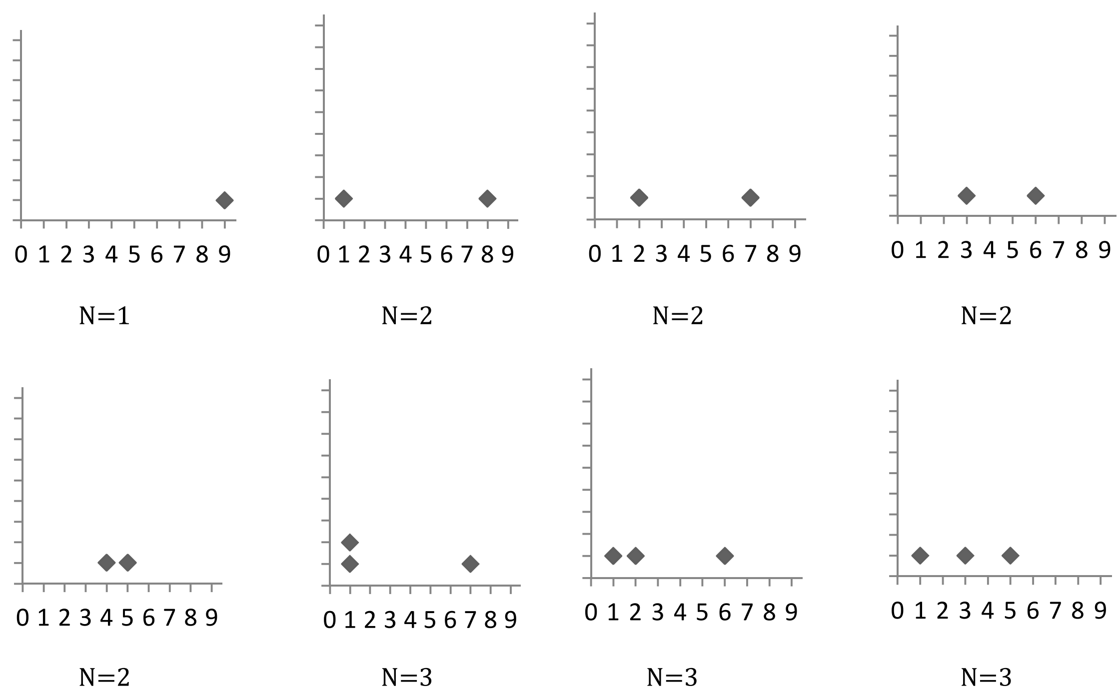
$$s = \langle q_1\rangle + \langle q_2\rangle + \langle q_3\rangle + \langle q_4\rangle + \langle q_5\rangle + \langle q_6\rangle + \langle q_7\rangle + \langle q_8\rangle$$
$$= \frac{1}{p_8}(p_7 + 3p_6 + 4p_5 + 7p_4 + 6p_3 + 12p_2 + 8p_1 + 15p_0) = 8\ (qu) \qquad (94)$$

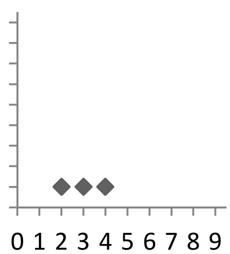
s = 9 quanta ; p_9 = 30 states

Eq.(14) reads

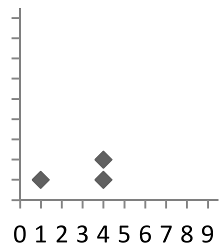
$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 + 8n_8 + 9n_9 = 9 \qquad (95)$$

Diagrams





N=3



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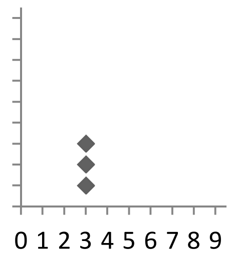

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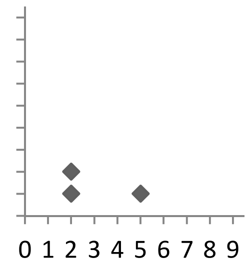

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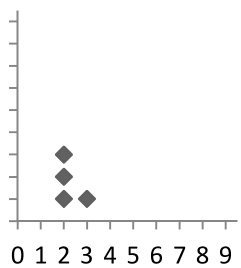
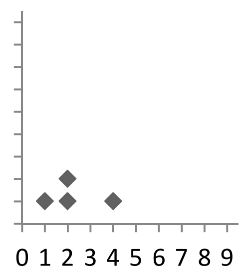


N=4



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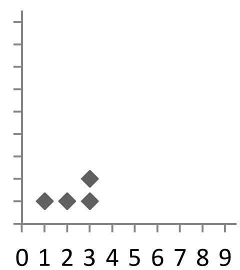

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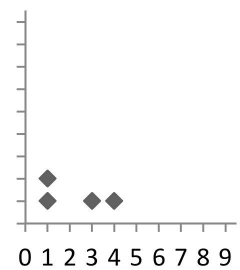

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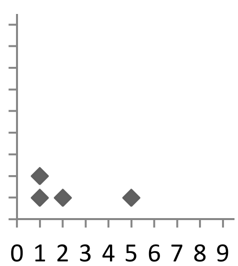

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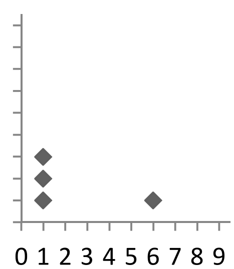

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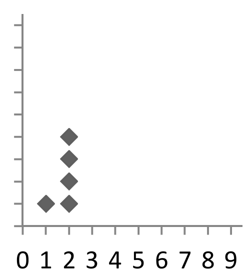
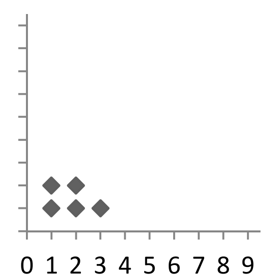


N=5



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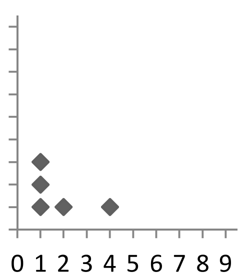

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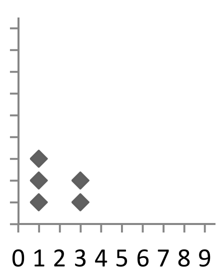

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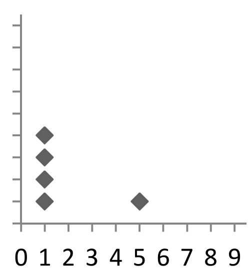
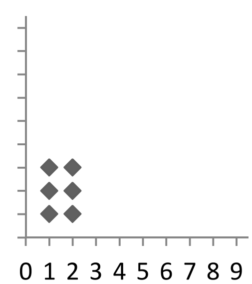
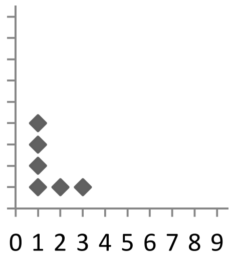


N=5



N=6



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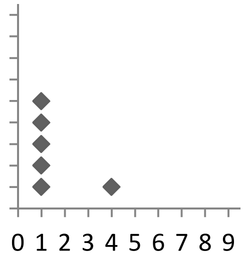
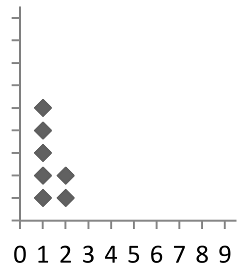
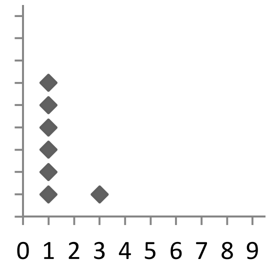


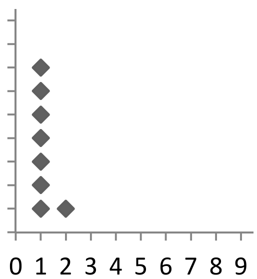
N=6



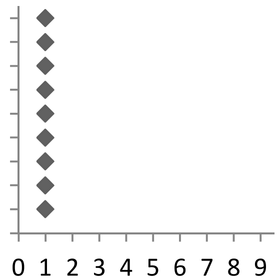
N=7



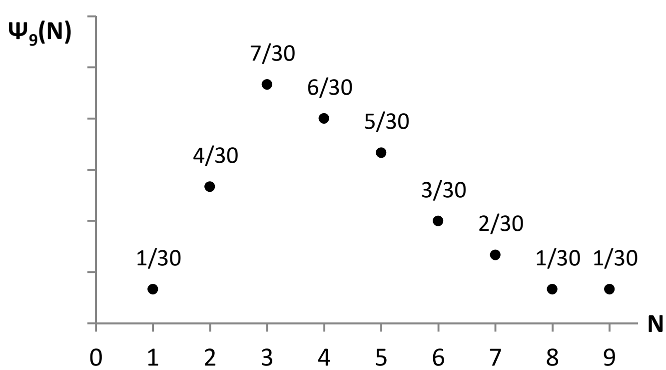
N=7



N=8



N=9



From Eqs (19,20) we have

$$\langle N\rangle_9 = 1\cdot\frac{1}{30}+2\cdot\frac{4}{30}+3\cdot\frac{7}{30}+4\cdot\frac{6}{30}+5\cdot\frac{5}{30}+6\cdot\frac{3}{30}+7\cdot\frac{2}{30}+8\cdot\frac{1}{30}+9\cdot\frac{1}{30}=\frac{128}{30}\,(ph) \qquad (96)$$

$$\langle\kappa\rangle_9 = 9\left\{\frac{1}{1}\cdot\frac{1}{30}+\frac{1}{2}\cdot\frac{4}{30}+\frac{1}{3}\cdot\frac{7}{30}+\frac{1}{4}\cdot\frac{6}{30}+\frac{1}{5}\cdot\frac{5}{30}+\frac{1}{6}\cdot\frac{3}{30}+\frac{1}{7}\cdot\frac{2}{30}+\frac{1}{8}\cdot\frac{1}{30}+\frac{1}{9}\cdot\frac{1}{30}\right\}$$

$$=\frac{4463}{1680}\,(qu/ph) \qquad (97)$$

Conditional probabilities

Using the diagrams we calculate g(n/κ) ; κ=1,2,3,4,5,6,7,8,9 and express the results in terms of partitions:

κ=1

$$g\,(0/1) = \frac{p_9 - p_8}{p_9} = \frac{8}{30} \qquad g\,(1/1) = \frac{p_8 - p_7}{p_9} = \frac{7}{30} \qquad g\,(2/1) = \frac{p_7 - p_6}{p_9} = \frac{4}{30}$$

$$g\,(3/1) = \frac{p_6 - p_5}{p_9} = \frac{4}{30} \qquad g\,(4/1) = \frac{p_5 - p_4}{p_9} = \frac{2}{30} \qquad g\,(5/1) = \frac{p_4 - p_3}{p_9} = \frac{2}{30}$$

$$g\,(6/1) = \frac{p_3 - p_2}{p_9} = \frac{1}{30} \qquad g\,(7/1) = \frac{p_2 - p_1}{p_9} = \frac{1}{30} \qquad g\,(8/1) = \frac{p_1 - p_0}{p_9} = 0$$

$$g\,(9/1) = \frac{p_0}{p_9} = \frac{1}{30} \qquad \text{Norm.} \sum_{n=0}^{9} g(n/1) = 1 \tag{98}$$

$$\langle n_1 \rangle = \sum_{n=0}^{9} n\, g(n/1) = \frac{1}{p_9}(p_8 + p_7 + p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{67}{30}$$

$$\langle q_1 \rangle = 1 \cdot \langle n_1 \rangle = \frac{1}{p_9}(p_8 + p_7 + p_6 + p_5 + p_4 + p_3 + p_2 + p_1 + p_0) = \frac{67}{30}$$

κ=2

$$g\,(0/2) = \frac{p_9 - p_7}{p_9} = \frac{15}{30} \qquad g\,(1/2) = \frac{p_7 - p_5}{p_9} = \frac{8}{30} \qquad g\,(2/2) = \frac{p_5 - p_3}{p_9} = \frac{4}{30}$$

$$g\,(3/2) = \frac{p_3 - p_1}{p_9} = \frac{2}{30} \qquad g\,(4/2) = \frac{p_1}{p_9} = \frac{1}{30} \qquad \text{Norm.} \sum_{n=0}^{4} g(n/2) = 1 \tag{99}$$

$$\langle n_2 \rangle = \sum_{n=0}^{4} n\, g(n/2) = \frac{1}{p_9}(p_7 + p_5 + p_3 + p_1) = \frac{26}{30}$$

$$\langle q_2 \rangle = 2 \cdot \langle n_2 \rangle = \frac{2}{p_9}(p_7 + p_5 + p_3 + p_1) = \frac{52}{30}$$

κ=3

$$g\,(0/3) = \frac{p_9 - p_6}{p_9} = \frac{19}{30} \qquad g\,(1/3) = \frac{p_6 - p_3}{p_9} = \frac{8}{30} \qquad g\,(2/3) = \frac{p_3 - p_0}{p_9} = \frac{2}{30}$$

$$g\,(3/3) = \frac{p_0}{p_9} = \frac{1}{30} \qquad \text{Norm.} \sum_{n=0}^{3} g(n/3) = 1 \tag{100}$$

$$\langle n_3 \rangle \ = \ \sum_{n=0}^{3} n\, g(n/3) = \frac{1}{p_9}(p_6 + p_3 + p_0) = \frac{15}{30}$$

$$\langle q_3 \rangle \ = \ 3 \cdot \langle n_3 \rangle \ = \ \frac{3}{p_9}(p_6 + p_3 + p_0) = \frac{45}{30}$$

κ=4

$$\mathrm{g}\,(0/4) = \ \frac{p_9 - p_5}{p_9} = \frac{23}{30} \qquad \mathrm{g}\,(1/4) = \ \frac{p_5 - p_1}{p_9} = \frac{6}{30} \qquad \mathrm{g}\,(2/4) = \ \frac{p_1}{p_9} = \frac{1}{30}$$

$$\text{Norm.} \sum_{n=0}^{2} g(n/4) = 1 \tag{101}$$

$$\langle n_4 \rangle \ = \ \sum_{n=0}^{2} n\, g(n/4) = \frac{1}{p_9}(p_5 + p_1) = \frac{8}{30} \qquad \langle q_4 \rangle \ = \ 4 \cdot \langle n_4 \rangle \ = \ \frac{4}{p_9}(p_5 + p_1) = \frac{32}{30}$$

κ=5

$$\mathrm{g}\,(0/5) = \ \frac{p_9 - p_4}{p_9} = \frac{25}{30} \qquad \mathrm{g}\,(1/5) = \ \frac{p_4}{p_9} = \frac{5}{30} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/5) = 1 \tag{102}$$

$$\langle n_5 \rangle \ = \ \sum_{n=0}^{1} n\, g(n/5) = \frac{p_4}{p_9} = \frac{5}{30} \qquad \langle q_5 \rangle \ = \ 5 \cdot \langle n_5 \rangle \ = \ \frac{5p_4}{p_9} = \frac{25}{30}$$

κ=6

$$\mathrm{g}\,(0/6) = \ \frac{p_9 - p_3}{p_9} = \frac{27}{30} \qquad \mathrm{g}\,(1/6) = \ \frac{p_3}{p_9} = \frac{3}{30} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/6) = 1 \tag{103}$$

$$\langle n_6 \rangle \ = \ \sum_{n=0}^{1} n\, g(n/6) = \frac{p_3}{p_9} = \frac{3}{30} \qquad \langle q_6 \rangle \ = \ 6 \cdot \langle n_6 \rangle \ = \ \frac{6p_3}{p_9} = \frac{18}{30}$$

κ=7

$$\mathrm{g}\,(0/7) = \ \frac{p_9 - p_2}{p_9} = \frac{28}{30} \qquad \mathrm{g}\,(1/7) = \ \frac{p_2}{p_9} = \frac{2}{30} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/7) = 1 \tag{104}$$

$$\langle n_7 \rangle \ = \ \sum_{n=0}^{1} n\, g(n/7) = \frac{p_2}{p_9} = \frac{2}{30} \qquad \langle q_7 \rangle \ = \ 7 \cdot \langle n_7 \rangle \ = \ \frac{7p_2}{p_9} = \frac{14}{30}$$

κ=8

$$g\,(0/8) = \frac{p_9 - p_1}{p_9} = \frac{29}{30} \qquad g\,(1/8) = \frac{p_1}{p_9} = \frac{1}{30} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/8) = 1 \tag{105}$$

$$\langle n_8 \rangle = \sum_{n=0}^{1} n\, g(n/8) = \frac{p_1}{p_9} = \frac{1}{30} \qquad \langle q_8 \rangle = 8 \cdot \langle n_8 \rangle = \frac{8p_1}{p_9} = \frac{8}{30}$$

κ=9

$$g\,(0/9) = \frac{p_9 - p_0}{p_9} = \frac{29}{30} \qquad g\,(1/9) = \frac{p_0}{p_9} = \frac{1}{30} \qquad \text{Norm.} \sum_{n=0}^{1} g(n/9) = 1 \tag{106}$$

$$\langle n_9 \rangle = \sum_{n=0}^{1} n\, g(n/9) = \frac{p_0}{p_9} = \frac{1}{30} \qquad \langle q_9 \rangle = 9 \cdot \langle n_9 \rangle = \frac{9p_0}{p_9} = \frac{9}{30}$$

Also, we obtain the average number of photons existing in a system of s=9 quanta, given also by Eq.(96), as

$$\begin{aligned}\langle N \rangle_9 &= \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle + \langle n_4 \rangle + \langle n_5 \rangle + \langle n_6 \rangle + \langle n_7 \rangle + \langle n_8 \rangle + \langle n_9 \rangle \\ &= \frac{1}{p_9}(p_8 + 2p_7 + 2p_6 + 3p_5 + 2p_4 + 4p_3 + 2p_2 + 4p_1 + 3p_0) = \frac{128}{30}\ (ph)\end{aligned} \tag{107}$$

and as expected, the sum of the average number of quanta existing in energy levels 1,2,3,4,5,6,7,8,9 according to Eq.(24) reads

$$\begin{aligned}s &= \langle q_1 \rangle + \langle q_2 \rangle + \langle q_3 \rangle + \langle q_4 \rangle + \langle q_5 \rangle + \langle q_6 \rangle + \langle q_7 \rangle + \langle q_8 \rangle + \langle q_9 \rangle \\ &= \frac{1}{p_9}(p_8 + 3p_7 + 4p_6 + 7p_5 + 6p_4 + 12p_3 + 8p_2 + 15p_1 + 13p_0) = 9\ (qu)\end{aligned} \tag{108}$$

4. General theory for arbitrary number s of quanta and the limit $s \to \infty$

Extending the results of the previous section to a system containing an arbitrary number s of quanta, the leading terms of the conditional probability $g(n/\kappa)$ that there are n photons in energy level κ, can be expressed exactly in terms of partitions as shown below:

$\kappa \backslash n$	0	1	2	3
1	$\frac{p_s - p_{s-1}}{p_s}$	$\frac{p_{s-1} - p_{s-2}}{p_s}$	$\frac{p_{s-2} - p_{s-3}}{p_s}$	$\frac{p_{s-3} - p_{s-4}}{p_s}$
2	$\frac{p_s - p_{s-2}}{p_s}$	$\frac{p_{s-2} - p_{s-4}}{p_s}$	$\frac{p_{s-4} - p_{s-6}}{p_s}$	$\frac{p_{s-6} - p_{s-8}}{p_s}$
3	$\frac{p_s - p_{s-3}}{p_s}$	$\frac{p_{s-3} - p_{s-6}}{p_s}$	$\frac{p_{s-6} - p_{s-9}}{p_s}$	$\frac{p_{s-9} - p_{s-12}}{p_s}$
4	$\frac{p_s - p_{s-4}}{p_s}$	$\frac{p_{s-4} - p_{s-8}}{p_s}$	$\frac{p_{s-8} - p_{s-12}}{p_s}$	$\frac{p_{s-12} - p_{s-16}}{p_s}$

Table 2

Therefore, the leading terms of $g(n/\kappa)$ can be written compactly as

$$g\,(n/\kappa) = \frac{p_{s-n\kappa} - p_{s-(n+1)\kappa}}{p_s} \quad ; \quad n = 0,1,2,\dots \quad ; \quad \kappa = 1,2,3,\dots \tag{109}$$

Introducing the Hardy-Ramanujan formula [Eq. (16)] we get the behaviour of $g(n/\kappa)$ for large s :

$$g(n/\kappa) = \frac{\dfrac{a}{s-n\kappa} e^{b\sqrt{s-n\kappa}} - \dfrac{a}{s-(n+1)\kappa} e^{b\sqrt{s-(n+1)\kappa}}}{\dfrac{a}{s} e^{b\sqrt{s}}} \tag{110}$$

where $a = \frac{1}{4\sqrt{3}}$; $b = \pi\sqrt{2/3}$. For $s \gg 1$ we have

$$\sqrt{s-n\kappa} = \sqrt{s}\sqrt{1-\frac{n\kappa}{s}} \approx \sqrt{s}\left(1-\frac{n\kappa}{2s}\right) = \sqrt{s} - \frac{n\kappa}{2\sqrt{s}}$$

$$\sqrt{s-(n+1)\kappa} = \sqrt{s}\sqrt{1-\frac{(n+1)\kappa}{s}} \approx \sqrt{s}\left(1-\frac{(n+1)\kappa}{2s}\right) = \sqrt{s} - \frac{(n+1)\kappa}{2\sqrt{s}}$$

so that

$$g(n/\kappa) \approx \frac{\dfrac{a}{s\left(1-\dfrac{n\kappa}{s}\right)} e^{b\sqrt{s}-b\frac{n\kappa}{2\sqrt{s}}} - \dfrac{a}{s\left[1-\dfrac{(n+1)\kappa}{s}\right]} e^{b\sqrt{s}-b\frac{(n+1)\kappa}{2\sqrt{s}}}}{\dfrac{a}{s} e^{b\sqrt{s}}}$$

$$g(n/\kappa) \approx \left(1 - e^{-\frac{b}{2\sqrt{s}}\kappa}\right) e^{-\frac{b}{2\sqrt{s}}n\kappa} \qquad (111)$$

and since $b=\pi\sqrt{2/3}$ [Eq.(16)] we also obtain:

$$g(n/\kappa) = \left(1 - e^{-\frac{\pi}{\sqrt{6s}}\kappa}\right) e^{-\frac{\pi}{\sqrt{6s}}n\kappa} \qquad ; \; n = 0,1,2,\dots \qquad (112)$$

We observe that Eq.(112) coincides with Eq.(2) because according to Eq.(9) we have $\theta = T/\varepsilon_0 = \sqrt{6s}/\pi$. Therefore, the conditional probability $g(n/\kappa)$ [Eq.(2)] can be obtained without resorting to Boltzmann's law and to interactions between radiation and matter. Hence, Planck's distribution [Eq.(12)] may be further derived in 3-D as in the introduction, by using only conservation Eq.(14) and the principle of equal probabilities of quantum states.

Another generalization that can be obtained from the above theory, is that the average number of photons $\langle N \rangle_s$ of a 1-D system containing s quanta defined by Eq.(19), can be also expressed in terms of partitions [see Eqs (30, 38, 47, 57, 68, 80, 93, 107)] :

$$\langle N \rangle_s = \frac{1}{p_s}(\nu_1 p_{s-1} + \nu_2 p_{s-2} + \cdots + \nu_{s-1} p_1 + \nu_s p_0) \qquad (113)$$

where each coefficient ν_n ; n=1,2,…, s is equal to the *number of divisors of n* and has *universal* numerical values ν_n=(1, 2, 2, 3, 2, 4, 2, 4, 3,…..) independent of s. Note that in *number theory* [9] ν_n is denoted by $\sigma_0(n)$ or d(n).

We observe that ν_n can be obtained by the following triangular algorithm where the columns are well defined harmonic sequences:

n	ν_n		
1	1	=	1
2	2	=	$1+1$
3	2	=	$1+0+1$
4	3	=	$1+1+0+1$
5	2	=	$1+0+0+0+1$
6	4	=	$1+1+1+0+0+1$
7	2	=	$1+0+0+0+0+0+1$
8	4	=	$1+1+0+1+0+0+0+1$
9	3	=	$1+0+1+0+0+0+0+0+1$
10	4	=	$1+1+0+0+1+0+0+0+0+1$

. . . . (114)

Explicitly, the columns of the algorithm can be written as follows:

$$\tau_\kappa(n) = \frac{1}{\kappa}\sum_{l=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}l\right) \quad ; \quad n = \kappa, \kappa+1, \kappa+2, \dots \; ; \quad \kappa = 1,2,3, \dots \tag{115}$$

so that

$$\tau_1(n) = cos0 = (1, 1, 1, \dots) \;\; ; \;\; n = 1,2,3$$

$$\tau_2(n) = \frac{1}{2}\{1+\cos(\pi n)\} = (1, 0, 1, 0, \dots) \;\; ; \;\; n = 2,3,4, \dots$$

$$\tau_3(n) = \frac{1}{3}\left\{1+\cos\left(\frac{2\pi}{3}n\right)+\cos\left(\frac{4\pi}{3}n\right)\right\} = (1, 0, 0, 1, 0, 0, \dots) \;\; ; \;\; n = 3,4,5, \dots$$

. . . . (116)

Therefore, the coefficients ν_n of Eq.(113) are given exactly by the formula

$$\nu_n = \sum_{\kappa=1}^{n} \tau_\kappa(n) = \sum_{\kappa=1}^{n}\frac{1}{\kappa}\sum_{l=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}l\right) \quad ; \quad n = 1,2,3, \dots \tag{117}$$

Also, the sum of the average number of quanta existing in the energy levels 1,2,3, …, s is equal to s [Eq.(24)] and can be expressed in terms of partitions
[see Eqs (31, 39, 48, 58, 69, 81, 94, 108)]:

$$s = \frac{1}{p_s}(\lambda_1 p_{s-1} + \lambda_2 p_{s-2} + \cdots + \lambda_{s-1}p_1 + \lambda_s p_s) \tag{118}$$

where each coefficient λ_n ; n = 1, 2,..., s is equal to the *sum of divisors of n* and has *universal* numerical values λ_n = (1, 3, 4, 7, 6, 12, 8, 15, 13, ...) independent of s. Eq.(118) is well known in *number theory* [9] where λ_n is usually denoted by $\sigma_1(n)$ or $\sigma(n)$. Also, the coefficients λ_n were derived in ref. [10] using the following harmonic triangular algorithm:

$$
\begin{array}{cccl}
n & \lambda_n & & \\
1 & 1 & = & 1 \\
2 & 3 & = & 1+2 \\
3 & 4 & = & 1+0+3 \\
4 & 7 & = & 1+2+0+4 \\
5 & 6 & = & 1+0+0+0+5 \\
6 & 12 & = & 1+2+3+0+0+6 \\
7 & 8 & = & 1+0+0+0+0+0+7 \\
8 & 15 & = & 1+2+0+4+0+0+0+8 \\
9 & 13 & = & 1+0+3+0+0+0+0+0+9 \\
10 & 18 & = & 1+2+0+0+5+0+0+0+0+10 \\
\ldots & & & \ldots
\end{array}
\tag{119}
$$

As a result, the coefficients λ_n of Eq. (118) are given exactly by the formula :

$$\lambda_n = \sum_{\kappa=1}^{n} \sum_{l=0}^{\kappa-1} cos\left(2\pi \frac{n}{\kappa} l\right) \qquad ; \qquad n = 1,2,3, \ldots \tag{120}$$

Finally, the average number of photons $\langle n_\kappa \rangle$ occupying the energy levels κ=1,2,3,..., s can be expressed compactly in terms of partitions for arbitrary s, according to the results of section 3, as

$$\langle n_\kappa \rangle = \sum_{l=1}^{[s/\kappa]} \frac{p_{s-\kappa l}}{p_s} \tag{121}$$

Using next for large s the Hardy-Ramanujan formula [Eq.(16)], we also get

$$\langle n_\kappa \rangle = \sum_{l=1}^{\infty} \frac{\frac{a}{s-\kappa l}\, e^{b\sqrt{s-\kappa l}}}{\frac{a}{s}\, e^{b\sqrt{s}}} \approx \sum_{l=1}^{\infty} e^{-\frac{\pi}{\sqrt{6s}}\kappa l} = \frac{1}{e^{\frac{\pi}{\sqrt{6s}}\kappa} - 1} \tag{122}$$

where $b=\pi\sqrt{2/3}$. Again, due to Eq.(9), the above result coincides with Eq.(4) leading to Planck's distribution.

5. Conclusion

In the present paper it is shown that Planck's distribution of black-body radiation [Eq.(12)] can be derived by considering the cavity containing electromagnetic energy as a closed system without interactions with matter. In 1-D where the modes are standing over a length ℓ, the total energy $E = s\,\varepsilon_0$ is quantized according to Planck's hypothesis into s quanta of energy $\varepsilon_0 = hc/2\ell$, and the energy $\varepsilon = n\kappa\varepsilon_0$ of each mode is also quantized into n photons of energy $\kappa\,\varepsilon_0$ where κ is the energy level of the mode. The theory is based on the *principle of equal probabilities of quantum states* that are derived from conservation Eq.(14) taking into account that photons are indistinguishable particles obeying Bose statistics. In particular, each state containing s quanta of energy is represented by a diagram where the positions of the photons in the energy levels are shown explicitly. Since the total number of states of a 1-D photon system containing s quanta of energy, is equal to the number of partitions p_s of the integer s, we express exactly the conditional probability $g(n/\kappa)$ that n photons occupy level κ, in terms of partitions [Eq.(109)] and then study its behaviour for large s using the Hardy-Ramanujan formula [Eq.(16)]. Thus, Planck's distribution is derived without resorting to Boltzmann's law and to interactions between radiation and matter and the Bose statistics is fully justified. Note that the logic of the present work is similar to the one used in the derivation of the Maxwellian velocity distribution for an ideal gas in a closed system of constant energy, using the *Borel method* [11]. In this case, all points of the surface of the hypersphere created in 3N-D by the conservation of the total kinetic energy, are supposed to be equiprobable and consideration of a collision mechanism is not necessary in order to obtain the velocity distribution in thermal equilibrium.

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