ON THE NO-TRIVIAL ZEROS OF THE ZETA FUNCTION C(S)

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We research and explicitly expose example of an infinity of zeros (C(r+ic)=0) of RH (The Riemann hypothesis) in the critical line (having for real part r= 1/2). So there is infinity of no-trivial zeros of Riemann's zeta function which have the real part equal to 1/2, which shows (using simple mathematics baggage) Hardy and Littlewood Theorem and give as a hope that the Riemann's Conjecture would be true....
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1 Introduction

The Riemann hypothesis is a conjecture formulated in 1859 by the mathematician Bernhard Riemann, according to which the non-trivial and infinity zeros of Riemann's zeta function all have a real part equal to $\frac{1}{2}$.

Its demonstration would improve knowledge of the distribution of prime numbers and open up new areas for mathematics.

Riemann's article (see [1]) on the distribution of prime numbers is his only text dealing with number theory, he develops the properties of the function zeta $C(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and prove the prime number theorem by admitting to passing several results including what is now called RH the Riemann Hypothesis. After, Hardy says that there is an infinity of zeros on the critical line (see [2], [3]), this gives us an esperance that the RH would be true...

Here, we are going to look the zeros on the critical line, and we will explicitly determine some infinity zeros having the real part $\frac{1}{2}$. This will give us another and simple demonstration of Hardy result.

Let $s = r + ic = \frac{1}{2} + ic$, c and α such $\alpha = -\ln(2)c = \pm\frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$
We suppose (this is our proposal) that,

$$c = \frac{-1}{\ln(2)}\left(\pm\frac{\pi}{4} + 2k\pi\right), k \in \mathbb{Z} \text{ and } r = \frac{1}{2}$$

$$S = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = \sum_{n=1}^{+\infty} (-1)^n \frac{e^{-i\ln(n)c}}{\sqrt{n}} = C_1 - C_2$$

$$S = \sum_{n=1}^{+\infty} (-1)^n \frac{e^{i\alpha_n}}{\sqrt{n}} = R' + iI'$$

$$R' = \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(\alpha_n)}{\sqrt{n}}, I' = \sum_{n=1}^{+\infty} (-1)^n \frac{\sin(\alpha_n)}{\sqrt{n}}$$

with $\alpha_n = -\ln(n)c = \frac{\ln(n)}{\ln(2)}\left(\pm\frac{\pi}{4} + 2k\pi\right)$

$$C_1 = \sum_{n=1}^{+\infty} \frac{1}{(2n)^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{\sqrt{2n}}$$

$$C_1 = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n}}}{\sqrt{2n}} = R_1 + iI_1$$

and

$$C_2 = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n-1)c}}{\sqrt{2n-1}}$$

$$C_2 = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n-1}}}{\sqrt{2n-1}} = R_2 + iI_2$$

with

$$R_1 = \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n})}{\sqrt{2n}}$$

$$I_1 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n})}{\sqrt{2n}}$$

$$R_2 = \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n-1})}{\sqrt{2n-1}}$$

$$I_2 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n-1})}{\sqrt{2n-1}}$$

So,

$$C = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{\sqrt{n}} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_n}}{\sqrt{n}}$$

$$C = C_1 + C_2 = R + iI$$

$$R = \sum_{n=1}^{+\infty} \frac{\cos(\alpha_n)}{\sqrt{n}} = R_1 + R_2$$

$$I = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_n)}{\sqrt{2n}} = I_1 + I_2$$

$$R' = R_1 - R_2$$

$$I' = I_1 - I_2$$

$$e^{-i\ln(2)c} = e^{i\alpha} = e^{i\left(\pm\frac{\pi}{4}+2k\pi\right)} = \frac{1\pm i}{\sqrt{2}}$$

Therefore,

$$C_1 = \frac{e^{-i\ln(2)c}}{\sqrt{2}} \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{\sqrt{n}}$$

$$C_1 = \frac{e^{-i\ln(2)c}}{\sqrt{2}} C = \frac{e^{i\alpha}}{\sqrt{2}} C$$

$$C_1 = \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}} C = \left(\frac{1\pm i}{2}\right) C$$

$$C_1^2 = \frac{e^{\pm i\frac{\pi}{2}}}{2} C^2$$

$$C_1^2 = \pm\frac{i}{2}C^2$$

and

$$C_2 = C - C_1 = \left(1 - \frac{e^{i\alpha}}{\sqrt{2}}\right)C$$

$$C_2 = \left(1 - \frac{1\pm i}{2}\right)C = \left(\frac{1\mp i}{2}\right)C = \frac{e^{\mp i\frac{\pi}{4}}}{\sqrt{2}}C$$

$$C_2^2 = \frac{e^{\mp i\frac{\pi}{2}}}{2}C^2$$
$$C_2^2 = \mp\frac{i}{2}C^2$$

idem for S

$$S = C_1 - C_2 = \left(\frac{1\pm i}{2}\right)C - \left(\frac{1\mp i}{2}\right)C$$

$$S = \pm iC$$

$$\Longrightarrow$$
$$S^2 + C^2 = 0$$

Geometrically speaking,

$$S \perp C\text{and}|S| = |C|$$

We can also prove that:

$$|C_1| = \left|\frac{1\pm i}{2}\right||C| = \frac{1}{\sqrt{2}}|C| = \left|\frac{1\mp i}{2}\right||C|$$
$$|C_1| = |C_2|$$

such

$$S^2 + C^2 = (C_1 - C_2)^2 + (C_1 + C_2)^2$$
$$S^2 + C^2 = 2(C_1^2 + C_2^2)$$
$$S^2 + C^2 = 0$$

which implies that

$$C_1^2 + C_2^2 = 0$$

Therefore

$$C_1 = \pm iC_2$$

Thus

$$C_1 \perp C_2$$

Conclusion-1:

$$C_1^2 + C_2^2 = 0$$

$$S^2 + C^2 = 0$$
$$|C_1| = |C_2|$$
$$|S| = |C|$$
$$|S|^2 = 2|C_2|^2$$
$$|C|^2 = 2|C_1|^2$$

$S \perp C$ and $C_1 \perp C_2$

$$C_1^2 = \pm\frac{i}{2}C^2$$
$$C_2^2 = \pm\frac{i}{2}S^2$$

$C_1^4 = C_2^4, S^4 = C^4 = -4C_1^4 = -4C_2^4.$

2 The study of C & S.

Theorem 1 *Let* $r = \frac{1}{2}$ *and* $\alpha \equiv \pm\frac{\pi}{4}\ [2\pi]$
so

$$C^4 \in \mathbb{R} \& S^4 \in \mathbb{R}$$

Proof. Assuming that $C^2 \notin \mathbb{R}$,
so $(C, \overline{C})$ is a basis in $\mathbb{C}$.
So, let

$$C_1 = aC + b\overline{C}$$

$$C^4 = -4C_1^4$$
$$\Rightarrow C^4 = -4\left(aC + b\overline{C}\right)^4$$
$$\Longrightarrow -C^4 = 4\left(a^4C^4 + 4a^3b|C|^2C^2 + 6a^2b^2|C|^4 + 4ab^3|C|^2\overline{C}^2 + b^4\overline{C}^4\right)$$
$$\Longrightarrow$$
$$\left(a^4 + \frac{1}{4}\right)C^4 + 4a^3b|C|^2C^2 + 6a^2b^2|C|^4 + 4ab^3|C|^2\overline{C}^2 + b^4\overline{C}^4 = 0$$

$C^4 \neq 0$ (since $C^2 \notin \mathbb{R}$),
so $\left(a^4 + \frac{1}{4}\right)C^8 + 4a^3b|C|^2C^6 + 6a^2b^2|C|^4C^4 + 4ab^3|C|^6C^2 + b^4|C|^8 = 0$

$$\Longrightarrow \begin{cases} \left(a^4 + \frac{1}{4}\right)X^4 + 4a^3bX^3 + 6a^2b^2X^2 + 4ab^3X + b^4 = 0 \quad (E) \\ X = \dfrac{C^2}{|C|^2} \end{cases}$$

Let

$$P(X) = \left(a^4 + \frac{1}{4}\right)X^4 + 4a^3bX^3 + 6a^2b^2X^2 + 4ab^3X + b^4$$

We are going to study the function $P(X)$ on $\mathbb{R}[X]$.
Finding the roots of $P(X)$ on $\mathbb{R}$:
We have $a^4 + \frac{1}{4} \succ 0$
so

$$\lim_{X \to \pm\infty} P(X) = +\infty$$

We will find $X_0 \in \mathbb{R}$ the minimum of $P(X)$ and show that its image $P(X_0) \succ 0$:
We have

$$\Rightarrow P'(X) = 4\left[\left(a^4 + \frac{1}{4}\right)X^3 + 3a^3bX^2 + 3a^2b^2X + ab^3\right]$$

so

$$P'(X_0) = 0 \Leftrightarrow \left(a^4 + \frac{1}{4}\right)X_0^3 + 3a^3bX_0^2 + 3a^2b^2X_0 + ab^3 = 0$$

$$\Leftrightarrow \left(a^4 + \frac{1}{4}\right)X_0^3 = -[3a^3bX_0^2 + 3a^2b^2X_0 + ab^3]$$

$$ab \neq 0 \Rightarrow X_0 \neq 0$$

$$\Rightarrow \left(a^4 + \frac{1}{4}\right)X_0^4 = -[3a^3bX_0^3 + 3a^2b^2X_0^2 + ab^3X_0]$$

so

$$P(X_0) = -3a^3bX_0^3 - 3a^2b^2X_0^2 - ab^3X_0 + 4a^3bX_0^3 + 6a^2b^2X_0^2 + 4ab^3X_0 + b^4$$

$$P(X_0) = a^3bX_0^3 + 3a^2b^2X_0^2 + 3ab^3X_0 + b^4$$

$$P(X_0) = b[a^3X_0^3 + 3a^2bX_0^2 + 3ab^2X_0 + b^3]$$

$$\Longrightarrow$$

$$P(X_0) = b[aX_0 + b]^3 \quad (1)$$

Also

$$P'(X_0) = 0 \Leftrightarrow a^4X_0^3 + 3a^3bX_0^2 + 3a^2b^2X_0 + ab^3 = -\frac{X_0^3}{4}$$

$$\Leftrightarrow a[a^3X_0^3 + 3a^2bX_0^2 + 3ab^2X_0 + b^3] = -\frac{X_0^3}{4}$$

so $P(X_0) = b[a^3X_0^3 + 3a^2bX_0^2 + 3ab^2X_0 + b^3] \Longrightarrow$

$$aP(X_0) = -b\frac{X_0^3}{4} \quad (2)$$

as $b \neq 0$, from (1) & (2) $\Longrightarrow$

$$a[aX_0 + b]^3 = -\frac{X_0^3}{4}$$

$$\Longrightarrow 4a[aX_0 + b]^3 = -X_0^3$$

$\Longrightarrow \sqrt[3]{4a}[aX_0 + b] = -X_0$ (since $X_0 \in \mathbb{R}$)

$$\Longrightarrow$$

$$X_0 = -\frac{b\sqrt[3]{4a}}{1 + a\sqrt[3]{4a}}$$

Therfore, P has one and unique minimum.
We can also prove it from

$$P'(X) = 4\left[\left(a^4 + \frac{1}{4}\right)X^3 + 3a^3bX^2 + 3a^2b^2X + ab^3\right]$$

$$\Rightarrow P''(X) = 12\left[\left(a^4 + \frac{1}{4}\right)X^2 + 2a^3bX + a^2b^2\right]$$

$$\Delta = (2a^3b)^2 - 4a^2b^2\left(a^4 + \frac{1}{4}\right)$$

$$\Longrightarrow \Delta = -a^2b^2 \prec 0$$

$\Longrightarrow P'$ is strictly monotone,
P' is bijective (as P' is a continuous function)

so we have X_0 such $P'(X_0) = 0$ is unique,
$\Rightarrow \left(X_0, P(X_0)\right)$ is the overall-minimum,

$$aP(X_0) = -b\frac{X_0^3}{4} \Rightarrow P(X_0) = -\frac{bX_0^3}{4a}$$

$$\Rightarrow P(X_0) = -\frac{b\left(-\frac{b\sqrt[3]{4a}}{1+a\sqrt[3]{4a}}\right)^3}{4a}$$

$$\Rightarrow P(X_0) = \frac{b^4 4a}{4a\left(1+a\sqrt[3]{4a}\right)^3}$$

$$\Rightarrow P(X_0) = \frac{b^4}{\left(1+a\sqrt[3]{4a}\right)^3}$$

If $a \prec 0$ so $\sqrt[3]{4a} \prec 0$

$$\Longrightarrow a\sqrt[3]{4a} \succ 0 \Rightarrow 1 + a\sqrt[3]{4a} \succ 0$$

$$\Rightarrow \left(1+a\sqrt[3]{4a}\right)^3 \succ 0$$

and $b \neq 0 \Longrightarrow b^4 \succ 0 \Longrightarrow P(X_0) \succ 0$
If $a \succ 0$ so $\sqrt[3]{4a} \succ 0$

$$\Longrightarrow a\sqrt[3]{4a} \succ 0 \Rightarrow 1 + a\sqrt[3]{4a} \succ 0$$

$$\Rightarrow \left(1+a\sqrt[3]{4a}\right)^3 \succ 0$$

and $b \neq 0 \Longrightarrow b^4 \succ 0 \Longrightarrow P(X_0) \succ 0$
So

$$P(X_0) \succ 0 \;\; \forall (a,b) \in \mathbb{R}^2$$

So $P(X) \geq P(X_0) \succ 0 \; \forall (a,b) \in \mathbb{R}^2 \; \forall X \in \mathbb{R}$

$$\Rightarrow$$

$$\forall (a,b) \in \mathbb{R}^2 \forall X \in \mathbb{R} \quad P(X) \neq 0$$

So $P(X) = 0$ has non-real complexes as a solution.

Since $P \in \mathbb{R}[X]$ polynomial with real coefficients, the solutions will be $z_1, z_2, \overline{z}_1, \overline{z}_2$, and

$$P(X) = \left(a^4 + \frac{1}{4}\right)(X - z_1)(X - z_2)(X - \overline{z}_1)(X - \overline{z}_2)$$

$$P(X) = \left(a^4 + \frac{1}{4}\right)(X^2 - 2Re(z_1)X + |z_1|^2)(X^2 - 2Re(z_2)X + |z_2|^2)$$

$$X = \frac{C^2}{|C|^2} \Longrightarrow |X| = 1 \Longrightarrow |z_1| = |z_2| = 1 \Longrightarrow$$

$$P(X) = \left(a^4 + \frac{1}{4}\right)(X^2 - 2Re(z_1)X + 1)(X^2 - 2Re(z_2)X + 1)$$

$\frac{P(X)}{a^4+\frac{1}{4}} = X^4 - 2[Re(z_1) + Re(z_2)]X^3 + [2 + 4Re(z_1)Re(z_2)]X^2 - 2[Re(z_1) +$
$Rez2X+1$

since

$$P(X) = \left(a^4 + \frac{1}{4}\right)X^4 + 4a^3bX^3 + 6a^2b^2X^2 + 4ab^3X + b^4$$

we will have

$$\begin{cases} a^4 + \frac{1}{4} = b^4 \\ -2b^4[Re(z_1) + Re(z_2)] = 4a^3b \\ b^4[2 + 4Re(z_1)Re(z_2)] = 6a^2b^2 \\ -2b^4[Re(z_1) + Re(z_2)] = 4ab^3 \end{cases}$$
$$\Longrightarrow 4a^3b = 4ab^3 \Longrightarrow 4ab(a^2 - b^2) = 0$$
$$ab \neq 0 \Longrightarrow$$
$$a = \pm b$$

$$\Rightarrow C_1 = b\left(\pm C + \overline{C}\right)$$
$$\Rightarrow C_1^2 = b^2\left(C^2 + \overline{C}^2 \pm 2|C|^2\right) \in \mathbb{R}$$

$C_1^2 = \pm\frac{i}{2}C^2 \Longrightarrow C^2 \in i\mathbb{R}$

Conclusion:

$$C^2 \notin \mathbb{R} \Longrightarrow C^2 \in i\mathbb{R}$$

$\Longrightarrow C^2 \in i\mathbb{R}$ or $C^2 \in \mathbb{R}$

$$\Longrightarrow C^4 \in \mathbb{R}$$

$\Longrightarrow S^4 \in \mathbb{R}$ (since $C^4 = S^4$)

3 A Rotation in $\mathbb{C}$

We suppose that $C \neq 0$:
So $C^2 \neq 0$ & $|C| \neq 0$
$\Longrightarrow \exists r = |C| \neq 0, \exists \gamma = \arg(C)$ such $C^2 = r^2 e^{i2\gamma}$
$\Longrightarrow \left(Ce^{i\theta}\right)^2 = C^2 e^{i2\theta} = r^2 e^{i(2\gamma+2\theta)} \in \mathbb{R}$ such $2\gamma + 2\theta \equiv 0\ [\pi]$
So $\left(Ce^{i\theta}\right)^2 \notin \mathbb{R}$ such $\theta \not\equiv -\gamma \left[\frac{\pi}{2}\right]$
Let $\theta \in \mathbb{R}$ such $\theta \not\equiv -\gamma \left[\frac{\pi}{2}\right]$ and C', C_1', C_2', S' such

$$C' = Ce^{i\theta}$$

$$C_1' = C_1 e^{i\theta}$$

$$C_2' = C_2 e^{i\theta}$$

$$S' = Se^{i\theta}$$

$$\Longrightarrow$$
$$C'^2 \notin \mathbb{R}$$

We have

$$C_1' + C_2' = C'$$
$$C_1' - C_2' = S'$$

$|C_1'| = |C_1|$ and $|C_2'| = |C_2|$

$$|C_1| = |C_2| \Longrightarrow |C_1'| = |C_2'|$$

$|C'| = |C|\ , |S'| = |S|$

$$C_1'^2 + C_2'^2 = e^{i2\theta}(C_1^2 + C_2^2)$$

since $C_1^2 + C_2^2 = 0$
so

$$C_1'^2 + C_2'^2 = 0$$

and since $C^4 = -4C_1^4$, so

$$C'^4 = -4C_1'^4$$

$C'^2 \notin \mathbb{R} \Longrightarrow \left(C', \overline{C'}\right)$ is a base in $\mathbb{C}$.
As we have the same conditions as in the last theorem:
1) $C_1' = aC' + b\overline{C'}$
2) $C'^4 = -4C_1'^4$
we will have $C'^2 \notin \mathbb{R} \Longrightarrow C'^2 \in i\mathbb{R}$
So, $C'^2 \in \mathbb{R}$ or $C'^2 \in i\mathbb{R}$

$$\Longrightarrow$$

$$C'^4 \in \mathbb{R} \& S'^4 \in \mathbb{R} \forall \theta \in \mathbb{R}$$

$$\Longrightarrow C'^4 = e^{i4\theta} C^4 \in \mathbb{R}$$

$\Longrightarrow e^{i4\theta} \in \mathbb{R}\ \forall \theta \in IR - \left\{-\gamma + k\frac{\pi}{2} / k \in \mathbb{Z}\right\}$ (such $C^4 \in \mathbb{R}$)
absurd!!!,
unless

$$C^4 = 0$$

(because $e^{i4\theta} \in \mathbb{R} \Leftrightarrow \theta = k\frac{\pi}{4}, k \in \mathbb{Z}$)

Conclusion-3-

$$C = S = 0$$

4 Conclusion:

For $s = r + ic = \frac{1}{2} + ic$ such $c = -\left(\frac{1}{4} + 2k\right)\frac{\pi}{\ln(2)}$, $k \in \mathbb{Z}$ (idem for $c = -\left(-\frac{1}{4} + 2k\right)\frac{\pi}{\ln(2)}$, $k \in \mathbb{Z}$), we have

$$c = \left(\pm\frac{1}{4} + 2k\right)\frac{\pi}{\ln(2)}, k \in \mathbb{Z} \Longrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = 0.$$

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