Gravitational and Inertial Mass

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June 2024†

Abstract

Newtonian Gravitational fields have no kinetic energy. If they had, a moving mass would make fields variate around it and would give kinetic energy to space. In this paper we investigate the hypothesis that the energy of those varying fields could be responsible for the kinetic energy of a mass. To do that we have slightly modified Newtonian gravity to give kinetic energy to the fields and we have found that the integral of the kinetic energy of the varying fields around a moving mass is exactly $\frac{1}{2}mv^2$.

This leads somehow to the equivalence of gravitational and inertial mass because the very same fields responsible for masses to attract each other, are the same fields that give inertia to a mass.

Key Words: Newtonian gravity, equivalence principle.

1 Introduction

Two objects that gravitate in the same way, have the same inertia. This is known as the equivalence principle in its weak formulation. In this paper, we start from classical Newtonian gravity, where on one hand, perturbations in gravitational filed propagate at infinite speed and there is no Kinetic energy density associated to fields variation and, on the other hand, we know that in Classical Mechanics this is not the case and fields always have kinetic energy. However, it is easy to modify Newtonian gravity in a simple and natural way, such that fields propagate at a finite speed. This is done in **§2** and **§3** while in **§4** and **§5** we show that this modified Newtonian gravity leads immediately to the fact that it is possible to associate kinetic energy to fields variation.

Once fields have kinetic energy, we note immediately that a moving mass induces variation of fields in space because the fields generated by a mass move with it. The mast simple question we want to answer in this situation, is if the kinetic energy of a particle can be interpreted as the kinetic energy of the fields variation induced by its motion. To do that we integrate the kinetic energy of the fields in the whole space and we find exactly a value of $\frac{1}{2}mv^2$. This is done in **§6** in which we show that kinetic energy of fields gives inertia to a point mass and in **§7** where we perform the actual calculation and we find all equations of Newtonian Mechanics. In **§8** and **§9** we give some further consideration on the parameters of the model.

All the above reasoning leads immediately to the equivalence principle in its weak form because the very same fields responsible for masses to attract each other, are also responsible for their dynamics. This is the main point of this paper and it is made in **§7.4**.

2 Newtonian Gravity with no Instant Action at Distance

In Newtonian gravity, given a distribution of mass $\rho_m(\mathbf{x})$, the gravitational potential generated by this distribution of mass, in units where $4\pi G = 1$ and where G is the gravitational constant, is given by the Poisson's equation:

$$\nabla^2\phi(\mathbf{x},t) = \rho_m(\mathbf{x},t) \tag{1}$$

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†Posted at: https://vixra.org/abs/2406.0083 - Current version: v3 - Jan. 2025

where $\mathbf{x}$ is a point in 3-dimensional space. In particular for a point mass m in the origin, we have:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=m\delta(\mathbf{x}) \tag{2}$$

In this model, gravity is an instant action at distance. I we move a mass, the potential will change instantaneously in the whole universe. Modern physics has taught us that this is never the case and that perturbations in fields propagate at the speed of light. It is natural to add to our equation a term which takes into account how perturbations propagate at a limited speed. The Poisson's equation has second derivatives with respect to the special coordinates. The obvious thing to do is to add a second derivative with respect to time and turn the Poisson's equation into a wave equation. In units where also the speed of light $c = 1$ we have:

$$-\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}+\frac{\partial^2\phi(\mathbf{x},t)}{\partial x^2}+\frac{\partial^2\phi(\mathbf{x},t)}{\partial y^2}+\frac{\partial^2\phi(\mathbf{x},t)}{\partial z^2}=\rho_m(\mathbf{x},t) \tag{3}$$

which is also written as:

$$\Box\phi=\rho_m \tag{4}$$

Where $\Box$ is the d'Alembert operator (in this paper with signature $(-+++)$). For a moving point mass with coordinates $\sigma(t)=(x(t),\ y(t),\ z(t))$ we have:

$$\Box\phi=m\delta(\mathbf{x}-\sigma) \tag{5}$$

Being this a wave equation, clearly a "massless" perturbations (as it happens for massless particle) move into space at the speed of light.

In the next sections we will discuss what happens to "massive" perturbations where "massive" needs to be defined properly.

3 Length Contraction

Suppose a mass moves in space along the x axis with coordinates $\sigma(t)=(\sigma(t),\ 0,\ 0)$ where $\sigma(t)$ is a smooth function. We want to write Eq. (5) in the reference frame of the mass. To do that we use the change of coordinates (49) (see Appendix A.4) and we get an expression for the d'Alembert operator given by (56). Eq. (5) becomes:

$$-\phi_{tt}+2\dot{\sigma}\phi_{tx}+(1-\dot{\sigma}^2)\phi_{xx}+\phi_{yy}+\phi_{zz}=m\delta(\mathbf{x}) \tag{6}$$

By construction, in this reference frame the mass is always in the origin. Suppose now that $\sigma(t)=vt$ (i.e. the mass is moving with constant velocity along x), in this case Eq. (6) becomes:

$$-\phi_{tt}+2v\phi_{tx}+(1-v^2)\phi_{xx}+\phi_{yy}+\phi_{zz}=m\delta(\mathbf{x}) \tag{7}$$

We are looking for a solution which is stationary in the mass reference frame. To do that we set the derivatives of the potential with respect to time to vanish. We have:

$$\frac{1}{\gamma^2}\phi_{xx}+\phi_{yy}+\phi_{zz}=m\delta(\mathbf{x}) \tag{8}$$

where

$$\gamma=\frac{1}{\sqrt{1-v^2}}\ \xrightarrow{c\neq 1}\ \gamma=\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{9}$$

is the Lorentz factor and we have reintroduced the speed of light.

From Eq. (63) we see that the solution of Eq. (8) is:

$$\phi(\mathbf{x})=\tilde{\phi}(\gamma x,\ y,\ z) \tag{10}$$

where $\tilde{\phi}$ is the solution of Eq. (2) which in polar coordinates is:

$$\tilde{\phi}(r)=-\frac{1}{4\pi}\frac{m}{r}\ \xrightarrow{4\pi G\neq 1}\ \tilde{\phi}(r)=-\frac{Gm}{r} \tag{11}$$

and where we have reintroduced the gravitational constant. Note that a moving potential get contracted along the direction of motion by a factor $1/\gamma$ with respect to a not moving potential.

4 Analogy with a Classical Mechanics Model

To better understand the physical behaviour of our moving "massive" perturbation, we want to make an analogy with a Classical Mechanics system described by the same equation (3) of our Newtonian gravitational field propagating at a fixed velocity. We choose a model where space is just an elastic linear material and where propagating longitudinal pressure waves are the equivalent of propagating Newtonian gravity perturbations.

Longitudinal waves in an elastic material are described by the vectorial wave equation while Eq. (3) id the scalar wave equation. However, there is a subset of longitudinal elastic waves that are shier-less and that can be described by the scalar wave equation (see Appendix A.3).

What we are saying is that, in absence of a density mass distribution (empty space,) the propagation of gravitational waves is analogous to the propagation of longitudinal shierless waves in an elastic material described by the equation:

$$-\rho\frac{\partial^2\phi}{\partial t^2} + Y\nabla^2\phi = 0 \quad c = \sqrt{\frac{Y}{\rho}} \tag{12}$$

where ρ is the density of the material, Y is the Young modulus and $c = \sqrt{Y/\rho}$ is the velocity of the waves in the material. In this model, the scalar ϕ is the elastic potential (equivalent to gravitational potential) and $\mu = -\nabla\phi$ are the strains (with opposite sign) and are equivalent to the gravitational field.

We are not saying that space is an elastic material, but just that a class of waves in an elastic material are described by the same equation of Newtonian gravity with non instantaneous action at distance and therefore if we can find a characteristic of the elastic system (e.g. kinetic energy associated to variation of fields), this will apply also to gravity.

Now that we know how to describe massless perturbations (i.e. gravitational waves) we want to introduce in the model massive perturbations (i.e. point masses). These are point discontinuities in space which are surrounded by a spherical symmetric field of strains (equivalent to gravitational fields) that goes like $1/r^2$ with distance. We will use the idea of a **Space Deficiency**.

Let us consider a space filled with an elastic material, we remove a ball of material of radius R and we pull the remaining material, on the boundary of the hole, in such a way to reduce the radius of the empty space from R to η. At the equilibrium, the density in the material outside the sphere will decrease inducing strains. We want to evaluate those strains.

The figure below shows the amplitude of the displacements in the material as a function of r in a spherical coordinates systems.

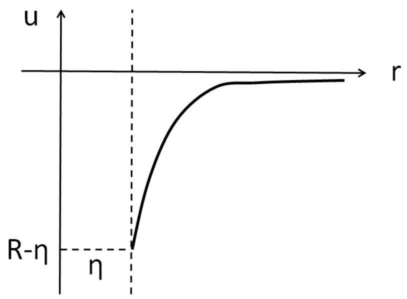

Figure 1: Displacements in a space deficiency

The equation for the equilibrium in the elastic material is given by (3) where in the static case we can ignore the derivatives with respect to time and we get the Poisson's equation. Note that (3) is valid for infinitesimal displacements and here displacements are finite. However, this equation is still valid for if we assume shierless displacements. We have:

$$\nabla^2\phi = 0 \tag{13}$$

where ϕ is the elastic potential and given that the displacements $\mathbf{u}$ have a spherical symmetry, we have that the potential $\phi = |\mathbf{u}|$ is equal to the amplitude of the displacements. This means that looking for a solution for ϕ or for a solution for $|\mathbf{u}|$ in this case is the same thing (see appendix A.3).

In spherical coordinates and for a spherical symmetric solution, the general solution for the Poisson equation is the following:

$$|u| = a\frac{1}{r} + br + c \quad \text{with} \quad |\mathbf{u}| = \phi \tag{14}$$

Given vanishing boundary condition at infinite we have $b = 0$, $c = 0$ and given the boundary condition in η where displacements/potential is equal to $-(R - \eta)$, we have:

$$|u(\eta)| = a\frac{1}{\eta} = -(R - \eta) \quad \Rightarrow \quad a = -\eta(R - \eta) \tag{15}$$

If we assume $R >> \eta$ than we have that the displacements in the material are given by :

$$\mathbf{u}(r) = -\frac{R\eta}{r}\hat{i}_r = -\frac{\mathfrak{m}}{r}\hat{i}_r \tag{16}$$

where

$$\mathfrak{m} = R\eta \tag{17}$$

Moreover, since the strains $\mu = -\nabla\phi$ in our model and $\phi = |\mathbf{u}|$, we have:

$$\mu(r) = -\nabla|u| = -\frac{\mathfrak{m}}{r^2}\hat{i}_r \tag{18}$$

From the above equations we see that a ball of removed material where the boundary are pulled to a radius η, which we call a **Space Deficiency**, is a good way to introduce point masses in our elastic model of Newtonian gravity where the mass of a space deficiency (e.g. a point mass) is given by $\mathfrak{m} = R\eta$ and it has unit of surface (i.e. $[m^2]$).

In this model we assume that the radius η is a constant of nature equal for all Space Deficiencies and therefore the value of the mass is given only by the radius R. This is a slightly odd but necessary assumption as it will be clear later on. Moreover, we want the Space Deficiency to be frictionless (e.g. free to move in space) and, being made of empty space, without inertia.

Note that with η that goes to zero, the required radius R for a given mass $\mathfrak{m}$ goes to infinity together with the energy required to shrink the boundary of empty space to zero. This is the reason why we shrink the spherical boundary to a fixed radius η and not to a point.

5 Energy Density Due to Variation of Gravitational Fields

In our elastic model of gravity, the elastic potential for displacements in empty regions of space (i.e. $\rho_m = 0$), is the solution of the following equation:

$$-\rho\frac{\partial^2\phi}{\partial t^2} + Y\nabla^2\phi = 0 \tag{19}$$

where ρ and Y are the density and the Young Modulus of the elastic material. The above equation holds for infinitesimal displacements in the general case and for finite and infinitesimal displacements if the solution is shierless, which is our case.

In some simple cases, for example plane waves as well as spherical or cylindrical symmetric solutions, displacements can be found simply using $|\mathbf{u}| = \phi$. In the most general cases, displacements can be reconstructed from ϕ by using Eq. (71) in Appendix A.3. From the elastic potential ϕ, we define the field $\mu = -\nabla\phi$, which represent strains with opposite sign and it is the equivalent of gravitational filed. The potential energy density $\mathcal{E}_V$ and the kinetic energy density $\mathcal{E}_T$ stored in the field of displacements are given by the following expressions:

$$\mathcal{E}_V = \frac{1}{2}Y|\mu|^2 \;\; ; \;\; \mathcal{E}_T = \frac{1}{2}\rho|\dot{\mathbf{u}}|^2 \tag{20}$$

Moreover we know that the velocity of the waves in the material is $c = \sqrt{Y/\rho}$.

Given the anology between gravity and the elastic model, it possible to find a field θ in gravity such that θ is the analogous of $\mathbf{u}$ in the elastic model. In this case from the known fact the potential energy stored in the gravitational field is given by the following expression (see [2]):

$$\mathcal{E}_V = \frac{1}{2}\overbrace{\left(\frac{1}{4\pi G}\right)}^{K_V}|g|^2 \tag{21}$$

We can evaluate the kinetic energy of the gravitational filed. if K_V and K_T are the constants associated to potential and kinetic energy in the gravitational fields, then our analogy with an elastic model tells us that $c = \sqrt{K_V/K_T}$ and we find easily:

$$\mathcal{E}_T = \frac{1}{2}\overbrace{\left(\frac{1}{4\pi G c^2}\right)}^{K_T}|\dot{\theta}|^2 \tag{22}$$

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), gravitational field must have kinetic energy associated to field variation.

6 Newton's First Law of Motion

In our Classical Mechanics model, gravitational fields associated to an the elementary volume dV, will have potential energy proportional to the square of the strains module $|\mu|^2$ (as already happens in classical instant action at distance Newtonian gravity) and kinetic energy proportional to square of the rate of variation of the displacements $|\dot{\mathbf{u}}|^2$ (and this is new) which are related to the velocity at which the volume dV is moving in the material.

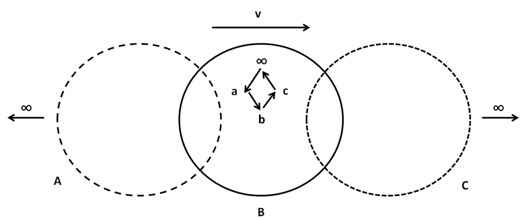

Figure 2: Orbits of Elementary Volumes

Now, if the discontinuity is moving with a velocity v to the right (see Fig. 1), points inside the area close to the discontinuity will readjust as the discontinuity passes by because the Space Deficiency will pull them. Elementary volumes will follow orbits and since the material has a mass density this will be associated with kinetic energy. However, as long as the velocity of the discontinuity moves with constant velocity the total kinetic energy in the material is constant and the total potential energy is also constant because it depends on the the shape of the potential which does not change.

When the discontinuity (i.e. our point mass) start moving pushed by a force, the kinetic energy of the field due to variation of the fields (i.e. $\dot{\mathbf{u}} \neq 0$) will increase and when we stop pushing the discontinuity, it is the conservation of this energy that gives inertia to it. Field variation is the flywheel of the systems.

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a mass has inertia meaning that it keeps moving at the same speed in absence of external forces.

This is the first Newton's law of motion.

7 Evaluation of the Energy

We want to evaluate the Lagrangian $\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma})$ of our model. We want a simplified quasi-static system where a mass is moving at a speed $v << 1$, all the perturbations flies away at speed of light and the shape of the potential readjust itself to a potential $\phi(\sigma)$ depending only on σ and moving rigidly with the mass. With this assumption, the shape of ϕ is independent from σ while its position depends on σ only. This will simplify the Lagrangian as follows:

$$\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma}) \to \mathcal{L}(\sigma(t), \dot{\sigma}(t)) \tag{23}$$

In order to do that, we consider a two mass System M and m in $\mathbb{R}^3$ with M nailed to the position $(-\delta, 0, 0)$ and m free to move along the x axis (see Fig. 2).

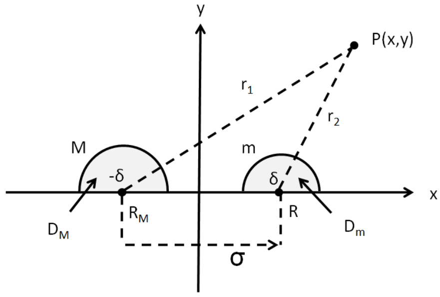

Figure 3: Two Mass System

To write the expression for the Lagrangian, we need to evaluate the potential and kinetic energy of the system. In our elastic model a mass m in units of $[kg]$, corresponds to a mass $\mathfrak{m} = R\eta$, in units of $[m^2]$, of a Space Deficiency of radius R with η a constant. Moreover the gravitational field is represented by the strains $\mu = -\phi$ and $\mathbf{u}$ are the displacements that can be evaluated by (71). The density of potential and kinetic energy is given by (20).

7.1 Potential Energy

To simplify the calculation we note that fields in a point P both lays on the plane passing through P and the axis x and they have a cylindrical symmetry with respect to the axis x. We will evaluate our fields on the (x, y) plane and we will get the field in space just rotating the (x, y) fields around the x axis. Given the linearity, the total strains are $\mu = \mu_M + \mu_n$. For the potential energy, on the (x, y) plane we have:

$$|\mu|^2 = |\mu^M + \mu^m|^2 = (\mu_1^M + \mu_1^m)^2 + (\mu_2^M + \mu_2^m)^2 \tag{24}$$

And we get two self-energy and one cross-energy terms. We give names to them:

$$|\mu|^2 = |\mu^M|^2 + |\mu^m|^2 + 2(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m) = I_M + I_m + 2I_{Mm} \tag{25}$$

We can evaluate the potential energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$V(\sigma) = \frac{1}{2}Y\int |\mu|^2 dxdydz = \frac{1}{2}Y\int_0^{2\pi} d\phi \int_\Gamma y(I_M + I_m + 2I_{Mm})dxdy \tag{26}$$

where Γ is the semi-plane (x, y^+) minus the two semi-disks D_M and D_m of radius R_M and R (see Fig. 3). The term I_M and I_m are just constant terms which do not depend on σ (because you can shift the function in order to have the disk in the origin regardless σ and integrating. They do not contribute to the equations of motion in the Lagrangian and can be discarded. We have:

$$V(\sigma) = 2\pi Y\int_\Gamma yI_{Mm}dxdy = \pi Y\int_\Gamma 2y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)dxdy \tag{27}$$

For the potential energy, given Eq. (92) in Appendix A.4 and Eq. (27) and in units where masses are measured in square meters, we have:

$$V(\sigma) = 4\pi Y\frac{\mathfrak{M}\mathfrak{m}}{\sigma} \tag{28}$$

7.2 Self Energy

The integrals I_M and I_m in Eq. (26) are useless to our calculation but they are important because they represent the self-energy of a particle. For example, given Eq. (83) in Appendix A.4 and Eq. (26) we have that the self energy of a mass, where mass is measured in square meters, is:

$$E_{Self} = \frac{1}{2}Y \int |\mu_m|^2 dxdydz = \pi Y \int_\Gamma y|\mu_m|^2 dxdy = \frac{1}{2}(4\pi Y \eta)\mathfrak{m} \tag{29}$$

which is proportional to the mass and it is not infinite as in Newtonian gravity.

7.3 Newton's Universal Law of Gravity

Given Eq. (28) and taking into account that force is the derivative of energy with respect to distance, in perfect agreement with Newton's gravitation law, we have that the force with which the two space deficiencies M and m, where masses are measured in square meters, attract each other is:

$$F = \left| -\frac{\partial V}{\partial \sigma} \right| = 4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{\sigma^2} \tag{30}$$

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), two bodies attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of their distance.

This is the universal Newton's law of gravitation.

7.4 Kinetic Energy

For the kinetic energy we have that $\frac{d}{dt}\mathbf{u}(x-\sigma, y, z) = -\frac{d\mathbf{u}^m}{dx}\dot{\sigma}$, where $\mathbf{u} = \mathbf{u}^M + \mathbf{u}^m$ are the total displacements. This is because time is present only in $\sigma(t)$ and σ is present only in $\mathbf{u}^m$, while $\frac{d\mathbf{u}^M}{dt} = 0$. As before we evaluate the field in a semi-plane and the final integral by rotating the solution around x:

$$\left|\frac{d\mathbf{u}}{dt}\right|^2 = \left| -\frac{d\mathbf{u}^m}{dx} \right| \dot{\sigma}^2 = [(u_{1x}^m)^2 + (u_{2x}^m)^2]\dot{\sigma} = |\mathbf{u}_x^m|^2 \dot{\sigma}^2 \tag{31}$$

Once again, we can evaluate the kinetic energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$T(\dot{\sigma}) = \frac{1}{2}\rho \int \left|\frac{d\mathbf{u}}{dt}\right|^2 dxdydz = \frac{1}{2}\rho \int_0^{2\pi} d\phi \int_\Gamma y|\mathbf{u}_x^m|^2 \dot{\sigma}^2 dxdy = \pi\rho \left(\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy \right) \dot{\sigma}^2 \tag{32}$$

For the kinetic energy, given Eq. (95) in Appendix A.4 and Eq. (32), in units where mass is measured in square meters, we have:

$$T(\dot{\sigma}) = \frac{1}{2}(4\pi\rho\eta)\mathfrak{m}\dot{\sigma}^2 \tag{33}$$

This is the equivalence principle in its weak formulation because, from Eq. (33) and Eq. (30), we see that the field responsible for the gravitational force is the very same fields that gives inertia to a mass.

7.5 Newton's Second Law of Motion

Given Eq. (28) and (33), we are now ready to write the Lagrangian:

$$\mathcal{L}(\sigma, \dot{\sigma}) = T(\dot{\sigma}) - V(\sigma) = 2\pi\rho\eta\mathfrak{m}\dot{\sigma}^2 - 4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{\sigma} \tag{34}$$

We can find the equations of motion by using the Euler-Lagrange equations. We have:

$$4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{\sigma^2} = 4\pi\rho\eta\mathfrak{m}\ddot{\sigma} \tag{35}$$

which, given Eq. (30), it can be written as:

$$F = (4\pi\rho\eta)\mathfrak{m}\ddot{\sigma} \tag{36}$$

Since ρ and η are parameters of our model, we can always choose them such that $4\pi\rho\eta = 1$ and, in units where mass is measured in square meters, we have:

$$F = \mathfrak{m}a \tag{37}$$

where a is the acceleration experienced by the mass under a force F.

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a body under a forces F experiences an acceleration proportional to the force and inversely proportional to its mass.

This is the Newton's Second Law of Motion.

8 Parameters of the Model

We need to introduce an additional parameter in our elastic model of gravity. If m in units of $[kg]$ is the mass of a particle in Newtonian gravity and $\mathfrak{m}$ in units of $[m^2]$ is the corresponding mass in our elastic model we define:

$$\Omega = \frac{m}{\mathfrak{m}} \ [kg \cdot m^{-2}] \tag{38}$$

The parameter Ω is the conversion factor of the mass in the two models.

Given Eq. (33), knowing that $T = \frac{1}{2}mv^2$ and using Eq. (38), we have:

$$T(v) = 2\pi\rho\eta\mathfrak{m}v^2 = \frac{1}{2}mv^2 \quad \Rightarrow \quad 4\pi\rho\eta = \Omega \tag{39}$$

From which we have:

$$\rho = \frac{\Omega}{4\pi\eta} \tag{40}$$

Given Eq. (30), knowing the universal low of gravitation and using Eq. (38), we have:

$$F = 4\pi Y\frac{\mathfrak{M}\mathfrak{m}}{d^2} = G\frac{Mm}{d^2} \quad \Rightarrow \quad 4\pi Y = G\Omega^2 \tag{41}$$

From which and given that $c^2 = Y/\rho$, we have:

$$\rho = \frac{G\Omega^2}{4\pi c^2} \tag{42}$$

Putting Eq. (40) and Eq. (42) together we find the expression for Ω first and then for the elastic parameters of our model, as a function of η:

$$\Omega = \frac{c^2}{G\eta} \ ; \ \ \rho = \frac{c^2}{4\pi G\eta^2} \ ; \ \ Y = \frac{c^4}{4\pi G\eta^2} \tag{43}$$

Now, given equation (29) for the Self-potential energy of a particle (i.e. the energy required to create it), we want to see if the relation $E = mc^2$ is valid for it, as required by special relativity. Using also Eq. (38), we have:

$$E_{Self} = \frac{1}{2}(4\pi Y\eta)\mathfrak{m} = mc^2 \quad \Rightarrow \quad 4\pi Y\eta = \Omega c^2 \tag{44}$$

and subtitling the expressions of Eq. (43) into the above equation, we find that:

$$4\pi\frac{c^4}{4\pi G\eta^2}\eta = \frac{c^2}{G\eta}c^2 \tag{45}$$

which is an identity, as expected.

9 Planck Mass

From Eq. (17) we have that $\eta = R/\mathfrak{m}$ and given Eq. (38) and (43), we have:

$$R = \frac{G}{c^2} m = \lambda m \tag{46}$$

with

$$\lambda = \frac{G}{c^2} = 7.41 \times 10^{-28} \left[\frac{m}{kg}\right] \; = \; 1.12 \times 10^{-57} \left[\frac{m}{MeV}\right] \tag{47}$$

Now, R is the radius of the removed space, $\eta \ll R$ and we see that even for the lightest particle of known mass, our model gives an R much smaller of the Planck length. And this is a problem for the model althought since this is a totally classical theory we may ignore it.

Finally, if we evaluate the radius R for a Plank mass object we have:

$$R = \lambda m_p = \frac{G}{c^2}\sqrt{\frac{\hbar c}{G}} = \sqrt{\frac{\hbar G}{c^3}} = l_P \tag{48}$$

and we get the Planck length.

Appendix

A.1 D'Alembert Operator - Change of Coordinates

We want to evaluate the d'Alembert operator in the following system of coordinates:

$$\begin{cases} t &= \bar{t} \\ x &= \bar{x} - \sigma(\bar{t}) \\ y &= \bar{y} \\ z &= \bar{z} \end{cases} \tag{49}$$

where $\sigma(\bar{t})$ is any smooth function and $\dot{\sigma}(\bar{t})$ is its derivative with respect to $\bar{t}$. We convert from the notation (t, x, y, z) to the notation x^μ and from the notation $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to the notation $\bar{x}^\mu$. The system of coordinates $\bar{x}^\mu$ we start from is a Minkowski space with a metric $\eta_{\mu\nu} = (-, +, +, +)$. The metric in the new system of coordinates x^μ is given by:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\nu} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \tag{50}$$

From which we get:

$$g_{\mu\nu} = \begin{pmatrix} -1+\dot{\sigma}^2 & -\dot{\sigma} & 0 & 0 \\ -\dot{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{51}$$

The above metric has determinant $g = -1$ and inverse metric equal to:

$$g^{\mu\nu} = \begin{pmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1-\dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{52}$$

The d'Alembert operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\Box\phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right) \tag{53}$$

From which we have:

$$\Box\phi = \frac{1}{\sqrt{1}} \partial_\mu \left(\sqrt{1} \begin{bmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1-\dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \phi \right) \tag{54}$$

and eventually:

$$\Box\phi = \left(-\partial_0^2 + 2\dot\sigma\partial_0\partial_1 + (1-\dot\sigma^2)\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \tag{55}$$

Converting the coordinates from the x^μ to the (t,x,y,z) notation we have:

$$\Box\phi = -\phi_{tt} + 2\dot\sigma\phi_{tx} + (1-\dot\sigma^2)\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{56}$$

A.2 Laplace Operator - Change of Coordinates

We want to evaluate the Laplace operator in the following system of coordinates:

$$(\bar x, \bar y, \bar z) \to (x = \gamma\bar x,\ y = \bar y,\ z = \bar z) \tag{57}$$

We proceed as in the previous section and using the notation x^i. The metric we start from is $\bar g_{ij} = (+,+,+)$. We have:

$$g_{ij} = \begin{pmatrix} \gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{58}$$

$$g^{ij} = \begin{pmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{59}$$

The Laplace operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\nabla^2\phi = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi\right) \tag{60}$$

From which we have:

$$\nabla^2\phi = \frac{1}{\sqrt{\gamma^2}}\partial_i\left(\sqrt{\gamma^2}\begin{bmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}\phi\right) \tag{61}$$

and eventually:

$$\nabla^2\phi = \left(\frac{1}{\gamma^2}\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \tag{62}$$

Converting the coordinates from the x^i to the (x,y,z) notation we have:

$$\nabla^2\phi = \frac{1}{\gamma^2}\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{63}$$

A.3 Scalar Longitudinal Elastic Waves

Longitudinal waves in an elastic material are solution of the vectorial wave equation equation (see [1] Eq. (22.11)):

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{u} + \nabla^2\mathbf{u} = 0 \tag{64}$$

where $\mathbf{u}$ are the infinitesimal displacements in the material and with the additional constrain that $\nabla\times\mathbf{u} = 0$. Moreover, strains in the material are given by (see [1] Eq. (1.5)):

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{65}$$

We are interested in a subset of these solution that can be described by the scalar wave equation. First of all we have to show that such solutions exist. Examples are plane waves and waves with spherical and cylindrical symmetry. For example, given a vector $\mathbf{k}$, called wave vector and a vector $\mathbf{u}_0$ such that $\mathbf{u}_0 \parallel \mathbf{k}$, a single-frequency longitudinal plane wave is given by the equation:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0 Re\{e^{j(\mathbf{k}\cdot\mathbf{x})-\omega t}\} \tag{66}$$

with $\omega = |\mathbf{k}|c$. Since Eq. (64) is rotation invariant, we can always rotate Eq. (66) such that the vector k is parallel to the x axis. By doing so, it is easy to show that Eq. (66) satisfies Eq. (64), it satisfies $\nabla \times \mathbf{u} = 0$, as required by longitudinal waves, and from Eq. (65) it is a shear-less solution meaning that ϵ_{ij} is a diagonal tensor. Moreover, any linear combination of Eq. (66) like solutions, with different wave vectors, satisfies the same properties.

We want to show now that we can map one-to-one solutions like Eq. (66) of Eq. (64) to solution of the scalar wave equation :

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi + \nabla^2\phi = 0 \tag{67}$$

we just take:

$$\phi(\mathbf{x}, t) = |\mathbf{u}| = |\mathbf{u}_0| Re\{e^{j(\mathbf{k}\cdot\mathbf{x})-\omega t}\} = \phi_0 Re\{e^{j(\mathbf{k}\cdot\mathbf{x})-\omega t}\} \tag{68}$$

and this is a one-to-one map because $\phi_0 = |\mathbf{u}_0|$ but given ϕ_0 and the direction $\mathbf{k}$, we can always reconstruct $\mathbf{u}_0$.

We conclude that there is a subset of solution of Eq. (64) that are shear-less longitudinal waves in an elastic material and that can be seen as solutions of the scalar wave equation in the parameter ϕ. For plane waves, we have simply that $|\mathbf{u}| = \phi$. In the general case, if we want to find the displacement $\mathbf{u}(\mathbf{x}, t)$ from any solution $\phi(\mathbf{x}, t)$, we have first to decompose the solution in plane waves by using the three-dimensional Fourier transform:

$$\phi(\mathbf{k}) = \mathcal{F}(\psi(\mathbf{x}, t)) \tag{69}$$

then, for each scalar component we can find the relative vector component as described above:

$$\phi(\mathbf{k}) \rightarrow \mathbf{u}(\mathbf{k}) \tag{70}$$

and finally we can find the displacements by summing over all the components by an anti-Fourier transform.

$$\mathbf{u}(\mathbf{x}, t) = \mathcal{F}^{-1}(\mathbf{u}(\mathbf{k})) \tag{71}$$

A.4 Integrals

The displacements relevant to a point mass (i.e. a discrete Space Deficiency or radius R), when the mass is in the origin and in spherical coordinates, are the following (see Eq. (16)):

$$\mathbf{u} = -\frac{\mathfrak{m}}{r} = -\frac{R\eta}{r} \tag{72}$$

Working on the (x,y) plane we have:

$$\phi = -|\mathbf{u}| = -\frac{R\eta}{\sqrt{x^2 + y^2}} \tag{73}$$

$$\mu_1 = -\frac{\partial\phi}{\partial x} = \frac{R\eta x}{(x^2 + y^2)^{\frac{3}{2}}} \tag{74}$$

$$\mu_2 = \frac{\partial\phi}{\partial y} = \frac{R\eta y}{(x^2 + y^2)^{\frac{3}{2}}} \tag{75}$$

Moreover:

$$\mathbf{u} = \left(-\frac{R\eta\cos\theta}{r}, -\frac{R\eta\sin\theta}{r}\right) \tag{76}$$

where θ is the angle between r and the x axis. We have:

$$\mathbf{u} = \left(-\frac{R\eta\cos\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2 + y^2}}, -\frac{R\eta\sin\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2 + y^2}}\right) \tag{77}$$

and finally for $x > 0$ (where the arctan is well defined):

$$\mathbf{u} = R\eta\left(\frac{-1}{\sqrt{\frac{y^2}{x^2} + 1}\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{\frac{y^2}{x^2} + 1}\, x\sqrt{x^2 + y^2}}\right) \tag{78}$$

We have to take into account of the limitation $x > 0$ when we integrate this functions by using the symmetry with respect to the y axis and doubling the integral in the first quadrant only. From the above we have:

$$u_{1x} = \frac{d}{dx}u_1 = \frac{R\eta(x^4 - y^4)}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^3 \left(x^2+y^2\right)^{\frac{3}{2}}} \tag{79}$$

$$u_{2x} = \frac{d}{dx}u_2 = \frac{2R\eta y}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^2 \sqrt{x^2+y^2}} \tag{80}$$

A.4.1 Integral of term $y|\mu|^2$

This term does not depend from both σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mu|^2 dxdy = \int_\Gamma y[(\mu_1)^2 + (\mu_2)^2]dxdy \tag{81}$$

Substituting the terms (74) and (75) we get:

$$\int_\Gamma y|\mu|^2 dxdy = R^2\eta^2 \int_\Gamma \frac{y(x^2+y^2)}{(x^2+y^2)^3}dxdy \tag{82}$$

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and D_m of radius R and R_M (see Fig. 3). However, when the two masses are far apart, we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, the fact that $y = r\sin\theta$ and given that $\mathfrak{m} = R\eta$ we have:

$$\int_\Gamma y|\mu|^2 dxdy = R^2\eta^2 \int_0^\pi \sin\phi d\phi \int_R^\infty \frac{r}{r^4} r dr = 2R\eta^2 = 2\eta\mathfrak{m} \tag{83}$$

A.4.2 Integral of term $y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)$

We have already evaluated this integral in a different context (see [2]). This term depends from both the two masses and the distance between them. Whatever the value of σ at a given time, we define $\delta = \sigma/2$ and we shift the integrand on the x axis such that the origin of the (x, y) plane is exactly in the middle between the two masses (see Fig. 2). The integral we want to evaluate is:

$$\mathcal{I}_\Gamma = \int_\Gamma y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)dxdy \tag{84}$$

Substituting the terms (74) and (75) and taking into account of their displacements along x, we get:

$$\mathcal{I}_\Gamma = \int_\Gamma y\left(\frac{R_M R\eta^2 (x-\delta)(x+\delta)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} + \frac{R_M R\eta^2 y^2}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}}\right)dxdy \tag{85}$$

Which can be simplified as:

$$\mathcal{I}_\Gamma = R_M R\eta^2 \int_\Gamma \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}}dxdy \tag{86}$$

It is possible to show that the disks D_M and D_m do not contribute to the above integral because:

$$\mathcal{I}_{D_M} = \mathcal{I}_{D_m} = 0 \quad \Rightarrow \quad \mathcal{I}_\Gamma = \mathcal{I}_{(x,y^+)} \tag{87}$$

Where $\mathcal{I}_{(x,y^+)}$ is the integral in the semi-plane $(x, y+)$, $\mathcal{I}_{D_M}$ is the integral in D_M and $\mathcal{I}_{D_m}$ is the integral in D_m. And therefore we have:

$$\mathcal{I}_\Gamma = \mathcal{I}_{(x,y^+)} = R_M R\eta^2 \int_{-\infty}^{\infty} dx \int_0^\infty \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}}dy \tag{88}$$

We use the symmetry of the function with respect to the y axis by integrating in the firsts quadrant only and doubling the result:

$$\mathcal{I}_\Gamma = 2R_M R\eta^2 \int_0^\infty dx \int_0^\infty \frac{y\left(y^2+x^2-\delta^2\right)}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} dy \tag{89}$$

We integrate with respect to the y variable first and we get:

$$\mathcal{I}_\Gamma = 2R_M R\eta^2 \int_0^\infty \frac{|x-\delta|+x-\delta}{4x^2(x-\delta)} dx \tag{90}$$

then we integrate with respect to the x variable. We have:

$$\mathcal{I}_\Gamma = 2R_M R\eta^2 \left(\int_0^\delta 0\, dx + \int_\delta^\infty \frac{1}{2x^2} dx \right) = \frac{R_M R\eta^2}{\delta} \tag{91}$$

since $\sigma = 2\delta$, $\mathfrak{M} = R_M\eta$ and $\mathfrak{m} = R\eta$, we have:

$$\int_\Gamma y(\mu_1^M \mu_1^m + \mu_2^M \mu_2^m) dxdy = \frac{2\mathfrak{M}\mathfrak{m}}{\sigma} \tag{92}$$

A.4.3 Integral of term $y|\mathbf{u}_x^m|^2$

This term does not depend both from σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = \int_\Gamma y[(u_{1x})^2 + (u_{2x})^2] dxdy \tag{93}$$

Substituting the terms (79) and (80) and simplifying we get:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^2\eta^2 \int_\Gamma \frac{y}{(x^2+y^2)^2} dxdy \tag{94}$$

Keep in mind that the terms (79) and (80) were valid only in the first quadrant of the semi-plane (x, y) due to the $\arctan(y/x)$ valid only for $x > 0$. However, from the symmetry of the integrand we see that the terms are actually valid in all the semi-plane and we can proceed.

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and Dm of radius R and R_M (see Fig. 3). However, when the two masses are far apart we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, the fact that $y = r\sin\theta$ and taking into account that $\mathfrak{m} = R\eta$, we have:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^2\eta^2 \int_0^\pi \sin\phi d\phi \int_R^\infty \frac{r}{r^4} rdr = 2R\eta^2 = 2\eta\mathfrak{m} \tag{95}$$

References

[1] L. D. Landau E. M. Lifshitz. *Course of Theoretical Physics Vol. 7 - Theory of Elasticity 2nd Edition* - Pergamon Press (1970).

[2] V. Nardozza. *Energy Stored in the Gravitational Field.* https://vixra.org/abs/1905.0515 (2019).