Gravitational and Inertial Mass

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Abstract

It is a law of nature that gravitational and inertial mass are the same. This is know as the equivalence principle in its weak formulation. In Newtonian gravity, fields act with an instant action at distance. In this paper, we show that by adding the assumption that fields perturbations propagate at a finite velocity, this lead somehow to the equivalence of gravitational and inertial mass.

Key Words: Newtonian gravity, equivalence principle.

1 Introduction

Two object that gravitate in the same way, have the same inertia. This is known as the equivalence principle in its weak formulation. In this paper, we start from classical Newtonian gravity, where bodies act with an instant action at distance, and we slightly change the equations of the theory to take into account that perturbations in gravitational filed propagate at a finite speed. From this simple assumption we derive a mechanical model that is equivalent to the modified version of the Newtonian gravity because it is described by the same equation. Finally, from this model, we derive the equivalence principle in its weak form.

2 Newtonian Gravity with no Instant Action at Distance

In Newtonian gravity, given a distribution of mass $\rho_m(\mathbf{x})$, the gravitational potential generated by this distribution of mass, in units where $4\pi G = 1$ and where G is the gravitational constant, is given by the Poisson's equation:

$$\nabla^2\phi(\mathbf{x}, t) = \rho_m(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x}$ is a point in 3-dimensional space. In particular for a point mass m in the origin, we have:

$$\nabla^2\phi(\mathbf{x}) = m\delta(\mathbf{x}) \tag{2}$$

In this model gravity is an instant action. I we move a mass, the potential will change instantaneously in the whole universe. Modern physics has taught us that this is never the case and that perturbations in fields propagate at the speed of light. It is natural to add to our equation a term which takes into how perturbations propagate at a limited speed. The Poisson's equation has second derivatives with respect to the special coordinates. The obvious thing to do is to add a second derivative with respect to time and turn the Poisson's equation into a wave equation. In units where also the speed of light $c = 1$ we have:

$$-\frac{\partial^2\phi(\mathbf{x}, t)}{\partial t^2} + \nabla^2\phi(\mathbf{x}, t) = \rho_m(\mathbf{x}, t) \tag{3}$$

which is also written as:

$$\Box\phi = \rho_m \tag{4}$$

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Where $\Box$ is the d'Alembert operator (with -1 in front of the time derivative). For a moving point mass with coordinates $\sigma(t) = (x(t),\ y(t),\ z(t))$ we have:

$$\Box\phi = m\delta(\mathbf{x} - \sigma) \tag{5}$$

Being this a wave equation, clearly a perturbation moves into space at the speed of light. This is compatible con the fact that massless particle travel at the speed of light. As a matter of fact a massless particle has energy, it curves space and it can be seen as a perturbation travelling in space.

In the next section we will discuss what happens to "massive" perturbations where "massive" needs to be defined properly.

3 Length Contraction

Suppose a mass moves in space along the x axis with coordinates $\sigma(t) = (\sigma(t),\ 0,\ 0)$ where $\sigma(t)$ is a smooth function. We want to write Eq. (5) in the reference frame of the mass. To do that we use the change of coordinates (41) and we get an expression for the d'Alembert operator given by (48). Eq. (5) becomes:

$$-\phi_{tt} + 2\dot{\sigma}\phi_{tx} + (1 - \dot{\sigma}^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{6}$$

By construction, in this reference frame the mass is always in the origin. Suppose now that $\sigma(t) = vt$ (i.e. the mass is moving with constant velocity along x), in this case Eq. (6) becomes:

$$-\phi_{tt} + 2v\phi_{tx} + (1 - v^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{7}$$

We are looking for a solution which is stationary in the mass reference frame. To do that we set the derivatives of the potential with respect to time to vanish. We have:

$$\frac{1}{\gamma^2}\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{8}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \quad \xrightarrow{c \neq 1} \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{9}$$

is the Lorentz factor and we have reintroduced the speed of light.

From Eq. (55) we see that the solution of Eq. (8) is:

$$\phi(\mathbf{x}) = \tilde{\phi}(\gamma x,\ y,\ z) \tag{10}$$

where $\tilde{\phi}$ is the solution of Eq. (2) which in polar coordinates is:

$$\tilde{\phi}(r) = -\frac{1}{4\pi}\frac{m}{r} \quad \xrightarrow{4\pi G \neq 1} \quad \tilde{\phi}(r) = -\frac{Gm}{r} \tag{11}$$

and where we have reintroduced the gravitational constant. Note that a moving potential get contracted along the direction of motion by a factor $1/\gamma$ with respect to a not moving potential.

4 Analogy with a Mechanical Model

To better understand the physical behaviour of our moving "massive" perturbation, we want to make an analogy with a mechanical system described by the same equation (3) of our Newtonian gravitational field propagating at a fixed velocity. We choose a model where space is just an elastic linear material and where propagating pressure waves are the equivalent of propagating Newtonian gravity perturbations.

We are not saying that space is an elastic material, we are saying that the elastic material is described by the same equation of Newtonian gravity with non instantaneous action at distance and therefore if we can find a characteristic of the elastic system (e.g. kinetic energy associated to variation of fields), this will apply also to gravity.

This mechanical model can be used not only to describe perturbations of the gravitation potential propagating in space. It can also be used to describe a concentrated mass. This can be done by using the idea of a **Space Deficiency** free to move in space (frictionless) to represent what we have called a "massive" perturbation at the beginning of this section.

A Space Deficiency is a ball B of elastic material removed from the space which leave the space with a hole and a boundary ∂B between the hole and the empty space. Then we apply a force orthogonal to ∂B in order to shrink the radius of the ball by several order of magnitude. The material will readjust creating strains with a radial configuration and amplitude that goes like $1/r^2$ which mimic the gravitational field generated by a concentrated mass. A precise definition of a Space Deficiency is given in Appendix A.3.

5 Energy Density Due to Variation of Gravitational Fields

In our elastic model of gravity, the elastic potential for infinitesimal displacements in empty regions of space (i.e. $\rho_m = 0$), is the solution of the following equation:

$$-\rho\frac{\partial^2\phi}{\partial t^2} + Y\nabla^2\phi = 0 \tag{12}$$

where ρ is the density and Y the Young Modulus of the elastic material.

In the case where the field of displacements has the property that $|\mathbf{u}| = \phi$ and given the field of strains with opposite sign $\mu = -\nabla\phi$, the potential energy density e_V and the kinetic energy density e_T stored in the field of displacements are given by the following expressions:

$$e_V = \frac{1}{2}Y|\mu|^2 \ \ ; \ \ e_T = \frac{1}{2}\rho|\dot{\mathbf{u}}|^2 \tag{13}$$

Moreover we know that the velocity of the waves in the material is $c = \sqrt{Y/\rho}$.

For finite displacements Eq. (12) does not hold any more, the phenomenon is described by a non linear equation and waves interact each other when they meet. However, in the quasi-static case where finite displacements are caused by discontinuities that move at a speed much lower than light and strains have vanishing curl (i.e we have $\nabla \times \mu = 0$), then all the above equations hold, the fields are described by a linear equation and the energy density is given by the same expressions relevant to infinitesimal displacements (see [1]). This is the case studied in this paper where the strains μ are linear combinations of radial fields that go like $1/r^2$ and are caused by point-like discontinuities.

For gravity, if we can find a field θ such that $|\theta| = \phi$ (θ is the analogous of $\mathbf{u}$), then $g = -\nabla\phi$ is the analogous of μ. Moreover, we know that the potential energy stored in the gravitational field is given by the following expression (see [2]):

$$e_V = \frac{1}{2}\overbrace{\left(\frac{1}{4\pi G}\right)}^{K_V}|\mu|^2 \tag{14}$$

Now, if K_V and K_T are the constants associated to potential and kinetic energy in the gravitational fields, then our analogy with an elastic model tells us that $c = \sqrt{K_V/K_T}$ and we find easily:

$$e_T = \frac{1}{2}\overbrace{\left(\frac{1}{4\pi G c^2}\right)}^{K_T}|\dot{\theta}|^2 \tag{15}$$

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), gravitational field must have "kinetic" energy associated to field variation.

6 Newton's First Law of Motion

In our mechanical model, gravitational fields associated to an the elementary volume dV, will have potential energy proportional to the square of the strains module $|\mu|^2$ (as already happens in classical instant action at distance Newtonian gravity) and kinetic energy proportional to square of the rate of variation of the displacements $|\dot{\mathbf{u}}|^2$ (and this is new) which are related to the velocity at which the volume dV is moving in the material.

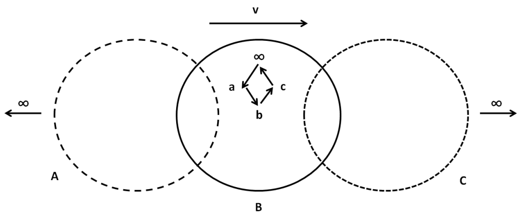

Figure 1: Orbits of Elementary Volumes

Now, if the discontinuity is moving with a velocity v to the right (see Fig. 1), points inside the area close to the discontinuity will readjust as the discontinuity passes by because the space deficiency will pull them. Elementary volumes will follow orbits and since the material has a mass density this will be associated with kinetic energy. However, as long as the velocity of the discontinuity moves with constant velocity the total kinetic energy is constant and the total potential energy is also constant because it depends on the the shape of the potential which does not change.

When the discontinuity (i.e. our point mass) start moving pushed by a force, the kinetic energy of the field due to variation of the fields (i.e. $\dot{\mathbf{u}} \neq 0$) will increase and when we stop pushing the discontinuity, it is the conservation of this energy that gives inertia to it. Field variation is the flywheel of the systems.

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a mass has inertia meaning that it keeps moving at the same speed in absence of external interferences.

This is the first Newton's law of motion.

7 Evaluation of the Energy

We want to evaluate the Lagrangian $\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma})$ of our model. We want a simplified quasi-static system where $v << 1$, all the perturbations flies away at speed of light and the shape of the potential readjust itself to a potential $\phi(\sigma)$ depending only on σ and moving rigidly. With this assumption, the shape of ϕ is independent from σ while its position depends on σ only. This will simplify the Lagrangian as follows:

$$\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma}) \to \mathcal{L}(\sigma(t), \dot{\sigma}(t)) \tag{16}$$

In order to do that, we consider a two mass System M and m in $\mathbb{R}^3$ with M nailed to the position $(-\delta, 0, 0)$ and m free to move along the x axis (see Fig. 2).

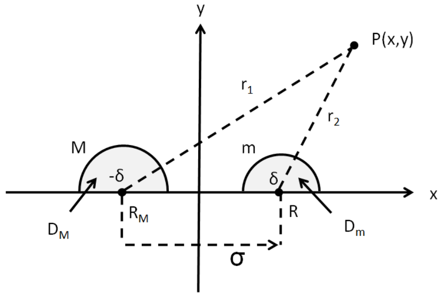

Figure 2: Two Mass System

To write the expression for the Lagrangian, we need to evaluate the potential and kinetic energy of the system. In our elastic model a mass m in units of $[kg]$, corresponds to a mass $\mathfrak{m} = R\eta$ in units of $[m^2]$, is a removed ball of space of radius R with η a constant (see Appendix A.3). Moreover the gravitational field is represented by the strains $\mu = -\nabla\mathbf{u}$, where $\mathbf{u}$ are the displacements and the density of potential and kinetic energy is given by Eq. (13).

7.1 Potential Energy

To simplify the calculation we note that fields in a point P both lays on the plane passing through P and the axis x and they have a cylindrical symmetry with respect to the axis x. We will evaluate our fields on the (x, y) plane and we will get the field in space just rotating the (x, y) fields around the x axis. For the potential energy, on the (x, y) plane we have:

$$|\mu|^2 = |\mu^M + \mu^m|^2 = (\mu_1^M + \mu_1^m)^2 + (\mu_2^M + \mu_2^m)^2 \tag{17}$$

And we get two self-energy and one cross-energy terms. We give names to them:

$$|\mu|^2 = |\mu^M|^2 + |\mu^m|^2 + 2(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m) = I_M + I_m + 2I_{Mm} \tag{18}$$

We can evaluate the potential energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$V(\sigma) = \frac{1}{2}Y\int|\mu|^2 dxdydz = \frac{1}{2}Y\int_0^{2\pi} d\phi\int_\Gamma y(I_M + I_m + 2I_{Mm})dxdy \tag{19}$$

where Γ is the semi-plane (x, y^+) minus the two semi-disks D_M and $D_m$of radius R_M and R (see Fig. 2). The term I_M and I_m are just constant terms which do not depend on σ (because you can shift the function in order to have the disk in the origin regardless σ and integrating. They do not contribute to the equations of motion in the Lagrangian and can be discarded. We have:

$$V(\sigma) = 2\pi Y\int_\Gamma yI_{Mm}dxdy = 2\pi Y\int_\Gamma y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)dxdy \tag{20}$$

For the potential energy, given Eq. (20) and (82) we have:

$$V(\sigma) = \frac{4\pi Y\mathfrak{M}\mathfrak{m}}{\sigma} \tag{21}$$

7.2 Self Energy

The integrals I_M and I_m in Eq. (19) are useless to our calculation but they are important because they represent the self-energy of a particle. For example, given Eq. (73) in Appendix A.4 and Eq. (19) we have that the self energy of a particle is:

$$E_{Self} = \frac{1}{2}Y\int|\mu_m|^2 dxdydz = 2\pi Y\int_\Gamma y|\mu_m|^2 dxdy = 4\pi^2 Y\eta\mathfrak{m} \tag{22}$$

which is proportional to the mass it is not infinite as in Newtonian gravity.

7.3 Kinetic Energy

For the kinetic energy we have that $\partial_t \mathbf{u}^m(x-\sigma, y, z) = -\partial_x \mathbf{u}\dot{\sigma}$. This is because time is present only in $\sigma(t)$ and σ is present only in $\mathbf{u}^m$. while $\partial_t \mathbf{u}^M = 0$. As before we evaluate the field in a semi-plane and the final integral by rotating the solution around x:

$$|\partial_t \mathbf{u}|^2 = |-\partial_x \mathbf{u}^M - \partial_x \mathbf{u}^m|\dot{\sigma}^2 = [(u^m_{1x})^2 + (u^m_{2x})^2]\dot{\sigma} = |\mathbf{u}^m_x|^2\dot{\sigma}^2 \tag{23}$$

Once again, we can evaluate the kinetic energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$T(\dot{\sigma}) = \frac{1}{2}\rho \int |\partial_t \mathbf{u}|^2 dxdydz = \frac{1}{2}\rho \int_0^{2\pi} d\phi \int_\Gamma y|\mathbf{u}^m_x|^2\dot{\sigma}^2 dxdy = \pi\rho \left(\int_\Gamma y|\mathbf{u}^m_x|^2 dxdy\right)\dot{\sigma}^2 \tag{24}$$

For the kinetic energy, given Eq. (24) and (85) we have:

$$T(\dot{\sigma}) = 2\pi^2\rho\eta\mathfrak{m}\dot{\sigma}^2 \tag{25}$$

7.4 Newton's Universal Low of Gravity

Given Eq. (21) and taking into account that force is the derivative of energy with respect to distance, in perfect agreement with Newton's gravitation law, we have that the force with which the two space deficiencies M and m attract each other is:

$$F = -\frac{\partial V}{\partial \sigma} = 4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{\sigma^2} \tag{26}$$

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a two bodies attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of their distance.

This is the universal Newton's law of gravitation.

7.5 Newton's Second Law of Motion

Given Eq. (21) and (25), we are now ready to write the Lagrangian:

$$\mathcal{L}(\sigma, \dot{\sigma}) = T(\dot{\sigma}) - V(\sigma) = 2\pi^2\rho\eta\mathfrak{m}\dot{\sigma}^2 - \frac{4\pi Y\mathfrak{M}\mathfrak{m}}{\sigma} \tag{27}$$

We can find the equations of motion by using the Euler-Lagrange equations. We have:

$$-4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{\sigma^2} = 4\pi^2\rho\eta\mathfrak{m}\ddot{\sigma} \tag{28}$$

which, given Eq. (26) and using a positive direction for force and motion when towards M, it can be written as:

$$F = (4\pi^2\rho\eta)\mathfrak{m}\ddot{\sigma} \tag{29}$$

Which is:

$$F \propto \mathfrak{m}a \tag{30}$$

where a is the acceleration experienced by the mass under a force F.

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a body under a forces F experiences an acceleration proportional to the force and inversely proportional to its mass.

This is the Newton's Second Law of Motion.

8 Parameters of the Model

We need to introduce an additional parameter in our elastic model o gravity. If m in units of $[kg]$ is the mass of a particle in Newtonian gravity and $\mathfrak{m}$ in units of $[m^2]$ is the corresponding mass in our elastic model we define:

$$\Omega = \frac{m}{\mathfrak{m}} \;\; [kg \cdot m^{-2}] \tag{31}$$

The parameter Ω is the conversion factor of the mass in the two models.

Given Eq. (25), knowing that $T = \frac{1}{2}mv^2$ and using Eq. (31), we have:

$$T(v) = 2\pi^2 \rho\eta\mathfrak{m}v^2 = \frac{1}{2}mv^2 \;\; \Rightarrow \;\; 4\pi^2\rho\eta = \Omega \tag{32}$$

From which we have:

$$\rho = \frac{\Omega}{4\pi^2\eta} \tag{33}$$

Given Eq. (26), knowing the universal low of gravitation and using Eq. (31), we have:

$$F = 4\pi Y \frac{\mathfrak{M}\mathfrak{m}}{d^2} = G\frac{Mm}{d^2} \;\; \Rightarrow \;\; 4\pi Y = G\Omega^2 \tag{34}$$

From which and given that $c^2 = Y/\rho$, we have:

$$\rho = \frac{G\Omega^2}{4\pi c^2} \tag{35}$$

Putting Eq. (33) and Eq. (35) together we find the expression for Ω first and then for the elastic parameters of our model, as a function of η:

$$\Omega = \frac{c^2}{\pi G\eta} \;\; ; \;\; \rho = \frac{c^2}{4\pi^3 G\eta^2} \;\; ; \;\; Y = \frac{c^4}{4\pi^3 G\eta^2} \tag{36}$$

Now, given equation (22) for the Self-potential energy of a particle (i.e. the energy required to create it), we want to see if the relation $E = mc^2$ is valid for it, as required by special relativity. Using also Eq. (31), we have:

$$E_{Self} = 4\pi^2 Y\eta\mathfrak{m} = mc^2 \;\; \Rightarrow \;\; 4\pi^2 Y\eta = \Omega c^2 \tag{37}$$

and subtitling the expressions of Eq. (36) into the above equation, we find that:

$$4\pi^2 \frac{c^4}{4\pi^3 G\eta^2}\eta = \frac{c^2}{\pi G\eta}c^2 \tag{38}$$

which is an identity, as expected.

Finally, from Eq. (60) we have that $\eta = R/\mathfrak{m}$ and given Eq. (31) and (36), we have:

$$R = \frac{\pi G m}{c^2} = K_m m \tag{39}$$

with

$$Km = \frac{\pi G}{c^2} = 2.33 \times 10^{-27} \left[\frac{m}{kg}\right] \; = \; 3.52 \times 10^{-57} \left[\frac{m}{MeV}\right] \tag{40}$$

Now, R is the radius of the removed space, $\eta \ll R$ and we see that even for the lightest particle of known mass, our model gives an R much smaller of the Planck length.

This is a problem and there are two ways of dealing with it. We may say the this is a classical mechanics theory and ignore the problem or we may say that this is unacceptable whatever physics we are doing and therefore our model has to be fixed in the framework of classical mechanics or introducing in the model general relativity or quantum mechanics.

Appendix

A.1 D'Alembert Operator - Change of Coordinates

We want to evaluate the d'Alembert operator in the following system of coordinates:

$$\begin{cases} t &= \bar{t} \\ x &= \bar{x} - \sigma(\bar{t}) \\ y &= \bar{y} \\ z &= \bar{z} \end{cases} \tag{41}$$

where $\sigma(\bar{t})$ is any smooth function and $\dot{\sigma}(\bar{t})$ is its derivative with respect to $\bar{t}$. We convert from the notation (t, x, y, z) to the notation x^μ and from the notation $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to the notation $\bar{x}^\mu$. The system of coordinates $\bar{x}^\mu$ we start from is a Minkowski space with a metric $\eta_{\mu\nu} = (-, +, +, +)$. The metric in the new system of coordinates x^μ is given by:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\nu} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \tag{42}$$

From which we get:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dot{\sigma}^2 & -\dot{\sigma} & 0 & 0 \\ -\dot{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

The above metric has determinant $g = -1$ and inverse metric equal to:

$$g^{\mu\nu} = \begin{pmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1 - \dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

The d'Alembert operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\Box\phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \tag{45}$$

From which we have:

$$\Box\phi = \frac{1}{\sqrt{1}} \partial_\mu \left(\sqrt{1} \begin{bmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1 - \dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \phi \right) \tag{46}$$

and eventually:

$$\Box\phi = \left(-\partial_0^2 + 2\dot{\sigma}\partial_0\partial_1 + (1 - \dot{\sigma}^2)\partial_1^2 + \partial_2^2 + \partial_3^2 \right) \phi \tag{47}$$

Converting the coordinates from the x^μ to the (t, x, y, z) notation we have:

$$\Box\phi = -\phi_{tt} + 2\dot{\sigma}\phi_{tx} + (1 - \dot{\sigma}^2)\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{48}$$

A.2 Laplace Operator - Change of Coordinates

We want to evaluate the Laplace operator in the following system of coordinates:

$$(\bar{x}, \bar{y}, \bar{z}) \to (x = \gamma\bar{x},\ y = \bar{y},\ z = \bar{z}) \tag{49}$$

We proceed as in the previous section and using the notation x^i. The metric we start from is $\bar{g}_{ij} = (+, +, +)$. We have:

$$g_{ij} = \begin{pmatrix} \gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{50}$$

$$g^{ij} = \begin{pmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{51}$$

The Laplace operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\nabla^2\phi = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi\right) \tag{52}$$

From which we have:

$$\nabla^2\phi = \frac{1}{\sqrt{\gamma^2}}\partial_i\left(\sqrt{\gamma^2}\begin{bmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}\phi\right) \tag{53}$$

and eventually:

$$\nabla^2\phi = \left(\frac{1}{\gamma^2}\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \tag{54}$$

Converting the coordinates from the x^i to the (x, y, z) notation we have:

$$\nabla^2\phi = \frac{1}{\gamma^2}\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{55}$$

A.3 Space Deficiency as a Model for a Concentrated Mass

Let us consider a space filled with an elastic material, we remove a ball of radius R and we apply a force on the spherical boundary of the removed ball, orthogonal to the surface and in such a way the sphere will reduce its radius from R to η. At the equilibrium, the density in the material outside the sphere will decrease. The figure below shown the amplitude of the displacements in the material in as a function of r in a spherical coordinates systems.

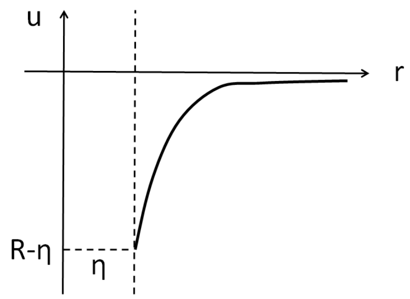

Figure 3: Displacements in a space deficiency

If the solution is quasi static and it is the superposition of solutions which have vanishing curl (i.e. no shiers stress), the equation for the equilibrium in the elastic material is the Poisson's equation (see [1]):

$$\nabla^2\phi = 0 \tag{56}$$

where ϕ is the potential and if we assume the displacement $\nabla \times \mathbf{u} = 0$ to have vanishing curl for any reason, the potential $\phi = |\mathbf{u}|$ is equal to the amplitude of the displacements.

In spherical coordinates and for a spherical symmetric solution, the general solution for the Poisson equation is the following:

$$|u| = a\frac{1}{r} + br \quad \text{with} \quad |\mathbf{u}| = \phi \tag{57}$$

For $r > \eta$ we have b=0 and:

$$|u(\eta)| = a\frac{1}{\eta} = -(R - \epsilon) \;\;\Rightarrow\;\; a = -\eta(R - \eta) \tag{58}$$

If we assume $R >> \eta$ than we have that the displacements in the material are given by :

$$\mathbf{u}(r) = -\frac{R\eta}{r}\hat{i}_r = -\frac{\mathfrak{m}}{r}\hat{i}_r \tag{59}$$

where

$$\mathfrak{m} = R\eta \tag{60}$$

Moreover, we define the μ to be the strains in the material with opposite sign:

$$\mu(r) = -\nabla|u| = -\frac{\mathfrak{m}}{r^2}\hat{i}_r \tag{61}$$

From the above equations we see that we can introduce a mass, in our Newtonian gravity elastic model, by means of a **space deficiency** (a removed ball of material followed by a contraction of the spherical boundary). By doing so, gravitational potential will be equivalent to amplitude of displacements $|\mathbf{u}|$ and gravitational field will be equivalent to strains with opposite sign μ.

Moreover in our model, the mass of a space deficiency (e.g. a particle) is given by $\mathfrak{m} = R\eta$, has unit of surface (i.e. $[m^2]$) and we assume that the radius η is a constant of nature equal for all space deficiency. The mass of a particle is therefore given only by the radius R of the space deficiency.

Note that with η that goes to zero, the required radius R for a given mass m goes to infinity together with the energy required to shrink the boundary of the spherical removed ball to zero. This is the reason why we shrink the spherical boundary to a fixed radius η and not to a point.

A.4 Integrals

The displacements relevant to a point mass (i.e. a discrete space deficiency or radius R), when the mass is in the origin and in spherical coordinates, are the following (see Eq. (59)):

$$\mathbf{u} = -\frac{\mathfrak{m}}{r} = -\frac{R\eta}{r} \tag{62}$$

Working on the (x,y) plane we have:

$$\phi = -|\mathbf{u}| = -\frac{R\eta}{\sqrt{x^2+y^2}} \tag{63}$$

$$\mu_1 = -\frac{\partial\phi}{\partial x} = \frac{R\eta x}{(x^2+y^2)^{\frac{3}{2}}} \tag{64}$$

$$\mu_2 = \frac{\partial\phi}{\partial y} = \frac{R\eta y}{(x^2+y^2)^{\frac{3}{2}}} \tag{65}$$

Moreover:

$$\mathbf{u} = \left(-\frac{R\eta\cos\theta}{r}, -\frac{R\eta\sin\theta}{r}\right) \tag{66}$$

where θ is the angle between r and the x axis. We have:

$$\mathbf{u} = \left(-\frac{R\eta\cos\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}, -\frac{R\eta\sin\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}\right) \tag{67}$$

and finally for ($x > 0$ where the arctan is well defined:

$$\mathbf{u} = R\eta\left(\frac{-1}{\sqrt{\frac{y^2}{x^2}+1}\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{\frac{y^2}{x^2}+1}\,x\sqrt{x^2+y^2}}\right) \tag{68}$$

We have to take into account of the limitation $x > 0$ when we integrate this functions by using the symmetry with respect to the y axis and doubling the integral in the first quadrant only. From the above we have:

$$u_{1x} = \partial_x u_1 = \frac{R\eta(x^4-y^4)}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^3\left(x^2+y^2\right)^{\frac{3}{2}}} \tag{69}$$

$$u_{2x} = \partial_x u_2 = \frac{2R\eta y}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^2\sqrt{x^2+y^2}} \tag{70}$$

A.4.1 Integral of term $y|\mu|^2$

This term does not depend from both σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mu|^2 dxdy = \int_\Gamma y[(\mu_1)^2 + (\mu_2)^2]dxdy \tag{71}$$

Substituting the terms (64) and (65) we get:

$$\int_\Gamma y|\mu|^2 dxdy = R^2\eta^2 \int_\Gamma \frac{y(x^2+y^2)}{(x^2+y^2)^3} dxdy \tag{72}$$

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and D_m of radius R and R_M (see Fig. 2). However, when the two masses are far apart, we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, the fact that $y = r\sin\theta$ and given that $\mathfrak{m} = R\eta$ we have:

$$\int_\Gamma y|\mu|^2 dxdy = R^2\eta^2 \int_0^\pi \sin\phi d\phi \int_R^\infty \frac{r}{r^4} rdr = 2\pi R\eta^2 = 2\pi\eta\mathfrak{m} \tag{73}$$

A.4.2 Integral of term $y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)$

We have already evaluated this integral in a different context (see [2]). This term depends from both the two masses and the distance between them. Whatever the value of σ at a given time, we define $\delta = \sigma/2$ and we shift the integrand on the x axis such that the origin of the (x, y) plane is exactly in the middle between the two masses (see Fig. 2). The integral we want to evaluate is:

$$\mathcal{I} = \int_\Gamma y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)dxdy \tag{74}$$

Substituting the terms (64) and (65) and taking into account of their displacements along x, we get:

$$\mathcal{I} = \int_\Gamma y\left(\frac{R_M R\eta^2 (x-\delta)(x+\delta)}{((x-\delta)^2+y^2)^{\frac{3}{2}}((x+\delta)^2+y^2)^{\frac{3}{2}}} + \frac{R_M R\eta^2 y^2}{((x-\delta)^2+y^2)^{\frac{3}{2}}((x+\delta)^2+y^2)^{\frac{3}{2}}}\right)dxdy \tag{75}$$

Which can be simplified as:

$$\mathcal{I} = R_M R\eta^2 \int_\Gamma \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}}dxdy \tag{76}$$

It is possible to show that the disks D_M and D_m do not contribute to the above integral because:

$$\mathcal{I}_{D_M} = \mathcal{I}_{D_m} = 0 \;\Rightarrow\; \mathcal{I}_\Gamma = \mathcal{I}_{(x,y^+)} \tag{77}$$

And therefore we have:

$$\mathcal{I} = \mathcal{I}_{(x,y^+)} = R_M R\eta^2 \int_{-\infty}^\infty dx \int_0^\infty \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}}dy \tag{78}$$

We use the symmetry of the function with respect to the y axis by integrating in the firsts quadrant only and doubling the result:

$$\mathcal{I} = 2R_M R\eta^2 \int_0^\infty dx \int_0^\infty \frac{y\left(y^2+x^2-\delta^2\right)}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}}dy \tag{79}$$

We integrate with respect to the y variable first and we get:

$$\mathcal{I} = 2R_M R\eta^2 \int_0^\infty \frac{|x-\delta|+x-\delta}{4x^2(x-\delta)}dx \tag{80}$$

then we integrate with respect to the x variable. We have:

$$\mathcal{I} = 2R_M R\eta^2 \left(\int_0^\delta 0\, dx + \int_\delta^\infty \frac{1}{2x^2} dx \right) = \frac{R_M R\eta^2}{\delta} \tag{81}$$

since $\sigma = 2\delta$, $\mathfrak{M} = R_M \eta$ and $\mathfrak{m} = R\eta$, we have:

$$\int_\Gamma y(\mu_1^M \mu_1^m + \mu_2^M \mu_2^m) dxdy = \frac{2\mathfrak{M}\mathfrak{m}}{\sigma} \tag{82}$$

A.4.3 Integral of term $y|\mathbf{u}_x^m|^2$

This term does not depend both from σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = \int_\Gamma y[(u_{1x})^2 + (u_{2x})^2] dxdy \tag{83}$$

Substituting the terms (69) and (70) and simplifying we get:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^2\eta^2 \int_\Gamma \frac{y}{(x^2+y^2)^2} dxdy \tag{84}$$

Keep in mind that the terms (69) and (70) were valid only in the first quadrant of the semi-plane (x, y) due to the $\arctan(y/x)$ valid only for $x > 0$. However, from the symmetry of the integrand we see that the terms are actually valid in all the semi-plane and we can proceed.

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and Dm of radius R and R_M (see Fig. 2). However, when the two masses are far apart we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, the fact that $y = r\sin\theta$ and taking into account that $\mathfrak{m} = R\eta$, we have:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^2\eta^2 \int_0^\pi \sin\phi d\phi \int_R^\infty \frac{r}{r^4} rdr = 2\pi R\eta^2 = 2\pi\eta\mathfrak{m} \tag{85}$$

References

[1] V. Nardozza. *A Geometrical Model of Gravity* - https://vixra.org/abs/1806.0251 (2018).

[2] V. Nardozza. *Energy Stored in the Gravitational Field.* https://vixra.org/abs/1905.0515 (2019).

[3] V. Nardozza. *A Classical Mechanism for Creation of Magnetic Moment in a Particle* - https://vixra.org/abs/2310.0065 (2023).