Gravitational and Inertial Mass

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Abstract

In nature, gravitational and inertial mass appear to be the same. This is know as the equivalence principle in its weak formulation. In this paper, we discuss a possible relationship between gravitational and inertial mass in a Newtonian framework.

Key Words: gravity, equivalence principle.

1 Introduction

WARNING: In this paper we propose a model for gravity to explain the equivalence between inertial and gravitational mass. The model is interesting but it fails its objectives. Before you read, please go to Eq. (32) to decide whether reading this paper is worth or it is a waste of time.

In nature, gravitational and inertial mass appear to be the same. If two object gravitate in the same way, they have the same inertia. This is know as the equivalence principle in its weak formulation. In this paper, we discuss a possible relationship between gravitational and inertial mass in a Newtonian framework.

In Newtonian gravity, given a distribution of mass $\rho(\mathbf{x})$, the gravitational potential generated by this distribution of mass, in units where $4\pi G = 1$ where G is the gravitational constant, is given by the Poisson's equation:

$$\nabla^2 \phi(\mathbf{x}, t) = \rho(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x}$ is a point in 3-dimensional space. In particular for a point mass m in the origin, we have:

$$\nabla^2 \phi(\mathbf{x}) = m\delta(\mathbf{x}) \tag{2}$$

In this model gravity is an instant action. I we move a mass, the potential will change instantaneously in the whole universe. Modern physics has taught us that this is never the case and that perturbations in fields propagate at the speed of light. It is natural to add to our equation a term which takes into account the limited speed perturbations propagate at. The obvious thing to do is to add a second derivative with respect to time and turn the Poisson's equation into a wave equation. In units where also the speed of light $c = 1$ we have:

$$-\frac{\partial^2 \phi(\mathbf{x}, t)}{\partial t^2} + \nabla^2 \phi(\mathbf{x}, t) = \rho(\mathbf{x}, t) \tag{3}$$

which is also written as:

$$\Box \phi = \rho \tag{4}$$

Where $\Box$ is the d'Alembert operator (with -1 in front of the time derivative). For a moving point mass with coordinates $\sigma(t) = (x(t),\ y(t),\ z(t))$ we have:

$$\Box \phi = m\delta(\mathbf{x} - \sigma) \tag{5}$$

Being this a wave equation, clearly a perturbation moves into space at the speed of light. This is compatible con the fact that massless particle travel at the speed of light. As a matter of fact a massless particle has energy, it curves space and it can be seen as a perturbation travelling in space.

In the next section we will discuss what happens to "massive" perturbations where "massive" needs to be defined properly.

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2 Newton's First Law of Motion

Suppose a mass moves in space along the x axis with coordinates $\sigma(t) = (\sigma(t),\ 0,\ 0)$ where $\sigma(t)$ is a smooth function. We want to write Eq. (5) in the reference frame of the mass. To do that we use the change of coordinates (33) and we get an expression for the d'Alembert operator given by (40). Eq. (5) becomes:

$$-\phi_{tt} + 2\dot{\sigma}\phi_{tx} + (1-\dot{\sigma}^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{6}$$

By construction, in this reference frame the mass is always in the origin. Suppose now that $\sigma(t) = vt$ (i.e. the mass is moving with constant velocity along x), in this case Eq. (6) becomes:

$$-\phi_{tt} + 2v\phi_{tx} + (1-v^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{7}$$

We are looking for a solution which is stationary in the mass reference frame. To do that we set the derivatives of the potential with respect to time to vanish. We have:

$$\frac{1}{\gamma^2}\phi_{xx} + \phi_{yy} + \phi_{zz} = m\delta(\mathbf{x}) \tag{8}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)}} \tag{9}$$

is the Lorentz factor and we have reintroduced the speed of light.

From Eq. (47) we see that the solution of Eq. (8) is:

$$\phi(\mathbf{x}) = \tilde{\phi}(\gamma x,\ y,\ z) \tag{10}$$

where $\tilde{\phi}$ is the solution of Eq. (2) which in polar coordinates is:

$$\tilde{\phi}(r) = -\frac{(1 = 4\pi G)}{4\pi}\frac{m}{r} = -\frac{Gm}{r} \tag{11}$$

and where we have reintroduced the gravitational constant. Note that a moving potential get contracted along the direction of motion by a factor $1/\gamma$ with respect to a not moving potential.

We note that the above solution is the gravitational potential $\phi(\mathbf{x})$ for a given function $\sigma(t)$. This does not mean that it is also a solution of Eq. (7) when both $\phi(\mathbf{x}, t)$ and $\sigma(t)$ are let evolve as independent functions from a given initial condition. In order to be so, $\phi(\mathbf{x}, t)$ and $\sigma(t)$ should be a stationary point of the Lagrangian $\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma})$, which is a mixed Lagrangian (i.e. a Lagrangian density with respect to the field ϕ and a Lagrangian with respect to the function σ). We will show that this is the case in the next sections and in a simplified set up.

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), a mass has inertia meaning that it keeps moving at the same speed in absence of external interferences.

This is the first Newton's law of motion.

3 Analogy with a Mechanical Model

To better understand the physical behaviour of our moving "massive" perturbation, we want to make an analogy with a mechanical system described by the same equation (3) of our Newtonian gravitational field propagating at a fixed velocity. We choose a model where space is just an elastic linear and shierless material where propagating pressure waves are associated with propagating Newtonian gravity perturbations.

Note that, we are not saying that space is an elastic material, we are saying that the elastic material has the same equation of Newtonian gravity with non instantaneous action at distance and therefore if we can find a characteristic of the elastic system (e.g. kinetic energy associated to varying fields)), this will apply also to gravity.

If we include massive perturbations, this system is described by Eq. (5) whit the potential equal to $\phi = |\mathbf{u}|$ where $\mathbf{u}$ are the displacements of the material and $\mu = -\nabla|\mathbf{u}| = -\nabla\phi$ are the strains with opposite sign associated to gravitational field $\mathbf{g}$. This analogy is fully described in [1] were, in particular, it is shown that a discontinuity representing a point mass can be represented by a space deficiency free to move (frictionless). With space deficiency we mean a ball of material which is removed and where the renaming boundary is identified to a point. This will create displacements and therefore strains in the material which are associated to gravitational field.

In this analogy the elementary volume dV, will have potential energy proportional to the square of the strains $|\partial_i\mathbf{u}|$ and kinetic energy proportional to square of the rate of variation of the displacements $|\partial_t\mathbf{u}|$ which are related to the velocity at which the volume dV is moving in the material.

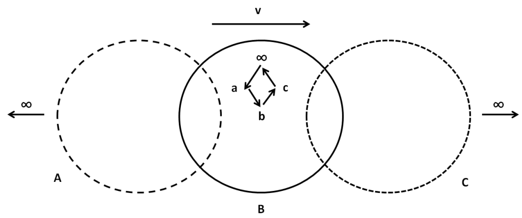

Figure 1: Orbits of Elementary Volumes

Now, if the discontinuity is moving with a velocity v to the right (see Fig. 1), points inside the area close to the discontinuity will readjust as the discontinuity pass by because the space deficiency will pull them. Elementary volumes will follow orbits and since the material has a mass density this will be associated with kinetic energy. However, as long as the velocity of the discontinuity moves with constant velocity the total kinetic energy is constant and the total potential energy is also constant because it depends on the the shape of the potential which does not change. This means that the trajectory is a stationary point of the Lagrangian.

When the discontinuity (i.e. our point mass) start moving pushed by a force, the kinetic energy of the field due to a $|\partial_t\mathbf{u}| \neq 0$ will increase and when we stop pushing the discontinuity, it is the conservation of this energy that gives inertia to it. Field variation is the flywheel of the systems.

4 Energy Density of the Gravitational Field

I our elastic model described in [1], where perturbations propagate according to the wave equation (3), we have that the potential energy density β_e and kinetic energy density τ_e stored in the field ϕ are equal to:

$$\beta_e = \frac{1}{2}Y|\mu|^2 \;\; ; \;\; \tau_e = \frac{1}{2}\rho|\dot{\mathbf{u}}|^2 \tag{12}$$

with the potential $\phi = |\mathbf{u}|$ where $\mathbf{u}$ are the displacements of the material and $\mu = -\nabla|\mathbf{u}| = -\nabla\phi$ are the strains with opposite sign, Y is the Young module, ρ is the density of the material. The kinetic energy is related to the rate of variation of the fields and the perturbation propagates at a speed $c = 1/\sqrt{Y/\rho}$. In this model displacements are finite (i.e. not infinitesimal), however there are no shier stress and therefore the displacements can be expressed by means of a potential and strains are not the derivatives of the displacements but are related to them by means of the potential (see [1]).

I analogy with the above model, we expect that the gravitational potential, where perturbations propagate according to the same wave equation (3), the potential energy density β_g and kinetic energy density τ_g stored in the field ϕ are equal to:

$$\beta_g = \frac{1}{2}K_\beta|\mathbf{g}|^2 \;\; ; \;\; \tau_g = \frac{1}{2}K_\tau|\partial_t\mathbf{g}|^2 \tag{13}$$

where $\mathbf{g} = -\nabla\phi$ is the gravitational field and $c = 1/\sqrt{K_\beta/K_\tau}$.

We know that (see [2]):

$$K_\tau = \frac{1}{4\pi G} \;\Rightarrow\; K_\beta = \frac{1}{c^2 K_\beta} = \frac{4\pi G}{c^2} \tag{14}$$

And we have eventually the expression for the energy stored in the gravitational field:

$$V_g = \frac{1}{2}\left(\frac{1}{4\pi G}\right)|\mathbf{g}|^2 \;\;;\;\; T_g = \frac{1}{2}\left(\frac{4\pi G}{c^2}\right)|\partial_t \mathbf{g}|^2 \tag{15}$$

We have just shown that adding the simple assumption that gravitational field propagates at the speed of light (or actually at any fixed speed), gravitational field mast have "kinetic" energy associated to field variation.

5 Newton's Second Law of Motion

We want to evaluate the Lagrangian $\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma})$ of our model. We want a simplified quasi-static system where $v << 1$, all the perturbations flies away at speed of light and the shape of the potential readjust itself to a potential $\phi(\sigma)$ depending only on σ and moving rigidly. With this assumption, the shape of ϕ is independent from σ while its position depends on σ only. This will simplify the Lagrangian as follows:

$$\mathcal{L}(\phi, \partial\phi, \sigma, \dot{\sigma}) \rightarrow \mathcal{L}(\sigma(t), \dot{\sigma}(t)) \tag{16}$$

In order to do that, we consider a two mass System M and m in $\mathbb{R}^3$ with M nailed to the position $(-\delta, 0, 0)$ and m free to move along the x axis (see Fig. 2).

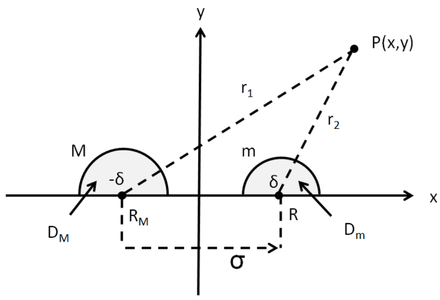

Figure 2: Two Mass System

To write the expression for the Lagrangian, we need to evaluate the potential and kinetic energy of the system. We will use our the elastic model for gravity given in [1]. In this elastic model, a mass m is a removed ball of space of radius R with the remaining boundaries identified to a point. This will generate displacement and strains which represent the gravitational field. The association is $m \rightarrow 2\pi R^2$ and:

$$\mathbf{g} = -\nabla\phi_g \;\;\rightarrow\;\; \mu = -\nabla\phi_e \tag{17}$$

where ϕ_g is the standard gravitational potential, ϕ_e is the elastic potential in our model, $\mathbf{g}$ is the gravitational field and μ are the strains with opposite sign (see [1]). Note also that in [1] v4 we have used a slightly different definition of μ. Ref. [1] may be updated to v5 to align to this paper.
In this model the density of potential and kinetic energy is given by Eq. (12). We have:

$$\beta_e = \frac{1}{2}Y|\mu|^2 \;\;;\;\; \tau_e = \frac{1}{2}\rho|\dot{\mathbf{u}}|^2 \tag{18}$$

5.1 Evaluation of the Energy

To simplify the calculation we note that fields in a point P both lays on the plane passing through P and the axis x and they have a cylindrical symmetry with respect to the axis x. We will evaluate our fields on the (x, y) plane and we will get the field in space just rotating the (x, y) fields around the x axis. For the potential energy, on the (x, y) plane we have:

$$|\mu|^2 = |\mu^M + \mu^m|^2 = (\mu_1^M + \mu_1^m)^2 + (\mu_2^M + \mu_2^m)^2 \tag{19}$$

And we get two self-energy and one cross-energy terms. We give names to them:

$$|\mu|^2 = |\mu^M|^2 + |\mu^m|^2 + 2(\mu_1^M \mu_1^m + \mu_2^M \mu_2^m) = I_M + I_m + 2I_{Mm} \tag{20}$$

We can evaluate the potential energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$V(\sigma) = \frac{1}{2} Y \int |\mu|^2 dxdydz = \frac{1}{2} Y \int_0^{2\pi} d\phi \int_\Gamma y(I_M + I_m + 2I_{Mm}) dxdy \tag{21}$$

where Γ is the semi-plane (x, y^+) minus the two semi-disks D_M and $D_m$of radius R_M and R (see Fig. 2). The term I_M and I_m are just constant terms which do not depend on σ (because you can shift the function in order to have the disk in the origin regardless σ and integrating. They do not contribute to the equations of motion in the Lagrangian and can be discarded. We have:

$$V(\sigma) = 2\pi Y \int_\Gamma y I_{Mm} dxdy = 2\pi Y \int_\Gamma y(\mu_1^M \mu_1^m + \mu_2^M \mu_2^m) dxdy \tag{22}$$

For the kinetic energy we have that $\partial_t \mathbf{u}^m(x - \sigma, y, z) = -\partial_x \mathbf{u}\dot{\sigma}$. This is because time is present only in $\sigma(t)$ and σ is present only in $\mathbf{u}^m$. while $\partial_t \mathbf{u}^M = 0$. As before we evaluate the field in a semi-plane and the final integral by rotating the solution around x:

$$|\partial_t \mathbf{u}|^2 = |-\partial_x \mathbf{u}^M - \partial_x \mathbf{u}^m|\dot{\sigma}^2 = [(u_{1x}^m)^2 + (u_{2x}^m)^2]\dot{\sigma} = |\mathbf{u}_x^m|^2 \dot{\sigma}^2 \tag{23}$$

Once again, we can evaluate the kinetic energy $V(\sigma)$ integrating the energy density in $\mathbb{R}^3$ as a revolution integral around the x axis of the field in the (x, y^+) semi-plane. We have:

$$T(\dot{\sigma}) = \frac{1}{2} \rho \int |\partial_t \mathbf{u}|^2 dxdydz = \frac{1}{2} \rho \int_0^{2\pi} d\phi \int_\Gamma y |\mathbf{u}_x^m|^2 \dot{\sigma}^2 dxdy = \pi\rho \left(\int_\Gamma y |\mathbf{u}_x^m|^2 dxdy \right) \dot{\sigma}^2 \tag{24}$$

The integrals I_M and I_m in Eq. (21) are useless to our calculation but they are important because they represent the self-energy of a particle. For example, given Eq. (59) in Appendix A.3 and Eq. (21) we have that the self energy of a particle is:

$$E_{Self} = \frac{1}{2} Y \int |\mu_m|^2 dxdydz = 2\pi Y \int_\Gamma y |\mu_m|^2 dxdy = 4\pi^2 Y R^3 \tag{25}$$

it goes as R^3 (i.e the volume of the space deficiency) and it is not infinite as in Newtonian gravity. In this theory mass goes as R^2 (see [1]) and self-energy of mass goes as R^3. It is still a classic theory not a relativistic one.
For the potential energy, given Eq. (22) and (68) we have:

$$V(\sigma) = \frac{\pi Y R_M^2 R^2}{\sigma} \tag{26}$$

Moreover, in perfect agreement with Newton's gravitation law, we have that the force experienced by the two space deficiency is:

$$F = -\frac{\partial V}{\partial \sigma} = \frac{\pi Y R_M^2 R^2}{\sigma^2} \tag{27}$$

For the kinetic energy, given Eq. (24) and (71) we have:

$$T(\dot{\sigma}) = 2\pi^2 \rho R^3 \dot{\sigma}^2 \tag{28}$$

5.2 The Equations of Motion

Given Eq. (26) and (28), we are now ready to write the Lagrangian:

$$\mathcal{L}(\sigma, \dot{\sigma}) = T(\dot{\sigma}) - V(\sigma) = 2\pi^2 \rho R^3 \dot{\sigma}^2 - \frac{\pi Y R_M^2 R^2}{\sigma} \tag{29}$$

We can find the equations of motion by using the Euler-Lagrange equations. We have:

$$-\frac{\pi Y R_M^2 R^2}{\sigma} = 4\pi^2 \rho R^3 \ddot{\sigma} \tag{30}$$

which, given Eq. (27) and using a positive direction for force and motion when towards M, it can be written as:

$$F = (4\pi^2 \rho) R^3 \ddot{\sigma} \tag{31}$$

We can now go back from our elastic model to Newtonian gravity. Since mass is proportional to R^2, we have:

$$F \propto m^{\frac{3}{2}} a \tag{32}$$

were a is the acceleration experienced by the mass under a force F. From the above equation we see that unfortunately our elastic model of gravity is not correct because it fails to explain the Newton's second law.

Appendix

A.1 D'Alembert Operator - Change of Coordinates

We want to evaluate the d'Alembert operator in the following system of coordinates:

$$\begin{cases} t &= \bar{t} \\ x &= \bar{x} - \sigma(\bar{t}) \\ y &= \bar{y} \\ z &= \bar{z} \end{cases} \tag{33}$$

where $\sigma(\bar{t})$ is any smooth function and $\dot{\sigma}(\bar{t})$ is its derivative with respect to $\bar{t}$. We convert from the notation (t, x, y, z) to the notation x^μ and from the notation $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to the notation $\bar{x}^\mu$. The system of coordinates $\bar{x}^\mu$ we start from is a Minkowski space with a metric $\eta_{\mu\nu} = (-, +, +, +)$. The metric in the new system of coordinates x^μ is given by:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\nu} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \tag{34}$$

From which we get:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dot{\sigma}^2 & -\dot{\sigma} & 0 & 0 \\ -\dot{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

The above metric has determinant $g = -1$ and inverse metric equal to:

$$g^{\mu\nu} = \begin{pmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1 - \dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

The d'Alembert operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\Box\phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \tag{37}$$

From which we have:

$$\Box\phi = \frac{1}{\sqrt{1}}\partial_\mu \left(\sqrt{1} \begin{bmatrix} -1 & \dot{\sigma} & 0 & 0 \\ \dot{\sigma} & 1-\dot{\sigma}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \phi \right) \tag{38}$$

and eventually:

$$\Box\phi = \left(-\partial_0^2 + 2\dot{\sigma}\partial_0\partial_1 + (1-\dot{\sigma}^2)\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \tag{39}$$

Converting the coordinates from the x^μ to the (t, x, y, z) notation we have:

$$\Box\phi = -\phi_{tt} + 2\dot{\sigma}\phi_{tx} + (1-\dot{\sigma}^2)\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{40}$$

A.2 Laplace Operator - Change of Coordinates

We want to evaluate the Laplace operator in the following system of coordinates:

$$(\bar{x}, \bar{y}, \bar{z}) \to (x = \gamma\bar{x},\ y = \bar{y},\ z = \bar{z}) \tag{41}$$

We proceed as in the previous section and using the notation x^i. The metric we start from is $\bar{g}_{ij} = (+, +, +)$. We have:

$$g_{ij} = \begin{pmatrix} \gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{42}$$

$$g^{ij} = \begin{pmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{43}$$

The Laplace operator in generic curvilinear coordinates can be evaluated using the following formula:

$$\nabla^2\phi = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi\right) \tag{44}$$

From which we have:

$$\nabla^2\phi = \frac{1}{\sqrt{\gamma^2}}\partial_i \left(\sqrt{\gamma^2} \begin{bmatrix} 1/\gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \phi \right) \tag{45}$$

and eventually:

$$\nabla^2\phi = \left(\frac{1}{\gamma^2}\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \tag{46}$$

Converting the coordinates from the x^i to the (x, y, z) notation we have:

$$\nabla^2\phi = \frac{1}{\gamma^2}\phi_{xx} + \phi_{yy} + \phi_{zz} \tag{47}$$

A.3 Integrals

The displacements relevant to a point mass (i.e. a discrete space deficiency or radius R), when the mass is in the origin and in spherical coordinates, are given by the following transformation (see [1]):

$$f : r \to r - \frac{R^2}{r} \tag{48}$$

Working on the (x,y) plane we have:

$$\phi = -|\mathbf{u}| = -\frac{R^2}{\sqrt{x^2 + y^2}} \tag{49}$$

$$\mu_1 = -\frac{\partial \phi}{\partial x} = \frac{R^2 x}{(x^2+y^2)^{\frac{3}{2}}} \tag{50}$$

$$\mu_2 = \frac{\partial \phi}{\partial y} = \frac{R^2 y}{(x^2+y^2)^{\frac{3}{2}}} \tag{51}$$

Moreover:

$$\mathbf{u} = \left(-\frac{R^2\cos\theta}{r}, \ -\frac{R^2\sin\theta}{r}\right) \tag{52}$$

where θ is the angle between r and the x axis. We have:

$$\mathbf{u} = \left(-\frac{R^2\cos\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}, \ -\frac{R^2\sin\left(\arctan\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}\right) \tag{53}$$

and finally for ($x > 0$ where the arctan is well defined:

$$\mathbf{u} = R^2\left(\frac{-1}{\sqrt{\frac{y^2}{x^2}+1}\sqrt{x^2+y^2}}, \ \frac{-y}{\sqrt{\frac{y^2}{x^2}+1}\,x\sqrt{x^2+y^2}}\right) \tag{54}$$

We have to take into account of the limitation $x > 0$ when we integrate this functions by using the symmetry with respect to the y axis and doubling the integral in the first quadrant only. From the above we have:

$$u_{1x} = \partial_x u_1 = \frac{R^2(x^4-y^4)}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^3 \left(x^2+y^2\right)^{\frac{3}{2}}} \tag{55}$$

$$u_{2x} = \partial_x u_2 = \frac{R^2 2y}{\left(\frac{y^2}{x^2}+1\right)^{\frac{3}{2}} x^2 \sqrt{x^2+y^2}} \tag{56}$$

A.3.1 Integral of term $y|\mu|^2$

This term does not depend from both σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mu|^2 dxdy = \int_\Gamma y[(\mu_1)^2 + (\mu_2)^2]dxdy \tag{57}$$

Substituting the terms (50) and (51) we get:

$$\int_\Gamma y|\mu|^2 dxdy = R^4\int_\Gamma \frac{y(x^2+y^2)}{(x^2+y^2)^3}dxdy \tag{58}$$

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and D_m of radius R and R_M (see Fig. 2). However, when the two masses are far apart we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, and the fact that $y = r\sin\theta$, we have:

$$\int_\Gamma y|\mu|^2 dxdy = R^4\int_0^\pi \sin\phi d\phi\int_R^\infty \frac{r}{r^4}rdr = 2\pi R^3 \tag{59}$$

A.3.2 Integral of term $y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)$

We have already evaluated this integral in a different context (see [2]). This term depends from both the two masses and the distance between them. Whatever the value of σ at a given time, we define $\delta = \sigma/2$ and we shift the integrand on the x axis such that the origin of the (x, y) plane is exactly in the middle between the two masses (see Fig. 2). The integral we want to evaluate is:

$$\mathcal{I} = \int_\Gamma y(\mu_1^M\mu_1^m + \mu_2^M\mu_2^m)dxdy \tag{60}$$

Substituting the terms (50) and (51) and taking into account of their displacements along x, we get:

$$\mathcal{I} = \int_\Gamma y \left(\frac{R_M^2 R^2 (x-\delta)(x+\delta)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}} \left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} + \frac{R_M^2 R^2 y^2}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}} \left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} \right) dxdy \tag{61}$$

Which can be simplified as:

$$\mathcal{I} = R_M^2 R^2 \int_\Gamma \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}} \left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} dxdy \tag{62}$$

It is possible to show that the disks D_M and D_m do not contribute to the above integral because:

$$\mathcal{I}_{D_M} = \mathcal{I}_{D_m} = 0 \Rightarrow \mathcal{I}_\Gamma = \mathcal{I}_{(x,y^+)} \tag{63}$$

And therefore we have:

$$\mathcal{I} = \mathcal{I}_{(x,y^+)} = R_M^2 R^2 \int_{-\infty}^{\infty} dx \int_0^\infty \frac{y\left(x^2+y^2-\delta^2\right)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}} \left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} dy \tag{64}$$

We use the symmetry of the function with respect to the y axis by integrating in the firsts quadrant only and doubling the result:

$$\mathcal{I} = 2R_M^2 R^2 \int_0^\infty dx \int_0^\infty \frac{y\left(y^2+x^2-\delta^2\right)}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}} \left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} dy \tag{65}$$

We integrate with respect to the y variable first and we get:

$$\mathcal{I} = 2R_M^2 R^2 \int_0^\infty \frac{|x-\delta|+x-\delta}{4x^2(x-\delta)} dx \tag{66}$$

then we integrate with respect to the x variable. We have:

$$\mathcal{I} = 2R_M^2 R^2 \left(\int_0^\delta 0\, dx + \int_\delta^\infty \frac{1}{2x^2} dx \right) = \frac{R_M^2 R^2}{\delta} \tag{67}$$

since $\sigma = 2\delta$, we have:

$$\int_\Gamma y(\mu_1^M \mu_1^m + \mu_2^M \mu_2^m) dxdy = \frac{2R_M^2 R^2}{\sigma} \tag{68}$$

A.3.3 Integral of term $y|\mathbf{u}_x^m|^2$

This term does not depend both from σ and the mass M and therefore we can move the mass m to the origin:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = \int_\Gamma y[(u_{1x})^2 + (u_{2x})^2] dxdy \tag{69}$$

Substituting the terms (55) and (56) and simplifying we get:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^4 \int_\Gamma \frac{y}{\left(x^2+y^2\right)^2} dxdy \tag{70}$$

Keep in mind that the terms (55) and (56) were valid only in the first quadrant of the semi-plane (x, y) due to the $\arctan(y/x)$ valid only for $x > 0$. However, from the symmetry of the integrand we see that the terms are actually valid in all the semi-plane and we can proceed.

The set Γ is the semi-plane $(x, y+)$ minus the semi-disks D_M and Dm of radius R and R_M (see Fig. 2). However, when the two masses are far apart we can ignore the contribution from removing the semi-disk D_M of radius R_M which is negligible.

In polar coordinates, given the transformation for the elementary surface $dxdy = rd\phi dr$, and the fact that $y = r \sin\theta$, we have:

$$\int_\Gamma y|\mathbf{u}_x^m|^2 dxdy = R^4 \int_0^\pi \sin\phi d\phi \int_R^\infty \frac{r}{r^4} r dr = 2\pi R^3 \tag{71}$$

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