Cobweb model with conformable fractional derivative in Liouville-Caputo sense

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ABSTRACT

In this paper, we formulate a continuous-time cobweb model expressed as a conformable fractional derivative in Liouville-Caputo sense, and a continuous-time cobweb model expressed as a beta-type conformable fractional derivative in Liouville-Caputo sense, and obtain an analytical solution of this model and analyze the properties of the solution.

We also compare the results of the previous cobweb model solutions with several examples.

Keywords: Fractional calculus, cobweb model, conformable derivative, fractional conformable derivative.

1. Introduction

Many natural phenomena as well as economic phenomena are modeled accurately by fractional differential equations [1-3]. The Cobweb model is a mathematical model that characterizes the relationship between demand and supply, and is widely used in the economic sector [4,5]. The fractional derivative describes the real phenomenon more accurately than the integral derivative. Hence, mathematical modeling with fractional differential equations has been widely performed in the last few decades [6,7].

The fractional derivatives used are the Riemann-Liouville fractional derivatives, Caputo fractional derivatives, and Grinwald-Letnikov fractional derivatives. These derivatives are called classical fractional derivatives.   However, these fractional derivative definitions do not reflect exactly the characteristics of the fractional derivative describing hysteresis due to local characteristics [8,9]. For this reason, several conformable derivatives have been proposed and used [10-12].

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In [13], an analytical solution of the cobweb model including the conformable derivative proposed in [10] and its stability criterion are proposed, and compared with the solution of the cobweb model including the integer derivative.

We formulate the cobweb model including the conformable fractional derivative in Liouville-Caputo sense, and the cobweb model including the beta-conformable fractional derivative in Liouville-Caputo sense, and find its analytical solution and analyze its properties. We also compared the results obtained in [13].

The rest of the paper is organized as follows.

Section 2 gives preliminaries and in Section 3, we analyze the analytical solution and stability of the cobweb model including fractional derivative in Liouville-Caputo sense and the cobweb model including beta-type fractional derivative in Liouville-Caputo sense , and in Section 4, we present the simulation results.

2. Background

1) Basic definitions

Definition 1. The Riemann-Liouville fractional derivative is defined as follows;

$$ {}_{t_0}^{R}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-x)^{n-\alpha-1} f(x) dx \tag{2.1} $$

Where $n-1<\alpha\le n, n\in N$.

Definition2 [10] . Assume $f:[a,\infty)\to R, \alpha\in(0,1)$. Then, The conformable derivative of $f(t)$ is defined as follows;

$$ {}_{a}T_t^{\alpha} f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.2} $$

Where $t>a$, $f^{\alpha}(0)=\lim_{\varepsilon\to 0+} f^{(\alpha)}(t)$.

The conformable integral of $f(t)$ is defined as follows;

$$ {}_{a}I_t^{\alpha} f(t) = \int_a^t \frac{f(x)}{(x-a)^{1-\alpha}} dx \ , 0<\alpha\le 1 \tag{2.3} $$

Also, the conformable fractional integral of $f(t)$ for $\beta\in C, \mathrm{Re}(\beta)>0$ is defined as follows;

$$ {}_{a}^{\beta}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \left(\frac{(t-a)^{\alpha} - (x-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(x)}{(x-a)^{1-\alpha}} dx \tag{2.4} $$

When $\alpha\in(0,1), \mathrm{Re}(\beta)\ge 0, n=[\mathrm{Re}(\beta)]+1, f\in C_{\alpha,0}^{n}([a,b])$, the conformable fractional derivative in Liouville-Caputo sense is defined as follows[11] :

$$ {}_{a}^{C\beta}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\beta)}\int_{0}^{t}\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{{}_{a}T_{x}^{\alpha}f(x)}{(x-a)^{1-\alpha}}dx={}_{a}^{n-\beta}I_{t}^{\alpha}\left({}_{a}^{n}T_{x}^{\alpha}f(t)\right) \quad (2.5) $$

Where ${}_{a}^{n}T_{t}^{\alpha}=\underbrace{{}_{a}T_{t}^{\alpha}\cdots{}_{a}T_{t}^{\alpha}}_{n-time}$

Definition 3 [12]. Assume $f:\left[-\frac{a}{\Gamma(\alpha)},\infty\right)\to R,\alpha\in(0,1)$. $\beta-$ type conformable integral is defined as follows;

$$ {}_{a}^{A}D_{t}^{\alpha}f(t)=\lim_{\varepsilon\to\infty}\frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)-f(t)}{\varepsilon} \quad (2.6) $$

The $\beta-$ type conformable integral is defined as follows;

$$ {}_{a}^{A}I_{t}^{\alpha}f(t)=\int_{-\frac{a}{\Gamma(\alpha)}}^{t}\frac{f(x)}{\left(x+\frac{a}{\Gamma(\alpha)}\right)^{1-\alpha}}dx\ ,\ 0<\alpha\leq 1 \quad (2.7) $$

The $\beta-$ type conformable fractional integral of $f:\left[-\frac{a}{\Gamma(\alpha)},\infty\right)\to R,\alpha\in(0,1)$, $\beta\in C,\mathrm{Re}(\beta)>0$ is defined as follows;

$$ {}_{a}^{A\beta}I_{t}^{\alpha}f(t)=\frac{1}{\Gamma(\beta)}\int_{-\frac{a}{\Gamma(\alpha)}}^{t}\left(\frac{\left(t+\frac{a}{\Gamma(\alpha)}\right)^{\alpha}-\left(x+\frac{a}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(x)}{\left(x+\frac{a}{\Gamma(\alpha)}\right)^{1-\alpha}}dx \quad (2.8) $$

When $\alpha\in(0,1),\mathrm{Re}(\beta)\geq 0,n=[\mathrm{Re}(\beta)]+1,f\in C_{\alpha,0}^{n}([a,b])$, the beta-type conformable fractional derivative in Liouville-Caputo sense is defined as follows :

$$ {}_{a}^{AC\beta}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\beta)}\int_{-\frac{a}{\Gamma(\alpha)}}^{t}\left(\frac{\left(t+\frac{a}{\Gamma(\alpha)}\right)^{\alpha}-\left(x+\frac{a}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{{}_{a}^{A^{n}}D_{x}^{\alpha}f(x)}{\left(x+\frac{a}{\Gamma(\alpha)}\right)^{1-\alpha}}dx \quad (2.9) $$

$$ ={}_{a}^{A^{n-\beta}}I_{t}^{\alpha}\left({}_{a}^{A^{n}}D_{t}^{\alpha}f(t)\right) $$

Where ${}_{a}^{A^{n}}D_{t}^{\alpha}=\underbrace{{}_{a}^{A}D_{t}^{\alpha}\cdots{}_{a}^{A}D_{t}^{\alpha}}_{n-time}$.

Lemma 1. When $f\in C_{\alpha,a}^{n}[a,b],\beta\in C$, we have

$$ {}_{a}^{\beta}I_{t}^{\alpha}\left({}_{a}^{AC\beta}D_{t}^{\alpha}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{{}_{a}^{k}T_{t}^{\alpha}f(a)(t-a)^{\alpha k}}{\alpha^{k}k!} \tag{2.10} $$

Where ${}_{a}^{\beta}I_{t}^{\alpha}$ is defined by Eq. (2.4), ${}_{a}^{k}T_{t}^{\alpha}=\underbrace{{}_{a}T_{t}^{\alpha}\cdots{}_{a}T_{t}^{\alpha}}_{k-time}$, and ${}_{a}T_{t}^{\alpha}$ is defined by Eq. (2.2).

2) cobweb model and its solution

(1) cobweb model with integer order derivative

The integer-order Cobweb model is defined as follows :

$$ \begin{cases} D(t)=a_1+b_1p(t+1) \\ S(t)=a_2+b_2p(t) \\ D(t)=S(t) \end{cases},\quad p(t_0)=p_0 \tag{2.11} $$

Where $a_1,b_1,a_2,b_2\in R$, $b_1\neq 0$.

The general solution of Eq. (2.6) is

$$ p(t)=(p_0-p_e)\left(\frac{b_2}{b_1}\right)^{t}+p_e \tag{2.12} $$

Where $p_0\in R$, and we call $p_e=\dfrac{a_2-a_1}{b_1-b_2}$ the equilibrium point.

Another type of model for the integer order Cobweb model (2.11) is

$$ \begin{cases} D(t)=a_1+b_1[p(t)+p'(t)] \\ S(t)=a_2+b_2p(t) \\ D(t)=S(t) \end{cases},\quad p(t_0)=p_0 \tag{2.13} $$

Where $a_1,b_1,a_2,b_2\in R$, $b_1\neq 0$. The general solution of model (2.13) is

$$ p(t)=(p_0-p_e)e^{\left(\frac{b_2}{b_1}-1\right)(t-t_0)}+p_e \tag{2.14} $$

Where $p_0\in R$, $p_e=\dfrac{a_2-a_1}{b_1-b_2}$.

(2) cobweb model with conformable derivative [13].

The cobweb model with conformable derivative is as follows;

$$ \begin{cases} D(t)=a_1+b_1\left[p(t)+{}_{t_0}T_{t}^{\alpha}(p)(t)\right] \\ S(t)=a_2+b_2p(t) \\ D(t)=S(t) \end{cases},\quad p(t_0)=p_0 \tag{2.15} $$

Where $p_e = (a_2 - a_1)/(b_1 - b_2)$, and ${}_{t_0}T_t^{\alpha}(f)$ is a conformable fractional derivative with $0 < \alpha < 1$.

The general solution for model (2.15) is as follows;

$$p(t) = (p_0 - p_e)e^{\left(\frac{b_2 - b_1}{\alpha b_1}\right)\left(t^{\alpha} - t_0^{\alpha}\right)} + p_e \qquad (2.16)$$

Where $a_1, b_1, a_2, b_2, p_0 \in R$, $b \neq 0, b_1 \neq b_2$.

If $b_2 / b_1 < 1$, the cost of model (2.15) with conformable derivative converges to the equilibrium point.

Another type of model for Cobweb model (2.15) with conformable derivatives is as follows;

$$\begin{cases} D(t) = a_1 + b_1 p(t) \\ S(t) = a_2 + b_2\left[p(t) + c\ {}_{t_0}T_t^{\alpha}(p)(t)\right], \ p(t_0) = p_0 \\ D(t) = S(t) \end{cases} \qquad (2.17)$$

The general solution to model (2.17) is as follows [5];

$$p(t) = (p_0 - p_e)e^{\left(\frac{b_1 - b_2}{\alpha c b_2}\right)\left(t^{\alpha} - t_0^{\alpha}\right)} + p_e \qquad (2.18)$$

Where $a_1, b_1, a_2, b_2, p_0 \in R$, $b \neq 0, b_1 \neq b_2$.

If $\frac{b_1 - b_2}{\alpha c b_2} < 0$, the cost of the conformable fractional model (2.13) converges to the equilibrium point p_e.

3. Main result

1) cobweb model with conformable fractional derivative in Liouville-Caputo sense.

The cobweb model with the conformable fractional derivative in Liouville Caputo is as follows;

$$\begin{cases} D(t) = a_1 + b_1[p(t) + {}_{t_0}^{C\beta}D_t^{\alpha}(p)(t)] \\ S(t) = a_2 + b_2 p(t) \\ D(t) = S(t) \end{cases}, \quad p(t_0) = p_0 \qquad (3.1)$$

where ${}_0^{C\beta}D_t^{\alpha}$ is the conformable fractional derivative operator in Liouville-Caputo.

Theorem 1. When $a_1, b_1, a_2, b_2, p_0 \in R, b_1 \neq 0, b_1 \neq b_2$, the solution of Eq. (3.1) is as follows;

$$p(t) = p_0 E_{\beta,1}\left(-\frac{b}{\alpha^{\beta}}(t - t_0)^{\alpha\beta}\right) + \frac{a(t - t_0)^{\alpha\beta}}{\alpha^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k b^k (t - t_0)^{k\alpha\beta}}{(k+1)\alpha^{k\beta}\beta\Gamma(k\beta + \beta)} \qquad (3.2)$$

Where $b_1 \neq 0$, $b = (b_1 - b_2)/b_1$, $a = (a_2 - a_1)/b_1$, $\Gamma(\cdot)$ is a gamma function and $E_{\alpha,\beta}(\cdot)$ is a Mittag-Leffler function of two kinds.

Proof. Equation (3.1) can be written as follows;

$$ {}_{t_0}^{C\beta}D_t^{\alpha}(p)(t) + bp(t) = a \ , \ p(t_0) = p_0 \tag{3.3} $$

By applying the inverse operator (2.4) on both sides of (3.3), we obtain

$$ {}_{t_0}^{\beta}I_t^{\alpha}\left({}_{t_0}^{C\beta}D_t^{\alpha}I(t)\right) + b\left({}_{t_0}^{\beta}I_t^{\alpha}p(t)\right) = \left({}_{t_0}^{\beta}I_t^{\alpha}a\right) $$

Considering the initial condition of Eq. (3.3) and Eq. (2.5), we obtain

$$ p(t) = p(t_0) + \left({}_{t_0}^{\beta}I_t^{\alpha}a\right) - b\left({}_{t_0}^{\beta}I_t^{\alpha}p(t)\right) $$

Then consider the following :

$$ p_{n+1}(t) = p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left({}_{t_0}^{\beta}I_t^{\alpha}p(t)\right) \ , \ n = 0,1,2,\cdots \tag{3.4} $$

For $n = 0$, the above equation is

$$ p_1(t) = p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left({}_{t_0}^{\beta}I_t^{\alpha}p_0(t)\right) \tag{3.5} $$

Where

$$ {}_{t_0}^{\beta}I_t^{\alpha}p_0 = \frac{p_0}{\Gamma(\beta)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\alpha} - (x-t_0)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{dx}{(x-t_0)^{1-\alpha}} \tag{3.6} $$

Using the new variable $u = \frac{(x-t_0)^{\beta}}{(t-t_0)^{\alpha}}$, the above expression is obtained as follows :

$$ {}_{t_0}^{\beta}I_t^{\alpha}p_0 = \frac{p_0(t-t_0)^{\alpha\beta}}{\Gamma(\beta)\alpha^{\beta}}\int_0^1(1-u)^{\beta-1}du = \frac{p_0(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} \tag{3.7} $$

Substituting Eq. (3.7) into Eq. (3.5), we obtain

$$ p_1 = p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} \tag{3.8} $$

When $n = 1$, Eq. (3.4) becomes

$$ \begin{aligned} p_2(t) &= p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left({}_{t_0}^{\beta}I_t^{\alpha}p_1(t)\right) \\ &= p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left[{}_{t_0}^{\beta}I_t^{\alpha}\left(p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}\right)\right] \\ &= p_0 + \frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{ab(t-t_0)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)} + \frac{b^2p_0}{\alpha^{\beta}\Gamma(\beta+1)}\left({}_{t_0}^{\beta}I_t^{\alpha}t^{\alpha\beta}\right) \end{aligned} \tag{3.9} $$

Where

$$\left({}_{0}^{\beta}I_{t}^{\alpha}(t-t_0)^{\alpha\beta}\right)=\frac{\Gamma(\beta+1)}{\alpha^{\beta}\Gamma(2\beta+1)}(t-t_0)^{2\alpha\beta} \tag{3.10}$$

Substituting the above expression into Eq. (3.9), we obtain

$$\begin{aligned} p_2(t) &= p_0+\frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-\frac{bp_0(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-\frac{ab(t-t_0)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}+\frac{b^2p_0(t-t_0)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)} \\ &= p_0\left[1-\frac{b(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}+\frac{b^2p_0(t-t_0)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}\right]+\frac{at^{\alpha\beta}}{\alpha^{\beta}}\left[\frac{1}{\beta\Gamma(\beta)}-\frac{b(t-t_0)^{\alpha\beta}}{2\alpha^{\beta}\beta\Gamma(2\beta)}\right] \end{aligned} \tag{3.11}$$

When the procedure shown above for $n=2,3,\cdots$ is repeated,

$$\begin{aligned} p_n(t) &= p_0\left[1-\frac{b(t-t_0)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}+\frac{b^2(t-t_0)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}-\cdots\right]+\frac{at^{\alpha\beta}}{\alpha^{\beta}}\left[\frac{1}{\beta\Gamma(\beta)}-\frac{b(t-t_0)^{\alpha\beta}}{2\alpha^{\beta}\beta\Gamma(2\beta)}+\cdots\right] \\ &= p_0\sum_{k=0}^{n}\frac{(-1)^k b^k(t-t_0)^{k\alpha\beta}}{\alpha^{k\beta}\Gamma(k\beta+1)}+\frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{n}\frac{(-1)^k b^k(t-t_0)^{k\alpha\beta}}{(k+1)\alpha^{k\beta}\beta\Gamma(k\beta+\beta)} \end{aligned}$$

(3.12) (30)

Then, $n\to\infty$, the above equation becomes

$$\begin{aligned} p(t) &= p_0\sum_{k=0}^{\infty}\frac{(-1)^k b^k(t-t_0)^{k\alpha\beta}}{\alpha^{k\beta}\Gamma(k\beta+1)}+\frac{at^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k b^k(t-t_0)^{k\alpha\beta}}{(k+1)\alpha^{k\beta}\beta\Gamma(k\beta+\beta)} \\ &= p_0E_{\beta,1}\left(-\frac{b}{\alpha^{\beta}}(t-t_0)^{\alpha\beta}\right)+\frac{a(t-t_0)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k b^k(t-t_0)^{k\alpha\beta}}{(k+1)\alpha^{k\beta}\beta\Gamma(k\beta+\beta)} \end{aligned}$$

□.

2) cobweb model with beta-type conformable fractional derivative in Liouville-Caputo sense.

The cobweb model with beta-type conformable fractional derivative in Liouville Caputo sense is

$$\begin{cases} D(t)=a_1+b_1\left[p(t)+{}_{t_0}^{AC\beta}D_t^{\alpha}(p)(t)\right] \\ S(t)=a_2+b_2p(t) \\ D(t)=S(t) \end{cases},\quad p(t_0)=p_0 \tag{3.13}$$

Where ${}_{t_0}^{AC\beta}D_t^{\alpha}$ is a beta-type conformable fractional derivative operator in Liouville-Caputo sense.

Theorem 2. When $a_1,b_1,a_2,b_2,p_0\in R, b_1\neq 0, b_1\neq b_2$, the solution of Eq. (31) is

$$p(t)=p_0E_{\beta,1}\left(-\frac{b}{\alpha^{\beta}}\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}\right)+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}(k+1)\beta\Gamma(k\beta+\beta)} \tag{3.14}$$

Where $b_1\neq 0$, $b=(b_1-b_2)/b_1$, $a=(a_2-a_1)/b_1$, $\Gamma(\cdot)$ is a gamma function, and $E_{\alpha,\beta}(\cdot)$ is a Mittag-Leffler function of two kinds.

Proof.

We can write (3.14) as follows.

$$ {}^{AC\beta}_{t_0}D_t^{\alpha}(p)(t)+bp(t)=a\ ,\ p(t_0)=p_0 \tag{3.15}$$

By applying the inverse operator ${}^{AC\beta}_{t_0}I_t^{\alpha}$ on both sides of Eq. (3.15), we have

$$ {}^{AC\beta}_{t_0}I_t^{\alpha}\left({}^{AC\beta}_{t_0}D_t^{\alpha}(p)(t)\right)+b\left({}^{AC\beta}_{t_0}I_t^{\alpha}p(t)\right)=\left({}^{AC\beta}_{t_0}I_t^{\alpha}a\right) \tag{3.16}$$

Considering Lemma 1 and initial conditions, we have

$$p(t)=p_0+\left({}^{AC\beta}_{t_0}I_t^{\alpha}a\right)-b\left({}^{AC\beta}_{t_0}I_t^{\alpha}p(t)\right) \tag{3.17}$$

Where,

$$p_{n+1}(t)=p_0+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-b\left({}^{AC\beta}_{t_0}I_t^{\alpha}p_n(t)\right),\ n=0,1,2,\cdots \tag{3.18}$$

For $n=0$,

$$p_1(t)=p_0+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-b\left({}^{AC\beta}_{t_0}I_t^{\alpha}p_0\right) \tag{3.19}$$

Where,

$$ {}^{AC\beta}_{t_0}I_t^{\alpha}p_0=\frac{p_0}{\Gamma(\beta)}\int_{t_0}^{t}\left(\frac{\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha}-\left(x+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{dx}{\left(x+\frac{t_0}{\Gamma(\alpha)}\right)^{1-\alpha}} \tag{3.20}$$

Introducing a new variable $u=\left(\dfrac{x+\frac{t_0}{\Gamma(\alpha)}}{t+\frac{t_0}{\Gamma(\alpha)}}\right)^{\alpha}$, the above equation becomes

$$ {}_{t_0}^{AC\beta}I_t^{\alpha} p_0 = \frac{p_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\Gamma(\beta)\alpha^{\beta}}\int_0^1 (1-u)^{\beta-1}dx = \frac{p_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} \tag{3.21} $$

Substituting Eq. (3.21) into Eq. (3.19), we have

$$ {}_{t_0}^{AC\beta}I_t^{\alpha} p_0 = \frac{p_0\left(t+\frac{\frac{t_0}{\Gamma(\alpha)}}{\Gamma(\beta)\alpha^{\beta}}\right)^{\alpha\beta}}{\Gamma(\beta)\alpha^{\beta}}\int_0^1 (1-u)^{\beta-1}du = \frac{p_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} \tag{3.22} $$

Substituting Eq. (3.22) into (3.19), we have

$$ p_1(t) = p_0 + \frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} \tag{3.23} $$

When $n=1$, it becomes

$$ \begin{aligned} p_2(t) &= p_0 + \frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left({}_{t_0}^{AC\beta}I_t^{\alpha} p_1(t)\right) \\ &= p_0 + \frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - b\left({}_{t_0}^{AC\beta}I_t^{\alpha}\left(p_0 + \frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}\right)\right) \\ &= p_0 + \frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{bp_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)} - \frac{ba\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)} \\ &\quad + \frac{b^2p_0}{\alpha^{\beta}\Gamma(\beta+1)}\left({}_{t_0}^{AC\beta}I_t^{\alpha}\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}\right) \end{aligned} \tag{3.24} $$

Where,

$$ \left({}_{t_0}^{AC\beta}I_t^{\alpha}\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}\right) = \frac{\Gamma(\beta+1)}{\alpha^{\beta}\Gamma(2\beta+1)}\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta} \tag{3.25} $$

Substituting Eq. (3.25) into Eq. (3.24), we have

$$p_2(t)=p_0+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-\frac{bp_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}-\frac{ab\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}+\frac{b^2p_0\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}$$

$$=p_0\left[1-\frac{b\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}+\frac{b^2\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}\right]+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\left[\frac{1}{\beta\Gamma(\beta)}-\frac{b\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}2\beta\Gamma(2\beta)}\right] \quad (3.26)$$

When the procedure shown above for $n=2,3,\cdots$ is repeated, the obtained is as follows;

$$p_n(t)=p_0\left[1-\frac{b\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}\Gamma(\beta+1)}+\frac{b^2\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{2\alpha\beta}}{\alpha^{2\beta}\Gamma(2\beta+1)}-\cdots\right]+$$

$$+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\left[\frac{1}{\beta\Gamma(\beta)}-\frac{b\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}2\beta\Gamma(2\beta)}+\cdots\right]$$

$$=p_0\sum_{k=0}^{n}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}\Gamma(k\beta+1)}+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{n}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}\beta\Gamma(k\beta+\beta)}$$

When $n\to\infty$, the above equation becomes

$$p(t)=p_0\sum_{k=0}^{\infty}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}\Gamma(k\beta+1)}+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}\beta\Gamma(k\beta+\beta)}$$

$$=\mathrm{p}_0E_{\beta,1}\left(-\frac{b}{\alpha^{\beta}}\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}\right)+\frac{a\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{\alpha\beta}}{\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k b^k\left(t+\frac{t_0}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}(k+1)\beta\Gamma(k\beta+\beta)}$$

□

4. simulation example

Consider the following model for $a_1=0.8, b_1=-0.4, a_2=-1, b_2=0.2$.

$$\begin{cases} D(t)=0.8-0.4[p(t)+D_t(p)(t)] \\ S(t)=-1+0.2p(t) \\ D(t)=S(t) \end{cases}, \quad p(0.5)=2 \quad (4.1)$$

Where, D_t is one of the three operators ${}_{t_0}T_t^{\alpha}, {}_{t_0}^{C\beta}D_t^{\alpha}, {}_{t_0}^{AC\beta}D_t^{\alpha}$.

When $D_t = {}_{t_0}T_t^{\alpha}, D_t = {}_{t_0}^{C\beta}D_t^{\alpha}$ and $D_t = {}_{t_0}^{AC\beta}D_t^{\alpha}$, the solution of Eq. (4.1) is respectively as follow;

$$p(t) = -e^{\left(-\frac{3}{\alpha}\right)\left(t^{\alpha}-2^{-\alpha}\right)} + 3 \tag{4.2}$$

$$p(t) = \frac{1}{2}E_{\beta,1}\left(-\frac{3}{2\alpha^{\beta}}\left(t-\frac{1}{2}\right)^{\alpha\beta}\right) + \frac{9(t-0.5)^{\alpha\beta}}{2\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k 1.5^k (t-0.5)^{k\alpha\beta}}{(k+1)\alpha^{k\beta}\beta\Gamma(k\beta+\beta)} \tag{4.3}$$

$$p(t) = \frac{1}{2}E_{\beta,1}\left(-\frac{3}{2\alpha^{\beta}}\left(t+\frac{0.5}{\Gamma(\alpha)}\right)^{\alpha\beta}\right) + \frac{9\left(t+\frac{0.5}{\Gamma(\alpha)}\right)^{\alpha\beta}}{2\alpha^{\beta}}\sum_{k=0}^{\infty}\frac{(-1)^k 1.5^k\left(t+\frac{0.5}{\Gamma(\alpha)}\right)^{k\alpha\beta}}{\alpha^{k\beta}(k+1)\beta\Gamma(k\beta+\beta)} \tag{4.4}$$

Figs. 1~3 compare the analytical solutions of the Cobweb model with the conformable derivative (CD), the conformable fractional derivative (CFD) in Liouville-Caputo sense, and the beta-type conformable fractional derivative (Beta-type CFD) in Liouville-Caputo sense.

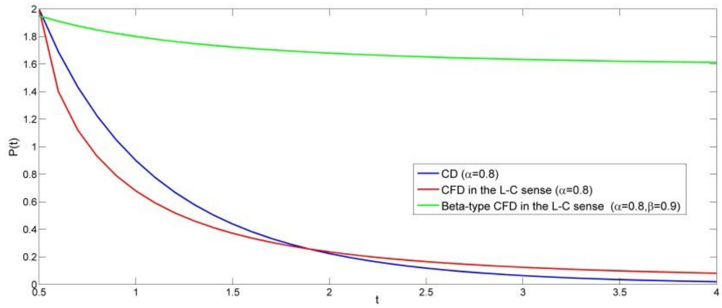

Fig. 1. Analytical solution of Cobweb model with $\alpha = 0.8, \beta = 0.9$.

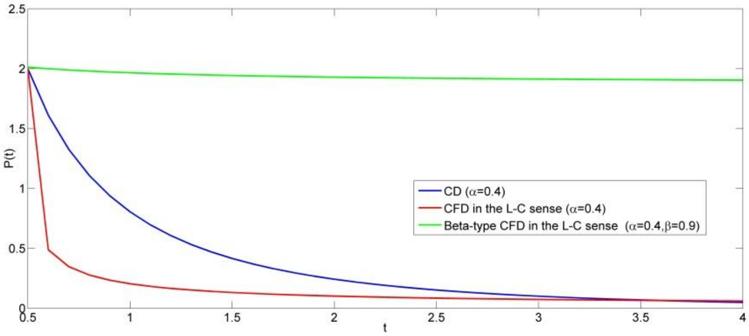

Fig. 2. Analytical solution of Cobweb model with $\alpha = 0.4, \beta = 0.9$

As can be seen in Figs. 1 and 2, when α decreases, the analytical solution of the Cobweb model with the conformable fractional derivative in Liouville-Caputo sense is closer to the equilibrium point, and the analytical solution of the Cobweb model with the beta-type conformable fractional derivative in Liouville-Caputo sense is slower. The effect of fractional order is shown in Fig. 3.

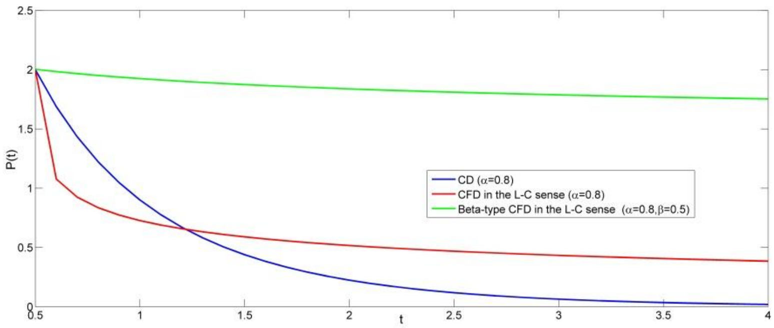

Fig. 3. Analytical solution of Cobweb model with $\alpha = 0.8, \beta = 0.5$.

5.Conclusions

We consider the case where the well-known supply-demand model 'cobweb' is used for the conformable fractional derivative in Liouville-Caputo sense, and the case where the beta-type conformable fractional derivative in Liouville-Caputo sense, is used.

Compared with the analytical solution of the Cobweb model with the conformable derivative, the analytical solution of the Cobweb model with the conformable fractional derivative in Liouville-Caputo sense and the analytical solution of the Cobweb model with the beta-type conformable fractional derivative in Liouville-Caputo sense are slowly approaching the equilibrium point.

References

[1] H. Alzer, Inequalities for the gamma and polygamma functions. *Abh. Math. Semin. Univ. Hamb*. 68(1998), 363–372.

[2] D.W. DeTemple, A quicker convergence to Euler's constant. *Am. Math. Mon.* 100(5) (1993), 468–470.

[3] C. P. Chen, C. Mortici, New sequence converging towards the Euler–Mascheroni constant. *Comput. Math. Appl.* 64(2012), 391–398.

[4] C. Mortici, C.P. Chen, On the harmonic number expansion by Ramanujan. *J. Inequal. Appl.* (2013)2013 222.

[5] D. W. Lu, Some new improved classes of convergence towards Euler's Constant, *Applied Mathematics and Computation* 243 (2014) 24–32.

[6] X. You, D.R. Chen, A new sequence convergent to Euler–Mascheroni constant, J. Inequal. Appl. (2018) 2018 75.

[7] C. Mortici, New approximations of the gamma function in terms of the digamma function, *Appl. Math. Lett.* 23(2010) 97–100.

[8] C. Mortici, Product approximations via asymptotic integration, *Amer. Math. Monthly* 117 (2010) 434–441.