There are infinitely many integers that can be expressed as the sum of four cubes of polynomials.

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Abstract

In this paper, we prove that there are infinitely many integers that can be expressed as the sum of four cubes of polynomials.

1. Introduction

As of now, the only integers that can be expressed as the sum of three cubes of polynomials are 1 and 2.

$$(1+6t^3)^3+(1-6t^3)^3+(-6t^2)^3=2.$$

$$(9t^4)^3+(3t-9t^4)^3+(1-9t^3)^3=1.$$

It appears that no identities have been published expressing an integer as a sum of four cubes of polynomials. Recently, Choudhry[1] proved that any integer expressible either as p^3+q^3 or $2(p^6-q^6)$ is expressible as a sum of four cubes of univariate polynomials.

In this paper, we prove that all integers of the form $3n^2-3n+1$ or $24n^2+2$ can be expressed as the sum of four cubes of polynomials.

2. $X^3+Y^3+Z^3+W^3=3n^2-3n+1$

$$X^3+Y^3+Z^3+W^3=3n^2-3n+1 \tag{1}$$

We prove that there are infinitely many integer solutions of $X^3+Y^3+Z^3+W^3=3n^2-3n+1$.
Let $X=x+m,\ Y=x+n,\ Z=-x+p,\ W=-x+q$, then LHS of equation (1) becomes

$$(3m+3n+3p+3q)x^2+(3m^2+3n^2-3p^2-3q^2)x+m^3+n^3+p^3+q^3. \tag{2}$$

Taking $x=-\dfrac{m^2+n^2-p^2-q^2}{m+n+p+q}$ and $q=1-m-n-p$, equation (2) becomes to

$$(-3m+3-3n)p^2-3(m-1+n)^2p-3(n-1)(-1+m)(m+n)+1. \tag{3}$$

Taking $m=1-n$, equation (3) becomes to

$$3n^2-3n+1. \tag{4}$$

Hence, we get a solution of equation (1).

$$X = n - 2n^2 + 2p^2,$$
$$Y = -1 + 3n - 2n^2 + 2p^2,$$
$$Z = 1 - 2n + 2n^2 - 2p^2 + p,$$
$$W = 1 - 2n + 2n^2 - 2p^2 - p.$$

n, p are any integers.
We give six examples below with $3n^2 - 3n + 1 < 100$.

$$(2p^2-1)^3 + (2p^2)^3 + (-2p^2+p+1)^3 + (-2p^2-p+1)^3 = 1,$$
$$(2p^2-6)^3 + (2p^2-3)^3 + (-2p^2+p+5)^3 + (-2p^2-p+5)^3 = 7,$$
$$(2p^2-15)^3 + (2p^2-10)^3 + (-2p^2+p+13)^3 + (-2p^2-p+13)^3 = 19,$$
$$(2p^2-28)^3 + (2p^2-21)^3 + (-2p^2+p+25)^3 + (-2p^2-p+25)^3 = 37,$$
$$(2p^2-45)^3 + (2p^2-36)^3 + (-2p^2+p+41)^3 + (-2p^2-p+41)^3 = 61,$$
$$(2p^2-66)^3 + (2p^2-55)^3 + (-2p^2+p+61)^3 + (-2p^2-p+61)^3 = 91.$$

Numerical examples for $p = 2 \cdots 10$.

$$7^3 + 8^3 + (-5)^3 + (-9)^3 = 1,$$
$$17^3 + 18^3 + (-14)^3 + (-20)^3 = 1,$$
$$31^3 + 32^3 + (-27)^3 + (-35)^3 = 1,$$
$$49^3 + 50^3 + (-44)^3 + (-54)^3 = 1,$$
$$71^3 + 72^3 + (-65)^3 + (-77)^3 = 1,$$
$$97^3 + 98^3 + (-90)^3 + (-104)^3 = 1,$$
$$127^3 + 128^3 + (-119)^3 + (-135)^3 = 1,$$
$$161^3 + 162^3 + (-152)^3 + (-170)^3 = 1,$$
$$199^3 + 200^3 + (-189)^3 + (-209)^3 = 1.$$

3. $X^3 + Y^3 + Z^3 + W^3 = 24n^2 + 2$

$$X^3 + Y^3 + Z^3 + W^3 = 24n^2 + 2 \tag{5}$$

We prove that there are infinitely many integer solutions of $X^3 + Y^3 + Z^3 + W^3 = 24n^2 + 2$.
Let $X = p + 2a^2,\ Y = q + 2a^2,\ Z = r - 2a^2 + a,\ W = r - 2a^2 - a$, then LHS of equation (5) becomes

$$(12p + 24r - 12 + 12q)a^4 + (6q^2 + 6p^2 - 12r^2 + 6r)a^2 + 2r^3 + p^3 + q^3. \tag{6}$$

Taking r $= -1/2p - 1/2q + 1/2$, then $6q^2 + 6p^2 - 12r^2 + 6r = 0$ becomes to

$$q^2 + p^2 - 2pq + p + q = 0. \tag{7}$$

To parameterize the equation (7), taking $p = -1 + t,\ q = 0 + n/mt$ then we get

$$(t,\ p,\ q) = \Big(-\frac{m(-m+3n)}{m^2-2nm+n^2},\ -\frac{n(m+n)}{(n-m)^2},\ -\frac{n(-m+3n)}{(n-m)^2}\Big).$$

Let $m = n-1$, then we get $(p,\ q,\ 2r^3+p^3+q^3) = (-n(2n-1),\ -n(2n+1),\ 3n^2+1/4)$.
Hence, we get a solution of equation (5).

$$\begin{aligned} X &= 4a^2-4n^2+2n,\\ Y &= 4a^2-4n^2-2n,\\ Z &= -4a^2+2a+4n^2+1,\\ W &= -4a^2-2a+4n^2+1. \end{aligned}$$

a, n are any integers.
We give two examples below with $24n^2+2 < 100$.

$$\begin{aligned} &(4a^2-2)^3+(4a^2-6)^3+(-4a^2+2a+5)^3+(-4a^2-2a+5)^3 = 26,\\ &(4a^2-12)^3+(4a^2-20)^3+(-4a^2+2a+17)^3+(-4a^2-2a+17)^3 = 98. \end{aligned}$$

4. $X^3+Y^3+2Z^3 = m$

We give several identities of $X^3+Y^3+2Z^3 = m$.

$$(3n^3+1)^3+(-3n^3+1)^3+2(-3n^2)^3 = 2 \tag{8}$$

$$(-x^2+3bx-b^2)^3+(-x^2+bx+b^2)^3+2(x^2-2bx+b^2)^3 = 2b^6 \tag{9}$$

To obtain the identity (8), we consider

$$(an^3+1)^3+(-an^3+1)^3+2(-bn^2)^3 = (6a^2-2b^3)n^6+2.$$

Taking $(a, b) = (3, 3)$, we get (8).

To obtain the identity (9), we consider

$$(-ax^2+b_1x-c)^3+(-ax^2+b_2x+c)^3+2(ax^2+(b_2-b_1)x+c)^3 = 0.$$

Let $b_1 = 3b_2,\ c = b_2^2/a,\ b = b_2$, and $a = 1$, then we get (9).

Taking $b = 1$, we get

$$(-x^2+3x-1)^3+(-x^2+x+1)^3+2(x^2-2x+1)^3 = 2.$$

References

[1] A. Choudhry, Expressing an integer as a sum of cubes of polynomials, https://arxiv.org/abs/2311.07325

[2] L. J. MORDELL, Diophantine Equations, Pure and Appl. Math., vol. 30, Academic Press, London and New York, 1969