Characteristics of primes within a limited number boundary

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Abstract

Primes less a given number n ($n \geq 2$) determines new primes within a limited area increased with a square (n^2) or decreased with a square root ($\sqrt{n}$). As the area is extended, the number of primes is also changed and controlled within an extended area boundary or number boundary, $n \sim n^2$ or $n \sim \sqrt{n}$. The structure of a number boundary is applied to the Euler product and helps to characterize the Euler's prime boundary between n and ($n^2 - 1$). The characterized Euler product is used to characterize the non-trivial zeroes derived in an elementary way of Riemann zeta function. Then, the characterized Euler product and non-trivial zeroes are discussed regarding their potential number boundaries. Overall, it is concluded that the characteristic of a number boundary can represent the characteristic of primes, especially the number of primes. As the number boundary is characterized by the increased or decreased exponent while the base or given number n is fixed, it is concluded that the pattern of exponent in the number boundary would be a key to understanding the pattern of primes.

Introduction

A prime number is a natural number greater than or equal to 2, and it has no divisors except 1 and itself (Stein 2000). Since Euclid proved the infinite number of primes, there have been efforts to identify a pattern in prime numbers, but a definitive governing rule for the distribution of primes is still lacking.

The number of primes within a given integer n, $\pi(n)$, had been estimated in an equation of $\pi(n) \approx \frac{n}{ln\,(n)}$, while Euler proved that the sum of all positive integers raised to a certain power of prime numbers - known as the Euler product (Euler 1737). The Euler product was continued to the complex plane and attempted to improve the accuracy of the prime-counting in the Riemann zeta function; as a result, the non-trivial zeroes were derived (Zagier 1977). Thus, the non-trivial zeroes have been expected to understand the pattern of prime numbers.

In this paper, the characteristic of primes was analyzed in three sections: 1. Application of sine wave analysis, 2. Analysis of Euler product, and 3. Analysis of non-trivial zeroes. In the first section, the general prime rule was defined using the sine wave analysis then applied to modify the Euler product in the second section. The structure of the modified Euler product was applied to reorganize the non-trivial zeroes in the last section. Since the non-trivial zeroes did not directly address primes, this was separated and independently analyzed in the discussion chapter.

Materials and Methods

The various calculations and trends analysis were performed using Excel (version 2016, Microsoft, Redmond, WA, USA). After the analysis of trends, the equations were visualized in the Desmos, online graphing calculator (www.desmos.com). Additional graph modification was performed in Illustrator (version CS6, Adobe, CA, USA).

The list of primes and non-trivial zeroes were identified in the online database, the Prime I.T. (www.compoasso.free.fr) and the L-functions and modular forms database (www.lmdfb.org), respectively.

Results and Conclusions

Section I. Application of sine wave function

The key point in this section is defining the number boundary that governs the general prime rules using the sine wave analysis. The overall procedure is similar to the Sieve of Eratosthenes, an algorithm for finding primes by deleting composites (Stein 2000). Thus, it is reasonable to consider that the sine wave function is a modified version of the Sieve of Eratosthenes.

The rhythm of numbers in the sine wave

Using the sine wave, $sin\ (\frac{180}{n} \cdot x)$, the rhythm of numbers is prepared at interval of n ($n \geq 1$).

$$y_1 = sin\ (\frac{180}{1} \cdot x),\ y_2 = sin\ (\frac{180}{2} \cdot x),\ y_3 = sin\ (\frac{180}{3} \cdot x),\ y_4 = sin\ (\frac{180}{4} \cdot x),\ y_5 = sin\ (\frac{180}{5} \cdot x),$$

$$y_6 = sin\ (\frac{180}{6} \cdot x),\ y_7 = sin\ (\frac{180}{7} \cdot x),\ y_8 = sin\ (\frac{180}{8} \cdot x),\ y_9 = sin\ (\frac{180}{9} \cdot x),\ y_{10} = sin\ (\frac{180}{10} \cdot x),\ \ldots$$

$$y_n = sin\ (\frac{180}{n} \cdot x)$$

Suppose that the sine wave, y_1, is divided by the rest of sine waves up to y_n.

$$y = \frac{y1}{y2 \cdot y3 \cdot y4 \cdot y5 \cdot y6 \cdot y7 \cdot y8 \cdot y9 \cdot y10 \ldots yn}$$

$$= \frac{sin(\frac{180}{1} \cdot x)}{sin(\frac{180}{2} \cdot x) \cdot sin(\frac{180}{3} \cdot x) \cdot sin(\frac{180}{4} \cdot x) \cdot sin(\frac{180}{5} \cdot x) \cdot sin(\frac{180}{6} \cdot x) \cdot sin(\frac{180}{7} \cdot x) \cdot sin(\frac{180}{8} \cdot x) \cdot sin(\frac{180}{9} \cdot x) \cdot sin(\frac{180}{10} \cdot x) \ldots \cdot sin\ (\frac{180}{n} \cdot x)}$$

If the interval n is composite with products of primes p and q in y_n, the prime wave y_p and y_q should be overlapped with y_n. As a result, y_n is removed in the denominator. In this way, the wave of composites is removed and only the wave of primes remains.

$$y = \frac{sin(180 \cdot x)}{sin(\frac{180}{2} \cdot x) \cdot sin(\frac{180}{3} \cdot x) \cdot sin(\frac{180}{5} \cdot x) \cdot sin(\frac{180}{7} \cdot x) \ldots \cdot sin\ (\frac{180}{prime} \cdot x)}$$

$$= \frac{sin(180 \cdot x)}{\prod_{p\ =\ prime}\ sin\ (\frac{180}{p} \cdot x)}$$

The characteristics of sine wave analysis are summarized below.

1. Suppose any composite number c in the *x-axis*. If there is a factor of c in the denominator, then it satisfies ' $y = \frac{sin(180 \cdot c)}{sin(\frac{180}{2} \cdot c) \cdot sin(\frac{180}{3} \cdot c) \cdot sin(\frac{180}{5} \cdot c) \cdot sin(\frac{180}{7} \cdot c) \ldots \cdot sin\ (\frac{180}{p} \cdot c)} \neq 0$'. Consequently, the value of c cannot be defined on the *x-axis* while $y = 0$, so the remaining values less than c are all primes.
2. In this way, the primes less than p determine new primes within a limited number boundary between p and p^2.
3. After p^2, the series of new primes are found until the first non-prime, q^2, is found, then q is the prime following p.
4. As a result, the series of primes less than p determine the new primes within a number boundary from $p \sim p^2$ (minimum) to $p \sim q^2$ (maximum).

Using the sine wave analysis, the real primes (≥ 2) can be determined within a limited but increased number boundary with a square (p^{2n}, where $n \geq 1$). Also, it is possible to consider the limited but decreased number boundary with a square root ($p^{\frac{1}{2n}}$) (Figure 1). Similar to real primes, the decreased boundary is thought to be determined by the decreased number of primes including unreal primes less than 2, referred to as 'hypothetical unreal primes' in this paper. The exponent controls the boundary limits while any given base number is fixed. Therefore, it is concluded that the pattern of the exponent characterizes the pattern of the number boundary. As a result, it affects the pattern of primes, especially the number of primes.

If each limited but increased (p^{2n}) or decreased ($p^{\frac{1}{2n}}$) number boundary from any base number, p^n, is divided by the respective logarithmic value, $ln(p^{2n})$ or $ln(p^{\frac{1}{2n}})$, it forms the prime-counting function:

estimated number of real primes in the increased boundary

$$\pi(p^{2n}) = \frac{p^{2n}}{\ln(p^{2n})} = \frac{p^{(2n-1)}}{2n} \cdot \frac{p}{ln(p)}, \text{ and}$$

estimated number of primes including hypothetical unreal primes in the decreased boundary

$$\pi\left(p^{\frac{1}{2n}}\right) = \frac{p^{\frac{1}{2n}}}{\ln(p^{\frac{1}{2n}})} = 2n \cdot p^{(\frac{1}{2n}-1)} \cdot \frac{p}{ln(p)}$$

, where $\pi(x)$ is the number of primes within a given number x.

The number of primes between increased and decreased boundaries are balanced while $n = \pm 0.5$. This means that the number of primes can be balanced while the base number is formed with $\sqrt[2]{p}$ or $\frac{1}{\sqrt[2]{p}}$, and its interpretation is identical to Figure 1. If $\sqrt[2]{p}$ or $\frac{1}{\sqrt[2]{p}}$ is infinitely regularized back into the form of $\lim_{n\to\infty} \sqrt[2n]{p}$ or $\frac{1}{\sqrt[2n]{p}}$, it converges to 1. Therefore, 1 is the root number of all number boundaries.

Proof of number boundary characterized in the sine wave analysis

Suppose that series of natural numbers between n and n^2, where n is greater than or equal to 2.

$$\boldsymbol{n}, (n+1), (n+2), (n+3), (n+4), \ldots, (n^2-4), (n^2-3), (n^2-2), (n^2-1), \boldsymbol{n^2}$$

The largest composite number less than n^2 is $(n^2 – 1)$.

$$(n^2 - 1) = (n - 1) \cdot (n + 1)$$

Let $(n - 1)$ and $(n + 1)$ be equal to positive integers a and b, respectively. Then,

$$(n - 1) \cdot (n + 1) = a \cdot \mathrm{b}\ (a < b).$$

If n is even and both a and b are primes, then the prime a is a factor of $(n^2 – 1)$ and its value is less than n.

If n is odd and/or prime, then both a and b are even. The even a should be factorized into primes and used as factors for $(n^2 – 1)$. Thus, the factorized primes from a are also less than n.

Overall, it is concluded that the composites within a limited number boundary between n and n^2 have at least one prime within n as a factor. It means that the waves of the series of primes less than n can remove the composites, as a result, the remaining numbers are all primes within a limited boundary between n and n^2.

Section 2. Analysis of Euler product

Primes in the Euler product are replaced with the number of primes accordingly to adapt the Euler product within a limited number boundary, and this is a key process in Section 2.

Euler (1774) proved that the sum of the reciprocals of all positive integers was equivalent to the product of primes (Theorem 7 and 8), and it can be written as follows

$$f(x) = \frac{2^x}{2^x-1} \cdot \frac{3^x}{3^x-1} \cdot \frac{5^x}{5^x-1} \cdot \frac{7^x}{7^x-1} \cdot \frac{11^x}{11^x-1} \cdot \frac{13^x}{13^x-1} \cdots \frac{p^x}{p^x-1} \cdots$$

, where p is prime.

Suppose prime, p, is replaced by the number of primes, $\pi(p)$, then the Euler product is modified below.

$$f(x) = \frac{\pi(2)^x}{\pi(2)^x-1} \cdot \frac{\pi(3)^x}{\pi(3)^x-1} \cdot \frac{\pi(5)^x}{\pi(5)^x-1} \cdot \frac{\pi(7)^x}{\pi(7)^x-1} \cdot \frac{\pi(11)^x}{\pi(11)^x-1} \cdot \frac{\pi(13)^x}{\pi(13)^x-1} \cdots$$

Since the Euler product is composed of a sequential p, it can be expressed with a series of natural numbers.

$$f(x) = \frac{1^x}{1^x-1} \cdot \frac{2^x}{2^x-1} \cdot \frac{3^x}{3^x-1} \cdot \frac{4^x}{4^x-1} \cdot \frac{5^x}{5^x-1} \cdot \frac{6^x}{6^x-1} \cdots$$

The first value in the multiplication is moved to the left side to prevent errors in the equation.

$$\frac{1^x-1}{1^x} \cdot f(x) = \frac{2^x}{2^x-1} \cdot \frac{3^x}{3^x-1} \cdot \frac{4^x}{4^x-1} \cdot \frac{5^x}{5^x-1} \cdot \frac{6^x}{6^x-1} \cdots$$

$$0 = \frac{2^x}{2^x-1} \cdot \frac{3^x}{3^x-1} \cdot \frac{4^x}{4^x-1} \cdot \frac{5^x}{5^x-1} \cdot \frac{6^x}{6^x-1} \cdots$$

$$= 2^x \cdot 3^x \cdot 4^x \cdot 5^x \cdot 6^x \cdot \ldots$$

After selecting any number of primes, $\pi(p) = n$ (≥ 2), the Euler product is formed by

$0 = ((2^x \ldots \cdot (n-3)^x \cdot (n-2)^x \cdot (n-1)^x \cdot \boldsymbol{n^x} \cdot (n+1)^x \cdot (n+2)^x \cdot (n+3)^x \ldots \cdot (2n-2)^x) \cdot$

$(2n-1)^x \cdot (2n)^x \cdot (2n+1)^x \cdot (2n+2)^x \cdot (2n+3)^x \cdot (2n+4)^x \ldots$.

After taking x root, the modified Euler product is

$$0 = \big(2 \cdot \ldots (n-3) \cdot (n-2) \cdot (n-1) \cdot \boldsymbol{n} \cdot (n+1) \cdot (n+2) \cdot (n+3) \ldots (2n-2)\big) \cdot$$

$$(2n-1) \cdot (2n) \cdot (2n+1) \cdot (2n+2) \cdot (2n+3) \cdot (2n+4) \ldots .$$

$$= (\boldsymbol{n} \cdot (n^2-1^2) \cdot (n^2-2^2) \cdot (n^2-3^2) \ldots 2(2n-2)\ ) \cdot$$

$$(2n \cdot ((2n)^2-1^2) \cdot (2n+2) \cdot (2n+3) \cdot (2n+4) \ldots .$$

The modified Euler product is composed of three number boundaries:

the first boundary is $(\boldsymbol{n} \cdot (n^2-1^2) \cdot (n^2-2^2) \cdot (n^2-3^2) \ldots 2(2n-2)) \cdot$

, the second boundary is $(2n \cdot ((2n)^2-1^2) \cdot$

, and the third boundary is $(2n+2) \cdot (2n+3) \cdot (2n+4) \cdot \ldots$.

The values greater than $2n$ in the second and third boundaries can be rewritten as $(an)^2$ - b^2 or $(an-b) \cdot (an+b)$, where a and b are positive integers ($a > b$). Now, the second and third boundaries can change the forms as

$$0 = first\ boundary \cdot\ (\ldots (an-3) \cdot (an-2) \cdot (an-1) \cdot \boldsymbol{an} \cdot (an+1) \cdot (an+2) \cdot (an+3) \ldots) \cdot \ldots$$

$$= first\ boundary \cdot\ (\boldsymbol{an} \cdot ((an)^2-1^2) \cdot ((an)^2-2^2) \cdot ((an)^2-3^2) \ldots) \cdot \ldots .$$

The second and third boundaries are satisfied while the value n is equal to $\frac{b}{a}$. As $a > b$, the value n is less than 1 and it does not meet the basic requirements of the modified Euler product. Thus, the first boundary only satisfies the modified Euler product. In addition, the values should be overlapped between the first and third boundaries; the value of first boundary increases by the exponent of *m*, where *m* is a multiple of 2.

$$0 = (\boldsymbol{n} \cdot (2(2n-2) \cdot \ldots \cdot (n^2-3^2) \cdot (n^2-2^2) \cdot (n^2-1^2))^m) \cdot \ldots \cdot second\ boundary \cdot third\ boundary$$

However, the modified Euler product is balanced with 0, so the original form of the first boundary can be maintained regardless of *m*.

Overall, it is concluded that the modified Euler product can be defined within a limited first boundary between *n* and (n^2-1^2) after selecting any number of primes, *n*.

$$0 = (\boldsymbol{n} \cdot 2(2n-2) \cdot \ldots \cdot (n^2-3^2) \cdot (n^2-2^2) \cdot (n^2-1^2))$$

Further analysis of the modified Euler product

The modified Euler product can be written as follows.

$$0 = \boldsymbol{n} \cdot 2(2n-2) \cdot \ldots \cdot (n^2-3^2) \cdot (n^2-2^2) \cdot (n^2-1^2)$$

$$= \boldsymbol{n} \cdot \prod_{m=1}^{n-1} (n^2-m^2)$$

As each parenthesis satisfies the modified Euler product, it can be treated as an array.

The subtraction of consecutive arrays is performed to investigate the potential patterns.

$$(n^2-(m-1)^2) - (n^2-m^2) = 2m-1$$

For example, if *n* is 16 and the subtraction values can be calculated and it forms the Euler's boundary between 256 and 31.

m	**2*m* - 1**	$(\boldsymbol{n^2-m^2})$
1	1	**256 (16^2)**
2	3	255
3	5	252
4	7	247
5	9	240

6	11	231
7	13	220
8	15	207
9	17	192
10	19	175
11	21	156
12	23	135
13	25	112
14	27	87
15	29	60
.	.	31

Then, the Euler's boundary can be standardized in the quadratic equation,

$$y = -0.25 \cdot x^2 + 0.5 \cdot x + (n^2 - 0.25)$$

, where *x-axis* is ($2m - 1$) and *y-axis* is ($n^2 - m^2$) (Figure 2).

In results, the Euler's boundary can be defined between ($3, n^2 - 1$) and ($2n$ - 1, $2n$ - 1) (shaded area in Figure 2). Along the parabola, the boundary can be extended except ($1, n^2$) as it is pointed at the vertex. While the parabola is shifted up or down due to *y*-intercept, ($0, n^2$- 0.25), it affects the angle of θ (Figure 3A to C). Thus, the potential pattern of Euler's number boundary would be analyzed between n and θ. For example, θ is limited between 0° (0) < θ < 90° ($\frac{\pi}{2}$ or 1.5707). If θ increases, it converges close to 1.5707 and the ratio between hypotenuse and opposite can be standardized with $\sin\theta$ (Figure 3A). If θ decreases close to 0, $\sin\theta$ can also be standardized (Figure 3B). While the *y*-intercept is 0, the value n is ±0.5 and $\sin\theta$ is 0.2425 (Figure 3C). It means that the angle of θ is balanced at 0.2527. Thus, it is concluded that the modified Euler product would be characterized by two different boundary patterns: increasing from (0.5, 0.2425) to (∞, 1) (bold red) in the flipped bell curve or decreasing from (0.5, 0.2425) to (0, 0) (bold blue) in the sigmoid (Figure 3D).

Discussion

Both Euler product and non-trivial zeroes are derived from primes (Zagier 1977), so it is reasonable to approach the non-trivial zeroes using the characteristic of the Euler product. In this discussion, the non-trivial zeroes are just treated as specialized numbers related to primes; mathematical concepts, techniques, and other principles relevant to the Riemann hypothesis are excluded.

Section 3. Analysis of non-trivial zeroes

Using the sine function, the general rule of primes was explained between the initial integer *n* and its increased or decreased number boundary, n^2 or $\sqrt{n}$. Between *n* and n^2, for example, the characteristic of primes was explained in a form of '... $(n-3),\ (n-2),\ (n-1),\ \boldsymbol{n},\ (n+1),\ (n+2),\ (n+3)$ ...' and each array was connected by multiplication in the modified Euler product.

The structure of non-trivial zeroes (NT_n) was similar with the Euler's number boundary and it was fixed from (0.5, $\pm NT_n\, i$). Thus, NT_n could be written following the Euler's boundary structure.

$$0 = \cdots (0.5 - NT_3 i) \cdot (0.5 - NT_2 i) \cdot (0.5 - NT_1 i) \cdot \mathbf{0.5} \cdot (0.5 + NT_1 i) \cdot (0.5 + NT_2 i) \cdot (0.5 + NT_3 i) \ldots$$

$$= \cdots (0.5 - 25.0109i) \cdot (0.5 - 21.0220i) \cdot (0.5 - 14.1347i) \cdot \mathbf{0.5} \cdot (0.5 + 14.1347i) \cdot (0.5 + 21.0220i) \cdot (0.5 + 25.0109i) \ldots$$

The role of imaginary function was transforming the negative sign into positive.

$$0 \ = \ \mathbf{0.5} \cdot (0.5^2 + 14.1347^2) \cdot (0.5^2 + 21.0220^2) \cdot (0.5^2 + 25.0109^2) \ldots$$

$$= \ \mathbf{0.5} \cdot \prod_{n=1}^{\infty} (0.5^2 + NT_n^2)$$

Due to similarity with Euler's boundary, each parenthesis could be treated as an array. Thus, the above equation could be simplified with the quadratic equation below (Figure 4).

$$f(NT_n) = (0.5^2 + NT_n^2)$$

As NT_n shifted in *x-axis*, the angle of θ was also shifted between 0° (0) < θ < 90° ($\frac{\pi}{2}$ *or* 1.5707) (Figure 4A). Thus, the parabola could be standardized with $\sin\theta$ (pattern IV), so that it could be compared with other $\sin\theta$ in the modified Euler product (patterns I, II, and III) (Figure 4B).

In the results, the role of actual NT_n was limited because NT_n was treated with its order rather than its value. After NT_n was treated similarly to n^{th} prime number, it showed the pattern of sigmoid (black line, IV) and intersected with Euler's flipped bell curve at point (1) (1, 0.707) (Figure 4B). From the point (1), it could systematically decrease either to (0.5, 0.242) between pattern I and II or (0, 0) between pattern II and IV. Similarly, it could also increase to (∞, ∞). Meanwhile, the *x-axis* of point (1) moved by multiplication of $\frac{1}{\sqrt{2x}}$ or $\sqrt{2x}$ along the curvature of modified Euler product or by multiplication of $\frac{1}{2x}$ or $2x$ along the curvature of NT_n. Considering the multiplication scales, it was possible to conclude that the modified Euler product and non-trivial zeroes could also be defined within a limited number boundary. As the number boundary was characterized by the exponent in Section 1, the patterns of potential number boundaries in Figure 4B would represent the patterns of exponents from the fixed base number or logarithmically treated base number.

The number boundary is controlled by the exponent while any given base number is fixed, so the pattern of the exponent would be a key to understanding the pattern of primes. Also, the exponent has a reciprocal relationship, and this would be how the increased and decreased boundaries are related (Figure 1). Seemingly, the reciprocal relationship is indirectly expressed in the curvature of the Euler product and non-trivial zeroes. For example, the points among (1) and (2), (-1) and (0), and (-3) and (-2) would be cases of reciprocal relationship between increased (pattern I, red) and decreased (pattern II, blue) Euler's curvatures in Figure 4B. Also, the curvature of non-trivial zeroes shows a similar relationship but their values are related by $\frac{1}{2x}$ or $2x$ in the *x-axis* (pattern IV, black). As long as the base number is logarithmically treated with the exponent of $\frac{1}{2x}$ or $2x$, it can be compared with values with the exponent of $\frac{1}{\sqrt{2x}}$ or $\sqrt{2x}$. Therefore, it is possible to estimate that the condition of the base number in a boundary would differ between the Euler product and non-trivial zeroes.

General Conclusions

In general, it is concluded that the characteristic of the number boundary is represented in the characteristic of primes, especially the number of primes. In Section 1, it is proved that the primes determine new primes within a limited but increased or decreased number boundary, and this is why the number of primes depends on the size of the number boundary. The dependent relationship between primes and number boundary characterizes the Euler product and non-trivial zeroes in Section 2 and 3 by suggesting their potential number boundaries.

Overall, it is concluded that the Euler product and non-trivial zeroes would also be controlled within limited number boundaries and they might be represented by the boundary's exponent pattern. In this paper, the modified Euler product and non-trivial zeroes were not addressed with primes, but in terms of the number of primes and the order of non-trivial zeroes. Therefore, the interpretation discussed above should be seen as one of the possible scenarios.

Figure 1. Visualization of number boundary from 2. A) Primes less than or equal to 2 determine the new primes in a boundary of 2^2. In this way, the number boundary continues to increase by determining the real primes. B) Similarly, the number boundary also continues to decrease with a square root and it is determined by the hypothetical unreal primes.

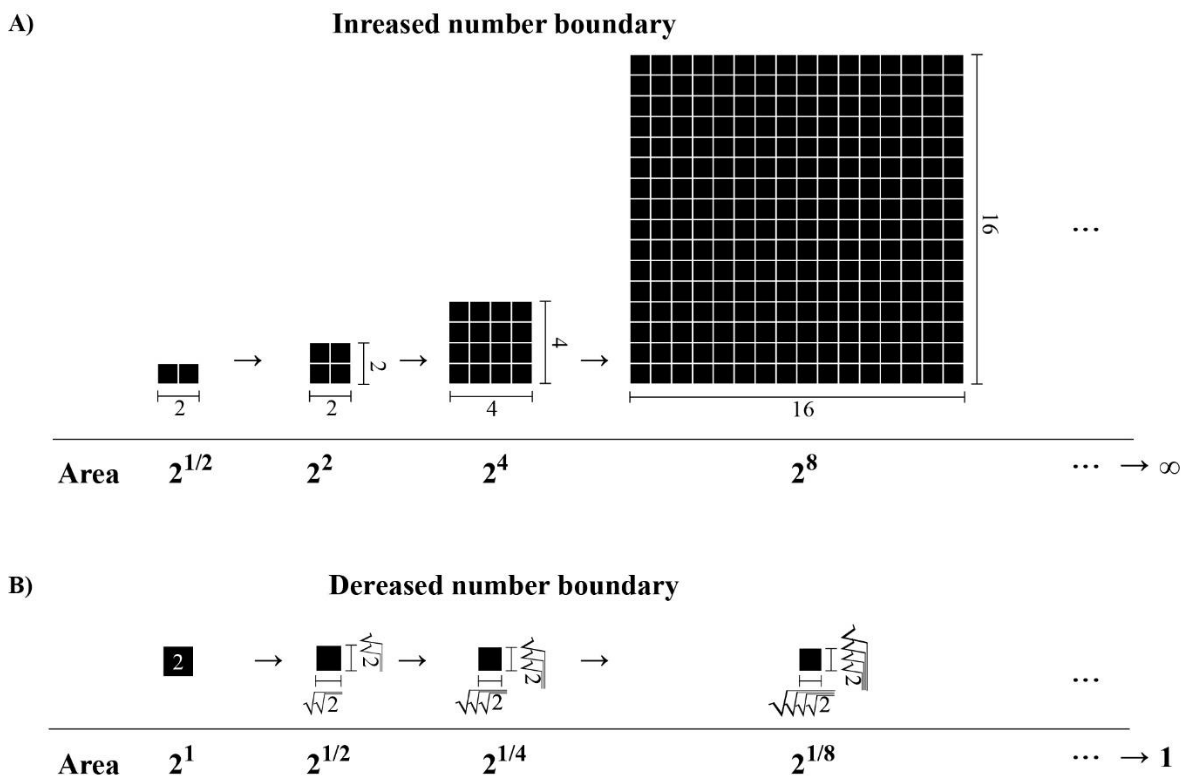


Figure 2. Visualized modified Euler product. The quadratic shows left-right symmetry from $(1, n^2)$, where n is number of primes, $\pi(p)$, less than prime, p. After selecting any number of primes, n, the modified Euler product is satisfied in the shaded area.

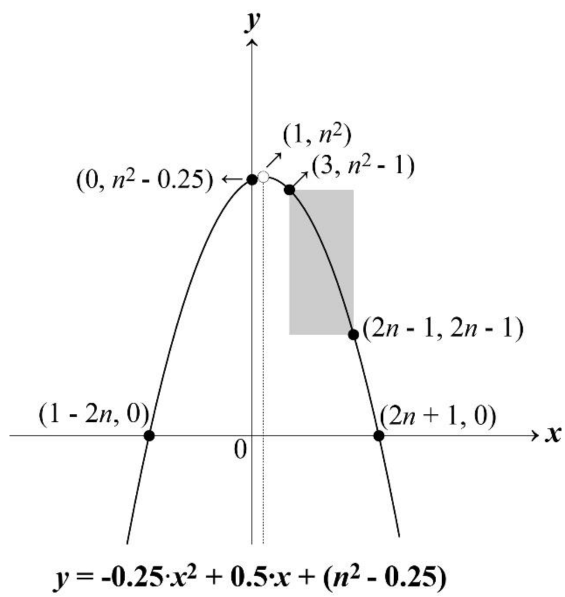


Figure 3. Further analysis of the modified Euler product. As the quadratic shifted to A) up or B) down, it affects the angle of θ and it is used to standardize the Euler product. While the number of primes is 0.5, C) the quadratic shows left-right symmetry. D) From the point (0.5, 0.2425), the modified Euler product decreases to (0, 0) with the sigmoid (bold blue) while increases to (∞, 1) with the flipped bell curve (bold red).

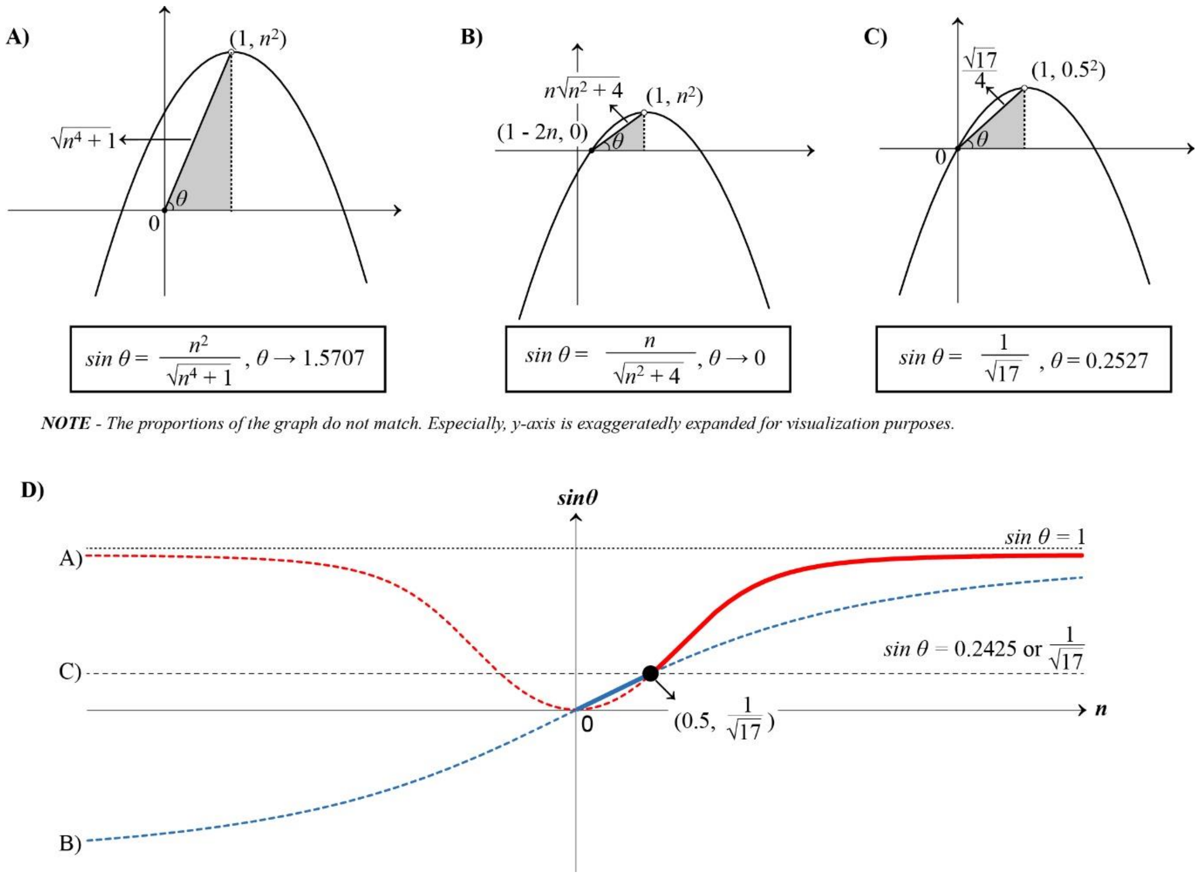
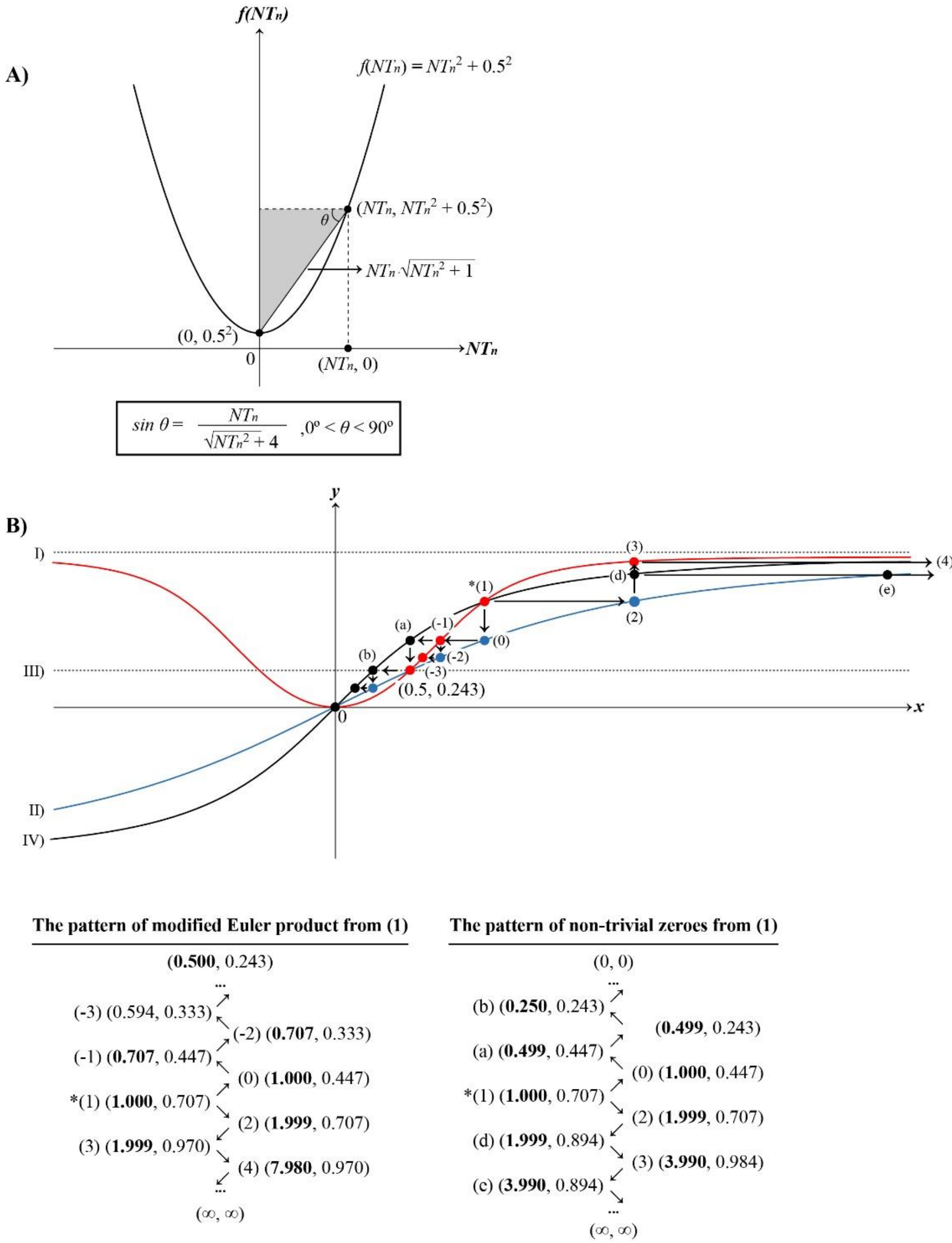


Figure 4. Analysis of the non-trivial zeroes (NT_n). Using the Euler's structure, A) NT_n was reorganized and standardized with the quadratic equation. As NT_n shifted along the *x-axis*, it affected the angle of θ between $0° < \theta < 90°$; B), *sin* θ (pattern IV) was calculated for comparison to Euler's *sin* θ (patterns I, II, and III). A limited but potentially increased or decreased number of number boundaries were estimated in the *x-axis* from the point *(1) (1.000, 0.707) in the Euler's *sin* θ, and it helped to estimate the potential boundary of NT_n.



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