On the Breakdown of Stationary Action in Field Theory

Ervin Goldfain

Ronin Institute, Montclair, New Jersey 07043

Email: ervin.goldfain@ronininstitute.org

Abstract

It is known that both classical and Quantum Field Theory (QFT) are built on the fundamental principle of stationary action. The goal of this introductory work is to analyze the breakdown of stationary action under *nonadiabatic conditions*. These conditions are presumed to develop far above the Standard Model scale and favor the onset of Hamiltonian chaos and fractal spacetime. The nearly universal transition to nonadiabatic behavior is illustrated using a handful of representative examples. If true, these findings are likely to have far-reaching implications for phenomena unfolding beyond the Standard Model scale and in early Universe cosmology.

Key words: field theory, decoherence, adiabatic invariance, nonintegrability, nonequilibrium statistical mechanics, Hamiltonian chaos, fractal spacetime.

1. Introduction

Action invariance is the bedrock of classical and quantum physics, as both Lagrangian and Hamiltonian formulations of the theory stem from the stationary action principle and its consequences. What was (and continues to be) largely overlooked is that the action principle ceases to hold in *strongly fluctuating settings*, in unstable systems *far from thermodynamic equilibrium*, as well as in systems approaching *critical behavior*. It is for this (deceptively) simple reason that extrapolations of field theories beyond their effective approximations are doomed to fail near continuous phase transitions and in the limit of exceedingly large *or* exceedingly low energy scales. We briefly review here the breakdown of stationary action in nonadiabatic conditions present in the deep ultraviolet (UV) sector of high-energy physics and of early Universe cosmology.

The paper is organized in the following way: Section two elaborates upon the breakdown of adiabatic invariance in Hamiltonians dependent on a generic parameter and evolving on ultrashort time scales. Next section illustrates the nearly universal transition to nonadiabatic behavior using several textbook examples. A summary discussion and concluding remarks are included in the last section.

The paper is based on the following couple of assumptions:

A1) *Decoherence* of quantum processes and the subsequent transition to classicality is expected to occur in the deep UV sector of energies, at some larger scale exceeding the Standard Model scale.

A2) Canonical variables of Hamiltonian theory (q, p) represent classical fields and their conjugate momenta, i.e. $(q, p) \Leftrightarrow (\phi, \pi)$.

Our analysis confirms that the breakdown of stationary action in non-adiabatic conditions overlaps with the onset of *nonintegrability*. The net effect of this scenario is the unavoidable emergence of *fractal spacetime* endowed with evolving deviations from four dimensions.

The reader is cautioned that our intent here is opening an unexplored research avenue that goes beyond the mainstream paradigms of foundational physics. Given its introductory nature, the paper must be approached with caution and a healthy dose of skepticism. Independent work needs to confirm, develop or reject the body of ideas detailed below.

2. Adiabatic invariants in Hamiltonian dynamics

Following [1, 6], consider a one-dimensional field model characterized by the Hamiltonian $H(q,p;\lambda)$, which is dependent on a time-varying parameter $\lambda=\lambda(t)$. Assume that the field undergoes a finite periodic motion with period T_0 and that the parameter λ is slowly varying during T_0, that is,

$$d\lambda/dt << \lambda/T_0 \tag{1}$$

If λ were constant, the motion of the field would be strictly periodic with a constant energy $E=E(T_0)$. Since λ is slowly varying, averaging the energy rate over T_0 yields the approximation,

$$\frac{\overline{dE}}{dt} = \frac{d\lambda}{dt}\frac{\overline{\partial H}}{\partial \lambda} \qquad (2)$$

For fixed E and λ, the canonical action of the system is the integral taken over the closed path C in phase space, namely,

$$I = \int_C \frac{p\,dq}{2\pi} = \iint \frac{dp\,dq}{2\pi} \qquad (3)$$

By (1) – (3), the rate of the action average is an adiabatic invariant defined by

$$\boxed{\frac{\overline{dI}}{dt} = 0} \qquad (4)$$

To fix ideas, consider a one-dimensional scalar field with parameter independent Hamiltonian,

$$H(p,q) = \frac{1}{2}(p^2 + \omega^2 q^2) = E \qquad (5)$$

The phase space trajectory of (5) is an ellipse, and the adiabatic invariant is simply,

$$\bar{I} = I = \frac{E}{\omega} = \frac{ET_0}{2\pi} \tag{6}$$

In this case, (1), (4), (6) are held by default, and the canonical action (6) recovers the *Lorentz invariant* of classical and QFT.

An interesting extension is the case of the one-dimensional scalar field whose Hamiltonian has a slowly varying frequency [10]

$$H(p,q,\tau) = \frac{1}{2}[p^2 + \omega^2(t)q^2] = \frac{1}{2}[p^2 + \frac{q^2}{\tau^2(t)}] \tag{7}$$

Here, the driving parameter is $\lambda(t) \equiv \omega(t) = [\tau(t)]^{-1}$. Introducing a new set of canonical variables

$$I = H\tau\,; \quad \varphi = \sin^{-1}\frac{q}{\sqrt{2H\tau^2}} \tag{8}$$

and relabeling the new action as

$$J = \frac{1}{2\pi}\int_0^{2\pi} I\,d\varphi \tag{9}$$

implies the condition

$$\left|J-I\right|=O(\frac{d\tau}{dt}) \tag{10}$$

Analysis shows that the rate of (9) takes the form [10]

$$\frac{dJ}{dt}=O[\frac{d^2\tau}{dt^2};(\frac{d\tau}{dt})^2] \tag{11}$$

It follows that, if the time rate of τ is adiabatically slow, integrating (11) leads to

$$\left|J(t)-J(0)\right|<<1 \tag{12}$$

and

$$\boxed{\left|I(t)-I(0)\right|<<1} \tag{13}$$

In summary, (4) and (13) show that, over *suitably defined extended periods of time*, action I acts as adiabatic invariant, a conclusion consistent with the concept of integrability in Hamiltonian dynamics [14]. The opposite conclusion is that, *over sufficiently short time scales* labeled as $t<<t_0=O(1)$, (4)

and (13) are not guaranteed to hold. Specifically, if the rate of the action is comparable with the parameter rate,

$$\boxed{\frac{dI}{dt} = O(\frac{\overline{dI}}{dt}) = O(\frac{d\lambda}{dt})} \tag{14}$$

the action I no longer plays the role of a Lorentz invariant. These considerations strongly hint that (14) has far-reaching implications for phenomena occurring beyond the Standard Model scale and in early Universe cosmology [12 – 13, 15 – 16].

3. Field theory under nonadiabatic conditions

The goal of this section is to expand the analysis of nonadiabatic behavior to more realistic parameter-driven models.

3.1) First off, consider a statistical mechanics context where the rate of parameter $\lambda(t)$ follows a Langevin-type equation [5],

$$\frac{d\lambda(t)}{dt} + \eta\lambda(t) = s(t) + f(t) \tag{15}$$

Here, η is the dissipation parameter, $s(t)$ and $f(t)$ denote the slow and fast fluctuations with constant and zero mean values, respectively,

$$\langle s(t) \rangle = s_0 << 1 \tag{16}$$

$$\langle f(t) \rangle = 0 \tag{17}$$

If the parameter $\lambda(t)$ is taken to be nearly-vanishing upon averaging, a minimal extension of adiabatic invariance embodied in (1), (2), (4) and (13) may be derived from

$$\left\langle \frac{d\lambda(t)}{dt} \right\rangle = s_0 - \eta\lambda_0 \tag{18}$$

where

$$\langle \lambda(t) \rangle = \lambda_0 << 1 \tag{19}$$

A typical hypothesis behind the Langevin equation is that the fast fluctuations $f(t)$ represent uncorrelated "white" noise, as in

$$\langle f(t_1) f(t_2) \rangle = g\,\delta(t_1 - t_2) \qquad (20)$$

Here, g is a characteristic strength factor given by

$$g = \frac{2\eta}{\beta}; \quad \beta = T^{-1} \qquad (22)$$

and T is the temperature, assumed to be a time-independent quantity. If

$$L(t) = \int_t \lambda(t)\, dt \qquad (21)$$

it can be shown that, in the long-time limit $t >> t_0 = O(1)$, the mean square of (21) assumes the form

$$\Delta^2 = \langle L^2(t) - L^2(0) \rangle \propto t \qquad (22)$$

which shows that parameter fluctuations are prone to grow over time and eventually destabilize the adiabatic invariance condition set by (4).

The takeaway point of this derivation is that long-term adiabatic invariance is likely to break down regardless of initial conditions (16) – (19). Yet, this is hardly surprising since, by construction, the Langevin equation describes the behavior of out-of-equilibrium statistical systems.

3.2) We next analyze the case where the dynamics contains two sets of canonical variables, each evolving on its own time scale. Consider the "slow-fast" Hamiltonian [2 - 3]

$$H(p,q,\Lambda,\lambda,\omega(t)) = \frac{p^2}{2} + \cos(q) + \cos(q-\lambda) + \omega(t)\Lambda \tag{23}$$

(23) describes a "pendulum-like" oscillator whose fast and slow conjugate variables are, respectively,

$$(p,q) \Rightarrow \textit{fast variables} \tag{24a}$$

$$(\Lambda,\lambda) \Rightarrow \textit{slow variables} \tag{24b}$$

Here, the external parameter λ represents an angle with a time-varying frequency $\omega(t)$. Note that, in the absence of slow variables, (23) reduces to the dynamics of the classical pendulum discussed, for instance, in [8].

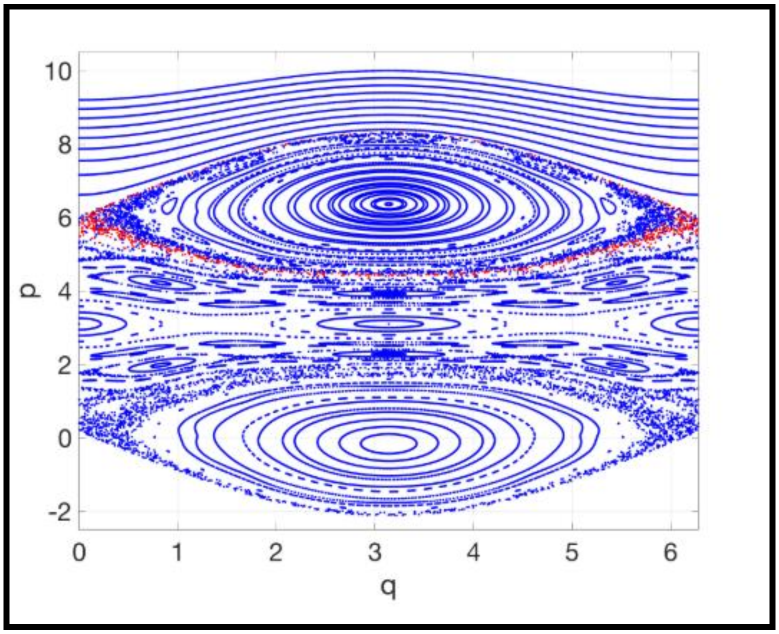

Fig. 1: Poincaré section of flow (23) for $\omega = 6.2$ [2].

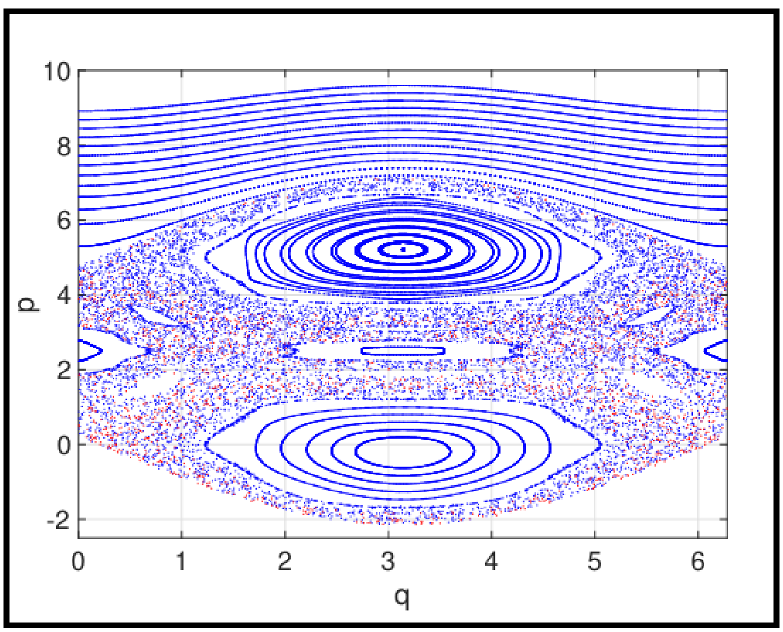

Fig. 2: Poincaré section of flow (23) for $\omega = 5$ [2].

Figs. 1 and 2 are phase space (Poincaré) sections of the flow induced by (23) for different values of ω. The interlaced structure of chaotic regions and islands of stability is a typical signature of Hamiltonian chaos and the emergence of fractal spacetime, per [7, 12 – 13.] Fig. 2 shows the structure of the flow almost entirely covered by chaotic regions.

3.3) Next model looks at the case where λ is a constant driving force acting on a nonlinear oscillator with cubic interaction. The equations of motion are given by [11]

$$\frac{dp}{dt} = q - q^3 + \lambda \cos \omega t \qquad (25a)$$

$$\frac{dq}{dt} = p \qquad (25b)$$

These equations describe a periodically driven *Duffing oscillator*. As in the previous model 3.2), the phase space plots clearly show the onset of Hamiltonian chaos and fractal spacetime.

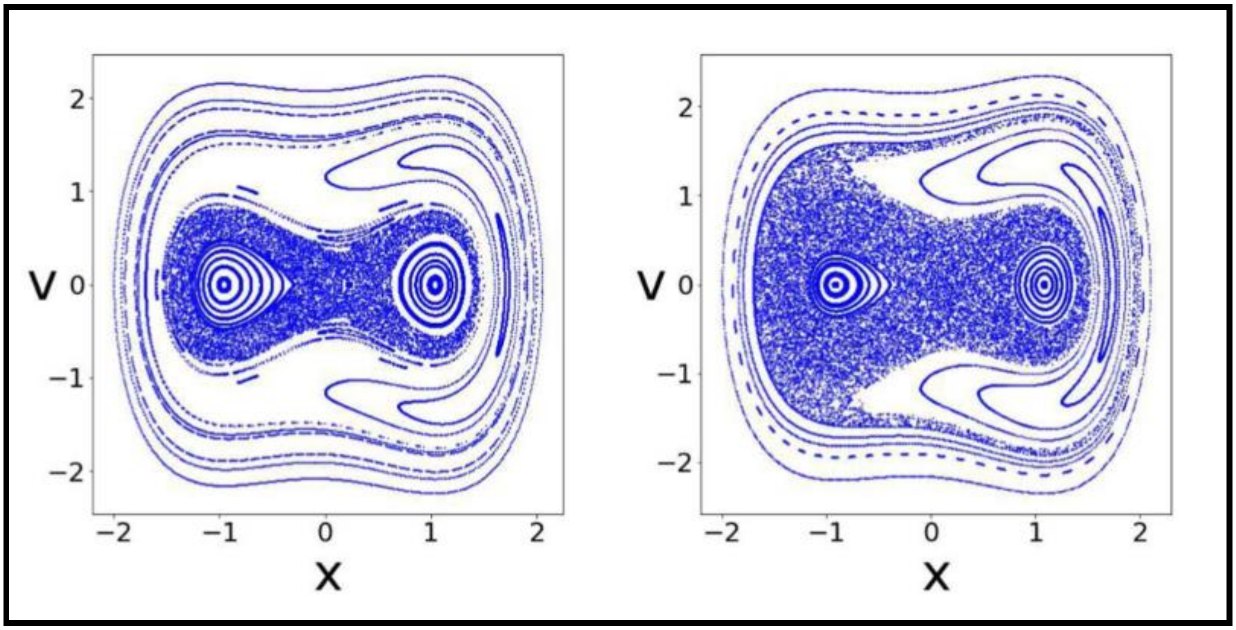

Fig 3: Duffing oscillator for different values of λ [11] ($x = q$; $p = v$).

3.4) The previous model can be generalized to the case where both periodic forces (λ) and dissipation (η) are added to a standard oscillator.

3.4.1) As traditional prototype of Hamiltonian chaos, the *conservative Standard Map* models a "kicked rotor" system whose evolution equations are given by [4, 7]

$$I' = I + \lambda \sin\theta \tag{26a}$$

$$\theta' = \theta + I + \lambda \sin\theta \tag{26b}$$

where I,θ are action-angle variables. Likewise, the *dissipative Standard Map* is defined by

$$I' = (1-\eta)I + \mu + \lambda \sin\theta \tag{27a}$$

$$\theta' = \theta + (1-\eta)I + \mu + \lambda \sin\theta \tag{27b}$$

in which μ represents a drift parameter, independent of λ and η.

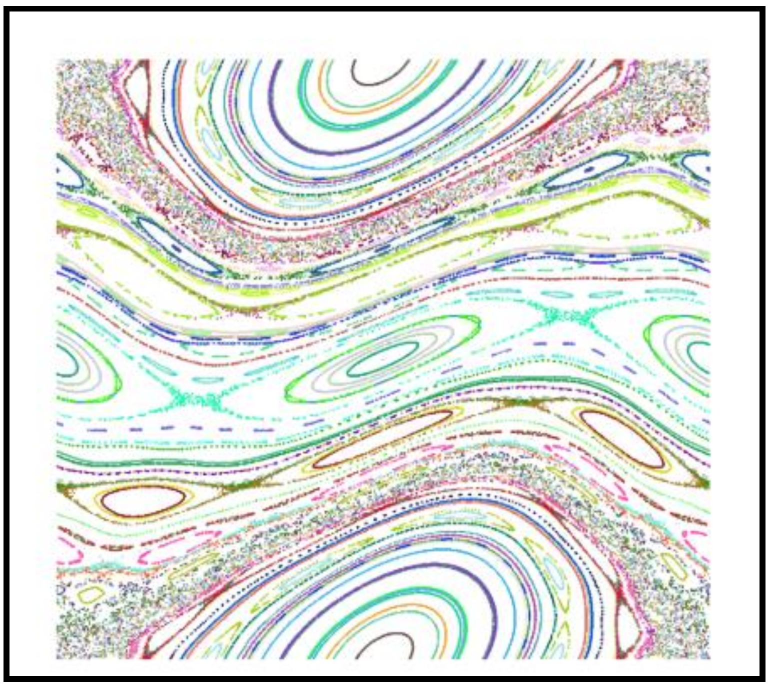

Fig. 4: The conservative Standard Map [4]

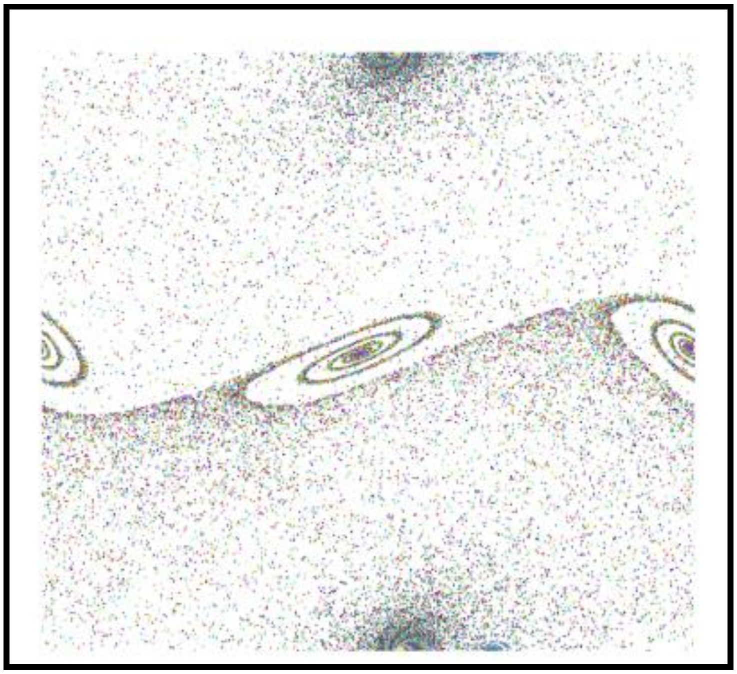

Fig. 5: Dissipative Standard Map with parameter drift [4]

Once again, both Figs. 4 and 5, illustrate the coexistence of unperturbed dynamics and Hamiltonian chaos.

3.4.2) Finally, consider the classical conservative pendulum with Hamiltonian

$$H(p,q) = \frac{p^2}{2} + (\cos q - 1) \tag{28}$$

Its phase space plot exhibits hyperbolic fixed points and cycles winding around the elliptic center (fig. 6). In the presence of dissipation (η) and periodic forcing (λ), the pendulum equations take the form [4],

$$\frac{dp}{dt} = -\eta p + \sin q + \lambda \sin t \tag{29a}$$

$$\frac{dq}{dt} = p \tag{29b}$$

Fig. 7 shows the genesis of a point attractor replacing the elliptic fixed point of Fig. 6.

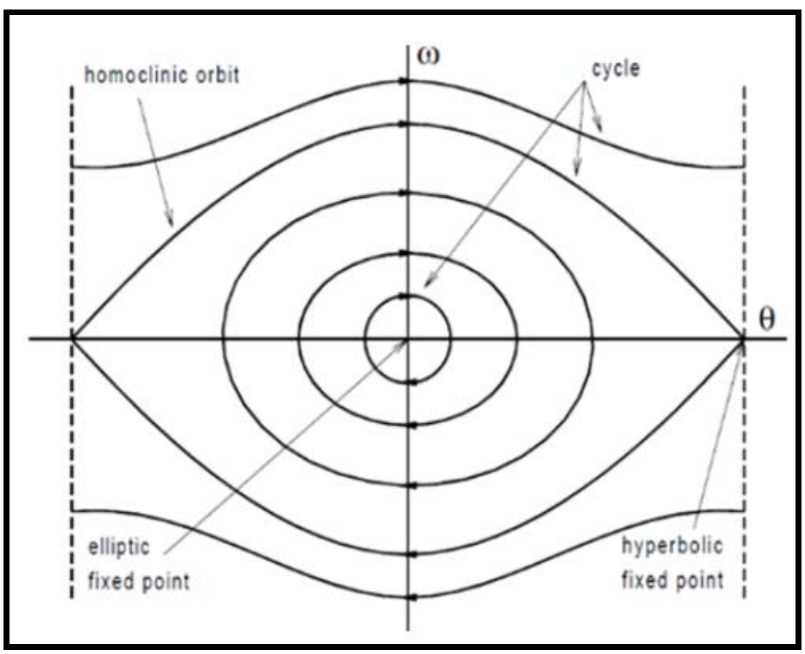

Fig. 6: Conservative pendulum [4, 7 - 9]

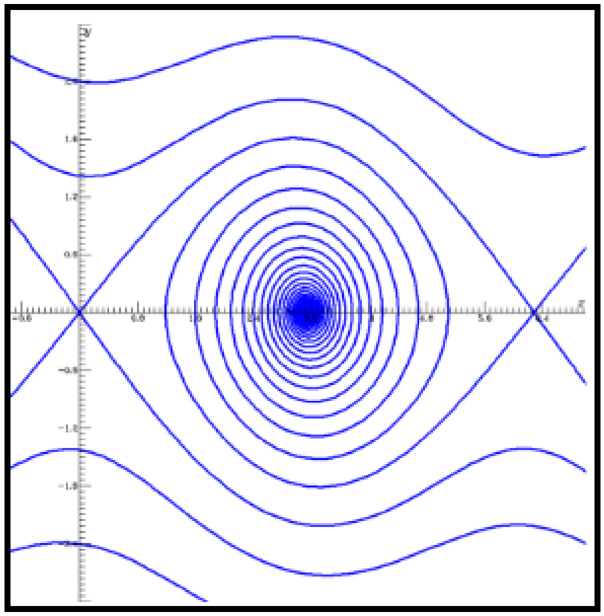

Fig. 7: Dissipative pendulum and the formation of a point attractor [4].

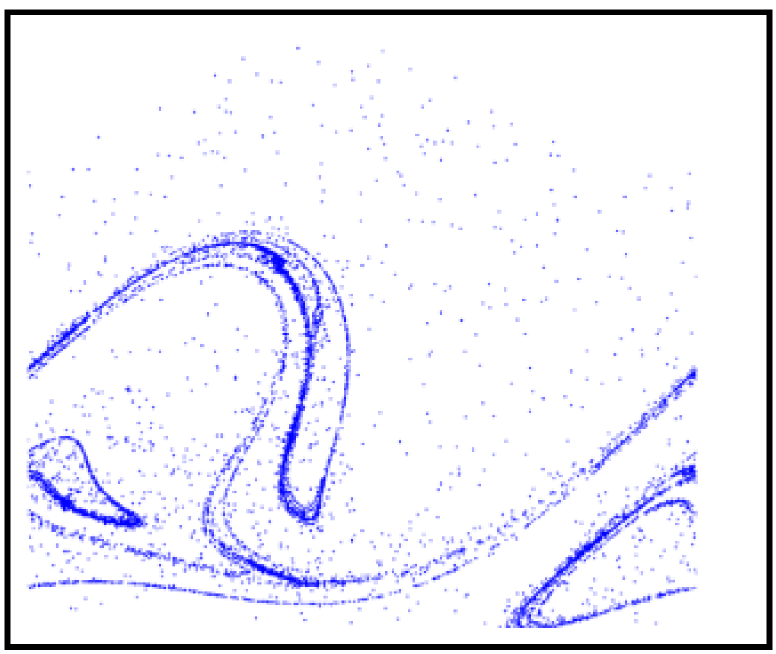

Fig. 8: Chaotic attractors under both dissipation and forcing [4]

A key observation is that, when dissipation and forcing are coexisting, the route to Hamiltonian chaos is NOT universal. Specifically, dissipation creates attractors that inhibit diffusion, while periodic forcing can overcome the effects of dissipation. The net effect is a competition between the formation of attractors and Hamiltonian chaos [4]. The dissipation losses are however expected to be negligible *over ultrashort time scales*, commensurate with the UV sector of high-energy physics and early Universe cosmology. It is in this regime that the onset of Hamiltonian chaos and fractal spacetime becomes a likely occurrence.

4. Summary and concluding remarks

Under certain conditions, the breakdown of the action invariance in non-adiabatic conditions overlaps with the onset of nonintegrability and chaos in Hamiltonian dynamics. The nearly universal transition to nonadiabatic behavior has been illustrated using several examples from nonlinear dynamics literature. As extensively discussed over the years, our findings may have far-reaching implications for the *complex behavior* of UV field theory and early Universe cosmology [17].

References

1. Landau L. D. and Lifschitz E. M., *Mechanics*, 3rd Edition, Butterworth Heinemann, 1976.

2. https://hal.science/hal-02016094/document

3. https://arxiv.org/pdf/0710.1576

4. https://arxiv.org/pdf/2206.14878

5. https://www.sciencedirect.com/topics/chemistry/langevin-equation

6. Corben, H. C. and Stehle, P. "Classical Mechanics", 2nd Edition, Dover Publications Inc., 1977.

7. Zaslavsky G. M., Hamiltonian Chaos and Fractional Dynamics, Oxford Univ. Press, 2005.

8. https://physics.stackexchange.com/questions/661027/limit-cycles-of-a-simple-pendulum

9. https://www.physicswithelliot.com/simple-pendulum-with-phase-space

10. https://arxiv.org/pdf/physics/0610084

11. http://theorphys.elte.hu/tel/pdf_pub/Chaos29.pdf

12. https://www.qeios.com/read/7ZYVDB

13. https://www.qeios.com/read/0KXQIL

14. https://arxiv.org/pdf/1012.4384

15. https://www.researchgate.net/publication/361644297

16. https://www.researchgate.net/publication/349476440

17. https://www.researchgate.net/profile/Ervin-Goldfain