Graphs and their symmetries

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

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Abstract. This is an introduction to graph theory, from a geometric viewpoint. A finite graph X is described by its adjacency matrix $d \in M_N(0,1)$, which can be thought of as a kind of discrete Laplacian, and we first discuss the basics of graph theory, by using d and linear algebra tools. Then we discuss the computation of the classical and quantum symmetry groups $G(X) \subset G^+(X)$, which must leave invariant the eigenspaces of d. Finally, we discuss similar questions for the quantum graphs, with these being again described by certain matrices $d \in M_N(\mathbb{C})$, but in a more twisted way.

Preface

Graph theory is a wide topic, and there are many possible ways of getting into it. Basically any question which is of "discrete" nature, with a finite number of objects N interacting between them, has an associated graph X, and the question can be reformulated, and hopefully solved, in terms of that graph X. A possible exception seems to come from discrete questions which are naturally encoded by a matrix, but isn't a $N\times N$ matrix A some kind of graph too, having $1,\ldots,N$ as vertices, and with any pair of vertices i,j producing the oriented edge $i\to j$, colored by corresponding the matrix entry A_{ij}.

Graphs appear as well in connection with continuous questions, via "discretization" methods. Discretizing is something commonplace in physics, and sometimes in pure mathematics too, with the continuous question, system or manifold M appearing as the $N\to\infty$ limit of its discrete versions, which typically correspond to graphs X_N, which are often easier to solve, and with this philosophy having produced good results all around the spectrum, from common sense science and engineering questions up to fairly advanced topics, such as quantum gravity, dark matter and teleportation.

The purpose of this book is to talk about graphs, viewed from this perspective, as being "discrete geometry" objects. To be more precise, we will be rather mathematical, and the cornerstone of our philosophy will be the commonly accepted fact that the basic objects of mainstream, continuous mathematics are the Riemannian manifolds M, together with their Laplacians Δ. In the discrete setting the analogues of these objects are the finite graphs X, together with their adjacency matrices d, and this will be our way to view the graphs, in this book, as being some kind of "discrete Riemannian manifolds".

This was for the philosophy. In practice, this will not prevent us of course for doing many elementary things, and our purpose will be to talk about graphs in a large sense, ranging from very basic to more advanced. All in all, the book covers what can be taught during a one-year course, in between the undergraduate and graduate level.

More in detail now, we will start in Part I with standard introductory material on graph theory, namely definition, main examples, and some computations and basic theorems too. Our presentation here will be for the most of spectral flavor, by translating everything in terms of the adjacency matrix d, and proving the results via linear algebra.

Then, we will discuss more theory and examples in the transitive case, and finally we will have a look at planar graphs, and more generally at genus $g \geq 0$ graphs.

Switching gears, in Part II we will decide that graph theory is most likely related to abstract algebra, and that what we are mostly interested in is the systematic study of the symmetry groups $G(X)$ of the finite graphs X. This will require of course some preliminaries on group theory, that we will include, by staying as intuitive and elementary as possible. Then, we will go on with the general theory of $G(X)$. The material here will be mostly upper undergraduate level, sometimes erring on the graduate side.

Escalating difficulties, in Part III we will include as well in our discussion the quantum symmetry groups $G^+(X)$ of the same finite graphs X, which are in general bigger than their classical counterparts $G(X)$, and with our motivation here coming from theoretical physics, and more specifically from statistical mechanics and quantum mechanics. The material here will be rather of graduate level, and quite often having a look into the recent research on the subject. There will be here a lot of probability theory as well.

Finally, in Part IV we will go back to a more normal pace, although still of graduate level for the most, talking here about various generalizations of the techniques that we learned, but with the graph X being now replaced by more general objects. Our first topic here will be the quantum graphs, that we will study via both $G(X)$ and $G^+(X)$. Then we will discuss various quantum algebraic generalizations of the finite graphs, and have a look as well at the random matrix model problematics for $G^+(X)$.

Part of this book is based on research work that I did some time ago on graphs and their symmetries, and I would like to thank my coworkers. Many thanks go as well to the young researchers in the area, who have recently pushed things to unexpected levels of deepness, that we will try to explain a bit here. Finally, many thanks go to my cats. When it comes to quick search over a graph, they are the kings of this universe.

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Up up to the sky

CHAPTER 1

Finite graphs

1a. Finite graphs

We will be interested in this book in graph theory, which is the same as discrete geometry. Or perhaps vice versa. Personally I prefer the "vice versa" viewpoint, geometry is something basic, coming first, and graphs, which are more specialized, come after. And this, I hope, will agree with you too. I can only imagine that you have some knowledge of geometry, for instance that round thing that you played with as a kid is a sphere, and that's quality geometry, called Riemannian, you're already a bit expert in that. While in what regards graphs, you've probably heard of them, but have no idea what they are good for, and you are here, starting this book, for getting introduced to them.

So, discrete geometry. And the first question here is of course, why? Is there something wrong with the usual, continuous geometry?

Good question, and as a first answer, I would argue that a good quality digital music record is as good as an old vinyl one. Of course some people might claim the opposite, but these people can be proved, scientifically, to be wrong, for the simple reason that the human ear has a certain resolution, and once your digital technology passes that resolution, records are 100% perfect to the human ear. And there is even more, because the same goes for vision, smell, taste and touch. In short, all our senses function like a computer, in a digital way, and we might be well living in a continuous world, but we will never be able to really sense this, and so fully benefit from that continuity.

Some further thinking on this leads to the scary perspective that our world might be actually discrete, at a resolution much finer than that of our human senses. You might argue that why not using some sharp scientific machinery instead, but that machinery has a certain resolution too. And so our scare is justified, and time to ask:

Question 1.1. *Is the world continuous, or quantized?*

Here we have used the word "quantized", which is a fancy scientific way of saying "discrete", as a matter of getting into serious physics. And here, hang on, opinions are split, but most physicists tend to favor the possibility that our world is indeed quantized. And at a resolution that is so fine, that is guaranteed to stay beyond the level of what can be directly observed with past, present and future scientific machinery.

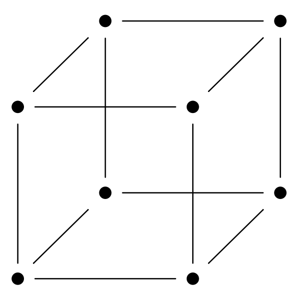
But probably enough talking, time to get to work, remember that we are here for doing math and computations. Let us summarize this discussion by formulating:

Conclusion 1.2. *Discrete geometry is worth a study, as a useful discretization of our usual continuous geometry. And why not as a replacement for it, in case it's wrong.*

In order now to get started, we first need to talk about continuous geometry. Generally speaking, that is about curves, surfaces and other shapes, called "manifolds" in $\mathbb{R}^N$. However, instead of getting into what a manifold exactly is, which can be a bit technical, let us just take this intuitively, M is by definition a "continuous shape" in $\mathbb{R}^N$.

Now for discretizing such a manifold $M \subset \mathbb{R}^N$, the idea is very simple, namely placing a sort of net on M. To be more precise, assume that we found a finite set of points $X \subset M$, with some edges between them, denoted $i - j$, and corresponding to paths on M, all having the same lenght $\varepsilon > 0$. Then, we have our discretization $X \subset M$.

As an example here, let us try to discretize a sphere $S \subset \mathbb{R}^3$. There are many ways of doing so, and a quite straightforward one is by using an inscribed cube, as follows:

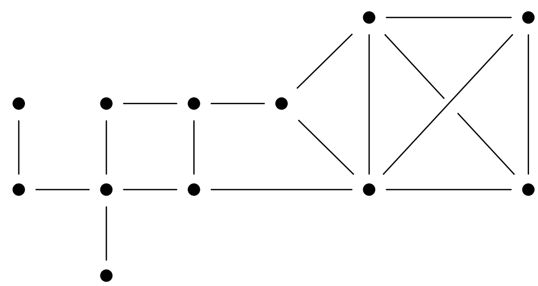


With this done, let us try now to forget about M itself, and think at X, taken alone. This beast X is a finite set of points, with edges between them. In addition, all edges have the same lenght $\varepsilon > 0$, but now that M is gone, we can assume if we want that we have $\varepsilon = 1$, or simply forget about ε. Thus, we are led into the following definition:

Definition 1.3. *A graph X is a finite set of points, with certain edges $i - j$ drawn between certain pairs of points (i, j).*

All this might seem overly simplified, and I can hear some of you saying hey, but deep mathematics needs very complicated definitions, that no one really understands, and so on. Wrong. The above definition is in fact the good one, and the graphs themselves are very interesting objects. That will keep us busy, for the rest of this book.

Before getting into mathematics, let us explore a bit Definition 1.3, and see how the graphs look like. We would need here some sort of "random graph", in order to see what types of phenomena can appear, and here is an example, which is quite illustrating:



We can see that there are several things going on here, and here is a list of observations that can be formulated, just by looking at this graph:

(1) First of all, in relation with our discretization philosophy, $X \subset M$, there is a bit of a mess with dimensions here, because the whole middle of the graph seems to discretize a surface, or 2D manifold, but the left part is rather 1D, and the right part, with that crossing edges, suggests either a 2D manifold in $\mathbb{R}^3$ or higher, or a 3D manifold.

(2) So, this is one problem that we will have to face, when working with Definition 1.3, at that level of generality there is no indication about the "dimension" of what our graph is supposed to describe. But this is not an issue, because when this will be really needed, with some further axioms we can divide the graphs into classes and so on.

(3) As another observation now, our graph above is not that "random", because it is connected. If this annoys you, please consider that the graph contains as well three extra points, that I can even draw for you, here they are, $\bullet\bullet\bullet$, not connected to the others. But will this bring something interesting to our formalism, certainly not.

(4) In short, we will usually not bother with disconnected graphs X, a bit like geometers won't bother with disconnected manifolds M. This being said, at some point of this book, towards the middle, when looking at graphs from a quantum perspective, we will reconsider this, because "quantum" allows all sorts of bizarre "jumps". More later.

(5) Finally, if you're an electric engineer you might be a bit deceived by all this, because what we're doing so far is obviously binary, and won't allow the installation of bulbs, resistors, capacitors and so on, having continuous parameters. Criticism accepted, and we will expand our formalism, once our theory using Definition 1.3 will be ripe.

As a last comment, we are of course allowed to draw graphs as we find the most appropriate, and no rules here, free speech as we like it. However, as an important point, if you are a bit familiar with advanced mathematics, and know the difference between geometry and topology, you might have the impression that what we are doing is topology. But this is wrong, what we will be doing is definitely geometry. More on this later.

Let us record the main conclusions from this discussion, as follows:

Conclusion 1.4. *Our definition for the graphs looks good, and appears as a good compromise between:*

(1) *More particular definitions, such as looking at connected graphs only, which is the main case of interest, geometrically speaking.*
(2) *More general definitions, such as allowing machinery and symbols to be installed, either at the vertices, or on the edges, or both.*

We will come back to this, whenever needed, and fine-tune our definition for the graphs, along the above lines, depending on what exact problems we are interested in.

Moving ahead, now that we have our objects of study, the graphs, time to do some mathematics. For this purpose, we will mostly use linear algebra, and a bit of calculus too. At some point later, we will need as well some basic knowledge of group theory, along with some basic functional analysis, and basic probability theory. All these can be learned from many places, and if looking for a compact package, talking a bit about everything that will be needed here, you can check my linear algebra book [5].

But let us start with some algebra and generalities. We surely know from Definition 1.3 what a graph is, and normally no need for more, we are good for starting some work. However, it is sometimes useful to have some alternative points of view on this. So, let us replace Definition 1.3 with the following definition, which can be quite useful:

Definition 1.5. *A graph X can be viewed as follows:*

(1) *As a finite set of points X, with certain edges $i - j$ drawn between certain pairs of distinct points (i, j).*
(2) *As a finite vertex set X, given with an edge set $E \subset X \times X$, which must be symmetric, and avoiding the diagonal.*
(3) *As a finite set X, given with a relation $i - j$ on it, which must be non-reflexive, meaning $i \not- i$, and symmetric.*

To be more precise, here the formulation in (1) is the one in our old Definition 1.3, with the remark that we forgot to say there that $i \neq j$, and with this coming from the fact that, geometrically speaking, self-edges $i - i$ look like a pathology, to be avoided.

Regarding now (2), this is something clearly equivalent to (1). It is sometimes convenient to use the notation $X = (V, E)$, with the vertex set denoted V, but in what concerns us, we will keep using our policy of calling X both the graph, and the vertex set.

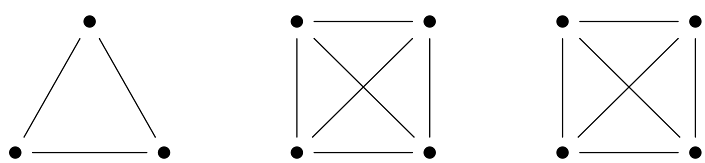
Finally, regarding (3), this is something equivalent to (1) and (2), and with "relation on X" there not meaning anything in particular, and more specifically, just meaning "subset of $X \times X$". Observe that nothing is said about the transitivity of $-$.

All this is nice, and as a first mathematical question for us, let us clarify what happens in relation with the transitivity of $-$. To start with, as our previous examples of graphs indicate, the relation $-$ is not transitive, in general. In fact, $-$ can never be transitive, unless for graphs of type $\bullet\bullet\bullet$, without edges at all, because once we have an edge $i-j$, by symmetry and then transitivity we are led to the following wrong conclusion:

$$i-j\ ,\ j-i\implies i-i$$

However, as a matter of recycling our question, we can ask if $-$, once completed with $i-i$ as to be symmetric, can be transitive. And the answer here is as follows:

PROPOSITION 1.6. *The graphs X having the property that $-$, once completed with $i-i$ as to be symmetric, is transitive, are exactly the graphs of type*



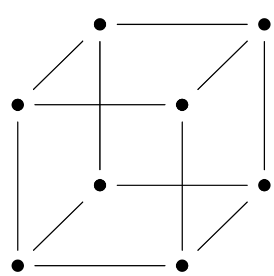
that is, are the disjoint unions of complete graphs.

PROOF. This is clear, because after thinking a bit, our question simply asks to draw the graph of an equivalence relation, and what you get is of course a picture as above, with various complete graphs corresponding to the various equivalence classes. $\square$

In practice now, all three formulations of the notion of graph from Definition 1.5 can be useful, for certain questions, but we will keep using by default the formulation (1) there. Indeed, here is what happens for the cube, and judge yourself:

PROPOSITION 1.7. *The cube graph is as follows:*

(1) *Viewed as a set, with edges drawn between points:*



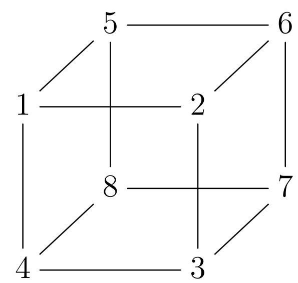
(2) *Viewed as a vertex set, plus edge set:*

$$X=\{1,2,3,4,5,6,7,8\}$$

$$E=\begin{Bmatrix}12,14,15,21,23,26,32,34,37,41,43,48\\51,56,58,62,65,67,73,76,78,84,85,87\end{Bmatrix}$$

(3) *Viewed as a set, with a relation on it: same as in* (2).

Proof. To start with, (1) is clear. For (2) however, we are already a bit in trouble, because in order to figure out what the edge set is, we must first draw the cube, as in (1), so failure with respect to (1), and then label the vertices with numbers, say $1, \ldots, 8$:



But with this done, we can then list the 12 edges, say in lexicographic order. Finally, (3) is obviously the same thing as (2), so failure too with respect to (1). □

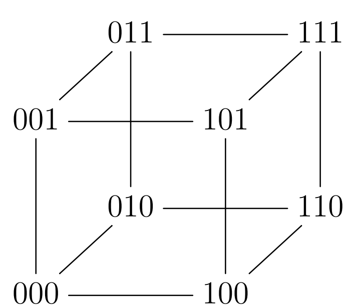
Summarizing, and hope you agree with me, Definition 1.5 (1), which is more or less our original Definition 1.3, is the best, for general graphs. This being said, the cube example in Proposition 1.7 might look quite harsh, to the point that you might now be wondering if Definition 1.5 (2) and Definition 1.5 (3) have any uses at all.

Good point, and in order to answer, let us first go back to the cube. With mea culpa to that cube, we can in fact do better when labelling the vertices, and we have:

Proposition 1.8. *The cube can be best viewed, via edge and vertex set, as follows:*

(1) *The vertices are the usual* 3D *coordinates of the vertices of the unit cube, that is,* 000, 001, 010, 011, 100, 101, 110, 111.
(2) *The edge set consists of the pairs* (abc, xyz) *of such coordinates having the property that* $abc \leftrightarrow xyz$ *comes by modifying exactly one coordinate.*

Proof. This is clear from definitions, and no need to draw a cube or anything for this, in fact the geometric cube is there, in what we said in (1) and (2). Here is however an accompanying picture, in case you have troubles in seeing this right away:



Thus, one way or another, we are led to the conclusion in the statement. □

As a comment here, we can further build along the above lines, with the ultimate conclusion being something which looks very good and conceptual, as follows:

ADVERTISEMENT 1.9. *The cube is the Cayley graph of $\mathbb{Z}_2^3$.*

Obviously, this looks like a quite deep statement, probably beating everything else that can be said, about the cube graph. We will talk about this later, when discussing in detail the Cayley graphs, but coming a bit in advance, some explanations on this:

(1) You surely know about the cyclic group $\mathbb{Z}_N$, having as elements the remainders $0, 1, \ldots, N-1$ modulo N. But at $N = 2$ this group is simply $\mathbb{Z}_2 = \{0, 1\}$, so in view of what we found in Proposition 1.8, and leaving now 3D geometry aside, for a trip into arithmetic, we can say that the vertices of the cube are the elements $g \in \mathbb{Z}_2^3$.

(2) Regarding now the edges $g - h$, we know from Proposition 1.8 that these appear precisely at the places where the passage $g \leftrightarrow h$ comes by modifying exactly one coordinate. But, and skipping some details here, with all this left to be discussed later in this book, this means precisely that the cube is the Cayley graph of $\mathbb{Z}_2^3$.

As an overall conclusion now, we have several equivalent definitions for the graphs, those in Definition 1.5, which can sometimes lead to interesting mathematics, and we have as well Conclusion 1.4, telling us how to fine-tune our graph formalism, when needed, depending on the precise questions that we are interested in. Very nice all this, foundations laid, and we are now good to go, for some tough mathematics on graphs.

1b. Adjacency matrix

As a first true result about graphs, and we will call this theorem, not because of the difficulty of its proof, but because of its beauty and usefulness, we have:

THEOREM 1.10. *A graph X, with vertices labeled $1, \ldots, N$, is uniquely determined by its adjacency matrix, which is the matrix $d \in M_N(0, 1)$ given by:*

$$d_{ij} = \begin{cases} 1 & \text{if } i - j \\ 0 & \text{if } i \not- j \end{cases}$$

Moreover, the matrices $d \in M_N(0, 1)$ which can appear in this way, from graphs, are precisely those which are symmetric, and have 0 on the diagonal.

PROOF. We have two things to be proved, the idea being as follows:

(1) Given a graph X, we can construct a matrix $d \in M_N(0, 1)$ as in the statement, and this matrix is obviously symmetric, and has 0 on the diagonal.

(2) Conversely, given a matrix $d \in M_N(0,1)$ which is symmetric, and has 0 on the diagonal, we can associate to it the graph X having as vertices the numbers $1,\ldots,N$, and with the edges between these vertices being defined as follows:

$$i - j \iff d_{ij} = 1$$

It is then clear that the adjacency matrix of this graph X is the matrix d itself. Thus, we have established a correspondence $X \leftrightarrow d$, as in the statement. $\square$

The above result is very useful for various purposes, a first obvious application being that we can now tell a computer what our favorite graph X is, just by typing in the corresponding adjacency matrix $d \in M_N(0,1)$, which is something that the computer will surely understand. In fact, computers like 0-1 data, that's the language that they internally speak, and when that data comes in matrix form, they even get very happy and excited, and start doing all sorts of computations. But more on this later.

Speaking conversion between graphs X and matrices d, this can work as well for the two of us. Here is my matrix d, and up to you to tell me what my graph X was:

$$d = \begin{pmatrix} 0&1&1\\ 1&0&1\\ 1&1&0 \end{pmatrix}$$

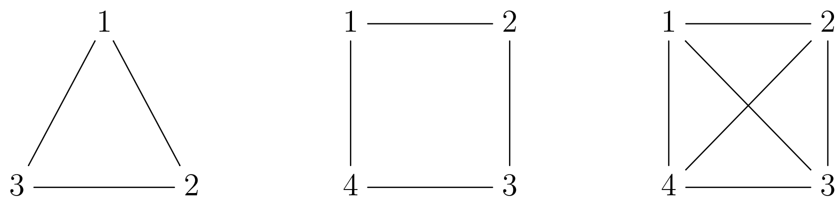
Problem solved I hope, with the graph in question being a triangle. Here is now another matrix d, and in the hope that you will see here a square:

$$d = \begin{pmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0 \end{pmatrix}$$

Here is now another matrix d, and in the hope that you will see the same square here, but this time with both its diagonals drawn, and you can call that tetrahedron too:

$$d = \begin{pmatrix} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{pmatrix}$$

By the way, talking exercises, please allow me to record the solutions to the above exercises, not of course for you, who are young and enthusiastic and must train hard, but rather for me and my colleagues, who are often old and tired. Here they are:



And to finish this discussion, at a more advanced level now, here is another matrix d, and in the hope that you will see something related to cycling here:

$$d=\begin{pmatrix}0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\end{pmatrix}$$

All this looks fun, and mathematically relevant too. Based on this, we will agree to sometimes speak directly in terms of d, which is quite practical. Especially for me, typing a matrix in Latex, the computer language used for typing math, and the present book, being easier than drawing a graph. Plus our computer friend will understand too.

Let us develop now some theory, for the general graphs. Once the number of vertices is fixed, $N \in \mathbb{N}$, what seems to most distinguish the graphs, for instance in connection with the easiness or pain of drawing them, is the total number of edges $|E|$. Equivalently, what distinguishes the graphs is the density of edges at each vertex, given by:

$$\rho=\frac{|E|}{N}$$

So, let us take a closer look at this quantity. It is convenient, for a finer study, to formulate our definition as follows:

Definition 1.11. *Given a graph X, with X from now on standing for both the vertex set, and the graph itself, the valence of each vertex i is the number of its neighbors:*

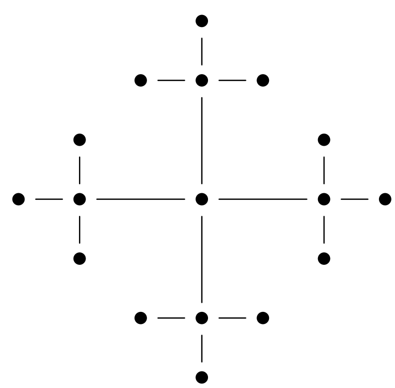
$$v_i=\#\left\{j\in X\Big|i-j\right\}$$

We call X regular when the valence function $v:X\to\mathbb{N}$ is constant. More specifically, in this case, we say that X is k-regular, with k being the common valence of vertices.

At the level of examples, all the graphs pictured above, with our usual convention that 0-1 matrix means picture, are regular, with the valence being as follows:

Graph	Regularity	Valence
triangle	yes	2
square	yes	2
tetrahedron	yes	3
hexagonal wheel	yes	3

Which leads us into the question, is there any interesting non-regular graph? Certainly yes, think for instance at all that beautiful pictures of snowflakes, such as:



Here the valence is 4 at the inner points, and 1 at the endpoints. Graphs as the above one are called in mathematics "trees", due to the fact that, thinking a bit, if you look at a usual tree from the above, say from a helicopter, what you see is a picture as above. But more on such graphs, which can be quite complicated, later.

For the moment, let us study the valence function, and the regular graphs. We can do some math, by using the adjacency matrix, as follows:

Proposition 1.12. *Given a graph X, with adjacency matrix d:*

1. *The valence function v is the row sum function for d.*
2. *Equivalently, $v = d\xi$, with ξ being the all-1 vector.*
3. *X is regular when d is stochastic, meaning with constant row sums.*
4. *Equivalently, X is k-regular, with $k \in \mathbb{N}$, when $d\xi = k\xi$.*

PROOF. All this looks quite trivial, but let us make some effort, and write down a complete mathematical proof. Regarding (1), this follows from:

$$\begin{aligned} v_i &= \sum_{j-i} 1 \\ &= \sum_{j-i} 1 + \sum_{j\neq i} 0 \\ &= \sum_j d_{ij} \end{aligned}$$

Regarding (2), this follows from (1), because for any matrix d we have:

$$d\xi = \begin{pmatrix} d_{11} & \ldots & d_{1N} \\ \vdots & & \vdots \\ d_{N1} & \ldots & d_{NN} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d_{11} + \ldots + d_{1N} \\ \vdots \\ d_{N1} + \ldots + d_{NN} \end{pmatrix}$$

Finally, (3) follows from (1), and (4) follows from (2). □

Observe that, in linear algebra terms, (4) above reformulates as follows:

THEOREM 1.13. *A graph is regular precisely when the all-1 vector is an eigenvector of the adjacency matrix. In this case, the valence is the corresponding eigenvalue.*

PROOF. This is clear indeed from what we know from Proposition 1.12 (4). As a philosophical comment here, you might wonder how something previously labeled Proposition can suddenly become a Theorem. Welcome to mathematics, which is not an exact science, the general rule being that everything that looks fancy enough can be called Theorem. In fact, we have already met this phenomenon in the context of Proposition 1.8, which got converted into Advertisement 1.9, and with that advertisement being, you guessed right, an advertisement for a future Theorem, saying exactly the same thing. □

The above result is quite interesting, and suggests systematically looking at the eigenvalues and eigenvectors of d. Which is indeed a very good idea, but we will keep this for later, starting with chapter 2, once we will have a bit more general theory.

1c. Walks on graphs

Have you ever played chess, or simply observed a cat in action. There are all sorts of paths and combinations that can be followed, and the problem is that of quickly examining all these possibilities, with the series of conclusions being typically something on type damn, damn, damn, damn, damn, yes. With the yes followed by some action.

Mathematically speaking, this brings us into walks on graphs. And here, as usual, we get into the adjacency matrix d, thanks to the following key result:

THEOREM 1.14. *Given a graph X, with adjacency matrix $d \in M_N(0,1)$, we have:*

$$(d^k)_{ij} = \#\left\{i = i_0 - i_1 - \ldots - i_{k-1} - i_k = j\right\}$$

That is, the k-th power of d describes the length k paths on X.

PROOF. According to the usual rule of matrix multiplication, the formula for the powers of the adjacency matrix $d \in M_N(0,1)$ is as follows:

$$\begin{aligned}
(d^k)_{i_0 i_k} &= \sum_{i_1,\ldots,i_{k-1}} d_{i_0 i_1} d_{i_1 i_2} \ldots d_{i_{k-1} i_k} \\
&= \sum_{i_1,\ldots,i_{k-1}} \delta_{i_0 - i_1} \delta_{i_1 - i_2} \ldots \delta_{i_{k-1} - i_k} \\
&= \sum_{i_1,\ldots,i_{k-1}} \delta_{i_0 - i_1 - \ldots - i_{k-1} - i_k} \\
&= \#\left\{i = i_0 - i_1 - \ldots - i_{k-1} - i_k = j\right\}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Of particular interest are the paths which begin and end at the same point. These are called loops, and in the case of loops, Theorem 1.14 particularizes as follows:

THEOREM 1.15. *Given a graph X, with adjacency matrix $d \in M_N(0,1)$, we have:*

$$(d^k)_{ii} = \#\left\{k - \text{loops based at } i \in I\right\}$$

Also, the total number of k-loops on X, at various vertices, is the number

$$Tr(d^k) = \sum_i (d^k)_{ii}$$

which can be computed by diagonalizing d.

PROOF. There are several things going on here, the idea being as follows:

(1) The first assertion follows from Theorem 1.14, which at $i = j$ gives the following formula, which translates into the first formula in the statement:

$$(d^k)_{ii} = \#\left\{i = i_0 - i_1 - \ldots - i_{k-1} - i_k = i\right\}$$

(2) Regarding now the second assertion, this follows from the first one, simply by summing over all the vertices $i \in X$, which gives, as desired:

$$Tr(d^k) = \sum_i \#\left\{k - \text{loops based at } i \in I\right\}$$

(3) Finally, the third assertion is something well-known from linear algebra, the idea being that once we diagonalize a matrix, $A = PDP^{-1}$, we have:

$$\begin{aligned} A = PDP^{-1} &\implies A^k = PD^kP^{-1} \\ &\implies Tr(A^k) = Tr(D^k) \end{aligned}$$

Thus, back now to our graph case, if we denote by $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ the eigenvalues of d, then the formula of the trace of d^k, that we are interested in, is as follows:

$$Tr(d^k) = \lambda_1^k + \ldots + \lambda_N^k$$

Here we have used the fact that d, which is a real symmetric matrix, is indeed diagonalizable, and with real eigenvalues. You might perhaps know this from linear algebra, and if not do not worry, we will discuss all this in detail in chapter 2. □

All the above is quite interesting, and adds to our investigations from the previous section, suggesting as well to get into the diagonalization question for the adjacency matrix $d \in M_N(0,1)$. We will do this as soon as possible, and more precisely starting from chapter 2 below. But before that, we still have to discuss some examples for all this, plus a number of other topics of general interest, not needing diagonalization.

Let us start with the examples. Unfortunately, here everything is quite complicated, because even for very simple graphs X, go count the loops directly, via recurrences and so on, that is a lot of combinatorics, which is invariably non-trivial. Or go compute the powers d^k without diagonalizing d, that is a lot of heavy work too.

So, as a first philosophical question, what are the simplest graphs X, that we can try to do some loop computations for? And here, we have 3 possible answers, as follows:

Fact 1.16. *The following are graphs X, with a distinguished vertex $0 \in X$:*

1. *The circle graph, having N vertices, with 0 being one of the vertices.*
2. *The segment graph, having N vertices, with 0 being the vertex at left.*
3. *The segment graph, having $2N+1$ vertices, with 0 being in the middle.*

So, let us start with these. For the circle, the computations are quite non-trivial, and you can try doing some, in order to understand what I am talking about. The problem comes from the fact that loops of length $k = 0, 2, 4, 6, \ldots$ are quite easy to count, but then, once we pass $k = N$, the loops can turn around the circle or not, and they can even turn several times, and so on, and all this makes the count too complicated. In addition, again due to loop turning, when N is odd, we have as well loops of odd length.

As for the two segment graphs, here the computations look again complicated, and even more complicated than for the circle, because, again, once we pass $k = N$ many things can happen, and this makes the count too complicated. And here, again you can try doing some computations, in order to understand what I am talking about.

So, shall we give up, in waiting for more advanced techniques, say coming from diagonalization? That would be a wise decision, but before that, let us pull an analysis trick, and formulate the following result, which is of course something informal, and modest:

THEOREM 1.17. *For the circle graph, having N vertices, the number of length k loops based at one of the vertices is approximately*

$$L_k \simeq \frac{2^k}{N}$$

in the $k \to \infty$ limit, when N is odd, and is approximately

$$L_k \simeq \begin{cases} \frac{2^{k+1}}{N} & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

also with $k \to \infty$, when N is even. However, in what regards the two segment graphs, we can expect here things to be more complicated.

PROOF. This is something not exactly trivial, and with the way the statement is written, which is clearly informal, witnessing for that. The idea is as follows:

(1) Consider the circle graph X, with vertices denoted $0, 1, \ldots, N-1$. Since each vertex has valence 2, any length k path based at 0 will consist of a binary choice at the beginning, then another binary choice afterwards, and so on up to a k-th binary choice at the end. Thus, there is a total of 2^k such paths, based at 0, and having length k.

(2) But now, based on the obvious "uniformity" of the circle, we can argue that, in the $k \to \infty$ limit, the endpoint of such a path will become random among the vertices $0, 1, \ldots, N-1$. Thus, if we want this endpoint to be 0, as to have a loop, we have $1/N$ chances for this to happen, so the total number of loops is $L_k \simeq 2^k/N$, as stated.

(3) With the remark, however, that the above argument works fine only when N is odd. Indeed, when N is even, the endpoint of a length k path will be random among $0, 2, \ldots, 2N-2$ when k is even, and random among $1, 3, \ldots, 2N-1$ when k is odd. Thus for getting a loop we must assume that k is even, and in this case the number of such loops is the total number of length k paths, namely 2^k, approximately divided by $N/2$, the number of points in $\{0, 2, \ldots, 2N-2\}$, which gives $L_k = 2^k/(N/2)$, as stated.

(4) All this was of course a bit borderline, I know, with respect to what rigorous mathematics is supposed to be, but honestly, I think that the argument is there, and good, in short I trust this proof. Needless to say, we will be back to all this later, with some better tools for attacking such problems, and with full rigor, at that time.

(5) Moving ahead now to the segment graphs, it is pretty much clear that for both, we lack the "uniformity" needed in (2), and this due to the 2 endpoints of the segment. In fact, thinking well, these graphs are no longer 2-valent, again due to the 2 endpoints, each having valence 1, and so even (1) must be fixed. And so, we will stop here. □

All this is obviously not very good news, and so again, as question, shall we give up, in waiting for more advanced techniques, say coming from diagonalization?

Well, instead of giving up, let us look face-to-face at the difficulties that we met. We are led this way, after analyzing the situation, to the following thought:

Thought 1.18. *The difficulties that we met, with the circle and the two segments, come from the fact that our loops are not "free to move",*

(1) *for the circle, because these can circle around the circle,*
(2) *for the segments, obviously because of the endpoints,*

and so our difficulties will dissapear, and we will be able to do our exact loop count, once we find a graph X where the loops are truly "free to move".

Thinking some more, all this definitely buries the first interval graph, where the vertex 0 is one of the endpoints. However, we can still try to recycle the circle, by unwrapping it, or extend our second interval graph up to ∞. But in both cases what we get is the graph $\mathbb{Z}$ formed by the integers. So, let us formulate the following definition:

Definition 1.19. *An infinite graph is the same thing as a finite graph, but now with an infinity of vertices, $|X|=\infty$. As basic example, we have $\mathbb{Z}$. We also have $\mathbb{N}$.*

Leaving aside now $\mathbb{N}$, which looks more complicated, let us try to count the length k paths on $\mathbb{Z}$, based at 0. At $k=1$ we have 2 such paths, ending at -1 and 1, and the count results can be pictured as follows, with everything being self-explanatory:

$$\begin{array}{ccccccccccccc}\circ&-&\circ&-&\circ&-&\bullet&-&\circ&-&\circ&-&\circ\\ &&&&1&&&&1&&&&\end{array}$$

At $k=2$ now, we have 4 paths, one of which ends at -2, two of which end at 0, and one of which ends at 2. The results can be pictured as follows:

$$\begin{array}{ccccccccccccc}\circ&-&\circ&-&\circ&-&\bullet&-&\circ&-&\circ&-&\circ\\ &&1&&&&2&&&&1&&\end{array}$$

At $k=3$ now, we have 8 paths, the distribution of the endpoints being as follows:

$$\begin{array}{ccccccccccccccccc}\circ&-&\circ&-&\circ&-&\circ&-&\bullet&-&\circ&-&\circ&-&\circ&-&\circ\\ &&1&&&&3&&&&3&&&&1&&\end{array}$$

As for $k=4$, here we have 16 paths, the distribution of the endpoints being as follows:

$$\begin{array}{ccccccccccccccccccccc}\circ&-&\circ&-&\circ&-&\circ&-&\circ&-&\bullet&-&\circ&-&\circ&-&\circ&-&\circ&-&\circ\\ &&1&&&&4&&&&6&&&&4&&&&1&&\end{array}$$

And good news, we can see in the above the Pascal triangle. Thus, eventually, we found the simplest graph ever, namely $\mathbb{Z}$, and we have the following result about it:

Theorem 1.20. *The paths on $\mathbb{Z}$ are counted by the binomial coefficients. In particular, the $2k$-paths based at 0 are counted by the central binomial coefficients*

$$\binom{2k}{k} \simeq \frac{4^k}{\sqrt{\pi k}}$$

with the estimate, in the $k \to \infty$ limit, coming from the Stirling formula.

Proof. This basically follows from the above discussion, as follows:

(1) In what regards the count, we certainly have the Pascal triangle, as discovered above, and the rest is just a matter of finishing. There are many possible ways here, a straightforward one being that of arguing that the number C_k^l of length k loops $0 \to l$ is subject, due to the binary choice at the end, to the following recurrence relation:

$$C_k^l = C_{k-1}^{l-1} + C_{k-1}^{l+1}$$

But this is exactly the recurrence for the Pascal triangle, so done with the count.

(2) In what regards the estimate, this follows indeed from Stirling, as follows:

$$\begin{aligned}
\binom{2k}{k} &= \frac{(2k)!}{k!k!} \\
&\simeq \left(\frac{2k}{e}\right)^{2k} \sqrt{4\pi k} \times \left(\frac{e}{k}\right)^{2k} \frac{1}{2\pi k} \\
&= \frac{4^k}{\sqrt{\pi k}}
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Not bad all this. We will be back to other graphs, such as $\mathbb{N}$, which is still left, or the circle, or the two segments, and many more, in chapter 2 below. But before that, we still have to discuss a number of other topics of general interest, including coloring.

1d. Coloring questions

There are actually two ways of coloring a graph, either by coloring the vertices, or by coloring the edges. Usually mathematicians are quite excited about coloring the vertices, and more on this in a moment. However, electric engineers for instance are more excited about coloring the edges, with machinery like bulbs, resistors and capacitors, or simply with arrows, describing the direction of the current through each edge. Although for machinery having several legs, such as transistors, we are back to vertex coloring.

In this book we will use the edge coloring convention, and this actually for rather mathematical reasons. To be more precise, recall from the very beginning of this book that we are philosophically interested in discretizing manifolds, $X \subset M$, and in fact this is how we run into graphs, by placing a net over such a manifold M.

But now imagine that our net is allowed to stretch. In this case no more graph, what we have is simply a subset $X \subset M$, which technically is a finite metric space. Thus, beyond usual graphs, what we would mostly like to cover with our formalism are the finite metric spaces X. But this can be done by calling "colored graph" a graph with the vertices still uncolored, but with the edges colored. Indeed, any finite metric space X, consisting of N points, can be viewed as the N-simplex, with each edge $i-j$ colored by its length $d_{ij} > 0$. So, this will be our colored graph formalism in this book:

Definition 1.21. *A colored graph is a usual graph X, which each edge $i-j$ colored by a symbol $d_{ij} \in C$, with C being a set, called color set. We call the matrix*

$$d \in M_N(C)$$

the adjacency matrix of such a colored graph X.

In practice, as already mentioned in the above, we are mostly interested in the case $C = \{-1, 0, 1\}$, as to cover the oriented graphs, and also in the case $C = \mathbb{R}_+$, as to cover the finite metric spaces. However, the best is to use Definition 1.21 as stated, with an abstract color set C, that will quite often be a set of reals, $C \subset \mathbb{R}$.

As a first statement, regarding our enlarged graph formalism, we have:

Theorem 1.22. *The following are covered by our colored graph formalism:*

1. *The usual graphs. In fact, the usual graphs are precisely the colored versions of the simplex, with color set $C = \{0, 1\}$.*
2. *The oriented graphs. In fact, the oriented graphs are precisely the colored versions of the simplex, with color set $C = \{-1, 0, 1\}$.*
3. *The finite metric spaces. These are the colored versions of the simplex, with color set $C = \mathbb{R}_+$, and with the coloring subject to the triangle inequality.*

Proof. All this is trivial, and self-explanatory, and the reason why we called this Theorem instead of Proposition only comes from its theoretical importance. $\square$

As an interesting remark now, with colored graphs we are in fact not that far from the usual graphs, and this due to the following simple fact:

Theorem 1.23. *Any formal matrix $d \in M_N(C)$ has a color decomposition,*

$$d = \sum_{c \in C} c d_c$$

with the color components $d_c \in M_N(0, 1)$ being constructed as follows:

$$(d_c)_{ij} = \begin{cases} 1 & \text{if } d_{ij} = c \\ 0 & \text{if } d_{ij} \neq c \end{cases}$$

If X is the colored graph having adjacency matrix d, these matrices d_c are the adjacency matrices of the color components of X, which are usual graphs X_c.

PROOF. As before with Theorem 1.22, this is something trivial, and self-explanatory, and called Theorem instead of Proposition just because its theoretical importance. □

Many things can be said about colored graphs, and especially about the oriented graphs and finite metric spaces from Theorem 1.22. We will be back to them, later.

All this being said, coloring the vertices of a graph X, and we will call such a beast a "vertex-colored graph", is something quite interesting too. Let us formulate:

DEFINITION 1.24. *A vertex-colored graph is a usual graph X, with each vertex $i \in X$ colored by a symbol $c_i \in C$, with C being a set, called color set.*

As a first question regarding such vertex-colored graphs, pick whatever black and white geographical map, with countries and boundaries between them, then pick 4 colored pencils, and try coloring that map. Can you do that? What is your algorithm?

Well, the point here is that we have a deep theorem, as follows:

THEOREM 1.25. *Any map can be colored with 4 colors.*

PROOF. This is something non-trivial, which took mankind a lot of time to prove, and whose final proof is something extremely long, and with computers involved, at several key places. So, theorem coming without proof, and sorry for this. But, we will be back to this in chapter 4 below, when discussing planar graphs, with at least some explanations on what the word "map" in the statement exactly means, mathematically speaking. □

As a conclusion now, it looks like with our graph formalism, and our adjacency matrix technology, sometimes modified a bit, as above, we can deal with everything in discrete mathematics. However, before moving ahead with more mathematics, a warning:

WARNING 1.26. *Not everything in discrete mathematics having a picture, and an adjacency matrix, is a graph.*

This is something quite important, worth discussing in detail, with a good example. So, have you heard about projective geometry? In case you didn't yet, the general principle is that "this is the wonderland where parallel lines cross". Which might sound a bit crazy, and not very realistic, but take a picture of some railroad tracks, and look at that picture. Do that parallel railroad tracks cross, on the picture? Sure they do. So, we are certainly not into abstractions here, but rather into serious science.

Mathematically now, here are some axioms, to start with:

DEFINITION 1.27. *A projective space is a space consisting of points and lines, subject to the following conditions:*

(1) *Each 2 points determine a line.*
(2) *Each 2 lines cross, on a point.*

As a basic example we have the usual projective space $P^2_{\mathbb{R}}$, which is best seen as being the space of lines in $\mathbb{R}^3$ passing through the origin. To be more precise, let us call each of these lines in $\mathbb{R}^3$ passing through the origin a "point" of $P^2_{\mathbb{R}}$, and let us also call each plane in $\mathbb{R}^3$ passing through the origin a "line" of $P^2_{\mathbb{R}}$. Now observe the following:

(1) Each 2 points determine a line. Indeed, 2 points in our sense means 2 lines in $\mathbb{R}^3$ passing through the origin, and these 2 lines obviously determine a plane in $\mathbb{R}^3$ passing through the origin, namely the plane they belong to, which is a line in our sense.

(2) Each 2 lines cross, on a point. Indeed, 2 lines in our sense means 2 planes in $\mathbb{R}^3$ passing through the origin, and these 2 planes obviously determine a line in $\mathbb{R}^3$ passing through the origin, namely their intersection, which is a point in our sense.

Thus, what we have is a projective space in the sense of Definition 1.27. More generally now, we can perform in fact this construction over an arbitrary field, as follows:

THEOREM 1.28. *Given a field F, we can talk about the projective space P^2_F, as being the space of lines in F^3 passing through the origin, having cardinality*

$$|P^2_F| = q^2 + q + 1$$

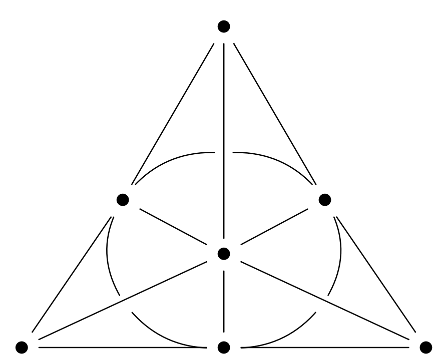
where $q = |F|$, in the case where our field F is finite.

PROOF. This is indeed clear from definitions, with the cardinality coming from:

$$|P^2_F| = \frac{|F^3 - \{0\}|}{|F - \{0\}|} = \frac{q^3 - 1}{q - 1} = q^2 + q + 1$$

Thus, we are led to the conclusions in the statement. $\square$

As an example, let us see what happens for the simplest finite field that we know, namely $F = \mathbb{F}_2$. Here our projective space, having $4 + 2 + 1 = 7$ points, and 7 lines, is a famous combinatorial object, called Fano plane, which looks as follows:



Here the circle in the middle is by definition a line, and with this convention, the projective geometry axioms from Definition 1.27 are satisfied, in the sense that any two points determine a line, and any two lines determine a point. And isn't this magic.

The question is now, in connection with Warning 1.26, as follows:

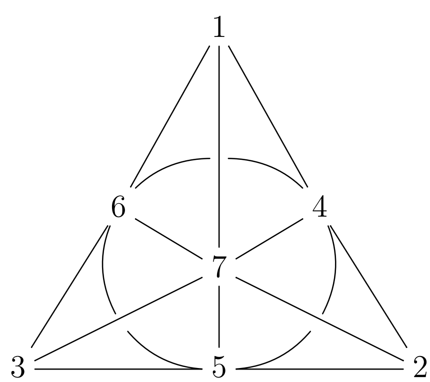
QUESTION 1.29. *What type of beast is the above Fano plane, with respect to graph theory, and its generalizations?*

To be more precise, the Fano plane looks like a graph, but is not a graph, because, save for the circle in the middle and its precise conventions, it matters whether edges are aligned or not, and with this being the whole point with the Fano plane.

However, not giving up, let us try to investigate this plane, inspired from what we know about graphs. A first interesting question is that of suitably labeling its vertices, and here we know from Theorem 1.28 and its proof that these are naturally indexed by the elements of the group $\mathbb{Z}_2^3$, with the neutral element $0 = 000$ excluded:

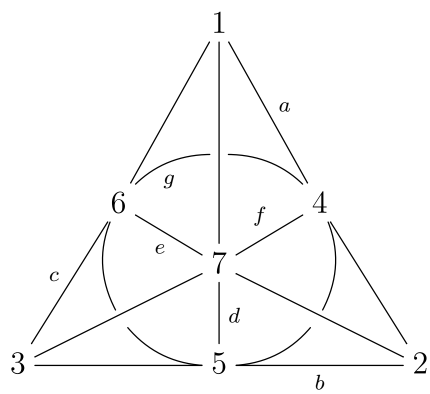
$$\mathbb{Z}_2^3 - \{000\} = \{001, 010, 011, 100, 101, 110, 111\}$$

However, there is some sort of too much symmetry between these symbols, which will not help us much, so instead let us just label the vertices $1, 2, \ldots, 7$, as follows:



Next comes the question of suitably labeling the lines, and with here, we insist, these lines being not "edges", because the Fano plane is not a graph, as explained above. This is again an interesting question, of group theory flavor, and among the conclusions that we can come upon, by thinking at all this, we have the quite interesting fact that, when interchanging the 7 points and the 7 lines, the Fano plane stays the same.

Which looks like something quite deep, but which also teaches us that the labeling question for the lines is the same as the labeling question for the points, and so, in view of the above, that we have to give up. So, let us simply label the lines by letters $a, b, \ldots, g$, in a somewhat random fashion, a bit as we did for the vertices, as follows:



Now with our Fano plane fully labeled, we can answer Question 1.29, as follows:

ANSWER 1.30. *The Fano plane can be described by the* 7×7 *incidence matrix recording the matches between points and lines, which is*

$$m=\begin{pmatrix}1&0&1&1&0&0&0\\1&1&0&0&1&0&0\\0&1&1&0&0&1&0\\1&0&0&0&0&1&1\\0&1&0&1&0&0&1\\0&0&1&0&1&0&1\\0&0&0&1&1&1&0\end{pmatrix}$$

with points on the vertical, and lines on the horizontal, labeled as above. However, unlike for graphs, this matrix is no longer symmetric, or having 0 *on the diagonal.*

So, question answered, rather defavorably for graph theory, and with this being said, shall we give up? Well, never underestimate the graphs. Indeed, these can strike back, and we have the following alternative answer to Question 1.29:

ANSWER 1.31. *The Fano plane can be described by the bipartite graph having as vertices the points and lines, and edges at matches, whose* 14×14 *adjacency matrix is*

$$d=\begin{pmatrix}0&m\\m^t&0\end{pmatrix}$$

with m *being the usual* 7×7 *incidence matrix between points and lines, constructed before, and with* m^t *being the transpose of* m*.*

Not bad all this. By the way, many other interesting things can be said, about the bipartite graphs, that is, about the graphs whose vertices can be divided into 2 classes, with no edges within the same class. For instance, it is clear that these are precisely the graphs having the property that, with a suitable labeling of the vertices, the adjacency matrix looks as follows, with m being a certain rectangular 0-1 matrix:

$$d=\begin{pmatrix}0&m\\ m^t&0\end{pmatrix}$$

Along the same lines, we can talk as well about tripartite graphs, with the adjacency matrices here being as follows, with m,n,p being certain rectangular 0-1 matrices:

$$d=\begin{pmatrix}0&m&n\\ m^t&0&p\\ n^t&p^t&0\end{pmatrix}$$

More generally, we can talk about N-partite graphs for any $N\in\mathbb{N}$, among others with the trivial remark that with $|X|=N$, our graph X is obviously N-partite.

We will be back to such graphs, and more specifically to the bipartite ones, which are the most useful and important, among multi-partite graphs, later in this book.

1e. Exercises

Here are some exercises, in relation with what we did in this chapter, quite often rather difficult, but hey, if looking for an easy book, many other choices available:

Exercise 1.32. *Learn some motivating physics, in relation with quantization.*

Exercise 1.33. *What Cayley graph should mean, based on what we know so far?*

Exercise 1.34. *Try doing some general spectral theory for the trees.*

Exercise 1.35. *Clarify our asymptotic estimate for the circle graph.*

Exercise 1.36. *Work out loop numerics for the circle, and the segment graphs.*

Exercise 1.37. *Fully clarify the loop count on $\mathbb{Z}$, including learning Stirling.*

Exercise 1.38. *Learn more about the* 4*-color theorem, and its proof.*

Exercise 1.39. *Learn more about projective spaces, and the Paley biplane too.*

As bonus exercise, learn some programming. Any graph expert knows some. So, find some free math software, download, and start playing with it.

CHAPTER 2

General theory

2a. Linear algebra

You probably know from linear algebra that the important question regarding any matrix $d\in M_N(\mathbb R)$ is its diagonalization. To be more precise, the first question is that of computing the eigenvalues, which are usually complex numbers, $\lambda\in\mathbb C$. Then comes the computation of the eigenvectors, and then the diagonalization problem.

In the case of the graphs, or rather of the adjacency matrices $d\in M_N(0,1)$ of the graphs, these notions are quite important and intuitive, as shown by:

Theorem 2.1. *The eigenvectors of $d\in M_N(0,1)$, with eigenvalue λ, can be identified with the functions f satisfying the following condition:*

$$\lambda f(i)=\sum_{i-j}f(j)$$

That is, the value of the function at each vertex i, when rescaled by λ, must equal the sum of the values of the function over the neighbors of i.

Proof. We have indeed the following computation, valid for any vector f:

$$\begin{aligned}(df)_i&=\sum_j d_{ij}f_j\\&=\sum_{i-j}d_{ij}f_j+\sum_{i\not-j}d_{ij}f_j\\&=\sum_{i-j}1\cdot f_j+\sum_{i\not-j}0\cdot f_j\\&=\sum_{i-j}f_j\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is quite interesting, and as an illustration, when assuming that our graph is k-regular, for the particular value $\lambda=k$, the eigenvalue condition reads:

$$f(i)=\frac{1}{k}\sum_{i-j}f(j)$$

Thus, we can see here a relation with harmonic functions. There are many things that can be said here, and we will be back to this later, when talking Laplace operators.

But let us pause now our study of graphs, and go back to linear algebra. Taking the entries to be complex numbers, which is something standard, here is a key result:

THEOREM 2.2. *Any matrix $d \in M_N(\mathbb{C})$ which is self-adjoint, $d = d^*$, is diagonalizable, with the diagonalization being of the following type,*

$$d = U\Lambda U^*$$

with $U \in U_N$, and with $\Lambda \in M_N(\mathbb{R})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse trivially holds, because if we take a matrix of the form $d = U\Lambda U^*$, with U unitary and Λ diagonal and real, then we have:

$$\begin{aligned} d^* &= (U\Lambda U^*)^* \\ &= U\Lambda^* U^* \\ &= U\Lambda U^* \\ &= d \end{aligned}$$

In the other sense now, assume that d is self-adjoint, $d = d^*$. Our first claim is that the eigenvalues are real. Indeed, assuming $dv = \lambda v$, we have:

$$\begin{aligned} \lambda < v, v > &= < \lambda v, v > \\ &= < dv, v > \\ &= < v, dv > \\ &= < v, \lambda v > \\ &= \bar{\lambda} < v, v > \end{aligned}$$

Thus we obtain $\lambda \in \mathbb{R}$, as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$dv = \lambda v \quad , \quad dw = \mu w$$

We have then the following computation, using $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned} \lambda < v, w > &= < \lambda v, w > \\ &= < dv, w > \\ &= < v, dw > \\ &= < v, \mu w > \\ &= \mu < v, w > \end{aligned}$$

Thus $\lambda \neq \mu$ implies $v \perp w$, as claimed. In order now to finish, it remains to prove that the eigenspaces span $\mathbb{C}^N$. For this purpose, we will use a recurrence method. Let us pick an eigenvector, $dv = \lambda v$. Assuming $v \perp w$, we have:

$$\begin{aligned} < dw, v > &= < w, dv > \\ &= < w, \lambda v > \\ &= \lambda < w, v > \\ &= 0 \end{aligned}$$

Thus, if v is an eigenvector, then the vector space $v^\perp$ is invariant under d. In order to do the recurrence, it still remains to prove that the restriction of d to the vector space $v^\perp$ is self-adjoint. But this comes from a general property of the self-adjoint matrices, that we will explain now. Our claim is that for any matrix $d \in M_N(\mathbb{C})$, we have:

$$d = d^* \iff < dv, v > \in \mathbb{R}, \forall v$$

Indeed, this follows from the following computation:

$$\begin{aligned} < dv, v > - \overline{< dv, v >} &= < dv, v > - < v, dv > \\ &= < dv, v > - < d^*v, v > \\ &= < (d - d^*)v, v > \\ &= \sum_{ij} (d - d^*)_{ij} \bar{v}_i v_j \end{aligned}$$

But this shows that the restriction of d to any invariant subspace, and in particular to $v^\perp$, is self-adjoint. Thus, we can proceed by recurrence, and we obtain the result. □

In what concerns us, in relation with our graph problems, we will rather need the real version of the above result, which is also something well-known, as follows:

THEOREM 2.3. *Any matrix $d \in M_N(\mathbb{R})$ which is symmetric, $d = d^t$, is diagonalizable, with the diagonalization being of the following type,*

$$d = U \Lambda U^t$$

with $U \in O_N$, and with $\Lambda \in M_N(\mathbb{R})$ diagonal. The converse holds too.

PROOF. As before, the converse trivially holds, because if we take a matrix of the form $d = U \Lambda U^t$, with U orthogonal, and Λ diagonal and real, then we have:

$$\begin{aligned} d^t &= (U \Lambda U^t)^t \\ &= U \Lambda^t U^t \\ &= U \Lambda U^t \\ &= d \end{aligned}$$

In the other sense now, this follows from Theorem 2.2, and its proof. Indeed, we know from there that the eigenvalues are real, and in what concerns the passage matrix, the arguments there carry over to the real case, and show that this matrix is real too. □

With the above results in hand, time now to get back to graphs. We have here the following particular case of Theorem 2.3, with the important drawback however that in what concerns the "converse holds too" part, that is unfortunately gone:

THEOREM 2.4. *The adjacency matrix $d \in M_N(0,1)$ of any graph is diagonalizable, with the diagonalization being of the following type,*

$$d = U\Lambda U^t$$

with $U \in O_N$, and with $\Lambda \in M_N(\mathbb{R})$ diagonal. Moreover, we have $Tr(\Lambda) = 0$.

PROOF. Here the first assertion follows from Theorem 2.3, because d is by definition real and symmetric. As for the last assertion, this deserves some explanations:

(1) Generally speaking, in analogy with the last assertions in Theorem 2.2 and Theorem 2.3, which are something extremely useful, we would like to know under which assumptions on a rotation $U \in O_N$, and on a diagonal matrix $\Lambda \in M_N(\mathbb{R})$, the real symmetric matrix $d = U\Lambda U^t$ has 0-1 entries, and 0 on the diagonal.

(2) Unfortunately, both these questions are obviously difficult, there is no simple answer to them, and things are like that. So, gone the possibility of a converse. However, as a small consolation, we can make the remark that, with $d = U\Lambda U^t$, we have:

$$Tr(d) = Tr(U\Lambda U^t) = Tr(\Lambda)$$

Thus we have at least $Tr(\Lambda) = 0$, as a necessary condition on (U,Λ), as stated. □

In view of the above difficulties with the bijectivity, it is perhaps wise to formulate as well the graph particular case of Theorem 2.2. The statement here is as follows:

THEOREM 2.5. *The adjacency matrix $d \in M_N(0,1)$ of any graph is diagonalizable, with the diagonalization being of the following type,*

$$d = U\Lambda U^t$$

with $U \in U_N$, and with $\Lambda \in M_N(\mathbb{R})$ diagonal. Moreover, we have $Tr(\Lambda) = 0$.

PROOF. This follows from Theorem 2.2, via the various remarks from the proof of Theorem 2.3 and Theorem 2.4. But the simplest is to say that the statement itself is just a copy of Theorem 2.4, with $U \in O_N$ replaced by the more general $U \in U_N$. □

All the above is useful, and we will use these results on a regular basis, in what follows. However, before getting into more concrete things, let us formulate:

PROBLEM 2.6. *Find a geometric proof of Theorem 2.4, or of Theorem 2.5, based on the interpretation of eigenvalues and eigenvectors from Theorem 2.1.*

This question looks quite reasonable, at a first glance, after all what we have in Theorem 2.1 is all nice and gentle material, so do we really need all the above complicated linear algebra machinery in order to deal with all this. However, at a second look, meaning after studying some examples, the problem suddenly looks very complicated. So, homework for you, in case I forget to assign this, to come back to this problem, later.

2b. The simplex

As an illustration for the above, let us diagonalize the adjacency matrix of the simplest graph that we know, namely the N-simplex. Let us start with:

PROPOSITION 2.7. *The adjacency matrix of the N-simplex, having 0 on the diagonal and 1 elsewhere, is in matrix form:*

$$d=\begin{pmatrix}0&1&\ldots&1&1\\1&0&\ldots&1&1\\\vdots&\vdots&&\vdots&\vdots\\1&1&\ldots&0&1\\1&1&\ldots&1&0\end{pmatrix}$$

We have the following formula for it, with $\mathbb{I}$ standing for the all-1 matrix, and with 1 standing for the identity matrix, both of size N:

$$d=\mathbb{I}-1$$

Equivalently, $d=NP-1$, with P being the projection on the all-1 vector $\xi\in\mathbb{R}^N$.

PROOF. Here the first assertion is clear from definitions, and the second assertion is clear too. As for the last assertion, observe first that with $P=\mathbb{I}/N$ we have:

$$P^2=\left(\frac{\mathbb{I}}{N}\right)^2=\frac{\mathbb{I}^2}{N^2}=\frac{N\mathbb{I}}{N^2}=\frac{\mathbb{I}}{N}=P$$

Thus P is a projection, and since we obviously have $P=P^t$, this matrix is an orthogonal projection. In order to find now the image, observe that for any vector $v\in\mathbb{R}^N$ we have the following formula, with $a\in\mathbb{R}$ being the average of the entries of v:

$$Pv=\begin{pmatrix}a\\\vdots\\a\end{pmatrix}$$

We conclude that the image of P is the vector space $\mathbb{R}\xi$, with $\xi\in\mathbb{R}^N$ being the all-1 vector, and so that P is the orthogonal projection on ξ, as claimed. □

The above is very nice, in particular with $d=NP-1$ basically diagonalizing d for us. However, thinking a bit, when it comes to explicitely diagonalize d, or, equivalently, P or $\mathbb{I}$, things are quite tricky, and we run into the following strange problem:

Problem 2.8. *In order to diagonalize d, we need solutions for*

$$x_1 + \ldots + x_N = 0$$

and there are no standard such solutions, over the reals.

So, this is the problem that we face, which might look a bit futile at a first glance, and in order for you to take me seriously here, let us work out some particular cases. At $N = 2$ things are quickly solved, with the diagonalization being as follows, and with the passage matrix being easy to construct, I agree with you on this:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix} \sim \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

However, things become suddenly complicated at $N = 3$, where I challenge you to find the passage matrix for the following diagonalization:

$$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix} \sim \begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

In the general case, $N \in \mathbb{N}$, the problem does not get any simpler, again with the challenge for you to find the passage matrix for the following diagonalization:

$$\begin{pmatrix}0&1&\ldots&1&1\\1&0&\ldots&1&1\\ \vdots&\vdots&&\vdots&\vdots\\1&1&\ldots&0&1\\1&1&\ldots&1&0\end{pmatrix} \sim \begin{pmatrix}N-1&&&&0\\&-1&&&\\&&\ddots&&\\&&&-1&\\0&&&&-1\end{pmatrix}$$

In short, you got my point, Problem 2.8 is something real. Fortunately the complex numbers come to the rescue, via the following standard and beautiful result:

Theorem 2.9. *The roots of unity, $\{w^k\}$ with $w = e^{2\pi i/N}$, have the property*

$$\frac{1}{N}\sum_{k=0}^{N-1}(w^k)^s = \delta_{N|s}$$

for any exponent $s \in \mathbb{N}$, where on the right we have a Kronecker symbol.

Proof. The numbers in the statement to be summed, when written more conveniently as $(w^s)^k$ with $k = 0, \ldots, N-1$, form a certain regular polygon in the plane P_s. Thus, if we denote by C_s the barycenter of this polygon, we have the following formula:

$$\frac{1}{N}\sum_{k=0}^{N-1} w^{ks} = C_s$$

Now observe that in the case $N \not| s$ our polygon P_s is non-degenerate, circling along the unit circle, and having center $C_s = 0$. As for the case $N|s$, here the polygon is degenerate, lying at 1, and having center $C_s = 1$. Thus, we have the following formula:

$$C_s = \delta_{N|s}$$

But this gives the formula in the statement. $\square$

Summarizing, we have the solution to our problem. In order now to finalize, let us start with the following definition, inspired by what happens in Theorem 2.9:

DEFINITION 2.10. *The Fourier matrix F_N is the following matrix, with $w = e^{2\pi i/N}$:*

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}$$

That is, $F_N = (w^{ij})_{ij}$, with indices $i, j \in \{0, 1, \dots, N-1\}$, taken modulo N.

Here the name comes from the fact that F_N is the matrix of the discrete Fourier transform, that over the cyclic group $\mathbb{Z}_N$, and more on this later, when talking Fourier analysis. As a first example now, at $N = 2$ the root of unity is $w = -1$, and with indices as above, namely $i, j \in \{0, 1\}$, taken modulo 2, our Fourier matrix is as follows:

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

At $N = 3$ now, the root of unity is $w = e^{2\pi i/3}$, and the Fourier matrix is:

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$$

At $N = 4$ now, the root of unit is $w = i$, and the Fourier matrix is:

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Finally, at $N=5$ the root of unity is $w=e^{2\pi i/5}$, and the Fourier matrix is:

$$F_5=\begin{pmatrix}1&1&1&1&1\\1&w&w^2&w^3&w^4\\1&w^2&w^4&w&w^3\\1&w^3&w&w^4&w^2\\1&w^4&w^3&w^2&w\end{pmatrix}$$

You get the point. Getting back now to the diagonalization problem for the flat matrix $\mathbb{I}$, this can be solved by using the Fourier matrix F_N, in the following way:

PROPOSITION 2.11. *The flat matrix $\mathbb{I}$ diagonalizes as follows,*

$$\begin{pmatrix}1&\ldots&\ldots&1\\\vdots&&&\vdots\\\vdots&&&\vdots\\1&\ldots&\ldots&1\end{pmatrix}=\frac{1}{N}\,F_N\begin{pmatrix}N&&&&0\\&0\\&&\ddots\\&&&0\\0&&&&0\end{pmatrix}F_N^*$$

with $F_N=(w^{ij})_{ij}$ being the Fourier matrix.

PROOF. According to the last assertion in Proposition 2.7, we are left with finding the 0-eigenvectors of $\mathbb{I}$, which amounts in solving the following equation:

$$x_0+\ldots+x_{N-1}=0$$

For this purpose, we can use the root of unity $w=e^{2\pi i/N}$, and more specifically, the following standard formula, coming from Theorem 2.9:

$$\sum_{i=0}^{N-1}w^{ij}=N\delta_{j0}$$

This formula shows that for $j=1,\ldots,N-1$, the vector $v_j=(w^{ij})_i$ is a 0-eigenvector. Moreover, these vectors are pairwise orthogonal, because we have:

$$<v_j,v_k>=\sum_i w^{ij-ik}=N\delta_{jk}$$

Thus, we have our basis $\{v_1,\ldots,v_{N-1}\}$ of 0-eigenvectors, and since the N-eigenvector is $\xi=v_0$, the passage matrix P that we are looking is given by:

$$P=\begin{bmatrix}v_0&v_1&\ldots&v_{N-1}\end{bmatrix}$$

But this is precisely the Fourier matrix, $P=F_N$. In order to finish now, observe that the above computation of $<v_i,v_j>$ shows that $F_N/\sqrt{N}$ is unitary, and so:

$$F_N^{-1}=\frac{1}{N}\,F_N^*$$

Thus, we are led to the diagonalization formula in the statement. $\square$

By substracting now -1 from everything, we can formulate a final result, as follows:

THEOREM 2.12. *The adjacency matrix of the simplex diagonalizes as follows,*

$$\begin{pmatrix}0&1&\ldots&1&1\\1&0&\ldots&1&1\\\vdots&\vdots&&\vdots&\vdots\\1&1&\ldots&0&1\\1&1&\ldots&1&0\end{pmatrix}=\frac{1}{N}\,F_N\begin{pmatrix}N-1&&&&0\\&-1&&&\\&&\ddots&&\\&&&-1&\\0&&&&-1\end{pmatrix}F_N^*$$

with $F_N=(w^{ij})_{ij}$ being the Fourier matrix.

PROOF. This follows as said above, from what we have in Proposition 2.11, by substracting -1 from everything. Alternatively, if you prefer a more direct proof, this follows from the various computations in the proof of Proposition 2.11. $\square$

The above result was something quite tricky, and we will come back regularly to such things, in what follows. For the moment, let us formulate an interesting conclusion:

CONCLUSION 2.13. *Theorem 2.5, telling us that d diagonalizes over $\mathbb{C}$, is better than the stronger Theorem 2.4, telling us that d diagonalizes over $\mathbb{R}$.*

And isn't this surprising. But after some thinking, after all no surprise, because the graphs, as we defined them in the beginning of chapter 1, are scalarless objects. So, when needing a field for studying them, we should just go with the mighty $F=\mathbb{C}$.

By the way, regarding complex numbers, time to recommend some reading. Mathematically the book of Rudin [79] is a must-read, and a pleasure to read. However, if you want to be really in love with complex numbers, and with this being an enormous asset, no matter what mathematics you want to do, nothing beats some physics reading.

The standard place for learning physics is the course of Feynman [40]. If you already know a bit about physics, you can go as well with the lovely books of Griffiths [44], [45], [46]. And if you know a bit more, good books are those of Weinberg [92], [93], [94]. In the hope that this helps, and I will not tell you of course what these books contain. Expect however lots of complex numbers, all beautiful, and used majestically.

2c. Catalan numbers

We can come back now to walks, that we know from chapter 1 to be related to the eigenvalues of the adjacency matrix $d\in M_N(0,1)$. Before that, however, we have unfinished business with the graph $\mathbb{N}$, which is infinite. To be more precise, we have seen in chapter 1 how to count loops on $\mathbb{Z}$, and the problem left was to do this as well for $\mathbb{N}$. So, let us solve this problem. At the experimental level, the result is as follows:

PROPOSITION 2.14. *The Catalan numbers C_k, counting loops on $\mathbb{N}$ based at 0,*

$$C_k = \#\Big\{0 = i_0 - i_1 - \ldots - i_{2k-1} - i_{2k} = 0\Big\}$$

are numerically $1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

PROOF. To start with, we have indeed $C_1 = 1$, the only loop here being $0 - 1 - 0$. Then we have $C_2 = 2$, due to two possible loops, namely:

$$0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0$$

Then we have $C_3 = 5$, the possible loops here being as follows:

$$0 - 1 - 0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 0 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 3 - 2 - 1 - 0$$

In general, the same method works, with $C_4 = 14$ being left to you, as an exercise, and with C_5 and higher to me, and I will be back with the solution, in due time. $\square$

Obviously, computing C_k is no easy task, and finding the formula of C_k, out of the data that we have, does not look as an easy task either. So, what to do. The first thought goes to our diagonalization methods, but these are valid for the finite graphs only, and better not to mess with that, because if you read some physics as previously recommended, you might have learned from there that infinite matrices are no easy business.

Summarizing, we are in front of a delicate combinatorial problem. And so, we will do what combinatorists do, let me teach you. The first step is to relax, then to look around, not with the aim of computing your numbers C_k, but rather with the aim of finding other objects counted by the same numbers C_k. With a bit of luck, among these objects some will be easier to count than the others, and this will eventually compute C_k.

This was for the strategy. In practice now, we first have the following result:

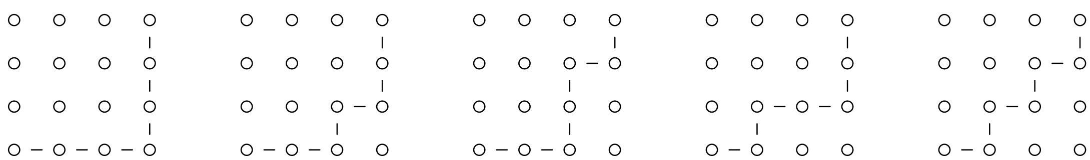
THEOREM 2.15. *The Catalan numbers C_k count:*

1. *The length $2k$ loops on $\mathbb{N}$, based at 0.*
2. *The noncrossing pairings of $1, \ldots, 2k$.*
3. *The noncrossing partitions of $1, \ldots, k$.*
4. *The length $2k$ Dyck paths in the plane.*

PROOF. All this is standard combinatorics, the idea being as follows:

(1) To start with, in what regards the various objects involved, the length $2k$ loops on $\mathbb{N}$ are the length $2k$ loops on $\mathbb{N}$ that we know, and the same goes for the noncrossing pairings of $1,\ldots,2k$, and for the noncrossing partitions of $1,\ldots,k$, the idea here being that you must be able to draw the pairing or partition in a noncrossing way.

(2) Regarding now the length $2k$ Dyck paths in the plane, these are by definition the paths from $(0,0)$ to (k,k), marching North-East over the integer lattice $\mathbb{Z}^2\subset\mathbb{R}^2$, by staying inside the square $[0,k]\times[0,k]$, and staying as well under the diagonal of this square. As an example, here are the 5 possible Dyck paths at $n=3$:



(3) Thus, we have definitions for all the objects involved, and in each case, if you start counting them, as we did in Proposition 2.14 with the loops on $\mathbb{N}$, you always end up with the same sequence of numbers, namely those found in Proposition 2.14:

$$1,2,5,14,42,132,429,1430,4862,16796,58786,\ldots$$

(4) In order to prove now that (1-4) produce indeed the same numbers, many things can be said. The idea is that, leaving aside mathematical brevity, and more specifically abstract reasonings of type $a=b,b=c\implies a=c$, what we have to do, in order to fully understand what is going on, is to etablish $\binom{4}{2}=6$ equalities, via bijective proofs.

(5) But this can be done, indeed. As an example here, the noncrossing pairings of $1,\ldots,2k$ from (2) are in bijection with the noncrossing partitions of $1,\ldots,k$ from (3), via fattening the pairings and shrinking the partitions. We will leave the details here as an instructive exercise, and exercise as well, to add (1) and (4) to the picture.

(6) However, matter of having our theorem formally proved, I mean by me professor and not by you student, here is a less elegant argument, which is however very quick, and does the job. The point is that, in each of the cases (1-4) under consideration, the numbers C_k that we get are easily seen to be subject to the following recurrence:

$$C_{k+1}=\sum_{a+b=k}C_aC_b$$

The initial data being the same, namely $C_1=1$ and $C_2=2$, in each of the cases (1-4) under consideration, we get indeed the same numbers. $\square$

Now we can pass to the second step, namely selecting in the above list the objects that we find the most convenient to count, and count them. This leads to:

Theorem 2.16. *The Catalan numbers are given by the formula*

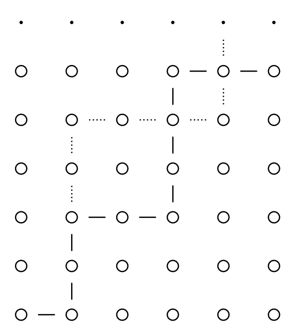
$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

with this being best seen by counting the length $2k$ Dyck paths in the plane.

Proof. This is something quite tricky, the idea being as follows:

(1) Let us count indeed the Dyck paths in the plane. For this purpose, we use a trick. Indeed, if we ignore the assumption that our path must stay under the diagonal of the square, we have $\binom{2k}{k}$ such paths. And among these, we have the "good" ones, those that we want to count, and then the "bad" ones, those that we want to ignore.

(2) So, let us count the bad paths, those crossing the diagonal of the square, and reaching the higher diagonal next to it, the one joining $(0,1)$ and $(k,k+1)$. In order to count these, the trick is to "flip" their bad part over that higher diagonal, as follows:



(3) Now observe that, as it is obvious on the above picture, due to the flipping, the flipped bad path will no longer end in (k,k), but rather in $(k-1,k+1)$. Moreover, more is true, in the sense that, by thinking a bit, we see that the flipped bad paths are precisely those ending in $(k-1,k+1)$. Thus, we can count these flipped bad paths, and so the bad paths, and so the good paths too, and so good news, we are done.

(4) To finish now, by putting everything together, we have:

$$\begin{aligned}
C_k &= \binom{2k}{k} - \binom{2k}{k-1}\\
&= \binom{2k}{k} - \frac{k}{k+1}\binom{2k}{k}\\
&= \frac{1}{k+1}\binom{2k}{k}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

We have as well another approach to all this, computation of the Catalan numbers, this time based on rock-solid standard calculus, as follows:

THEOREM 2.17. *The Catalan numbers have the following properties:*

(1) *They satisfy* $C_{k+1}=\sum_{a+b=k}C_aC_b$.
(2) *The series* $f(z)=\sum_{k\geq 0}C_kz^k$ *satisfies* $zf^2-f+1=0$.
(3) *This series is given by* $f(z)=\frac{1-\sqrt{1-4z}}{2z}$.
(4) *We have the formula* $C_k=\frac{1}{k+1}\binom{2k}{k}$.

PROOF. This is best viewed by using noncrossing pairings, as follows:

(1) Let us count the noncrossing pairings of $\{1,\ldots,2k+2\}$. Such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing of $\{2,\ldots,2a\}$, and a noncrossing pairing of $\{2a+2,\ldots,2k+2\}$. Thus we have, as claimed:

$$C_{k+1}=\sum_{a+b=k}C_aC_b$$

(2) Consider now the generating series of the Catalan numbers, $f(z)=\sum_{k\geq 0}C_kz^k$. In terms of this generating series, the above recurrence gives, as desired:

$$\begin{aligned}
zf^2 &= \sum_{a,b\geq 0}C_aC_bz^{a+b+1}\\
&= \sum_{k\geq 1}\sum_{a+b=k-1}C_aC_bz^k\\
&= \sum_{k\geq 1}C_kz^k\\
&= f-1
\end{aligned}$$

(3) By solving the equation $zf^2-f+1=0$ found above, and choosing the solution which is bounded at $z=0$, we obtain the following formula, as claimed:

$$f(z)=\frac{1-\sqrt{1-4z}}{2z}$$

(4) In order to compute this function, we use the generalized binomial formula, which is as follows, with $p\in\mathbb{R}$ being an arbitrary exponent, and with $|t|<1$:

$$(1+t)^p=\sum_{k=0}^\infty\binom{p}{k}t^k$$

To be more precise, this formula, which generalizes the usual binomial formula, holds indeed due to the Taylor formula, with the binomial coefficients being given by:

$$\binom{p}{k}=\frac{p(p-1)\ldots(p-k+1)}{k!}$$

(5) For the exponent $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\ldots(3/2-k)}{k!}\\ &= (-1)^{k-1}\frac{1\cdot 3\cdot 5\ldots(2k-3)}{2^k k!}\\ &= (-1)^{k-1}\frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!}\\ &= \frac{(-1)^{k-1}}{2^{2k-1}}\cdot\frac{1}{k}\binom{2k-2}{k-1}\\ &= -2\left(\frac{-1}{4}\right)^k\cdot\frac{1}{k}\binom{2k-2}{k-1}\end{aligned}$$

(6) Thus the generalized binomial formula at exponent $p = 1/2$ reads:

$$\sqrt{1+t} = 1 - 2\sum_{k=1}^{\infty}\frac{1}{k}\binom{2k-2}{k-1}\left(\frac{-t}{4}\right)^k$$

With $t = -4z$ we obtain from this the following formula:

$$\sqrt{1-4z} = 1 - 2\sum_{k=1}^{\infty}\frac{1}{k}\binom{2k-2}{k-1}z^k$$

(7) Now back to our series f, we obtain the following formula for it:

$$\begin{aligned}f(z) &= \frac{1-\sqrt{1-4z}}{2z}\\ &= \sum_{k=1}^{\infty}\frac{1}{k}\binom{2k-2}{k-1}z^{k-1}\\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}z^k\end{aligned}$$

(8) Thus the Catalan numbers are given by the formula the statement, namely:

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

So done, and note in passing that I kept my promise, from the proof of Proposition 2.14. Indeed, with the above final formula, the numerics are easily worked out. □

Many other things can be said about the Catalan numbers, as a continuation of the above, and there is as well a relation with the central binomial coefficients, that we found in chapter 1 in connection with the walks on $\mathbb{Z}$. We will be back to this.

2d. Spectral measures

With the above done, we can come back now to walks on finite graphs, that we know from chapter 1 to be related to the eigenvalues of the adjacency matrix $d \in M_N(0,1)$. But here, we are led to the following philosophical question, to start with:

QUESTION 2.18. *What are the most important finite graphs, that we should do our computations for?*

Not an easy question, you have to agree with me, with the answer to this obviously depending on your previous experience with mathematics, or physics, or chemistry, or computer science, or other branch of science that you are interested in, and also, on the specific problems that you are the most in love with, in that part of science.

So, we have to be subjective here. And with me writing this book, and doing some sort of complicated quantum physics, as daytime job, I will choose the ADE graphs. It is beyond our scope here to explain where these ADE graphs exactly come from, and what they are good for, but as a piece of advertisement for them, we have:

ADVERTISEMENT 2.19. *The ADE graphs classify the following things:*
(1) *Basic Lie groups and algebras.*
(2) *Subgroups of SU_2 and of SO_3.*
(3) *Singularities of algebraic manifolds.*
(4) *Basic invariants of knots and links.*
(5) *Subfactors and planar algebras of small index.*
(6) *Subgroups of the quantum permutation group S_4^+.*
(7) *Basic quantum field theories, and other physics beasts.*

Which sounds exciting, doesn't it. So, have a look at this, and with the comment that some heavy learning work is needed, in order to understand how all this works. And with the extra comment that, in view of (7), tough physics, no one really understands how all this works. A nice introduction to all this is the book by Jones [55].

Getting to work now, we first need to know what the ADE graphs are. The A graphs, which are the simplest, are as follows, with the distinguished vertex being denoted $\bullet$, and with A_n having $n \geq 2$ vertices, and $\tilde{A}_{2n}$ having $2n \geq 2$ vertices:

$$A_n = \bullet - \circ - \circ \cdots \circ - \circ - \circ \qquad\qquad A_\infty = \bullet - \circ - \circ - \circ \cdots$$

$$\begin{matrix} \circ - \circ - \circ \cdots \circ - \circ - \circ \\ | \qquad\qquad\qquad\qquad\quad | \\ \tilde{A}_{2n} = \bullet - \circ - \circ - \circ - \circ - \circ \end{matrix} \qquad\qquad \begin{matrix} \circ - \circ - \circ - \circ \cdots \\ | \qquad\qquad\qquad \\ \tilde{A}_\infty = \bullet - \circ - \circ - \circ \cdots \end{matrix}$$

These A graphs do not actually look that scary, because we already met all of them in the above, and as a comment on them, summarizing the situation, we have:

COMMENT 2.20. *With the A graphs we are not really lost into quantum physics, because all these graphs are quite familiar to us, as follows:*

(1) A_n *is the segment.*
(2) A_∞ *is the* $\mathbb{N}$ *graph.*
(3) $\tilde{A}_{2n}$ *is the circle.*
(4) $\tilde{A}_\infty$ *is the* $\mathbb{Z}$ *graph.*

You might probably say, why not stopping here, and doing our unfinished business for the segment and the circle, with whatever new ideas that we might have. Good point, but in answer, these ideas will apply as well, with minimal changes, to the D graphs, which are as follows, with D_n having $n\geq 3$ vertices, and $\tilde{D}_n$ having $n+1\geq 5$ vertices:

$$D_n=\begin{array}{ccccccccc} &&&&&&\circ&&\\ &&&&&&|&&\\ \bullet&-&\circ&-&\circ\cdots\circ&-&\circ&-&\circ \end{array}$$

$$\tilde{D}_n=\begin{array}{ccccccccc} &&\circ&&&&\circ&&\\ &&|&&&&|&&\\ \bullet&-&\circ&-&\circ\cdots\circ&-&\circ&-&\circ \end{array}$$

$$D_\infty=\begin{array}{ccccccc} &&\circ&&&&\\ &&|&&&&\\ \bullet&-&\circ&-&\circ&-&\circ\cdots \end{array}$$

As mentioned above, it is beyond our scope here to explain what the ADE graphs really stand for, but as an informal comment on these latter D graphs, we have:

COMMENT 2.21. *The D graphs are not that scary either, and they can be thought of as being certain technical versions of the A graphs.*

So, this is the situation, you have to trust me here, and for more on all this, check for instance the book of Jones [55]. In what concerns us, we will just take the above D graphs as they come, and do our loop count work for them, without questions asked.

As another comment, the labeling conventions for the AD graphs, while very standard, can be a bit confusing. The first graph in each series is by definition as follows:

$$A_2=\bullet-\circ\qquad\tilde{A}_2=\begin{array}{c}\circ\\ ||\\ \bullet\end{array}\qquad D_3=\begin{array}{ccc}\circ&&\\ |&&\\ \bullet&-&\circ\end{array}\qquad \tilde{D}_4=\begin{array}{ccccc}&\circ&&\circ&\\ &\backslash&&/&\\ \bullet&-&\circ&-&\circ\end{array}$$

Finally, there are also a number of exceptional ADE graphs. First we have:

$$\begin{array}{llllllllll} & & & & & \circ & & & & \\ & & & & & | & & & & \\ E_6 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

$$\begin{array}{llllllllllll} & & & & & & & \circ & & & & \\ & & & & & & & | & & & & \\ E_7 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

$$\begin{array}{llllllllllllll} & & & & & & & & & \circ & & & & \\ & & & & & & & & & | & & & & \\ E_8 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

Then, we have extended versions of the above exceptional graphs, as follows:

$$\begin{array}{llllllllll} & & & & & \circ & & & & \\ & & & & & | & & & & \\ & & & & & \circ & & & & \\ & & & & & | & & & & \\ \tilde{E}_6 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

$$\begin{array}{llllllllllllll} & & & & & & & \circ & & & & & & \\ & & & & & & & | & & & & & & \\ \tilde{E}_7 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

$$\begin{array}{llllllllllllllll} & & & & & & & & & & & \circ & & & & \\ & & & & & & & & & & & | & & & & \\ \tilde{E}_8 = & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

And good news, that is all. Hard job for me to come now with a comment on these latter E graphs, along the lines of Comments 2.20 and 2.21, and here is what I have:

COMMENT 2.22. *The E graphs naturally complement the AD series, by capturing the combinatorics of certain "exceptional" phenomena in mathematics and physics.*

So long for difficult definitions and related informal talk, and as already mentioned in the above, for more on all this, have a look at the book of Jones [55]. Getting now to work, we have some new graphs, and here is the problem that we would like to solve:

PROBLEM 2.23. *How to count loops on the ADE graphs?*

In answer, as mentioned in Comment 2.20, we are already familiar with two of the ADE graphs, namely A_∞ and $\tilde{A}_\infty$, which are respectively the graphs that we previously called $\mathbb{N}$ and $\mathbb{Z}$. So, based on our work for these graphs, where the combinatorics naturally led us into generating series, let us formulate the following definition:

Definition 2.24. *The Poincaré series of a rooted bipartite graph X is*

$$f(z)=\sum_{k=0}^\infty L_kz^k$$

where L_k is the number of $2k$-loops based at the root.

To be more precise, observe that all the above ADE graphs are indeed bipartite, because they are all trees, which are naturally bipartite, in the obvious way, with the only exception being the graph $\tilde{A}_{2n}$, which is bipartite too, again in the obvious way.

Now the point is that, for a bipartite graph, the loops based at any point must have even length. Thus, in order to study the loops on the ADE graphs, based at the root, we just have to count the above numbers L_k. And then, considering the generating series $f(z)$ of these numbers, and calling this Poincaré series, is something very standard.

Before getting into computations, let us introduce as well:

Definition 2.25. *The positive spectral measure μ of a rooted bipartite graph X is the real probability measure having the numbers L_k as moments:*

$$\int_\mathbb{R}x^kd\mu(x)=L_k$$

Equivalently, we must have the Stieltjes transform formula

$$f(z)=\int_\mathbb{R}\frac{1}{1-xz}\,d\mu(x)$$

where f is the Poincaré series of X.

Here the existence of μ, and the fact that this is indeed a positive measure, meaning a measure supported on $[0,\infty)$, comes from the following simple fact:

Theorem 2.26. *The positive spectral measure of a rooted bipartite graph X is given by the following formula, with d being the adjacency matrix of the graph,*

$$\mu=law(d^2)$$

and with the probabilistic computation being with respect to the expectation

$$A\to<A>$$

with $<A>$ being the $(*,*)$-entry of a matrix A, where $*$ is the root.

PROOF. With the above conventions, we have the following computation:

$$\begin{aligned} f(z) &= \sum_{k=0}^\infty L_k z^k \\ &= \sum_{k=0}^\infty \left\langle d^{2k} \right\rangle z^k \\ &= \left\langle \frac{1}{1-d^2z} \right\rangle \end{aligned}$$

But this shows that we have $\mu = law(d^2)$, as desired. $\square$

The above result shows that computing μ might be actually a simpler problem than computing f, and in practice, this is indeed the case. So, in what follows we will rather forget about loops and Definition 2.24, and use Definition 2.25 instead, with our computations to follow being based on the concrete interpretation from Theorem 2.26.

However, even with this probabilistic trick in our bag, things are not exactly trivial. So, following now [**14**], let us introduce as well the following notion:

DEFINITION 2.27. *The circular measure ε of a rooted bipartite graph X is given by*

$$d\varepsilon(q) = d\mu((q+q^{-1})^2)$$

where μ is the associated positive spectral measure.

To be more precise, we know from Theorem 2.26 that the positive measure μ is the spectral measure of a certain positive matrix, $d^2 \geq 0$, and it follows from this, and from basic spectral theory, that this measure is supported by the positive reals:

$$supp(\mu) \subset \mathbb{R}_+$$

But then, with this observation in hand, we can define indeed the circular measure ε as above, as being the pullback of μ via the following map:

$$\mathbb{R} \cup \mathbb{T} \to \mathbb{R}_+ \quad , \quad q \to (q+q^{-1})^2$$

As a basic example for this, to start with, assume that μ is a discrete measure, supported by n positive numbers $x_1 < \ldots < x_n$, with corresponding densities $p_1, \ldots, p_n$:

$$\mu = \sum_{i=1}^n p_i \delta_{x_i}$$

For each $i \in \{1, \ldots, n\}$ the equation $(q+q^{-1})^2 = x_i$ has then four solutions, that we can denote $q_i, q_i^{-1}, -q_i, -q_i^{-1}$. And with this notation, we have:

$$\varepsilon = \frac{1}{4} \sum_{i=1}^n p_i \left(\delta_{q_i} + \delta_{q_i^{-1}} + \delta_{-q_i} + \delta_{-q_i^{-1}} \right)$$

In general, the basic properties of ε can be summarized as follows:

THEOREM 2.28. *The circular measure has the following properties:*

(1) *ε has equal density at $q, q^{-1}, -q, -q^{-1}$.*
(2) *The odd moments of ε are 0.*
(3) *The even moments of ε are half-integers.*
(4) *When X has norm ≤ 2, ε is supported by the unit circle.*
(5) *When X is finite, ε is discrete.*
(6) *If K is a solution of $d = K + K^{-1}$, then $\varepsilon = law(K)$.*

PROOF. These results can be deduced from definitions, the idea being that (1-5) are trivial, and that (6) follows from the formula of μ from Theorem 2.26. $\square$

Getting now to computations, remember our struggle from chapter 1, with the circle graph? We can now solve this question, majestically, as follows:

THEOREM 2.29. *The circular measure of the basic index 4 graph, namely*

$$\tilde{A}_{2n} = \begin{array}{l} \circ - \circ - \circ \cdots \circ - \circ - \circ \\ | \qquad\qquad\qquad\qquad\quad | \\ \bullet - \circ - \circ - \circ - \circ - \circ \end{array}$$

is the uniform measure on the $2n$-roots of unity.

PROOF. Let us identify the vertices of $X = \tilde{A}_{2n}$ with the group $\{w^k\}$ formed by the $2n$-th roots of unity in the complex plane, where $w = e^{\pi i/n}$. The adjacency matrix of X acts then on the functions $f \in C(X)$ in the following way:

$$df(w^s) = f(w^{s-1}) + f(w^{s+1})$$

But this shows that we have $d = K + K^{-1}$, where K is given by:

$$Kf(w^s) = f(w^{s+1})$$

Thus we can use Theorem 2.26 and Theorem 2.28 (6), and we get:

$$\varepsilon = law(K)$$

But this is the uniform measure on the $2n$-roots of unity, as claimed. $\square$

All this is very nice, so, before going ahead with more computations, let us have an excursion into subfactor theory, and explain what is behind this trick. Following Jones [57], we can introduce the theta series of a graph X, as a version of the Poincaré series, via the change of variables $z^{-1/2} = q^{1/2} + q^{-1/2}$, as follows:

DEFINITION 2.30. *The theta series of a rooted bipartite graph X is*

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

where f is the Poincaré series.

The theta series can be written as $\Theta(q) = \sum a_r q^r$, and it follows from the above formula, via some simple manipulations, that its coefficients are integers:

$$a_r \in \mathbb{Z}$$

In fact, we have the following explicit formula from Jones' paper [57], relating the coefficients of $\Theta(q) = \sum a_r q^r$ to those of the Poincaré series $f(z) = \sum c_k z^k$:

$$a_r = \sum_{k=0}^{r} (-1)^{r-k} \frac{2r}{r+k} \binom{r+k}{r-k} c_k$$

As an important comment now, in the case where X is the principal graph of a subfactor $A_0 \subset A_1$ of index $N > 4$, it is known from [57] that the numbers a_r are certain multiplicities associated to the planar algebra inclusion $TL_N \subset P$, as explained there. In particular, the coefficients of the theta series are in this case positive integers:

$$a_r \in \mathbb{N}$$

In relation now with the circular measure, the result here, which is quite similar to the Stieltjes transform formula from Definition 2.25, is as follows:

THEOREM 2.31. *We have the Stieltjes transform type formula*

$$2\int \frac{1}{1-qu^2}\, d\varepsilon(u) = 1 + T(q)(1-q)$$

where the T series of a rooted bipartite graph X is by definition given by

$$T(q) = \frac{\Theta(q) - q}{1-q}$$

with Θ being the associated theta series.

PROOF. This follows by applying the change of variables $q \to (q+q^{-1})^2$ to the fact that f is the Stieltjes transform of μ. Indeed, we obtain in this way:

$$\begin{aligned} 2\int \frac{1}{1-qu^2}\, d\varepsilon(u) &= 1 + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right) \\ &= 1 + \Theta(q) - q \\ &= 1 + T(q)(1-q) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have a whole menagery of subfactor, planar algebra and bipartite graph invariants, which come in several flavors, namely series and measures, and which can be linear or circular, and which all appear as versions of the Poincaré series.

In order to discuss all this more systematically, let us introduce as well:

Definition 2.32. *The series of the form*

$$\xi(n_1,\ldots,n_s:m_1,\ldots,m_t)=\frac{(1-q^{n_1})\ldots(1-q^{n_s})}{(1-q^{m_1})\ldots(1-q^{m_t})}$$

with $n_i,m_i\in\mathbb{N}$ are called cyclotomic.

It is technically convenient to allow as well $1+q^n$ factors, to be designated by n^+ symbols in the above writing. For instance we have, by definition:

$$\xi(2^+:3)=\xi(4:2,3)$$

Also, it is convenient in what follows to use the following notations:

$$\xi'=\frac{\xi}{1-q}\quad,\quad\xi''=\frac{\xi}{1-q^2}$$

The Poincaré series of the ADE graphs are given by quite complicated formulae. However, the corresponding T series are all cyclotomic, as follows:

Theorem 2.33. *The T series of the ADE graphs are as follows:*

(1) *For A_{n-1} we have $T=\xi(n-1:n)$.*
(2) *For D_{n+1} we have $T=\xi(n-1^+:n^+)$.*
(3) *For $\tilde{A}_{2n}$ we have $T=\xi'(n^+:n)$.*
(4) *For $\tilde{D}_{n+2}$ we have $T=\xi''(n+1^+:n)$.*
(5) *For E_6 we have $T=\xi(8:3,6^+)$.*
(6) *For E_7 we have $T=\xi(12:4,9^+)$.*
(7) *For E_8 we have $T=\xi(5^+,9^+:15^+)$.*
(8) *For $\tilde{E}_6$ we have $T=\xi(6^+:3,4)$.*
(9) *For $\tilde{E}_7$ we have $T=\xi(9^+:4,6)$.*
(10) *For $\tilde{E}_8$ we have $T=\xi(15^+:6,10)$.*

Proof. These formulae were obtained in [14], by counting loops, and then by making the following change of variables, and factorizing the resulting series:

$$z^{-1/2}=q^{1/2}+q^{-1/2}$$

An alternative proof for these formulae can be obtained by using planar algebra methods, along the lines of the paper of Jones [57]. For details here, see [14]. □

Our purpose now will be that of converting the above technical results, regarding the T series, into some final results, regarding the corresponding circular measures ε. In order to formulate our results, we will need some more theory. First, we have:

Definition 2.34. *A cyclotomic measure is a probability measure ε on the unit circle, having the following properties:*

(1) *ε is supported by the $2n$-roots of unity, for some $n\in\mathbb{N}$.*
(2) *ε has equal density at $q,q^{-1},-q,-q^{-1}$.*

As a first observation, it follows from Theorem 2.28 and from Theorem 2.33 that the circular measures of the finite ADE graphs are supported by certain roots of unity, hence are cyclotomic. We will be back to this in a moment, with details, and computations.

At the general level now, let us introduce as well the following notion:

DEFINITION 2.35. *The T series of a cyclotomic measure ε is given by*

$$1+T(q)(1-q)=2\int\frac{1}{1-qu^2}\,d\varepsilon(u)$$

with ε being as usual the circular spectral measure.

Observe that this formula is nothing but the one in Theorem 2.31, written now in the other sense. In other words, if the cyclotomic measure ε happens to be the circular measure of a rooted bipartite graph, then the T series as defined above coincides with the T series as defined before. This is useful for explicit computations.

Good news, with this technology in hand, and with a computation already done, in Theorem 2.29, we are now ready to discuss the circular measures of all ADE graphs.

The idea will be that these measures are all cyclotomic, of level ≤ 3, and can be expressed in terms of the basic polynomial densities of degree ≤ 6, namely:

$$\alpha = Re(1-q^2)$$

$$\beta = Re(1-q^4)$$

$$\gamma = Re(1-q^6)$$

To be more precise, we have the following final result on the subject, with α,β,γ being as above, with d_n being the uniform measure on the $2n$-th roots of unity, and with $d_n'=2d_{2n}-d_n$ being the uniform measure on the odd $4n$-roots of unity:

THEOREM 2.36. *The circular measures of the ADE graphs are given by:*

(1) $A_{n-1}\to\alpha_n$.
(2) $\tilde{A}_{2n}\to d_n$.
(3) $D_{n+1}\to\alpha_n'$.
(4) $\tilde{D}_{n+2}\to(d_n+d_1')/2$.
(5) $E_6\to\alpha_{12}+(d_{12}-d_6-d_4+d_3)/2$.
(6) $E_7\to\beta_9'+(d_1'-d_3')/2$.
(7) $E_8\to\alpha_{15}'+\gamma_{15}'-(d_5'+d_3')/2$.
(8) $\tilde{E}_{n+3}\to(d_n+d_3+d_2-d_1)/2$.

PROOF. This is something which can be proved in three steps, as follows:

(1) For the simplest graph, namely the circle $\tilde{A}_{2n}$, we already have the result, from Theorem 2.29, with the proof there being something elementary.

(2) For the other non-exceptional graphs, that is, of type A and D, the same method works, namely direct loop counting, with some matrix tricks. See [14].

(3) In general, this follows from the T series formulae in Theorem 2.33, via some manipulations based on the general conversion formulae given above. See [14]. □

We refer to [14] and the subsequent literature for more on all this. Also, let us point out that all this leads to a more conceptual understanding of what we did before, for the graphs $\mathbb{N}$ and $\mathbb{Z}$. Indeed, even for these very basic graphs, using the unit circle and circular measures as above leads to a better understanding of the combinatorics.

2e. Exercises

We had a tough chapter here, mixing linear algebra with calculus, and with some probability too. Here are some exercises, in relation with the above:

EXERCISE 2.37. *Learn the spectral theorem for general normal matrices.*

EXERCISE 2.38. *Furher build on the spectral characterization of graphs.*

EXERCISE 2.39. *Find a geometric proof of the spectral theorem, for graphs.*

EXERCISE 2.40. *View the Fourier matrix as matrix of a Fourier transform.*

EXERCISE 2.41. *Work out all details for the bijections leading to Catalan numbers.*

EXERCISE 2.42. *Find some further interpretations of the Catalan numbers.*

EXERCISE 2.43. *Clarify all the details, in relation with our Dyck path counting.*

EXERCISE 2.44. *Compute the spectral measures of exceptional graphs.*

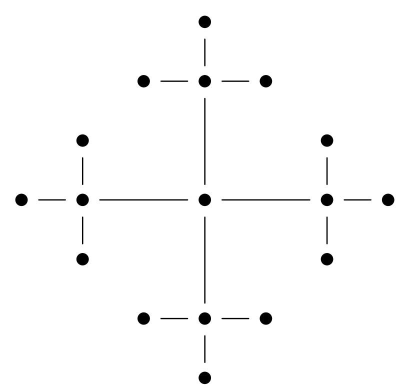
As bonus exercise, learn some subfactor theory. More generally, learn, or at least get to know about, all mathematical theories featuring an ADE classification result.

CHAPTER 3

Transitive graphs

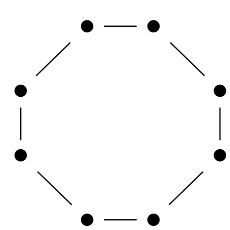
3a. Circulant graphs

You might have noticed already, some graphs look good, and some other look bad. This is of course a matter of taste, and there are several possible notions for what "good" should mean, and with this being a potential source of serious mathematical matter. As an example, we have already met a particularly beautiful graph before, namely:



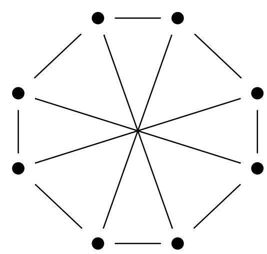
Such a graph is called a tree, because when looking at a tree from the above, what you see is something like this. In general, trees can be axiomatized as follows:

Definition 3.1. *A tree is a graph having no cycles. That is, there is no loop*

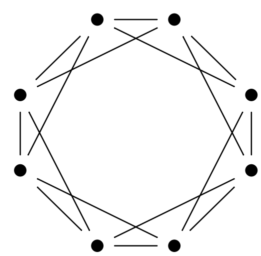


having length ≥ 3, and distinct vertices, inside the graph.

And aren't trees beautiful, hope you agree with me. But it is not about trees that we want to talk about here, these are in fact quite complicated mathematical objects, and we will keep them for later. As an alternative to them, we have the circulant graphs, which are equally beautiful, but in a somewhat opposite sense. Here is one, which is actually the most important graph in the history of mankind, along with the fire graph:



Here is another circulant graph, again with 8 vertices, again with the picture suggesting the name “circulant”, and of course, again beautiful as well:

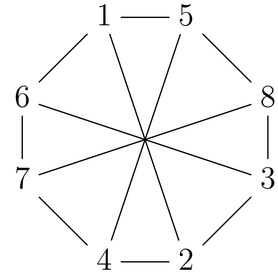


You get the point with these graphs, we are trying here to get in a sense which is opposite to that of Definition 3.1, with the cycles being not only welcome, but somehow mandatory. In general, the circulant graphs can be axiomatized as follows:

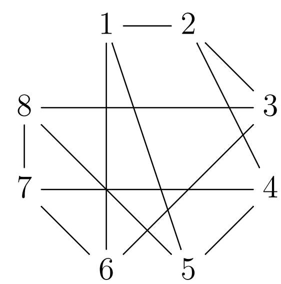
Definition 3.2. *A graph is called circulant if, when drawn with its vertices on a circle, equally spaced, it is invariant under rotations.*

To be more precise, this is the definition of the circulant graphs, when the vertices are labeled in advance $1, \ldots, N$. In general, when the vertices are not labeled in advance, the convention is that the graph is called circulant when it is possible to label the vertices $1, \ldots, N$, as for the graph to become circulant in the above sense.

In order to understand this, let us pick a graph which is obviously circulant, such as the wheel graph above, and mess up the labeling of the vertices, see what we get. For this purpose, let us put some random labels $1, 2, \ldots, 8$ on our wheel graph, say as follows:



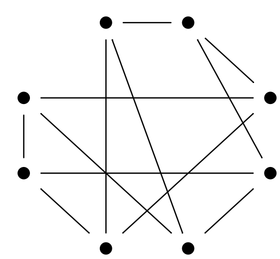
Now let us redraw this graph, with the vertices $1, 2, \ldots, 8$ ordered on a circle, equally spaced, as Definition 3.2 requires. We get something not very beautiful, as follows:



So, here is the point. This graph, regarded as a graph with vertices labeled $1, 2, \ldots, 8$ is obviously not circulant, in the sense of Definition 3.2. However, when removing the labels, this graph does become circulant, as per our conventions above.

All this might seem a bit confusing, when first seen, and you are probably in this situation, so here is a precise statement in this sense, coming with a full proof:

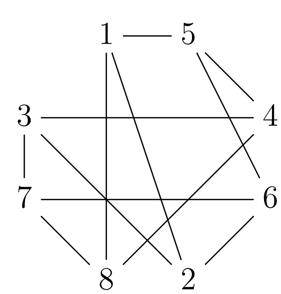
Proposition 3.3. *The following graph is circulant, despite its bad look,*



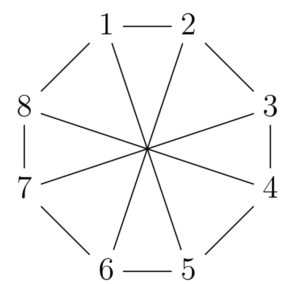
in the sense that it can be put in circulant form, with a suitable labeling of the vertices.

Proof. As already mentioned, this normally follows from the above discussion, but let us prove this as well directly. The idea is as follows:

(1) Let us label the vertices of our graph as follows, and I will explain in moment where this tricky labeling choice comes from:



Now let us redraw this graph, with the vertices $1,2,\ldots,8$ ordered on a circle, equally spaced, as Definition 3.2 requires. We get something very nice, as follows:



Thus, our original graph was indeed circulant, as stated.

(2) In order for everything to be fully clarified, we still must explain where the tricky labeling choice in (1) comes from. For this purpose, let us recall where the graph in the statement came from. Well, this graph was obtained by messing up the labeling of the vertices of the wheel graph, by using the following permutation:

$$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&8&3&2&4&7&6\end{pmatrix}$$

The point now is that, if we want to unmess our graph, we must use the inverse of the above permutation, obtained by reading things upside-down, which is given by:

$$\sigma^{-1}=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&4&6&2&8&7&3\end{pmatrix}$$

Thus, we must label our vertices $1,5,4,6,2,8,7,3$, precisely as done in (1). $\square$

As a conclusion to all this, deciding whether a graph is circulant or not is not an easy business. In what follows we will rather focus on the graphs which come by definition in circulant form, and leave decision problems for arbitrary graphs for later.

As basic examples now of circulant graphs, we have the triangle, the square, the pentagon, the hexagon and so on. As usual, let us record the formula of the adjacency matrix for such graphs. In the general case of the N-gon, this is as follows:

Proposition 3.4. *The adjacency matrix of the N-gon is*

$$d_{ij}=\delta_{|i-j|,1}$$

with the indices taken modulo N.

Proof. This is clear indeed from definitions, because with the indices taken modulo N, and arranged on a circle, being neighbor means $i=j\pm1$, and so $|i-j|=1$. $\square$

Now observe that the adjacency matrix computed above best looks when written in usual matrix form, with its circulant nature being quite obvious. As an illustration for this, here is the adjacency matrix of the hexagon, which is obviously circulant:

$$d=\begin{pmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{pmatrix}$$

Passed the N-gons, we can have more examples of circulant graphs by adding spokes to the N-gons. Here is a hexagonal wheel, that we already met in chapter 1:

$$d=\begin{pmatrix}0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\end{pmatrix}$$

Of course, not all circulant graphs appear in this way. As an example, here is another sort of "hexagonal wheel", without spokes, namely the Star of David:

$$d=\begin{pmatrix}0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\\0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\end{pmatrix}$$

But probably enough examples, you got the point, and time now to do some theory. Inspired by the above examples of explicit adjacency matrices, we have:

Proposition 3.5. *A graph is circulant precisely when its adjacency matrix is circulant, in the sense that it is of the following form, with the indices taken modulo N:*

$$d_{ij}=\gamma_{j-i}$$

In practice, and with indices $0,1,\ldots,N-1$, taken modulo N, this means that d consists of a row vector γ, sliding downwards and to the right, in the obvious way.

Proof. This is something quite obvious. Indeed, at $N = 4$ for instance, our assumption that X is circulant means that the adjacency matrix must look as follows:

$$d = \begin{pmatrix} x & y & z & t \\ t & x & y & z \\ z & t & x & y \\ y & z & t & x \end{pmatrix}$$

Now let us call γ the first row vector, (x, y, z, t). With matrix indices $0, 1, 2, 3$, taken modulo 4, as indicated in the statement, our matrix is then given by:

$$d = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 & \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_0 \end{pmatrix}$$

In the general case, $N \in \mathbb{N}$, the situation is similar, and this leads to the result. □

Observe that the above result is not the end of the story, because we still have to discuss when a circulant matrix is symmetric, and has 0 on the diagonal. But this is something easy to do, and our final result is then as follows:

Theorem 3.6. *The adjacency matrices of the circulant graphs are precisely the matrices of the following form, with indices* $0, 1, \ldots, N-1$*, taken modulo* N*,*

$$d_{ij} = \gamma_{j-i}$$

with the vector $\gamma \in (0,1)^N$ *being symmetric,* $\gamma_i = \gamma_{-i}$*, and with* $\gamma_0 = 0$*.*

Proof. This follows from our observations in Proposition 3.5, because:

(1) We know from there that we have $d_{ij} = \gamma_{j-i}$.

(2) The symmetry of d translates into the condition $\gamma_i = \gamma_{-i}$.

(3) The fact that d has 0 on the diagonal translates into the condition $\gamma_0 = 0$. □

All this is very nice, and with the circulant graphs being the same as the vectors $\gamma \in (0,1)^N$, subject to the simple assumptions above, this suggests that this might be the end of the story. Error. Recall that in the previous chapter we studied the N-simplex from a spectral point of view, and got into non-trivial things. But this N-simplex is, perhaps in rivalry with the N-gon, the simplest example of a circulant graph.

So, let us first recall our findings from chapter 2. We have seen there that the adjacency matrix of the simplex, with a copy of the identity added for simplifying things, diagonalizes as follows, with $F_N=(w^{ij})_{ij}$ with $w=e^{2\pi i/N}$ being the Fourier matrix:

$$\begin{pmatrix}1&\ldots&\ldots&1\\ \vdots&&&\vdots\\ \vdots&&&\vdots\\ 1&\ldots&\ldots&1\end{pmatrix}=\frac{1}{N}\,F_N\begin{pmatrix}N\\ &0\\ &&\ddots\\ &&&0\end{pmatrix}F_N^*$$

More generally now, going towards the case of the general circulant graphs, we have the following result, which is something standard in discrete Fourier analysis:

THEOREM 3.7. *For a matrix $A\in M_N(\mathbb C)$, the following are equivalent,*

(1) *A is circulant, $A_{ij}=\xi_{j-i}$, for a certain vector $\xi\in\mathbb C^N$,*
(2) *A is Fourier-diagonal, $A=F_NQF_N^*$, for a certain diagonal matrix Q,*

and if so, $\xi=F_N^*q$, where $q\in\mathbb C^N$ is the vector formed by the diagonal entries of Q.

PROOF. This follows from some basic computations with roots of unity, as follows:

(1) $\implies$ (2) Assuming $A_{ij}=\xi_{j-i}$, the matrix $Q=F_N^*AF_N$ is indeed diagonal, as shown by the following computation:

$$\begin{aligned}Q_{ij}&=\sum_{kl}w^{-ik}A_{kl}w^{lj}\\ &=\sum_{kl}w^{jl-ik}\xi_{l-k}\\ &=\sum_{kr}w^{j(k+r)-ik}\xi_r\\ &=\sum_r w^{jr}\xi_r\sum_k w^{(j-i)k}\\ &=N\delta_{ij}\sum_r w^{jr}\xi_r\end{aligned}$$

(2) $\implies$ (1) Assuming $Q=diag(q_1,\ldots,q_N)$, the matrix $A=F_NQF_N^*$ is indeed circulant, as shown by the following computation:

$$A_{ij}=\sum_k w^{ik}Q_{kk}w^{-jk}=\sum_k w^{(i-j)k}q_k$$

To be more precise, in this formula the last term depends only on $j-i$, and so shows that we have $A_{ij}=\xi_{j-i}$, with ξ being the following vector:

$$\xi_i=\sum_k w^{-ik}q_k=(F_N^*q)_i$$

Thus, we are led to the conclusions in the statement. $\square$

The above result is something quite powerful, and useful, and suggests doing everything in Fourier, when dealing with the circulant matrices. And we can use here:

THEOREM 3.8. *The various basic sets of $N\times N$ circulant matrices are as follows, with the convention that associated to any $q\in\mathbb C^N$ is the matrix $Q=diag(q_1,\ldots,q_N)$:*

(1) *The set of all circulant matrices is:*
$$M_N(\mathbb C)^{circ}=\left\{F_NQF_N^*\Big|q\in\mathbb C^N\right\}$$

(2) *The set of all circulant unitary matrices is:*
$$U_N^{circ}=\left\{\frac{1}{N}F_NQF_N^*\Big|q\in\mathbb T^N\right\}$$

(3) *The set of all circulant orthogonal matrices is:*
$$O_N^{circ}=\left\{\frac{1}{N}F_NQF_N^*\Big|q\in\mathbb T^N,\bar{q}_i=q_{-i},\forall i\right\}$$

In addition, in this picture, the first row vector of $F_NQF_N^*$ is given by $\xi=F_N^*q$.

PROOF. All this follows from Theorem 3.7, as follows:

(1) This assertion, along with the last one, is Theorem 3.7 itself.

(2) This is clear from (1), and from the fact that the rescaled matrix $F_N/\sqrt{N}$ is unitary, because the eigenvalues of a unitary matrix must be on the unit circle $\mathbb T$.

(3) This follows from (2), because the matrix is real when $\xi_i=\bar{\xi}_i$, and in Fourier transform, $\xi=F_N^*q$, this corresponds to the condition $\bar{q}_i=q_{-i}$. □

There are many other things that can be said about the circulant matrices, and all this is quite interesting, in relation with the circulant graphs. We will be back to this.

3b. Transitive graphs

We have seen so far that the circulant graphs are quite interesting objects. Our purpose now is to extend the theory that we have for them, to a more general class of graphs, the "transitive" ones. Let us start with a basic observation, as follows:

PROPOSITION 3.9. *A graph X, with vertices labeled $1,2,\ldots,N$, is circulant precisely when for any two vertices $i,j\in X$ there is a permutation $\sigma\in S_N$ such that:*

(1) *σ maps one vertex to another, $\sigma(i)=j$.*
(2) *σ is cyclic, $\sigma(k)=k+s$ modulo N, for some s.*
(3) *σ leaves invariant the edges, $k-l\iff\sigma(k)-\sigma(l)$.*

PROOF. This is obvious from definitions, and with the remark that the number s appearing in (2) is uniquely determined by (1), as being $s=j-i$, modulo N. □

The point now is that, with this picture of the circulant graphs in mind, it is quite clear that if we remove the assumption (2), that our permutation is cyclic, we will reach to a quite interesting class of graphs, generalizing them. So, let us formulate:

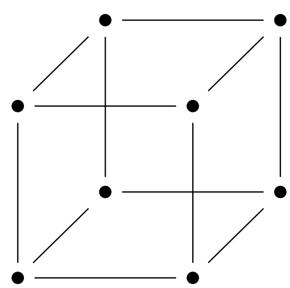
DEFINITION 3.10. *A graph X, with vertices labeled $1, 2, \ldots, N$, is called transitive when for any two vertices $i, j \in X$ there is a permutation $\sigma \in S_N$ such that:*

(1) *σ maps one vertex to another, $\sigma(i) = j$.*
(2) *σ leaves invariant the edges, $k - l \iff \sigma(k) - \sigma(l)$.*

In short, what we did here is to copy the statement of Proposition 3.9, with the assumption (2) there removed, and call this a Definition. In view of this, obviously, any circulant graph is transitive. But, do we have other interesting examples?

As a first piece of answer to this question, which is very encouraging, we have:

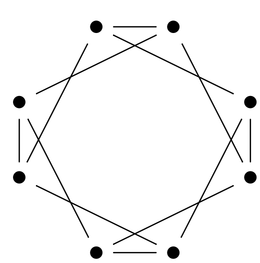
THEOREM 3.11. *The cube graph, namely*



is transitive, but not circulant.

PROOF. The fact that the cube is transitive is clear, because given any two vertices $i, j \in X$, we can certainly rotate the cube in 3D, as to have $i \to j$. As for the fact that the cube is not circulant, this is something more tricky, as follows:

(1) As a first observation, when trying to draw the cube on a circle, in a somewhat nice and intuitive way, as to have it circulant, we reach to the following picture:



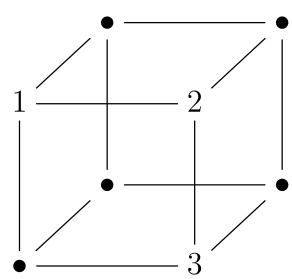
Thus, our cube is indeed not circulant, or at least not in an obvious way.

(2) However, this does not stand for a proof, and the problem of abstractly proving that the cube is not circulant remains. Normally this can be done by attempting to label

the vertices in a circulant way. Indeed, up to some discussion here, that we will leave as an instructive exercise, we can always assume that $1,2$ are connected by an edge:

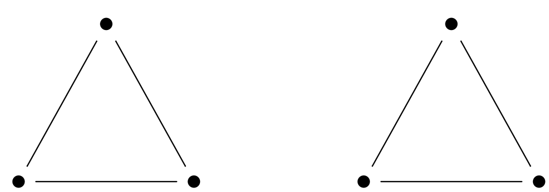
$$1-2$$

(3) But with this in hand, we can now start labeling the vertices of the cube, in a circulant way. Since $1-2$ implies via our circulant graph assumption $2-3$, $3-4$, and so on, in order to start our labeling, we must pick one vertex, and then follow a path on the cube, emanating from there. But, by some obvious symmetry reasons, this means that we can always assume that our first three vertices $1,2,3$ are as follows:

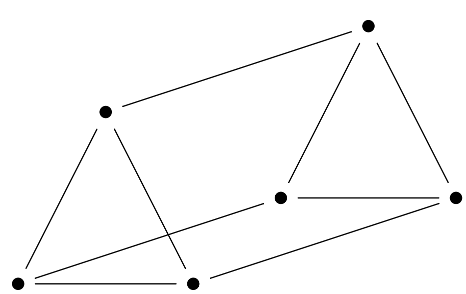


(4) So, the question comes now, where the vertex 4 can be, as for all this to lead, in the end, to a circulant graph. And the point is that, among the two possible choices for the vertex 4, as new neighbors of 3, none works. Thus, our cube is indeed not circulant, and we will leave the remaining details here as an instructive exercise. □

Let us look now for some more examples. Thinking a bit, it is in fact not necessary to go up to the cube, which is a rather advanced object, in order to have an example of a transitive, non-circulant graph, and this because two triangles will do too:

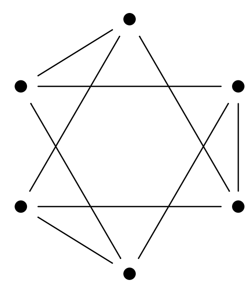


However, this latter graph is not connected, and so is not very good, as per our usual geometric philosophy. But, we can make it connected, by adding edges, as follows:



What we got here is a prism, and it is convenient now, for aesthetical and typographical reasons, to draw this prism on a circle, a bit like we did for the cube, at the beginning of the proof of Theorem 3.11. We are led in this way to the following statement:

THEOREM 3.12. *The prism, which is as follows when drawn on a circle,*



is transitive and non-circulant too, exactly as the cube was.

PROOF. The fact that the prism is indeed transitive follows from the above discussion, but it is convenient to view this as well directly on the above picture. Indeed:

– The prism as drawn above has 3 obvious symmetry axes, allowing us to do many of the $i \to j$ operations required by the definition of transitivity.

– In addition, the prism is invariant as well by the 120° and 240° rotations, and when combining this with the above 3 symmetries, we have all that we need.

Finally, the fact that the prism is indeed not circulant is quite clear, intuitively speaking, and this can be proved a bit as for the cube, as in the proof of Theorem 3.11. □

Summarizing, we have interesting examples, and our theory of transitive graphs seems worth developing. In order now to reach to something more conceptual, it is pretty much clear that we must get into group theory. So, let us formulate the following definition:

DEFINITION 3.13. *A group of permutations is a subset $G \subset S_N$ which is stable under the composition of permutations, and under their inversion. We say that:*

1. *G acts transitively on the set $\{1,\ldots,N\}$ if for any two points i,j we can find $\sigma \in G$ mapping one point to another, $\sigma(i)=j$.*
2. *G acts on a graph X with vertices labeled $1,\ldots,N$ when each $\sigma \in G$ leaves invariant the edges, $k-l \iff \sigma(k)-\sigma(l)$.*

Also, we say that G acts transitively on a graph X with vertices labeled $1,\ldots,N$ when it acts on X in the sense of (2), and the action is transitive in the sense of (1).

All this might seem a bit heavy, but as we will soon discover, is worth the effort, because group theory is a powerful theory, and having it into our picture will be certainly a good thing, a bit similar to the update from the Stone Age to the Bronze Age. Or perhaps to the update from the Bronze Age to the Iron Age, because what we did so far in this book was sometimes non-trivial, and can be counted as Bronze Age weaponry.

As a first good surprise, once Definition 3.13 formulated and digested, our definition of the circulant and transitive graphs becomes something very simple, as follows:

THEOREM 3.14. *The following happen, for a graph X having N vertices:*

(1) *X is circulant when we have an action $\mathbb{Z}_N \curvearrowright X$.*
(2) *X is transitive when we have a transitive action $G \curvearrowright X$.*

PROOF. This is something trivial and self-explanatory, and with the remark that in (1) we do not have to say something about transitivity, because the subgroup $\mathbb{Z}_N \subset S_N$ is transitive, in the sense of Definition 3.13. As usual, we have called this statement Theorem instead of Proposition simply due to its theoretical importance. □

As a second good surprise, our previous transitivity considerations regarding the cube and the prism take now a very simple form, in terms of groups, as follows:

PROPOSITION 3.15. *The following are transitive graphs:*

(1) *The cube, due to an action $\mathbb{Z}_2^3 \curvearrowright X$.*
(2) *The prism, due to an action $\mathbb{Z}_2 \times \mathbb{Z}_3 \curvearrowright X$.*

PROOF. As before with Theorem 3.14, this is trivial and self-explanatory, with the actions being the obvious ones, coming from our previous study of the cube and prism. □

Many things can be said about the transitive graphs, in general, but thinking well, what we would mostly like to have would be an extension of what we did in the previous section for the circulant graphs, including the Fourier transform material, which was something highly non-trivial and powerful, perhaps to a class of graphs smaller than that of the general transitive graphs. So, let us formulate the following definition:

DEFINITION 3.16. *A finite graph X is called generalized circulant when it has a transitive action $G \curvearrowright X$, with G being a finite abelian group.*

And this looks like a very good definition. Indeed, as examples we have the circulant graphs, but also the cube, and the prism, since products of abelian groups are obviously abelian. So, no interesting transitive graphs lost, when assuming that G is abelian.

In order now to further build on this definition, and in particular to develop our generalized Fourier transform machinery, as hoped in the above, let us temporarily leave aside the graphs X, and focus on the finite abelian groups G. We first have:

PROPOSITION 3.17. *Given a finite abelian group G, the group morphisms*

$$\chi : G \to \mathbb{T}$$

with $\mathbb{T}$ being the unit circle, called characters of G, form a finite abelian group $\widehat{G}$.

PROOF. There are several things to be proved here, the idea being as follows:

(1) Our first claim is that $\widehat{G}$ is a group, with the pointwise multiplication, namely:

$$(\chi\rho)(g) = \chi(g)\rho(g)$$

Indeed, if χ, ρ are characters, so is $\chi\rho$, and so the multiplication is well-defined on $\widehat{G}$. Regarding the unit, this is the trivial character, constructed as follows:

$$1 : G \to \mathbb{T} \quad , \quad g \to 1$$

Finally, we have inverses, with the inverse of $\chi : G \to \mathbb{T}$ being its conjugate:

$$\bar{\chi} : G \to \mathbb{T} \quad , \quad g \to \overline{\chi(g)}$$

(2) Our next claim is that $\widehat{G}$ is finite. Indeed, given a group element $g \in G$, we can talk about its order, which is smallest integer $k \in \mathbb{N}$ such that $g^k = 1$. Now assuming that we have a character $\chi : G \to \mathbb{T}$, we have the following formula:

$$\chi(g)^k = 1$$

Thus $\chi(g)$ must be one of the k-th roots of unity, and in particular there are finitely many choices for $\chi(g)$. Thus, there are finitely many choices for χ, as desired.

(3) Finally, the fact that $\widehat{G}$ is abelian follows from definitions, because the pointwise multiplication of functions, and in particular of characters, is commutative. □

The above construction is quite interesting, and we have:

THEOREM 3.18. *The character group operation $G \to \widehat{G}$ for the finite abelian groups, called Pontrjagin duality, has the following properties:*

1. *The dual of a cyclic group is the group itself, $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$.*
2. *The dual of a product is the product of duals, $\widehat{G \times H} = \widehat{G} \times \widehat{H}$.*
3. *Any product of cyclic groups $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ is self-dual, $G = \widehat{G}$.*

PROOF. We have several things to be proved, the idea being as follows:

(1) A character $\chi : \mathbb{Z}_N \to \mathbb{T}$ is uniquely determined by its value $z = \chi(g)$ on the standard generator $g \in \mathbb{Z}_N$. But this value must satisfy:

$$z^N = 1$$

Thus we must have $z \in \mathbb{Z}_N$, with the cyclic group $\mathbb{Z}_N$ being regarded this time as being the group of N-th roots of unity. Now conversely, any N-th root of unity $z \in \mathbb{Z}_N$ defines a character $\chi : \mathbb{Z}_N \to \mathbb{T}$, by setting, for any $r \in \mathbb{N}$:

$$\chi(g^r) = z^r$$

Thus we have an identification $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$, as claimed.

(2) A character of a product of groups $\chi : G \times H \to \mathbb{T}$ must satisfy:

$$\chi(g,h) = \chi\left[(g,1)(1,h)\right] = \chi(g,1)\chi(1,h)$$

Thus χ must appear as the product of its restrictions $\chi_{|G}, \chi_{|H}$, which must be both characters, and this gives the identification in the statement.

(3) This follows from (1) and (2). Alternatively, any character $\chi : G \to \mathbb{T}$ is uniquely determined by its values $\chi(g_1), \ldots, \chi(g_k)$ on the standard generators of $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k}$, which must belong to $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k} \subset \mathbb{T}$, and this gives $\widehat{G} = G$, as claimed. $\square$

At a more advanced level now, we have the following result:

THEOREM 3.19. *The finite abelian groups are the following groups,*

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

and these groups are all self-dual, $G = \widehat{G}$.

PROOF. This is something quite tricky, the idea being as follows:

(1) In order to prove our result, assume that G is finite and abelian. For any prime number $p \in \mathbb{N}$, let us define $G_p \subset G$ to be the subset of elements having as order a power of p. Equivalently, this subset $G_p \subset G$ can be defined as follows:

$$G_p = \left\{ g \in G \Big| \exists k \in \mathbb{N}, g^{p^k} = 1 \right\}$$

(2) It is then routine to check, based on definitions, that each G_p is a subgroup. Our claim now is that we have a direct product decomposition as follows:

$$G = \prod_p G_p$$

(3) Indeed, by using the fact that our group G is abelian, we have a morphism as follows, with the order of the factors when computing $\prod_p g_p$ being irrelevant:

$$\prod_p G_p \to G \quad , \quad (g_p) \to \prod_p g_p$$

Moreover, it is routine to check that this morphism is both injective and surjective, via some simple manipulations, so we have our group decomposition, as in (2).

(4) Thus, we are left with proving that each component G_p decomposes as a product of cyclic groups, having as orders powers of p, as follows:

$$G_p = \mathbb{Z}_{p^{r_1}} \times \ldots \times \mathbb{Z}_{p^{r_s}}$$

But this is something that can be checked by recurrence on $|G_p|$, via some routine computations, and we are led to the conclusion in the statement.

(5) Finally, the fact that the finite abelian groups are self-dual, $G = \widehat{G}$, follows from the structure result that we just proved, and from Theorem 3.18 (3). $\square$

In relation now with Fourier analysis, the result is as follows:

THEOREM 3.20. *Given a finite abelian group G, we have an isomorphism as follows, obtained by linearizing/delinearizing the characters,*

$$C^*(G)\simeq C(\widehat{G})$$

where $C^*(G)$ is the algebra of functions $\varphi:G\to\mathbb{C}$, with convolution product, and $C(\widehat{G})$ is the algebra of functions $\varphi:\widehat{G}\to\mathbb{C}$, with usual product.

PROOF. There are many things going on here, the idea being as follows:

(1) Given a finite abelian group G, we can talk about the complex algebra $C(G)$ formed by the complex functions $\varphi:G\to\mathbb{C}$, with usual product, namely:

$$(\varphi\psi)(g)=\varphi(g)\psi(g)$$

Observe that we have $C(G)\simeq\mathbb{C}^N$ as an algebra, where $N=|G|$, with this being best seen via the basis of $C(G)$ formed by the Dirac masses at the points of G:

$$C(G)=\left\{\sum_{g\in G}\lambda_g\delta_g\Big|\lambda_g\in\mathbb{C}\right\}$$

(2) On the other hand, we can talk as well about the algebra $C^*(G)$ formed by the same functions $\varphi:G\to\mathbb{C}$, but this time with the convolution product, namely:

$$(\varphi*\psi)(g)=\sum_{h\in G}\varphi(gh^{-1})\psi(h)$$

Since we have $\delta_k*\delta_l=\delta_{kl}$ for any $k,l\in G$, as you can easily check by using the above formula, the Dirac masses $\delta_g\in C^*(G)$ behave like the group elements $g\in G$. Thus, we can view our algebra as follows, with multiplication given by $g\cdot h=gh$, and linearity:

$$C^*(G)=\left\{\sum_{g\in G}\lambda_g g\Big|\lambda_g\in\mathbb{C}\right\}$$

(3) Now that we know what the statement is about, let us go for the proof. The first observation is that we have a morphism of algebras as follows:

$$C^*(G)\to C(\widehat{G})\quad,\quad g\to[\chi\to\chi(g)]$$

Now since on both sides we have vector spaces of dimension $N=|G|$, it is enough to check that this morphism is injective. But this is best done via Theorem 3.19, which shows that the characters $\chi\in\widehat{G}$ separate the points $g\in G$, as desired. $\square$

In practice now, we can clearly feel that Theorem 3.20 is related to Fourier analysis, and more specifically to the Fourier transforms and series that we know from analysis, but also to the discrete Fourier transform from the beginning of this chapter. However, all this remains a bit difficult to clarify, and we have here the following statement:

Fact 3.21. *The following happen, regarding the locally compact abelian groups:*

(1) *What we did in the finite case, namely group characters, and construction and basic properties of the dual, can be extended to them.*
(2) *As basic examples of this, besides what we have in the finite case, and notably $\widehat{\mathbb{Z}}_N=\mathbb{Z}_N$, we have $\widehat{\mathbb{Z}}=\mathbb{T}$, $\widehat{\mathbb{T}}=\mathbb{Z}$, and also $\widehat{\mathbb{R}}=\mathbb{R}$.*
(3) *With some care for analytic aspects, $C^*(G)\simeq C(\widehat{G})$ remains true in this setting, and in the case $G=\mathbb{R}$, this isomorphism is the Fourier transform.*

Obviously, all this is a bit heavy, but you get the point, there are 3 types of Fourier analysis in life, namely the "standard" one, that you might know from advanced calculus, corresponding to $G=\mathbb{R}$, then the "Fourier series" one, that you might know from advanced calculus too, corresponding to $G=\mathbb{Z},\mathbb{T}$, and finally the "discrete" one that we started to learn in this book, over $G=\mathbb{Z}_N$ and other finite abelian groups.

In practice, all this is a bit complicated, and back now to the finite abelian groups, let us work out a softer version of all the above, which is what is really needed, in practice, when doing discrete Fourier analysis. We have here the following result:

Theorem 3.22. *Given a finite abelian group G, with dual group $\widehat{G}=\{\chi:G\to\mathbb{T}\}$, consider the corresponding Fourier coupling, namely:*

$$\mathcal{F}_G:G\times\widehat{G}\to\mathbb{T}\quad,\quad(i,\chi)\to\chi(i)$$

(1) *Via the standard isomorphism $G\simeq\widehat{G}$, this Fourier coupling can be regarded as a usual square matrix, $F_G\in M_G(\mathbb{T})$.*
(2) *This matrix $F_G\in M_G(\mathbb{T})$ is complex Hadamard, in the sense that its entries are on the unit circle, and its rows are pairwise orthogonal.*
(3) *In the case of the cyclic group $G=\mathbb{Z}_N$ we obtain in this way, via the standard identification $\mathbb{Z}_N=\{1,\ldots,N\}$, the Fourier matrix F_N.*
(4) *In general, when using a decomposition $G=\mathbb{Z}_{N_1}\times\ldots\times\mathbb{Z}_{N_k}$, the corresponding Fourier matrix is given by $F_G=F_{N_1}\otimes\ldots\otimes F_{N_k}$.*

Proof. This follows indeed by using the above finite abelian group theory:

(1) Via the identification $G\simeq\widehat{G}$, we have indeed a square matrix, given by:

$$(F_G)_{i\chi}=\chi(i)$$

(2) The scalar products between distinct rows are indeed zero, as shown by:

$$<R_i,R_j>=\sum_\chi\chi(i)\overline{\chi(j)}=\sum_\chi\chi(i-j)=|G|\cdot\delta_{ij}$$

(3) This follows from the well-known and elementary fact that, via the identifications $\mathbb{Z}_N=\widehat{\mathbb{Z}_N}=\{1,\ldots,N\}$, the Fourier coupling here is as follows, with $w=e^{2\pi i/N}$:

$$(i,j)\to w^{ij}$$

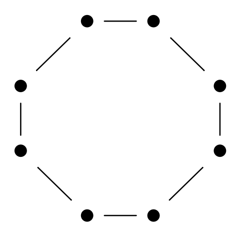
(4) Observe first that $\widehat{H \times K} = \widehat{H} \times \widehat{K}$ gives, at the level of the Fourier couplings, $F_{H\times K} = F_H \otimes F_K$. Now by decomposing G into cyclic groups, as in the statement, and using (3) for the cyclic components, we obtain the formula in the statement. □

So long for circulant graphs, finite group actions, transitive graphs, finite abelian groups, discrete Fourier analysis, and Hadamard matrices. We will be back to all this in Part II of the present book, where we will systematically investigate such things.

3c. Dihedral groups

As something more concrete now, which is a must-know, let us try to compute the dihedral group. This is a famous group, constructed as follows:

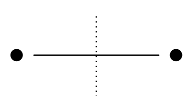
Definition 3.23. *The dihedral group D_N is the symmetry group of*



that is, of the regular polygon having N vertices.

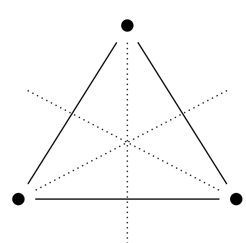
In order to understand how this works, here are the basic examples of regular N-gons, at small values of the parameter $N \in \mathbb{N}$, along with their symmetry groups:

<u>$N = 2$</u>. Here the N-gon is just a segment, and its symmetries are obviously the identity id, plus the symmetry τ with respect to the middle of the segment:



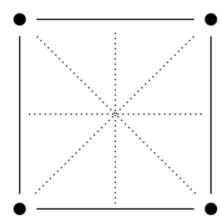
Thus we have $D_2 = \{id, \tau\}$, which in group theory terms means $D_2 = \mathbb{Z}_2$.

<u>$N = 3$</u>. Here the N-gon is an equilateral triangle, and we have 6 symmetries, the rotations of angles 0°, 120°, 240°, and the symmetries with respect to the altitudes:



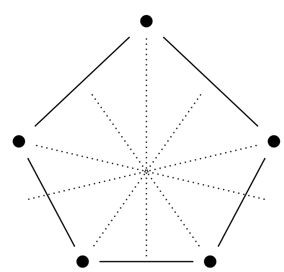
Alternatively, we can say that the symmetries are all the $3! = 6$ possible permutations of the vertices, and so that in group theory terms, we have $D_3 = S_3$.

<u>$N = 4$</u>. Here the N-gon is a square, and as symmetries we have 4 rotations, of angles $0°, 90°, 180°, 270°$, as well as 4 symmetries, with respect to the 4 symmetry axes, which are the 2 diagonals, and the 2 segments joining the midpoints of opposite sides:

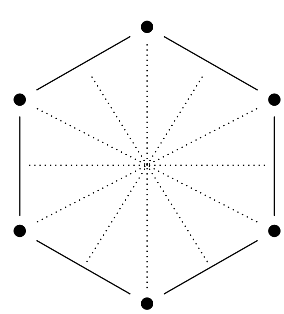


Thus, we obtain as symmetry group some sort of product between $\mathbb{Z}_4$ and $\mathbb{Z}_2$. Observe however that this product is not the usual one, our group being not abelian.

<u>$N = 5$</u>. Here the N-gon is a regular pentagon, and as symmetries we have 5 rotations, of angles $0°, 72°, 144°, 216°, 288°$, as well as 5 symmetries, with respect to the 5 symmetry axes, which join the vertices to the midpoints of the opposite sides:



<u>$N = 6$</u>. Here the N-gon is a regular hexagon, and we have 6 rotations, of angles $0°, 60°, 120°, 180°, 240°, 300°$, and 6 symmetries, with respect to the 6 symmetry axes, which are the 3 diagonals, and the 3 segments joining the midpoints of opposite sides:



We can see from the above that the various dihedral groups D_N have many common features, and that there are some differences as well. In general, we have:

Proposition 3.24. *The dihedral group D_N has $2N$ elements, as follows:*

(1) *We have N rotations $R_1,\ldots,R_N$, with R_k being the rotation of angle $2k\pi/N$. When labelling the vertices $1,\ldots,N$, the formula is $R_k:i\to k+i$.*
(2) *We have N symmetries $S_1,\ldots,S_N$, with S_k being the symmetry with respect to the Ox axis rotated by $k\pi/N$. The symmetry formula is $S_k:i\to k-i$.*

Proof. Our group is indeed formed of N rotations, of angles $2k\pi/N$ with $k=1,\ldots,N$, and then of the N symmetries with respect to the N possible symmetry axes, which are the N medians of the N-gon when N is odd, and are the $N/2$ diagonals plus the $N/2$ lines connecting the midpoints of opposite edges, when N is even. $\square$

With the above result in hand, we can talk about D_N abstractly, as follows:

Theorem 3.25. *The dihedral group D_N is the group having $2N$ elements, $R_1,\ldots,R_N$ and $S_1,\ldots,S_N$, called rotations and symmetries, which multiply as follows,*

$$R_kR_l=R_{k+l}\quad,\quad R_kS_l=S_{k+l}$$
$$S_kR_l=S_{k-l}\quad,\quad S_kS_l=R_{k-l}$$

with all the indices being taken modulo N.

Proof. With notations from Proposition 3.24, the various compositions between rotations and symmetries can be computed as follows:

$$R_kR_l\ :\ i\to l+i\to k+l+i\quad,\quad R_kS_l\ :\ i\to l-i\to k+l-i$$
$$S_kR_l\ :\ i\to l+i\to k-l-i\quad,\quad S_kS_l\ :\ i\to l-i\to k-l+i$$

But these are exactly the formulae for $R_{k+l},S_{k+l},S_{k-l},R_{k-l}$, as stated. Now since a group is uniquely determined by its multiplication rules, this gives the result. $\square$

Observe that D_N has the same cardinality as $E_N=\mathbb{Z}_N\times\mathbb{Z}_2$. We obviously don't have $D_N\simeq E_N$, because D_N is not abelian, while E_N is. So, our next goal will be that of proving that D_N appears by "twisting" E_N. In order to do this, let us start with:

Proposition 3.26. *The group $E_N=\mathbb{Z}_N\times\mathbb{Z}_2$ is the group having $2N$ elements, $r_1,\ldots,r_N$ and $s_1,\ldots,s_N$, which multiply according to the following rules,*

$$r_kr_l=r_{k+l}\quad,\quad r_ks_l=s_{k+l}$$
$$s_kr_l=s_{k+l}\quad,\quad s_ks_l=r_{k+l}$$

with all the indices being taken modulo N.

Proof. With the notation $\mathbb{Z}_2=\{1,\tau\}$, the elements of the product group $E_N=\mathbb{Z}_N\times\mathbb{Z}_2$ can be labelled $r_1,\ldots,r_N$ and $s_1,\ldots,s_N$, as follows:

$$r_k=(k,1)\quad,\quad s_k=(k,\tau)$$

These elements multiply then according to the formulae in the statement. Now since a group is uniquely determined by its multiplication rules, this gives the result. $\square$

Let us compare now Theorem 3.25 and Proposition 3.26. In order to formally obtain D_N from E_N, we must twist some of the multiplication rules of E_N, namely:

$$s_k r_l = s_{k+l} \to s_{k-l} \quad , \quad s_k s_l = r_{k+l} \to r_{k-l}$$

Informally, this amounts in following the rule "τ switches the sign of what comes afterwards", and we are led in this way to the following definition:

Definition 3.27. *Given two groups A, G, with an action $A \curvearrowright G$, the crossed product*

$$P = G \rtimes A$$

is the set $G \times A$, with multiplication $(g,a)(h,b) = (gh^a, ab)$.

Now with this technology in hand, by getting back to the dihedral group D_N, we can improve Theorem 3.25, into a final result on the subject, as follows:

Theorem 3.28. *We have a crossed product decomposition as follows,*

$$D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$$

with $\mathbb{Z}_2 = \{1, \tau\}$ acting on $\mathbb{Z}_N$ via switching signs, $k^\tau = -k$.

Proof. We have an action $\mathbb{Z}_2 \curvearrowright \mathbb{Z}_N$ given by the formula in the statement, namely $k^\tau = -k$, so we can consider the corresponding crossed product group:

$$P_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$$

In order to understand the structure of P_N, we follow Proposition 3.26. The elements of P_N can indeed be labelled $\rho_1, \ldots, \rho_N$ and $\sigma_1, \ldots, \sigma_N$, as follows:

$$\rho_k = (k, 1) \quad , \quad \sigma_k = (k, \tau)$$

Now when computing the products of such elements, we basically obtain the formulae in Proposition 3.26, perturbed as in Definition 3.27. To be more precise, we have:

$$\rho_k \rho_l = \rho_{k+l} \quad , \quad \rho_k \sigma_l = \sigma_{k+l}$$

$$\sigma_k \rho_l = \sigma_{k+l} \quad , \quad \sigma_k \sigma_l = \rho_{k+l}$$

But these are exactly the multiplication formulae for D_N, from Theorem 3.25. Thus, we have an isomorphism $D_N \simeq P_N$ given by $R_k \to \rho_k$ and $S_k \to \sigma_k$, as desired. $\square$

3d. Cayley graphs

We have kept the best for the end. We have the following notion, that we already met in a vague form in chapter 1, and which is something quite far-reaching:

Definition 3.29. *Associated to any finite group $G = < S >$, with the generating set S assumed to satisfy $1 \notin S = S^{-1}$, is its Cayley graph, constructed as follows:*

1. *The vertices are the group elements $g \in G$.*
2. *Edges $g - h$ are drawn when $h = gs$, with $s \in S$.*

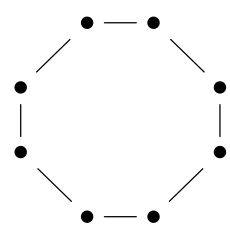
As a first observation, the Cayley graph is indeed a graph, because our assumption $S = S^{-1}$ on the generating set shows that we have $g - h \implies h - g$, as we should, and also because our assumption $1 \notin S$ excludes the self-edges, $g \not- g$.

We will see in what follows that most of the graphs that we met so far are Cayley graphs, and in addition, with G being abelian in most cases. Before starting with the examples, however, let us point out that the Cayley graph depends, in a crucial way, on the generating set S. Indeed, if we choose for instance $S = G - \{1\}$, we obtain the complete graph with $N = |G|$ vertices, and this regardless of what our group G is.

Thus, the Cayley graph as constructed above is not exactly associated to the group G, but rather to the group G viewed as finitely generated group, $G =< S >$.

In order to construct now examples, let us start with the simplest finite groups that we know. The simplest such group is the cyclic group $\mathbb{Z}_N$, and we have:

Proposition 3.30. *The N-gon graph, namely*

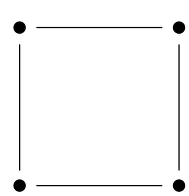


is the Cayley graph of $\mathbb{Z}_N =< \pm 1 >$, with $\mathbb{Z}_N$ being written here additively.

Proof. This is clear, because in additive notation, our condition for the edges $g - h$ reads $g = h \pm 1$, so we are led to the N-gon graph in the statement. □

The next thing that we can do is to look at the products of cyclic groups. In the simplest case here, that of a product of the group $\mathbb{Z}_2$ with itself, we obtain:

Theorem 3.31. *The square graph, namely*



is the Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$, with generating set $S = \{(1,0),(0,1)\}$.

PROOF. This is something which requires a bit of thinking. We must first draw the 4 elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and this is best done by using their coordinates, as follows:

$$\begin{matrix} 10 & & 11 \\ & & \\ & & \\ 00 & & 10 \end{matrix}$$

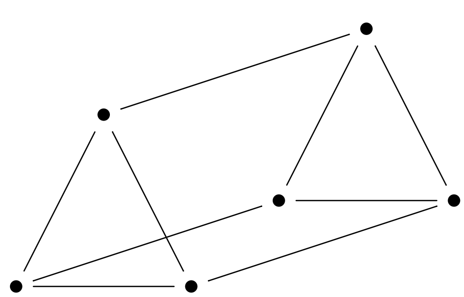
Now let us construct the edges. In additive notation, our condition for the edges $(a,b) - (c,d)$ reads either $(a,b) = (c,d) + (1,0)$ or $(a,b) = (c,d) + (0,1)$, which amounts in saying that the passage $(a,b) \to (c,d)$ appears by modifying one coordinate, and keeping the other coordinate fixed. We conclude from this that the edges are as follows:

$$\begin{tikzcd} 10 \arrow[r, dash] \arrow[d, dash] & 11 \arrow[d, dash] \\ 00 \arrow[r, dash] & 10 \end{tikzcd}$$

Now by removing the vertex labels, we obtain the usual square, as claimed. $\square$

What comes next? Obviously, more complicated products of cyclic groups. Here the situation ramifies a bit, and as our next basic example, we have:

PROPOSITION 3.32. *The prism graph, namely*

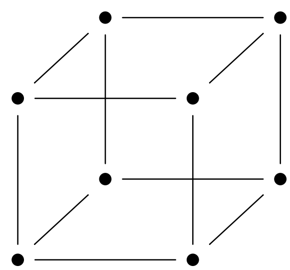


appears as Cayley graph of the group $\mathbb{Z}_2 \times \mathbb{Z}_3$.

PROOF. This follows a bit as before, for the square, and we will leave the details here, including of course finding the correct generating set S, as an instructive exercise. $\square$

As a main example now, obtained by staying with $\mathbb{Z}_2$, we have:

THEOREM 3.33. *The cube graph, namely*



is the Cayley graph of $\mathbb{Z}_2^3$, with generating set $S=\{(1,0,0),(0,1,0),(0,0,1)\}$.

PROOF. This is clear, as before for the square. In fact, we have already met this in chapter 1, with details, and this is our come-back to the subject, long overdue. $\square$

Looking at what we have so far, it is pretty much clear that Theorem 3.31 and Theorem 3.33 are of the same nature. The joint extension of these theorems, along with the computation of the corresponding symmetry group, leads to the following statement:

THEOREM 3.34. *The Cayley graph of the group $\mathbb{Z}_2^N$ is the hypercube*

$$\square_N\subset\mathbb{R}^N$$

and the symmetry group of $\square_N$, called hyperoctahedral group, is given by

$$H_N=\mathbb{Z}_2^N\rtimes S_N$$

with the permutations acting via $\sigma(e_1,\ldots,e_k)=(e_{\sigma(1)},\ldots,e_{\sigma(k)})$.

PROOF. Regarding the Cayley graph assertion, this follows as for the square, or for the cube. As for the computation of H_N, this can be done as follows:

(1) Consider the standard cube in $\mathbb{R}^N$, centered at 0, and having as vertices the points having coordinates ± 1. With this picture in hand, it is clear that the symmetries of the cube coincide with the symmetries of the N coordinate axes of $\mathbb{R}^N$.

(2) In order to count now these symmetries, observe first that we have as examples the $N!$ permutations of the N coordinate axes of $\mathbb{R}^N$. But each of these permutations $\sigma\in S_N$ can be further "decorated" by a sign vector $e\in\{\pm 1\}^N$, consisting of the possible ± 1 flips which can be applied to each coordinate axis, at the arrival. Thus, we have:

$$|H_N|=|S_N|\cdot|\mathbb{Z}_2^N|=N!\cdot 2^N$$

(3) Now observe that at the level of the cardinalities, the above formula gives:

$$|H_N|=|\mathbb{Z}_2^N\times S_N|$$

To be more precise, given an element $g\in H_N$, the element $\sigma\in S_N$ is the corresponding permutation of the N coordinate axes, regarded as unoriented lines in $\mathbb{R}^N$, and $e\in\mathbb{Z}_2^N$ is the vector collecting the possible flips of these coordinate axes, at the arrival. Now observe that the product formula for two such pairs $g = (e, \sigma)$ is as follows, with the permutations $\sigma \in S_N$ acting on the elements $f \in \mathbb{Z}_2^N$ as in the statement:

$$(e, \sigma)(f, \tau) = (ef^\sigma, \sigma\tau)$$

(4) Thus, we are precisely in the framework of Definition 3.27, and we conclude that we have a crossed product decomposition $H_N = \mathbb{Z}_2^N \rtimes S_N$, as in the statement. □

Many more things can be said, and more on all this in chapter 5 and afterwards.

3e. Exercises

We had a rather easy and relaxing chapter here, and as exercises, we have:

Exercise 3.35. *How to decide if an unlabeled graph is circulant or not?*

Exercise 3.36. *Learn more about permutation groups, their orbits and orbitals.*

Exercise 3.37. *Learn about Cayley embeddings $G \subset S_{|G|}$, and about $S_N \subset O_N$ too.*

Exercise 3.38. *Work out all details for the result on the finite abelian groups.*

Exercise 3.39. *Develop some theory for the generalized circulant graphs.*

Exercise 3.40. *Write down all possible decompositions of the group D_4.*

Exercise 3.41. *Compute the Cayley graphs of some non-abelian groups.*

Exercise 3.42. *If you teach, ask your students to compute $|H_3|$.*

As bonus exercise, learn some group theory. We will need this, for Part II.

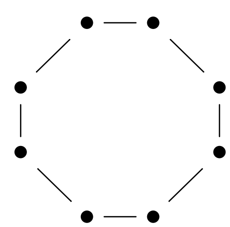
CHAPTER 4

Planar graphs

4a. Trees, counting

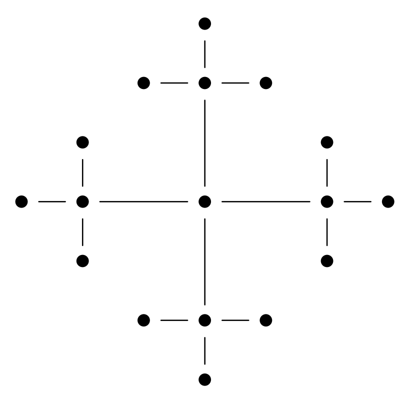
Trees, eventually. We have kept this topic for the end of this introductory Part I, because it is in fact quite tricky. As starting point, we have the following definition:

Definition 4.1. *A tree is a connected graph with no cycles. That is, there is no loop*



having length ≥ 3, and distinct vertices, inside the graph.

As a basic example here, which is quite illustrating for how trees look, in general, we have the following beautiful graph, that we already met in chapter 3:



As already mentioned in chapter 3, such a graph is called indeed a tree, because when looking at a tree from the above, what you see is something like this.

Getting to work now, we can see right away, from Definition 4.1, where the difficulty in dealing with trees comes from. Indeed, with respect to what we did so far in this book, problems investigated, and solutions found for them, the situation is as follows:

(1) Most of our results so far were based on the correspondence between the graphs X and their adjacency matrices d, which is something well understood, in general. However, for trees this correspondence is something quite tricky, because the loops of length k are counted by the diagonal entries of d^k, and the condition that these entries must come only from trivial loops is obviously something complicated, that we cannot control well.

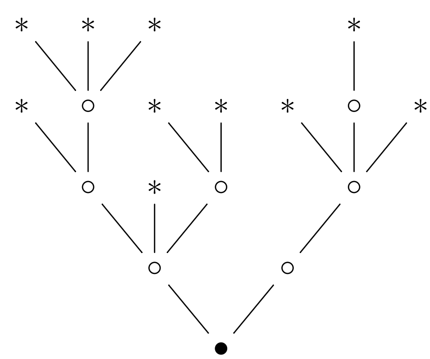
(2) Another thing that we did quite a lot, in the previous chapters, which was again of rather analytic type, in relation with the associated adjacency matrix d, was that of counting walks on our graphs. But here, for trees, things are again a bit complicated, as we have seen in chapter 1 for the simplest tree, namely $\mathbb{Z}$, and then in chapter 2 for the next simplest tree, namely $\mathbb{N}$. So, again, we are led into difficult questions here.

In view of this, we will take a different approach to the trees, by inventing and developing first the mathematics which is best adapted to them. As a first observation, both $\mathbb{N}$, $\mathbb{Z}$ and the snowflake graph above are special kinds of trees, having a root, like normal, real-life trees do. It is possible to talk about rooted trees in general, and also about leaves, cherries, and so on, and do many other tree-inspired things. Let us record some:

Definition 4.2. *In relation with our notion of tree:*

1. *We call rooted tree a tree with a distinguished vertex.*
2. *We call leaf of a tree a vertex having exactly* 1 *neighbor.*
3. *With the convention that when the tree is rooted, the root is not a leaf.*

In practice, it is often convenient to choose a root, and draw the tree oriented upwards, as the usual trees grow. Here is an example, with the root and leaves highlighted:



Along the same lines, simple facts that can be said about trees, we have as well:

Proposition 4.3. *A tree has the following properties:*

1. *It is naturally a bipartite graph.*
2. *The number of edges is the number of vertices minus* 1.
3. *Each two vertices are connected by a unique minimal path.*

Proof. All this is elementary, and is best seen by choosing a root, and then drawing the tree oriented upwards, as above, the idea being as follows:

(1) We can declare the root, lying on the ground 0, to be of type "even", then all neighbors of the root, lying at height 1, to be of type "odd", then all neighbors of these neighbors, at height 2, to be of type "even", and so on. Thus, our tree is bipartite.

(2) This is again clear by choosing a root, and then drawing our tree oriented upwards, as above. Indeed, in this picture any vertex, except for the root, comes from the edge below it, and this correspondence between non-root vertices and edges is bijective.

(3) Indeed, in our picture, traveling from one vertex to another, via the shortest path, is done uniquely, by going down as long as needed, and then up, in the obvious way. □

Getting started for good now, let us look at the snowflake graph pictured before, with the convention that the graph goes on, in a 4-valent way, in each possible direction. So, where does this graph come from? A bit of thinking leads to the following answer:

Theorem 4.4. *Consider the free group F_N, given by definition by the following formula, with $\emptyset$ standing for the lack of relations between generators:*

$$F_N = \left\langle g_1, \ldots, g_N \,\middle|\, \emptyset \right\rangle$$

The Cayley graph of F_N, with respect to the generating set $S = \{g_i, g_i^{-1}\}$, is then the infinite rooted tree having valence $2N$ everywhere.

Proof. This is something quite straightforward, the idea being as follows:

(1) At $N = 1$ our free group is by definition $F_1 =< g\,|\,\emptyset >$, which means group generated by a variable g without relations of type $g^s = 1$, between this variable and itself, preventing our group to be cyclic, and so means $F_1 = \mathbb{Z}$. Thus the Cayley graph is the usual $\mathbb{Z}$ graph, that we know well, with edges drawn between the neighbors:

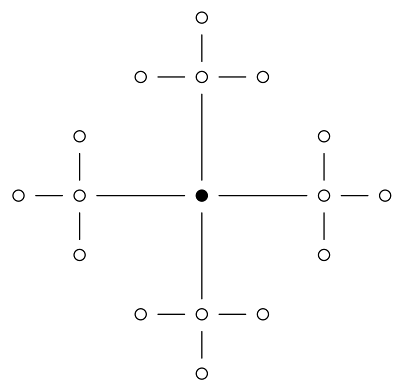
$$\cdots - \circ - \circ - \circ - \bullet - \circ - \circ - \circ - \cdots$$

Here we have used, as before, a solid circle for the root. Now this being obviously the unique rooted tree having valence 2 everywhere, we are done with the $N = 1$ case.

(2) At $N = 2$ now, our free group is by definition $F_2 =< g, h\,|\,\emptyset >$, which means group generated by two variables g, h, without relations between them. This is of course something a bit abstract, but algebrically speaking, this is in fact something very simple, because we can list the group elements, say according to their length, as follows:

$$\begin{gathered} 1 \\ g, h, g^{-1}, h^{-1} \\ g^2, gh, gh^{-1}, hg, h^2, hg^{-1}, g^{-1}h, g^{-2}, g^{-1}h^{-1}, h^{-1}g, h^{-1}g^{-1}, h^{-2} \\ \vdots \end{gathered}$$

Observe that we have one element of length 0, namely the unit 1, then 4 elements of length 1, then $16 - 4 = 12$ elements of length 2, then $64 - 16 = 48$ elements of length 3, and so on. Now when drawing the Cayley graph, the picture is as follows, with the convention that the graph goes on, in a 4-valent way, in each possible direction:



This graph being obviously the unique infinite rooted tree having valence 4 everywhere, we are done with the $N = 2$ case too.

(3) At an arbitrary $N \in \mathbb{N}$ now, the situation is pretty much similar to what we have seen in the above at $N = 1, 2$, and this leads to the conclusion in the statement. $\square$

Many other things can be said about free groups, and other free products of groups, and their Cayley graphs. We will be back to these topics on several occasions in what follows, later in this chapter, with a key occurrence of F_N in topology, and then on a more systematic basis when doing groups, starting from chapter 5 below.

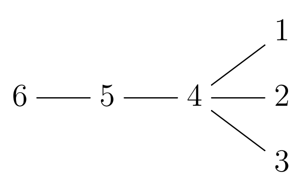
Moving ahead, let us discuss now counting questions for trees. We first have:

THEOREM 4.5. *Each tree with vertices labeled* $1, \ldots, N$ *can be encoded by its Prüfer sequence,* $(a_1, \ldots, a_{N-2})$ *with* $a_i \in \{1, \ldots, N\}$*, constructed as follows,*

(1) *at step* i*, remove the leaf with the smallest label,*
(2) *add the label of that leaf's neighbor to your list,*
(3) *and stop when you have only* 2 *vertices left,*

with this correspondence being bijective, between trees and lists as above.

PROOF. This is something quite self-explanatory, and as an illustration for how the Prüfer encoding works, consider for instance the following tree:



– Following the algorithm, we first have to remove 1, and put 4 on our list.

– Then we have to remove 2, and then 3 too, again by putting 4 on our list, twice.

– Then, when left with $4,5,6$, we have to remove 4, and put 5 on our list.

– And with this we are done, because we have only 2 vertices left, so we stop.

As a conclusion to this, for the above tree the Prüfer sequence is as follows:

$$p=(4,4,4,5)$$

So, this is how the Prüfer encoding works, and in what regards the last assertion, bijectivity, we will leave the proof here as an instructive exercise, and with the comment that we will come back to this in a moment, with details in a finer setting. $\square$

As a first consequence of Theorem 4.5, we have the following famous formula:

THEOREM 4.6. *The number of trees with vertices labeled* $1,\ldots,N$ *is*

$$T_N=N^{N-2}$$

called Cayley formula.

PROOF. This is clear from the bijection in Theorem 4.5, because the number of possible Prüfer sequences, $(a_1,\ldots,a_{N-2})$ with $a_i\in\{1,\ldots,N\}$, that is, sequences obtained by picking $N-2$ times numbers from $\{1,\ldots,N\}$, is obviously $T_N=N^{N-2}$.

As a number of comments, however, which have to be made here, we have:

(1) First of all, in order to avoid confusions, what we just counted are trees up to obvious isomorphism. For instance at $N=2$ we only have 1 tree, namely:

$$1-2$$

Indeed, the other possible tree, namely $2-1$, is clearly isomorphic to it.

(2) At $N=3$ now, the trees are sequences of type $a-b-c$, with $\{a,b,c\}=\{1,2,3\}$. Now since such a tree is uniquely determined, up to graph isomorphism, by the middle vertex b, we have 3 exactly trees up to isomorphism, namely:

$$1-2-3\quad,\quad 2-3-1\quad,\quad 3-1-2$$

(3) At $N=4$, we first have $2\binom{4}{2}=12$ bivalent trees, which can be listed by choosing the ordered outer vertices, and putting the smallest label left at position #2:

$$1-3-4-2\quad,\quad 1-2-4-3\quad,\quad 1-2-3-4$$
$$2-3-4-1\quad,\quad 2-1-4-3\quad,\quad 2-1-3-4$$
$$3-2-4-1\quad,\quad 3-1-4-2\quad,\quad 3-1-2-4$$
$$4-2-3-1\quad,\quad 4-1-3-2\quad,\quad 4-1-2-3$$

And then, for completing the count, we have $\binom{4}{1} = 4$ trees with a trivalent vertex, which again, by using some sort of lexicographic order, look as follows:

$$2 - 1 <^3_4 \quad , \quad 1 - 2 <^3_4 \quad , \quad 1 - 3 <^2_4 \quad , \quad 1 - 4 <^2_3$$

(4) And so on, and it is actually instructive here to try yourself listing the $5^3 = 125$ trees at $N = 5$, in order to convince you that the Cayley formula is something quite subtle, hiding inside plenty of binomials and factorials.

(5) Which leads us to the question, what is the simplest proof of the Cayley formula. Well, there are many possible proofs here, for all tastes, with the above one, using Prüfer sequences, being probably the simplest one. We will present as well a second proof, a bit later, due to Kirchoff, which is based on linear algebra, and is non-trivial as well. □

Moving ahead, as a more specialized consequence of Theorem 4.5, we have:

THEOREM 4.7. *The number of trees with vertices labeled* $1, \ldots, N$*, having respective valences* $v_1, \ldots, v_N$*, is the multinomial coefficient*

$$T_{v_1,\ldots,v_N} = \binom{N-2}{v_1 - 1, \ldots, v_N - 1}$$

and with this implying as well the Cayley formula, $T_N = N^{N-2}$*.*

PROOF. This follows again from Theorem 4.5, the idea being as follows:

(1) As a first observation, the formula in the statement makes sense indeed, with what we have there being indeed a multinomial coefficient, and this because by using the fact that the number of edges is $E = N - 1$, that we know from Proposition 4.3, we have:

$$\begin{aligned}
\sum_i (v_i - 1) &= \left(\sum_i v_i\right) - N \\
&= 2E - N \\
&= 2(N-1) - N \\
&= N - 2
\end{aligned}$$

(2) In what regards now the proof, this follows from Theorem 4.5, the point being that in the Prüfer sequence, each number $i \in \{1, \ldots, N\}$ appears exactly $v_i - 1$ times.

(3) As for the last assertion, this comes from this, and the multinomial formula:

$$\begin{aligned}
T_N &= \sum_{v_1,\ldots,v_N} T_{v_1,\ldots,v_N}\\
&= \sum_{v_1,\ldots,v_N} \binom{N-2}{v_1-1,\ldots,v_N-1}\\
&= \sum_{v_1,\ldots,v_N} \binom{N-2}{v_1-1,\ldots,v_N-1} \times 1^{v_1-1}\ldots 1^{v_N-1}\\
&= (1+\ldots+1)^{N-2}\\
&= N^{N-2}
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

4b. Kirchoff formula

Let us discuss now another proof of the Cayley formula, due this time to Kirchoff, using linear algebra techniques, of analytic flavor. Let us start with:

Definition 4.8. *We call Laplacian of a graph X the matrix*

$$L = v - d$$

with v being the diagonal valence matrix, and d being the adjacency matrix.

This definition is inspired by differential geometry, or just by multivariable calculus, and more specifically by the well-known Laplace operator there, given by:

$$\Delta f = \sum_{i=1}^N \frac{d^2 f}{dx_i^2}$$

More on this in a moment, but as a word regarding terminology, this is traditionally confusing in graph theory, and impossible to fix in a decent way, according to:

Warning 4.9. *The graph Laplacian above is in fact the negative Laplacian,*

$$L = -\Delta$$

with our preference for it, negative, coming from the fact that it is positive, $L \geq 0$.

Which sounds like a bad joke, but this is how things are, and more on this a moment. In practice now, the graph Laplacian is given by the following formula:

$$L_{ij} = \begin{cases} v_i & \text{if } i = j\\ -1 & \text{if } i - j\\ 0 & \text{otherwise}\end{cases}$$

Alternatively, we have the following formula, for the entries of the Laplacian:

$$L_{ij}=\delta_{ij}v_i-\delta_{i-j}$$

With these formulae in hand, we can formulate, as our first result on the subject:

PROPOSITION 4.10. *A function on a graph is harmonic, $Lf=0$, precisely when*

$$f_i=\frac{1}{v_i}\sum_{i-j}f_j$$

that is, when the value at each vertex is the average over the neighbors.

PROOF. We have indeed the following computation, for any function f:

$$\begin{aligned}(Lf)_i&=\sum_j L_{ij}f_j\\&=\sum_j(\delta_{ij}v_i-\delta_{i-j})f_j\\&=v_if_i-\sum_{i-j}f_j\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Summarizing, we have some good reasons for calling L the Laplacian, because the solutions of $Lf=0$ satisfy what we would expect from a harmonic function, namely having the "average over the neighborhood" property. With the remark however that the harmonic functions on graphs are something trivial, due to the following fact:

PROPOSITION 4.11. *A function on a graph X is harmonic in the above sense precisely when it is constant over the connected components of X.*

PROOF. This is clear from the equation that we found in Proposition 4.10, namely:

$$f_i=\frac{1}{v_i}\sum_{i-j}f_j$$

Indeed, based on this, we can say for instance that f cannot have variations over a connected component, and so must be constant on these components, as stated. □

At a more advanced level now, let us try to understand the relation with the usual Laplacian from analysis Δ, which is given by the following formula:

$$\Delta f=\sum_{i=1}^N\frac{d^2f}{dx_i^2}$$

In one dimension, $N = 1$, the Laplacian is simply the second derivative, $\Delta f = f''$. Now let us recall that the first derivative of a one-variable function is given by:

$$f'(x) = \lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

We deduce from this, or from the Taylor formula at order 2, to be fully correct, that the second derivative of a one-variable function is given by the following formula:

$$\begin{aligned} f''(x) &= \lim_{t\to 0} \frac{f'(x+t) - f'(x)}{t} \\ &= \lim_{t\to 0} \frac{f(x+t) - 2f(x) + f(x-t)}{t^2} \end{aligned}$$

Now since $\mathbb{R}$ can be thought of as appearing as the continuum limit, $t \to 0$, of the graphs $t\mathbb{Z} \simeq \mathbb{Z}$, this suggests defining the Laplacian of $\mathbb{Z}$ by the following formula:

$$\Delta f(x) = \frac{f(x+1) - 2f(x) + f(x-1)}{1^2}$$

But this is exactly what we have in Definition 4.8, up to a sign switch, the graph Laplacian of $\mathbb{Z}$, as constructed there, being given by the following formula:

$$Lf(x) = 2f(x) - f(x+1) - f(x-1)$$

Summarizing, we have reached to the formula in Warning 4.9, namely:

$$L = -\Delta$$

In arbitrary dimensions now, everything generalizes well, and we have:

THEOREM 4.12. *The Laplacian of graphs is compatible with the usual Laplacian,*

$$\Delta f = \sum_{i=1}^{N} \frac{d^2 f}{dx_i^2}$$

via the following formula, showing that our L is in fact the negative Laplacian,

$$L = -\Delta$$

via regarding $\mathbb{R}^N$ as the continuum limit, $t \to 0$, of the graphs $t\mathbb{Z}^N \simeq \mathbb{Z}^N$.

PROOF. This is something that we know from the above at $N = 1$, and the proof in general is similar. Indeed, at $N = 2$, to start with, the formula that we need, coming from standard multivariable calculus, or just from the $N=1$ formula, is as follows:

$$\begin{aligned}\Delta f(x,y) &= \frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}\\ &= \lim_{t\to0}\frac{f(x+t,y)-2f(x,y)+f(x-t,y)}{t^2}\\ &+ \lim_{t\to0}\frac{f(x,y+t)-2f(x,y)+f(x,y-t)}{t^2}\\ &= \lim_{t\to0}\frac{f(x+t,y)+f(x-t,y)+f(x,y+t)+f(x,y-t)-4f(x,y)}{t^2}\end{aligned}$$

Now since $\mathbb{R}^2$ can be thought of as appearing as the continuum limit, $t\to0$, of the graphs $t\mathbb{Z}^2\simeq\mathbb{Z}^2$, this suggests defining the Laplacian of $\mathbb{Z}^2$ as follows:

$$\Delta f(x)=\frac{f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)-4f(x,y)}{1^2}$$

But this is exactly what we have in Definition 4.8, up to a sign switch, the graph Laplacian of $\mathbb{Z}^2$, as constructed there, being given by the following formula:

$$Lf(x)=4f(x,y)-f(x+1,y)-f(x-1,y)-f(x,y+1)-f(x,y-1)$$

At higher $N\in\mathbb{N}$ the proof is similar, and we will leave this as an exercise. $\square$

All this is quite interesting, and suggests doing all sorts of other geometric and analytic things, with our graphs and their Laplacians. For instance, we can try to review the above with $\mathbb{R}^N$ replaced by more general manifolds, having a certain curvature.

Also, importantly, we can now do PDE over our graphs, by using the negative Laplacian that we constructed above. Skipping some details, we have here:

Fact 4.13. *The main equations of physics, namely the heat and wave ones,*

$$\dot{f}=\alpha\Delta f\quad,\quad\ddot{f}=v^2\Delta f$$

appear as the continuum limit, $t\to0$, of their discrete analogues, over the lattice $t\mathbb{Z}^N$.

All this is extremely interesting, and we insist here, from both the mathematical and physical perspective on all this, for two very good reasons, namely:

(1) From a theoretical point of view. Indeed, we reach in this way to a rock-solid justification for the heat and wave equations, via very simple lattice models.

(2) From a practical point of view. Indeed, the study of discrete PDE, usually called "finite element method", can be often simpler, when done with a computer.

As before with other interesting things, that we will not get into, in this book, exercise for you to learn about all this, at least a little bit. In what regards (1), what you need there is a good book on introductory physics, or PDE, or lattice models. As for (2), what you need there is again a good book on PDE, or a book on the finite element method.

Now back to our general graph questions, and to Definition 4.8 as it is, the Laplacian of graphs as constructed there has the following basic properties:

THEOREM 4.14. *The graph Laplacian $L = v - d$ has the following properties:*
(1) *It is symmetric, $L = L^t$.*
(2) *It is positive definite, $L \geq 0$.*
(3) *It is bistochastic, with row and column sums 0.*
(4) *It has 0 as eigenvalue, with the other eigenvalues being positive.*
(5) *The multiplicity of 0 is the number of connected components.*
(6) *In the connected case, the eigenvalues are $0 < \lambda_1 \leq \ldots \leq \lambda_{N-1}$.*

PROOF. All this is straightforward, the idea being as follows:

(1) This is clear from $L = v - d$, both v, d being symmetric.

(2) This follows from the following computation, for any function f on the graph:

$$\begin{aligned} < Lf, f > &= \sum_{ij} L_{ij} f_i f_j \\ &= \sum_{ij} (\delta_{ij} v_i - \delta_{i-j}) f_i f_j \\ &= \sum_i v_i f_i f_j - \sum_{i-j} f_i f_j \\ &= \sum_{i\sim j} f_i^2 - \sum_{i-j} f_i f_j \\ &= \frac{1}{2} \sum_{i-j} (f_i - f_j)^2 \\ &\geq 0 \end{aligned}$$

(3) This is again clear from $L = v - d$, and from the definition of v, d.

(4) Here the first assertion comes from (3), and the second one, from (2).

(5) Given an arbitrary graph, we can label its vertices inceasingly, over the connected components, and this makes the adjacency matrix d, so the Laplacian L as well, block diagonal. Thus, we are left with proving that for a connected graph, the multiplicity of 0 is precisely 1. But this follows from the formula from the proof of (2), namely:

$$< Lf, f >= \frac{1}{2} \sum_{i-j} (f_i - f_j)^2$$

Indeed, this formula shows in particular that we have the following equivalence:

$$Lf = 0 \iff f_i = f_j, \forall i - j$$

Now since our graph was assumed to be connected, as per the above beginning of proof, the condition on the right means that f must be constant. Thus, the 0-eigenspace of the Laplacian follows to be 1-dimensional, spanned by the all-1 vector, as desired.

(6) This follows indeed from (4) and (5), and with the remark that in fact we already proved this, in the proof of (5), with the formulae there being very useful in practice. □

We can now state the Kirchoff theorem, which is something quite tricky, as follows:

Theorem 4.15. *Given a connected graph X, with vertices labeled $1, \ldots, N$, the number of spanning trees inside X, meaning tree subgraphs using all vertices, is*

$$T_X = (-1)^{i+j} \det(L^{ij})$$

with this being independent on the chosen cofactor. Alternatively, we have the formula

$$T_X = \frac{\lambda_1 \ldots \lambda_{N-1}}{N}$$

where $\lambda_1, \ldots, \lambda_{N-1}$ are the nonzero eigenvalues of the Laplacian $L = v - d$.

Proof. This is something non-trivial, the idea being as follows:

(1) As a first observation, any connected graph has indeed a spanning tree, meaning a tree subgraph, making use of all vertices. This is something which is very intuitive, clear on pictures, and we will leave the formal proof, which is not difficult, as an exercise. In what concerns us, this will follow of course from $T_X > 0$, that we will prove below.

(2) Let us begin with some linear algebra observations. We know from Theorem 4.14 that the Laplacian L is bistochastic, with row and column sums 0. Due to this fact, the minors of L can be transformed one into another, via simple operations, namely adding rows or columns, switching rows or columns, or multiplying rows or columns by -1.

(3) But this shows that the cofactors of L will be all equal, up to certain signs. Now, thinking well, these signs will be all $+$, so the cofactors of L are all equal. In other words, we have a common formula as follows, with $T \in \mathbb{Z}$ being a certain number:

$$(-1)^{i+j} \det(L^{ij}) = T$$

(4) Our claim now, which will prove the first assertion, and then the second assertion too, via some standard eigenvalue discussion, is that the number of spanning trees is precisely given by this common cofactor. That is, our claim is that we have:

$$T_X = \det(L^{11})$$

(5) In order to prove our claim, which is non-trivial, we use a trick. We define the ordered incidence matrix of our graph, which is a rectangular matrix, with the vertices i as row indices, and the edges $e=(ij)$ as column indices, by the following formula:

$$E_{ie}=\begin{cases}1 & \text{if } e=(ij)\\ -1 & \text{if } e=(ji)\\ 0 & \text{otherwise}\end{cases}$$

The point is that, in terms of this matrix, the Laplacian decomposes as follows:

$$L=EE^t$$

Indeed, this formula is clear from the definition of the Laplacian, $L=v-d$. Note in passing that this gives another proof of $L\geq 0$, that we know from Theorem 4.14.

(6) Getting now towards minors, if we denote by F the submatrix of E obtained by deleting the first row, the one regarding the vertex 1, we have the following formula:

$$L^{11}=FF^t$$

(7) The point now is that, in order to compute the determinant of this matrix, we can use the Cauchy-Binet formula from linear algebra. We are led in this way to the following formula, with S ranging over the subsets of the edge set having size $N-1$, and with F_S being the corresponding square submatrix of E, having size $(N-1)\times(N-1)$, obtained by restricting the attention to the columns indexed by S:

$$\begin{aligned}\det(L^{11}) &= \det(FF^t)\\ &= \sum_S \det(F_S)\det(F_S^t)\\ &= \sum_S \det(F_S)^2\end{aligned}$$

(8) Now comes the combinatorics. The sets S appearing in the above computation specify in fact $N-1$ edges of our graph, and so specify a certain subgraph X_S. But, in this picture, it is routine to check that we have the following formula:

$$\det(F_S)=\begin{cases}\pm 1 & \text{if } X_S \text{ is a spanning tree}\\ 0 & \text{otherwise}\end{cases}$$

By putting now everything togeher, we are led to the following formula, with T_X being the number of spanning trees, which proves our claim in (4), and the first assertion:

$$\det(L^{11})=T_X$$

(9) With this done, it remains to do some linear algebra discussion, in order to reach to the second formula in the statement. So, let $0<\lambda_1\leq\ldots\leq\lambda_{N-1}$ be the eigenvalues

of the Laplacian. The determinant of L is then the product of these eigenvalues:

$$\det L = 0\lambda_1 \ldots \lambda_{N-1} = 0$$

But, this formula can be recovered as well by developing over the first row, and by using the fact, that we found in (3), that the cofactors all coincide, as follows:

$$\det L = v_1 \det(L^{11}) - \sum_{i-j} \det(L^{11}) = 0$$

(10) Now let us transform the matrix L, as for the first row and column to consist of 0 entries. In this process, the formula $\det L = \det L$ that we found in (9), with of course forgetting the actual value of this number, which happens to be 0, becomes:

$$\det(L^{11}) = \frac{\lambda_1 \ldots \lambda_{N-1}}{N}$$

Thus, we are led to the conclusion in the statement. □

As a basic application of the Kirchoff formula, let us apply it to the complete graph K_N. We are led in this way to another proof of the Cayley formula, as follows:

THEOREM 4.16. *The number of spanning trees of the complete graph K_N is*

$$T_{K_N} = N^{N-2}$$

in agreement with the Cayley formula.

PROOF. This is something which is clear from the Kirchoff formula, but let us prove this slowly, as an illustration for the various computations above:

(1) At $N = 2$ the Laplacian of the segment K_2 is given by the following fomula:

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Thus the common cofactor is 1, which equals the number of spanning trees, $2^0 = 1$.

(2) At $N = 3$ the Laplacian of the triangle K_3 is given by the following fomula:

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Thus the common cofactor is 3, which equals the number of spanning trees, $3^1 = 3$.

(3) At $N = 4$ the Laplacian of the tetrahedron K_4 is given by the following fomula:

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Here the cofactor is $27 - 11 = 16$, which is the number of spanning trees, $4^2 = 16$.

(4) In general, for the complete graph K_N, the Laplacian is as follows:

$$\Delta = \begin{pmatrix} N-1 & -1 & \ldots & -1 & -1 \\ -1 & N-1 & \ldots & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & \ldots & N-1 & -1 \\ -1 & -1 & \ldots & -1 & N-1 \end{pmatrix}$$

Thus, the common cofactor is N^{N-2}, in agreement with the Cayley formula. $\square$

Finally, let us mention that in what regards the counting of trees having N vertices, this time without labeled vertices, things here are far more complicated, and there is no formula available, for the number of such trees. We refer here to the literature.

4c. Genus, planarity

Remember the discussion from the opening of chapter 3, regarding good and bad graphs. We have seen there that a nice and fruitful notion of "goodness", coming from simple and beautiful things like the circulant graphs, is the notion of transitivity.

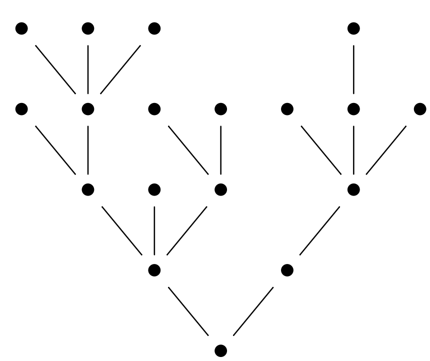
On the other hand, we have just seen that trees look beautiful and good as well, from an opposite aesthetic to that of the circulant graphs. So, question now for the two of us, subjective as they come: thinking deeply, what makes the beauty of a tree?

You might agree with this or not, but here is an answer to this question:

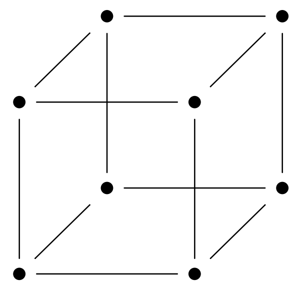
Answer 4.17. *Graphs fall into three classes:*

1. *Trees and other graphs which can be drawn without crossings are good.*
2. *If we can still do this, but on a torus, the graph is bad.*
3. *And the rest is evil.*

Here the fact that trees are indeed planar is obvious, and as an illustration, here is some sort of "random" tree, which is clearly planar, no question about it:

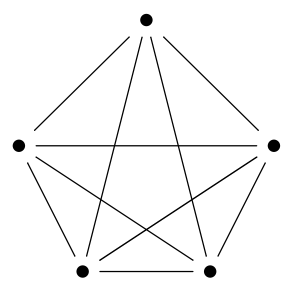


Of course, there are many other interesting examples of planar graphs, as for instance our beloved cube graph, and up to you to tell me why this graph is planar:



However, not all graphs are planar. In order to find basic examples of non-planar graphs, we can look at simplices, and we are led to the following result:

PROPOSITION 4.18. *When looking at simplices, the segment K_2, the triangle K_3 and the tetrahedron K_4 are planar. However, the next simplex K_5, namely*

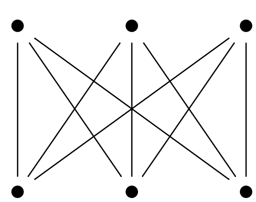


is not planar. Nor are the higher simplices, K_N with $N \geq 6$, planar.

PROOF. Here the planarity of K_2, K_3, K_4 is clear from definitions, and the non-planarity of K_5 and of higher K_N graphs is clear too, by thinking a bit. We will leave the formal proof of this latter fact as an instructive exercise, and of course, we will come back in a moment to such questions, with some tools for dealing with them. $\square$

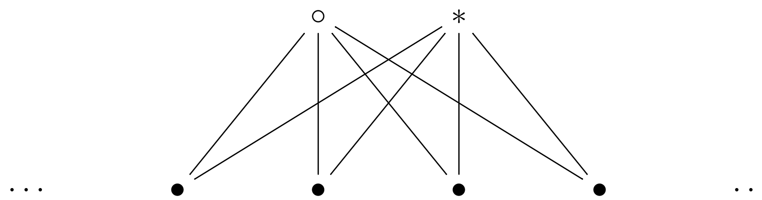
In order to find some further examples of non-planar graphs, we can look as well at the bipartite simplices, and we are led to the following result:

PROPOSITION 4.19. *When looking at bipartite simplices, the square $K_{2,2}$ is planar, and so are all the graphs $K_{2,N}$. However, the next such graph, namely $K_{3,3}$,*



called "utility graph" is not planar. Nor are planar the graphs $K_{M,N}$, for any $M, N \geq 3$.

Proof. In what regards the first assertion, the bipartite simplex $K_{2,N}$ looks at follows, making it clear that at $N=2$ we obtain a square, and also that at any $N\in\mathbb{N}$ we have something planar, as seen by pulling the vertex $*$ downwards, in the obvious way:



As for the second assertion, since $K_{M,N}$ with $M,N\geq 3$ contains $K_{3,3}$, it is enough to prove that this latter graph is not planar. But here, as before with K_5, the result is quite clear by thinking a bit, and we will leave the formal proof of this as an instructive exercise, with of course the promise to come back to this, with appropriate tools. $\square$

As a first main result now about the planar graphs, we have:

Theorem 4.20. *The fact that a graph X is non-planar can be checked as follows:*

(1) *Kuratowski criterion: X contains a subdivision of K_5 or $K_{3,3}$.*
(2) *Wagner criterion: X has a minor of type K_5 or $K_{3,3}$.*

Proof. This is obviously something quite powerful, when thinking at the potential applications, and non-trivial to prove as well, the idea being as follows:

(1) Regarding the Kuratowski criterion, the convention is that "subdivision" means graph obtained by inserting vertices into edges, e.g. replacing $\circ-\circ$ with $\circ-\circ-\circ$.

(2) Regarding the Wagner criterion, the convention there is that "minor" means graph obtained by contracting certain edges into vertices.

(3) Finally, regarding the proofs, the Kuratowski and Wagner criteria are more or less equivalent, and their proof is via standard recurrence methods, exercise for you. $\square$

As a second main result now about the planar graphs, we have:

Theorem 4.21. *For a connected planar graph we have the Euler formula*

$$v-e+f=2$$

with v,e,f being the number of vertices, edges and faces.

Proof. Given a connected planar graph, drawn in a planar way, without crossings, we can certainly talk about v and e, as for any graph, and also about f, as being the number of faces that our graph has, in our picture, with these including by definition the outer face too, the one going to ∞. As an example here, for a triangle we have $v=e=3$ and $f=2$, and we conclude that the Euler formula holds indeed, as:

$$3-3+2=2$$

More generally now, the Euler formula holds for any N-gon graph, as:

$$N - N + 2 = 2$$

But this shows that the Euler formula holds at $f = 2$, and by a standard recurrence on f, we conclude that this formula is valid at any $f \in \mathbb{N}$, as desired. □

As a third main result now about the planar graphs, we have:

THEOREM 4.22. *Any planar graph has the following properties:*

(1) *It is vertex* 4*-colorable.*
(2) *It is a* 4*-partite graph.*

PROOF. This is something that we talked about in chapter 1, by calling there "maps" the planar graphs, and with the comment that the proof is something extremely complicated. So, you might perhaps expect, with our graph learning doing quite well, that we might have some new things to say, about this. Unfortunately, no, this is definitely very difficult, beyond our reach, but do not hesitate to look it up, and learn more about it. □

As you can see, the theory of planar graphs can vary a lot, with Theorem 4.20 being something tricky, Theorem 4.21 being something quite trivial, and Theorem 4.22 being something of extreme difficulty. Quite fascinating all this, hope you agree with me.

Switching topics now, let us get into the following question:

QUESTION 4.23. *What are the graphs which are not planar, but can be however drawn on a torus? Also, what about graphs which can be drawn on higher surfaces, having $g \geq 2$ holes, instead of the $g = 0$ holes of the sphere, and the $g = 1$ hole of the torus?*

All this looks quite interesting, but before going head-first into such questions, let us pause our study of graphs, and look instead at surfaces, and other manifolds. The same questions make sense here, does our manifold have holes, and how many holes, of which type, and so on. Let us start with something that we know well, namely:

DEFINITION 4.24. *A topological space X is called path connected when any two points $x, y \in X$ can be connected by a path. That is, given any two points $x, y \in X$, we can find a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

The problem is now, given a connected space X, how to count its "holes". And this is quite subtle problem, because as examples of such spaces we have:

(1) The sphere, the donut, the double-holed donut, the triple-holed donut, and so on. These spaces are quite simple, and intuition suggests to declare that the number of holes of the N-holed donut is, and you guessed right, N.

(2) However, we have as well as example the empty sphere, I mean just the crust of the sphere, and while this obviously falls into the class of "one-holed spaces", this is not the same thing as a donut, its hole being of different nature.

(3) As another example, consider again the sphere, but this time with two tunnels drilled into it, in the shape of a cross. Whether that missing cross should account for 1 hole, or for 2 holes, or for something in between, I will leave it up to you.

Summarizing, things are quite tricky, suggesting that the "number of holes" of a topological space X is not an actual number, but rather something more complicated. Now with this in mind, let us formulate the following definition:

Definition 4.25. *The homotopy group $\pi_1(X)$ of a connected space X is the group of loops based at a given point $* \in X$, with the following conventions,*

1. *Two such loops are identified when one can pass continuously from one loop to the other, via a family of loops indexed by $t \in [0,1]$,*
2. *The composition of two such loops is the obvious one, namely is the loop obtained by following the first loop, then the second loop,*
3. *The unit loop is the null loop at $*$, which stays there, and the inverse of a given loop is the loop itself, followed backwards,*

with the remark that the group $\pi_1(X)$ defined in this way does not depend on the choice of the given point $* \in X$, where the loops are based.

Here the fact that $\pi_1(X)$ defined in this way is indeed a group is obvious, and obvious as well is the fact that, since X is assumed to be connected, this group does not depend on the choice of the given point $* \in X$, where the loops are based.

As basic examples, for spaces having "no holes", such as $\mathbb{R}$ itself, or $\mathbb{R}^N$, and so on, we have $\pi_1 = \{1\}$. In fact, having no holes can only mean, by definition, $\pi_1 = \{1\}$:

Definition 4.26. *A space is called simply connected when:*

$$\pi_1 = \{1\}$$

That is, any loop inside our space must be contractible.

So, this will be our starting definition, for the considerations in this section. As further illustrations for Definition 4.25, here are now a few basic computations:

Theorem 4.27. *We have the following computations of homotopy groups:*

1. *For the circle, we have $\pi_1 = \mathbb{Z}$.*
2. *For the torus, we have $\pi_1 = \mathbb{Z} \times \mathbb{Z}$.*
3. *For the disk minus 2 points, we have $\pi_1 = F_2$.*
4. *In fact, for the disk minus N points, we have $\pi_1 = F_N$.*

Proof. These results are all standard, as follows:

(1) The first assertion is clear, because a loop on the circle must wind $n \in \mathbb{Z}$ times around the center, and this parameter $n \in \mathbb{Z}$ uniquely determines the loop, up to the

identification in Definition 4.25. Thus, the homotopy group of the circle is the group of such parameters $n \in \mathbb{Z}$, which is of course the group $\mathbb{Z}$ itself.

(2) In what regards now the second assertion, the torus being a product of two circles, we are led to the conclusion that its homotopy group must be some kind of product of $\mathbb{Z}$ with itself. But pictures show that the two standard generators of $\mathbb{Z}$, and so the two copies of $\mathbb{Z}$ themselves, commute, $gh = hg$, so we obtain the product of $\mathbb{Z}$ with itself, subject to commutation, which is the usual product $\mathbb{Z} \times \mathbb{Z}$.

(3) This is quite clear, because the homotopy group is generated by the 2 loops around the 2 missing points, which are obviously free, algebrically speaking. Thus, we obtain a free product of the group $\mathbb{Z}$ with itself, which is the free group on 2 generators F_2.

(4) This is again clear, because the homotopy group is generated by the N loops around the N missing points, which are free, algebrically speaking. Thus, we obtain a N-fold free product of $\mathbb{Z}$ with itself, which is the free group on N generators F_N. $\square$

There are many other interesting things that can be said about homotopy groups. Also, another thing that can be done with the arbitrary topological spaces X, again in relation with studying their "shape", is that of looking at the fiber bundles over them, again up to continuous deformation. We are led in this way into a group, called $K_0(X)$. Moreover, both $\pi_1(X)$ and $K_0(X)$ have higher analogues $\pi_n(X)$ and $K_n(X)$ as well, and the general goal of algebraic topology is that of understanding all these groups, along with some other groups, of similar flavor, which can be constructed as well.

Moving ahead with some further topology, we have:

FACT 4.28. *We can talk about the genus of a surface*

$$g \in \mathbb{N}$$

as being its number of holes.

At this point, I am sure that you are wondering what the genus exactly is, mathematically speaking. There are many answers here, ranging from elementary to advanced, depending on how much geometric you want to be, and the best answer, which is a bit complicated, involves complex analysis, and the notion of Riemann surface.

We will not further insist on this, but as a key result here, let us mention:

THEOREM 4.29. *For a connected graph of genus $g \in \mathbb{N}$ we have the Euler formula*

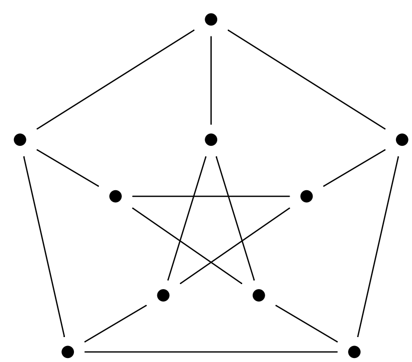
$$v - e + f = 2 - 2g$$

with v, e, f being the number of vertices, edges and faces.

PROOF. This comes as a continuation of Theorem 4.21, dealing with the case $g = 0$, and assuming that you have worked out all the details of the proof there, you will certainly have no troubles now in understanding the present extension, to genus $g \in \mathbb{N}$. $\square$

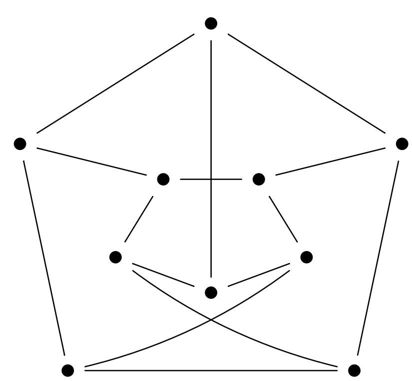
But all this might seem a bit abstract. In practice, passed the planar graphs, $g = 0$, that we understand quite well, the next problem comes in understanding the toral graphs, $g = 1$, with a main example here being the Petersen graph, which is as follows:

THEOREM 4.30. *The Petersen graph, namely*



is not planar, but is toral.

PROOF. The fact that this graph is not planar can be best seen by using the Wagner criterion from Theorem 4.20, with both K_5 and $K_{3,3}$ being minors of it. As for the torus assertion, observe first that our graph can be drawn with only 2 crossings, as follows:



But with a bit of thinking, we can see here a torus, as desired. □

Many other things can be said about the Petersen graph, and about other toral graphs, of similar type, or of general type. Exercise of course, read a bit here, if interested.

4d. Knot invariants

Let us go back now to topology, as a continuation of our various considerations regarding the homotopy groups, and other invariants. Leaving the general manifolds aside, let us focus now on the simplest objects of topology, which are the knots:

Definition 4.31. *A knot is a smooth closed curve in $\mathbb{R}^3$, regarded modulo smooth transformations.*

And isn't this a beautiful definition. We are here at the core of everything that can be called "geometry", and in fact, thinking a bit on how knots can be tied, in so many fascinating ways, we are led to the following philosophical conclusion:

Conclusion 4.32. *Knots are to geometry and topology what prime numbers are to number theory.*

At the level of questions now, once we have a closed curve, say given via its equations, can we decide if is tied or not, and if tied, how complicated is it tied, how to untie it, and so on? Also, experience with cables and ropes shows that a random closed curve is usually tied, but can we prove this? But these are, obviously, quite difficult questions.

Fortunately, the graphs come to the rescue, via the following simple observation:

Fact 4.33. *The plane projection of a knot is something similar to a graph with 4-valent vertices, except for the fact that we have some extra data at the vertices, telling us, about the 2 strings crossing there, which goes up and which goes down.*

Based on this, let us try now to construct some knot invariants. A natural idea is that of defining the invariant on the 2D picture of the knot, that is, on a plane projection of the knot, and then proving that the invariant is indeed independent on the chosen plane. This method rests on the following technical result, which is well-known:

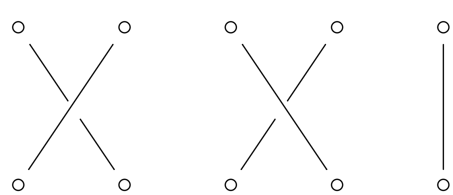
Proposition 4.34. *Two pictures correspond to plane projections of the same knot precisely when they differ by a sequence of Reidemeister moves, namely:*

(1) *Moves of type I, given by $\propto \leftrightarrow |$.*
(2) *Moves of type II, given by $\between \leftrightarrow)($.*
(3) *Moves of type III, given by $\triangle \leftrightarrow \nabla$.*

Proof. This is somewhat clear from definitions, and in practice, this can be done by some sort of cut and paste procedure, or recurrence if you prefer. Up to you to figure out all this, and also to draw more precise pictures for the above type I, II, III moves. $\square$

At a more advanced level now, we will need the following key observation, making the connection with group theory, and algebra in general, due to Alexander:

Proposition 4.35. *Any knot can be thought of as being the closure of a braid,*



with the braids forming a group B_k, called braid group.

PROOF. Again, this is something quite self-explanatory, with the closing operation in question consisting in adding semicircles at right, and with the formal proof of the result done by some sort of cut and paste procedure, or recurrence if you prefer. □

As an interesting observation here, in relation with Theorem 4.27, we have the following formula, with $\Delta\subset\mathbb C^k$ standing for the points z satisfying $z_i=z_j$ for some $i\neq j$:

$$B_k=\pi_1\left((\mathbb C^k-\Delta)/S_k\right)$$

Finally, also at the algebraic level, we will need as well the following key fact:

PROPOSITION 4.36. *The Temperley-Lieb algebra of index $N\in[1,\infty)$, defined as*

$$TL_N(k)=span\,(NC_2(k,k))$$

with product by vertical concatenation, with the rule $\bigcirc=N$, has the following properties:

1. *We have a representation $B_k\to TL_N(k)$, mapping generators to generators.*
2. *We have a trace $tr:TL_N(k)\to\mathbb C$, obtained by closing the diagrams.*

PROOF. Again, this is something quite intuitive, with the generators in (1) being the standard ones, on both sides, and with the closing operation in (2) being similar to the one for braids, from Proposition 4.35. Exercise for you, to figure out all this. □

We can now put everything together, and we obtain, following Jones [52], [53]:

THEOREM 4.37. *We can define the Jones polynomial of a knot as being the image of the corresponding braid producing it via the map*

$$tr:B_k\to TL_N(k)\to\mathbb C$$

suitably normalized, and with the change of variables $N=q^{1/2}+q^{-1/2}$, we obtain a Laurent polynomial in $q^{1/2}$, which is an invariant, up to planar isotopy.

PROOF. There is a long story here, the idea being as follows:

(1) To start with, the result follows indeed by combining the above ingredients, the idea being that the various algebraic properties of $tr:TL_N(k)\to\mathbb C$ are exactly what is needed for the above composition, up to a normalization, to be invariant under the Reidemeister moves of type I, II, III, and so to produce indeed a knot invariant.

(2) In practice now, far more things can be said. Observe for instance that the change of variables $N=q^{1/2}+q^{-1/2}$ is precisely the one that we used in chapter 2, when counting loops on the ADE graphs, with all this being intimately related to subfactor theory.

(3) From a purely topological perspective, however, nothing beats the skein relation interpretation of the Jones polynomial $V_L(t)$, which is as follows, with L_+,L_-,L_0 being knots, or rather links, differing at exactly 1 crossing, in the 3 possible ways:

$$q^{-1}V_{L_+}-qV_{L_-}=(q^{1/2}+q^{-1/2})V_{L_0}$$

(4) So, up to you to learn all this, and its generalizations too, with link polynomials defined more generally via relations of type $xP_{L_+} + yP_{L_-} + zP_{L_0} = 0$, or equivalently, via versions of the Temperley-Lieb algebra. Check here the papers of Jones [**52**], [**53**].

(5) However, with all this pure mathematics interpretation digested, physics strikes back, via a very interesting relation with statistical mechanics, happening in 2D as well, the idea being that "interactions happen at crossings", and it is these interactions that produce the knot invariant, as a kind of partition function. See Jones [**54**].

(6) Quite remarkably, the above invariants can be directly understood in 3D as well, in a purely geometric way, with elegance, and no need for 2D projection. But this is a more complicated story, involving ideas from quantum field theory. See Witten [**98**]. □

4e. Exercises

We had a very exciting chapter, our policy for this book being to have exciting chapters at the end of each Part. As a downside, however, many exercises left for you:

Exercise 4.38. *Apart from free groups F_N, how can you get trees as Cayley graphs?*

Exercise 4.39. *Details for the Prüfer sequences, and write a code here, too.*

Exercise 4.40. *Read about the discretized heat and wave equations.*

Exercise 4.41. *Clarifiy all the details in the proof of the Kirchoff formula.*

Exercise 4.42. *Clarify the Kuratowski and Wagner criteria, and their applications.*

Exercise 4.43. *Learn about genus, in all its flavors, including for Riemann surfaces.*

Exercise 4.44. *Work out the proof of the Euler formula, at $g = 0$, and in general.*

Exercise 4.45. *Reidemeister moves, braid group, Temperley-Lieb, Jones polynomial.*

As bonus exercise, learn some systematic topology, and more specifically some knot theory, from the papers of Jones [**52**], [**53**], [**54**], and from the paper of Witten [**98**].

Part II

Symmetry groups

Lady, hear me tonight
Cause my feeling is just so right
As we dance by the moonlight
Can't you see you're my delight

CHAPTER 5

Symmetry groups

5a. Finite groups

We have seen in Part I that the study of the graphs X is usually best done via the study of their adjacency matrices $d \in M_N(0,1)$, and with this latter study being typically a mix of linear algebra, combinatorics, calculus, probability, and some computer programming too. We will be back of course to this general principle, on numerous occasions.

However, as another interesting thing that we discovered in Part I, the group actions $G \curvearrowright X$ are something important too, making various group theoretical tools, coming from the study of the permutation groups $G \subset S_N$, useful for our graph questions. Remember for instance how many things we could say in the case where we have a transitive action $G \curvearrowright X$ of a cyclic or abelian group G, by using discrete Fourier analysis tools.

In this chapter, and in this whole Part II of this book, and in fact in the remainder of the whole book, we will take such things very seriously, and systematically develop the study of the group actions $G \curvearrowright X$, our aim being to solve the following question:

Question 5.1. *Given a graph X, which groups G act on it, $G \curvearrowright X$, and what can be said about X itself, coming from the group theory of G?*

As already mentioned, we have a whole 100 pages, and later even more, for dealing with this question. So, we will go very slowly, at least in the beginning.

Let us start with some generalities. We already met groups G and actions $G \curvearrowright X$ in Part I, and notably in chapter 3, which was dedicated to the transitive graphs. So, we are certainly not new to this subject. However, since there is no hurry with anything, let us first have a crash course in group theory. The starting point is of course:

Definition 5.2. *A group is a set G endowed with a multiplication operation*

$$(g,h) \to gh$$

which must satisfy the following conditions:

(1) *Associativity: we have $(gh)k = g(hk)$, for any $g,h,k \in G$.*
(2) *Unit: there is an element $1 \in G$ such that $g1 = 1g = g$, for any $g \in G$.*
(3) *Inverses: for any $g \in G$ there is $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.*

As a first comment, the multiplication law is not necessarily commutative. In the case where it is, $gh = hg$ for any $g, h \in G$, we call G abelian, and we usually denote its multiplication, unit and inverse operation in an additive way, as follows:

$$(g, h) \to g + h$$

$$0 \in G$$

$$g \to -g$$

However, this is not a general rule, and rather the converse is true, in the sense that if a group is denoted as above, this means that the group must be abelian.

Let us first look at the abelian groups. Here as basic examples we have various groups of numbers, such as $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, with the addition operation $+$. Observe that we have as well as $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$, and the unit circle $\mathbb{T}$ too, with the multiplication operation $\times$.

In order to reach to some theory, let us look into the finite group case, $|G| < \infty$. Here as basic examples we have the cyclic groups, constructed as follows:

Proposition 5.3. *The following constructions produce the same group, denoted $\mathbb{Z}_N$, which is finite and abelian, and is called cyclic group of order N:*

(1) *$\mathbb{Z}_N$ is the set of remainders modulo N, with operation $+$.*
(2) *$\mathbb{Z}_N \subset \mathbb{T}$ is the group of N-th roots of unity, with operation $\times$.*

Proof. Here the equivalence between (1) and (2) is obvious. More complicated, however, is the question of finding the good philosophy and notation for this group. In what concerns us, we will be rather geometers, as usual, and we will often prefer the interpretation (2). As for the notation, we will use $\mathbb{Z}_N$, which is very natural. $\square$

As a comment here, some algebraists prefer to reserve the notation $\mathbb{Z}_N$ for the ring of p-adic integers, when $N = p$ is prime, and use other notations for the cyclic group. Personally, although having a long love story with number theory, back in the days, I consider that the cyclic group is far more important, and gets the right to be denoted $\mathbb{Z}_N$. As for the p-adic integers, a reasonable notation for them, which does the job, is $\mathbf{Z}_N$.

As a basic thing to be known, about the abelian groups, still in the finite case, we can construct further examples of such groups by making products between various cyclic groups $\mathbb{Z}_N$. Quite remarkably, we obtain in this way all the finite abelian groups:

Theorem 5.4. *The finite abelian groups are precisely the products of cyclic groups:*

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

Moreover, there are technical extensions of this result, going beyond the finite case.

PROOF. This is something that we already discussed in chapter 3, with the main ideas of the proof, and for further details on this, and for the technical extensions to the infinite groups as well, we recommend a solid algebra book, such as Lang [62]. □

Moving forward, let us look now into the general, non-abelian case. The first thought goes here to the $N \times N$ matrices with their multiplication, but these do not form a group, because we must assume $\det A \neq 0$ in order for our matrix to be invertible.

So, let us call $GL_N(\mathbb{C})$ the group formed by these latter matrices, with nonzero determinant, with GL standing for "general linear". By further imposing the condition $\det A = 1$ we obtain a subgroup $SL_N(\mathbb{C})$, with SL standing for "special linear", and then we can talk as well about the real versions of these groups, and also intersect everything with the group of unitary matrices U_N. We obtain in this way 8 groups, as follows:

THEOREM 5.5. *We have groups of invertible matrices as follows,*

$$\begin{tikzcd}
& GL_N(\mathbb{R}) \arrow[rr] & & GL_N(\mathbb{C}) \\
O_N \arrow[ur] \arrow[rr] & & U_N \arrow[ur] & \\
& SL_N(\mathbb{R}) \arrow[uu] \arrow[rr] & & SL_N(\mathbb{C}) \arrow[uu] \\
SO_N \arrow[uu] \arrow[ur] \arrow[rr] & & SU_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

with S standing for "special", meaning having determinant 1.

PROOF. This is clear indeed from the above discussion. As a comment, we can talk in fact about $GL_N(F)$ and $SL_N(F)$, once we have a ground field F, but in what regards the corresponding orthogonal and unitary groups, things here are more complicated. □

There are many other groups of matrices, besides the above ones, as for instance the symplectic groups $Sp_N \subset U_N$, appearing at $N \in 2\mathbb{N}$. Generally speaking, the theory of Lie groups and algebras is in charge with the classification of such beasts.

Finally, and getting now to the point, let us discuss the general finite groups. As basic example here you have the symmetric group S_N, whose main properties are as follows:

THEOREM 5.6. *The permutations of $\{1,\ldots,N\}$ form a group, denoted S_N, and called symmetric group. This group has $N!$ elements. The signature map*

$$\varepsilon : S_N \to \mathbb{Z}_2$$

can be regarded as being a group morphism, with values in $\mathbb{Z}_2 = \{\pm 1\}$.

Proof. The group property is indeed clear, and the count is clear as well, by recurrence on $N \in \mathbb{N}$. As for the last assertion, recall here the following formula for the signature of the permutations, which is something elementary to establish:

$$\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$$

But this tells us precisely that ε is a group morphism, as stated. □

As graph theorists, we will be particularly interested in the subgroups $G \subset S_N$. We have already met a few such subgroups, as follows:

Proposition 5.7. *We have finite non-abelian groups, as follows:*

(1) S_N, *the group of permutations of* $\{1, \ldots, N\}$.
(2) $A_N \subset S_N$, *the permutations having signature* 1.
(3) $D_N \subset S_N$, *the group of symmetries of the regular* N*-gon.*

Proof. The fact that we have indeed groups is clear from definitions, and the non-abelianity of these groups is clear as well, provided of course that in each case N is chosen big enough, and with exercise for you to work out all this, with full details. □

For constructing further examples of finite non-abelian groups, the best is to "look up", by regarding S_N as being the permutation group of the N coordinate axes of $\mathbb{R}^N$. Indeed, this suggests looking at the symmetry groups of other geometric beasts inside $\mathbb{R}^N$, or even $\mathbb{C}^N$, and we end up with a whole menagery of groups, as follows:

Theorem 5.8. *We have groups of unitary matrices as follows,*

$$\begin{tikzcd}
 & H_N \arrow[rr] & & K_N \\
S_N \arrow[ur] \arrow[rr] & & S_N \arrow[ur] & \\
 & SH_N \arrow[uu] \arrow[rr] & & SK_N \arrow[uu] \\
A_N \arrow[uu] \arrow[ur] \arrow[rr] & & A_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

for the most finite, and non-abelian, called complex reflection groups.

Proof. This statement is of course something informal, and here are explanations on all this, including definitions for all the groups involved:

(1) To start with, S_N is the symmetric group S_N that we know, but regarded now as permutation group of the N coordinate axes of $\mathbb{R}^N$, and so as subgroup $S_N \subset O_N$.

(2) Similarly, A_N is the alternating group A_N that we know, but coming now geometrically, as $A_N = S_N \cap SO_N$, with the intersection being computed inside O_N.

(3) Regarding $H_N \subset O_N$, this is a famous group, called hyperoctahedral group, appearing as the symmetry group of the hypercube $\square_N \subset \mathbb{R}^N$.

(4) Regarding $K_N \subset U_N$, this is the complex analogue of H_N, consisting of the unitary matrices $U \in U_N$ having exactly one nonzero entry, on each row and each column.

(5) We have as well on our diagram the groups SH_N, SK_N, with S standing as usual for "special", that is, consisting of the matrices in H_N, K_N having determinant 1.

(6) In what regards now the diagram itself, sure I can see that S_N, A_N appear twice, but nothing can be done here, after thinking a bit, at how the diagram works.

(7) Let us mention too that the groups $\mathbb{Z}_N, D_N$ have their place here, in N-dimensional geometry, but not exactly on our diagram, as being the symmetry groups of the oriented cycle, and unoriented cycle, with vertices at the simplex $X_N = \{e_i\} \subset \mathbb{R}^N$.

(8) Finally, in what regards the finiteness, non-abelianity, and also the name "complex reflection groups", many things to be checked here, left to you as an exercise. $\square$

Very nice all this. Let us summarize this group theory discussion as follows:

CONCLUSION 5.9. *All groups, or almost, are best seen as being groups of matrices. And even as groups of unitary matrices, in most cases.*

Observe in particular that this justifies our choice in Definition 5.2, for the group operation to be denoted multiplicatively, $\times$. Indeed, in most cases, in view of the above general principle, that abstract multiplication is in fact a matrix multiplication.

Thinking a bit at the above principle, that is so useful in practice, taking us away from the abstraction of Definition 5.2, that we would like to have it as a theorem:

QUESTION 5.10. *Can we make our general "most groups are in fact groups of matrices" principle, into an abstract theorem?*

In general, this is a quite subtle question, related to advanced group theory, such as Lie groups and algebras, and representation theory for them, and we will have a taste of such things in chapter 7 below, when developing the Peter-Weyl theory for the finite groups, that we will need at that time, in connection with our regular graph business.

The point however is that, in the finite group case that we are interested in, the answer to Question 5.10 is indeed "yes", thanks to two well-known theorems, which are both elementary. First we have the Cayley embedding theorem, which is as follows:

THEOREM 5.11. *Given a finite group G, we have an embedding as follows,*

$$G \subset S_N \quad , \quad g \to (h \to gh)$$

with $N = |G|$. Thus, any finite group is a permutation group.

PROOF. Given a group element $g \in G$, we can associate to it the following map:

$$\sigma_g : G \to G \quad , \quad h \to gh$$

Since $gh = gh'$ implies $h = h'$, this map is bijective, and so is a permutation of G, viewed as a set. Thus, with $N = |G|$, we can view this map as a usual permutation, $\sigma_G \in S_N$. Summarizing, we have constructed a map as follows:

$$G \to S_N \quad , \quad g \to \sigma_g$$

Our first claim is that this is a group morphism. Indeed, this follows from:

$$\sigma_g \sigma_h(k) = \sigma_g(hk) = ghk = \sigma_{gh}(k)$$

It remains to prove that this group morphism is injective. But this follows from:

$$\begin{aligned} g \neq h \quad &\implies \quad \sigma_g(1) \neq \sigma_h(1) \\ &\implies \quad \sigma_g \neq \sigma_h \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a comment, the above Cayley embedding theorem, while being certainly very beautiful at the theoretical level, has two weaknesses. First is the fact that the embedding $G \subset S_N$ that we constructed depends on a particular writing $G = \{g_1, \ldots, g_N\}$, which is needed in order to identify the permutations of G with the elements of the symmetric group S_N. Which in practice, is of course something which is not very good.

As a second point of criticism, the Cayley theorem often provides us with a "wrong embedding" of our group. Indeed, as an illustration here, for the basic examples of groups that we know, the Cayley theorem provides us with embeddings as follows:

$$\mathbb{Z}_N \subset S_N \quad , \quad D_N \subset S_{2N} \quad , \quad S_N \subset S_{N!} \quad , \quad H_N \subset S_{2^N N!}$$

And, compare this with the "good embeddings" of these groups, which are:

$$\mathbb{Z}_N \subset S_N \quad , \quad D_N \subset S_N \quad , \quad S_N \subset S_N \quad , \quad H_N \subset S_{2N}$$

Thus, the first Cayley embedding is the good one, the second one is not the best possible one, but can be useful, and the third and fourth embeddings are obviously useless. So, as a conclusion, the Cayley theorem remains something quite theoretical.

Nevermind. We will fix this, once we will know more. Going ahead now, as previously mentioned, the Cayley theorem is just half of the story, the other half being:

THEOREM 5.12. *We have a group embedding as follows, obtained by regarding S_N as permutation group of the N coordinate axes of $\mathbb{R}^N$,*

$$S_N \subset O_N$$

which makes a permutation $\sigma \in S_N$ correspond to the matrix having 1 on row i and column $\sigma(i)$, for any i, and having 0 entries elsewhere.

PROOF. The first assertion is clear, because the permutations of the N coordinate axes of $\mathbb{R}^N$ are isometries. Regarding now the explicit formula, we have by definition:

$$\sigma(e_j) = e_{\sigma(j)}$$

Thus, the permutation matrix corresponding to σ is given by:

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

Thus, we are led to the formula in the statement. □

We can combine the above result with the Cayley theorem, and we obtain the following result, which is something very nice, of obvious theoretical importance:

THEOREM 5.13. *Given a finite group G, we have an embedding as follows,*

$$G \subset O_N \quad , \quad g \to (e_h \to e_{gh})$$

with $N = |G|$. Thus, any finite group is an orthogonal matrix group.

PROOF. The Cayley theorem gives an embedding as follows:

$$G \subset S_N \quad , \quad g \to (h \to gh)$$

On the other hand, Theorem 5.12 provides us with an embedding as follows:

$$S_N \subset O_N \quad , \quad \sigma \to (e_i \to e_{\sigma(i)})$$

Thus, we are led to the conclusion in the statement. □

The same remarks as for the Cayley theorem apply. First, the embedding $G \subset O_N$ that we constructed depends on a particular writing $G = \{g_1, \ldots, g_N\}$. And also, for the basic examples of groups that we know, the embeddings that we obtain are as follows:

$$\mathbb{Z}_N \subset O_N \quad , \quad D_N \subset O_{2N} \quad , \quad S_N \subset O_{N!} \quad , \quad H_N \subset O_{2^N N!}$$

And, compare this with the "good embeddings" of these groups, which are:

$$\mathbb{Z}_N \subset O_N \quad , \quad D_N \subset O_N \quad , \quad S_N \subset O_N \quad , \quad H_N \subset O_N$$

As before with the abstract group embeddings $G \subset S_N$ coming from Cayley, the first abstract embedding is the good one, the second one is not the best possible one, but can be useful, and the third and fourth abstract embeddings are obviously useless.

The problem is now, how to fix this, as to have a theorem providing us with good embeddings? And a bit of thinking at all the above leads to the following conclusion:

CONCLUSION 5.14. *The weak point in all the above is Cayley.*

So, leaving aside now Cayley, or rather putting it in our pocket, matter of having it there, handy when meeting abstract questions, let us focus on Theorem 5.12, which is the only reasonable theorem that we have. In relation with the basic groups, we have:

THEOREM 5.15. *We have the following finite groups of matrices:*

(1) $\mathbb{Z}_N \subset O_N$, *the cyclic permutation matrices.*
(2) $D_N \subset O_N$, *the dihedral permutation matrices.*
(3) $S_N \subset O_N$, *the permutation matrices.*
(4) $H_N \subset O_N$, *the signed permutation matrices.*

PROOF. This is something self-explanatory, the idea being that Theorem 5.12 provides us with embeddings as follows, given by the permutation matrices:

$$\mathbb{Z}_N \subset D_N \subset S_N \subset O_N$$

In addition, looking back at the definition of H_N, this group inserts into the embedding on the right, $S_N \subset H_N \subset O_N$. Thus, we are led to the conclusion that all our 4 groups appear as groups of suitable "permutation type matrices". To be more precise:

(1) The cyclic permutation matrices are by definition the matrices as follows, with 0 entries elsewhere, and form a group, which is isomorphic to the cyclic group $\mathbb{Z}_N$:

$$U = \begin{pmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{pmatrix}$$

(2) The dihedral matrices are the above cyclic permutation matrices, plus some suitable symmetry permutation matrices, and form a group which is isomorphic to D_N.

(3) The permutation matrices, which by Theorem 5.12 form a group which is isomorphic to S_N, are the $0-1$ matrices having exactly one 1 on each row and column.

(4) Finally, regarding the signed permutation matrices, these are by definition the $(-1)-0-1$ matrices having exactly one nonzero entry on each row and column, and by Theorem 5.8 these matrices form a group, which is isomorphic to H_N. $\square$

The above groups are all groups of orthogonal matrices. When looking into general unitary matrices, we led to the following interesting class of groups:

DEFINITION 5.16. *The complex reflection group $H_N^s \subset U_N$, depending on parameters*

$$N \in \mathbb{N} \quad , \quad s \in \mathbb{N} \cup \{\infty\}$$

is the group of permutation-type matrices with s-th roots of unity as entries,

$$H_N^s = M_N(\mathbb{Z}_s \cup \{0\}) \cap U_N$$

with the convention $\mathbb{Z}_\infty = \mathbb{T}$, at $s = \infty$.

Observe that at $s < \infty$, the above groups are finite. Also, at $s = 1, 2, \infty$ we obtain the following groups, that we already met in the above:

$$H_N^1 = S_N \quad , \quad H_N^2 = H_N \quad , \quad H_N^\infty = K_N$$

We will be back later in this chapter with more details about H_N^s. However, at the philosophical level, we have extended our basic series of finite groups, as follows:

$$\mathbb{Z}_N \subset D_N \subset S_N \subset H_N \subset H_N^4 \subset H_N^8 \subset \ldots\ldots \subset U_N$$

But all this looks a bit complicated, so for being even more philosophers, let restrict the attention to the cases $s = 1, 2, \infty$, with $H_N^\infty = K_N$ having already been adopted, as a kind of "almost" finite group. With this convention, the conclusion is that we have extended our series of basic finite groups, regarded as unitary groups, as follows:

$$\mathbb{Z}_N \subset D_N \subset S_N \subset H_N \subset K_N \subset U_N$$

And good news, this will be our final say on the subject. Time now to formulate a grand conclusion to what we did so far in this chapter, as follows:

Grand conclusion 5.17. *Group theory can be understood as follows:*

1. *Most interesting groups are groups of matrices, $G \subset GL_N(\mathbb{C})$.*
2. *Quite often, these matrices can be chosen to be unitaries, $G \subset U_N$.*
3. *The Cayley theorem tells us that $G \subset S_N \subset O_N$, with $N = |G|$.*
4. *Most interesting finite groups appear as $G \subset U_N$, with $N << |G|$.*

Which is very nice, at least we know one thing, and with this type of talisman, we can now safely navigate through the abstractions of group theory. Of course, we will be back to this, once we will know more about groups, and regularly update our conclusions.

5b. Graph symmetries

Moving now towards graph theory, that we are interested in, and which will in fact confirm and fine-tune the above conclusions, we have the following construction:

Theorem 5.18. *Given a finite graph X, with vertices denoted $1, \ldots, N$, the symmetries of X, which are the permutations $\sigma \in S_N$ leaving invariant the edges,*

$$i - j \implies \sigma(i) - \sigma(j)$$

form a subgroup of the symmetric group, as follows, called symmetry group of X:

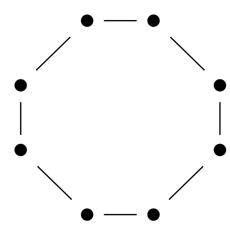
$$G(X) \subset S_N$$

As basic examples, for the empty graph, or for the simplex, we have $G(X) = S_N$.

Proof. Here the first assertion, regarding the group property of $G(X)$, is clear from definitions, because the symmetries of X are stable under composition. The second assertion, regarding the empty graph and the simplex, is clear as well. So done, everything being trivial, and we have called this Theorem instead of Proposition because the construction $X \to G(X)$ will keep us busy, for the remainder of this book. $\square$

Let us work out now some more examples. As a first result, dealing with the simplest graph ever, passed the empty graphs and the simplices, we have:

PROPOSITION 5.19. *The symmetry group of the regular N-gon*



is the dihedral group $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$.

PROOF. This is something that we know well from chapter 3, and with the remark, which is something new, that the notation D_N for the group that we get, which is the correct one, is justified by the general group theory discussion before, with N standing for the natural "dimensionality" of this group. To be more precise, geometrically speaking, the regular N-gon is best viewed in $\mathbb{R}^N$, with vertices $1, \ldots, N$ at the standard basis:

$$1 = (1, 0, 0, \ldots, 0, 0)$$

$$2 = (0, 1, 0, \ldots, 0, 0)$$

$$\vdots$$

$$N = (0, 0, 0, \ldots, 0, 1)$$

But, with this interpretation in mind, we are led to an embedding as follows:

$$D_N \subset S_N \subset O_N$$

We conclude from this that N is the correct dimensionality of our group, and so is the correct label to be attached to the dihedral symbol D. Of course, you might find this overly philosophical, or even a bit futile, but listen to this, there are two types of mathematicians in this world, those who use D_N and those who use D_{2N}, and do not ask me why, but it is better to be in the first category, mathematicians using D_N. $\square$

Moving ahead, the problem is now, is Proposition 5.19 good news, or bad news? I don't know about you, but personally I feel quite frustrated by the fact that the computation there leads to $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$, instead to $\mathbb{Z}_N$ itself. I mean, how can a theory be serious, if there is no room there, or even an Emperor's throne, for the cyclic group $\mathbb{Z}_N$.

So, let us fix this. It is obvious that the construction in Theorem 5.18 will work perfectly well for the oriented graphs, or for the colored graphs, so let us formulate:

Definition 5.20. *Given a generalized graph X, with vertices denoted $1,\ldots,N$, the symmetries of X, which are the permutations $\sigma\in S_N$ leaving invariant the edges,*

$$i-j\implies\sigma(i)-\sigma(j)$$

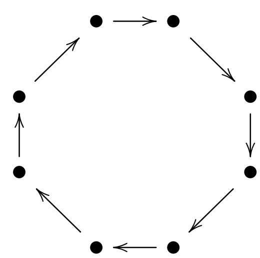
with their orientations and colors, form a subgroup of the symmetric group

$$G(X)\subset S_N$$

called symmetry group of X.

Here, as before with the construction in Theorem 5.18, the fact that we obtain indeed a group is clear from definitions. Now with this convention in hand, we have:

Proposition 5.21. *The symmetry group of the oriented N-gon*



is the cyclic group $\mathbb Z_N$.

Proof. This is clear from definitions, because once we choose a vertex i and denote its image by $\sigma(i)=i+k$, the permutation $\sigma\in S_N$ leaving invariant the edges, with their orientation, must map $\sigma(i+1)=i+k+1$, $\sigma(i+2)=i+k+2$ and so on, and so must be an element of the cyclic group, in remainder modulo N notation $\sigma=k\in\mathbb Z_N$. □

With this done, and the authority of $\mathbb Z_N$ restored, let us work out some general properties of the construction $X\to G(X)$. For simplicity we will restrict the attention to the usual graphs, as in Theorem 5.18, but pretty much everything will extend to the case of oriented or colored graphs. In fact, our policy in what follows will be that of saying nothing when things extend, and making a comment, when things do not extend.

As a first general result, coming as a useful complement to Theorem 5.18, we have:

Theorem 5.22. *Having a group action on a graph $G\curvearrowright X$ is the same as saying that the action of G leaves invariant the adjacency matrix d, in the sense that:*

$$d_{ij}=d_{g(i)g(j)}\quad,\quad\forall g\in G$$

Equivalently, the action must preserve the spectral projections of d:

$$d=\sum_\lambda\lambda P_\lambda\implies(P_\lambda)_{ij}=(P_\lambda)_{g(i)g(j)}$$

Thus, the symmetry group $G(X)\subset S_N$ is the subgroup preserving the eigenspaces of d.

PROOF. As before with Theorem 5.18, a lot of talking in the statement, with everything being trivial, coming from definitions, and with the statement itself being called Theorem instead of Proposition just due to its theoretical importance. □

Observe that Theorem 5.22 naturally leads us into colored graphs, because while the adjacency matrix is symmetric and binary, $d \in M_N(0,1)^{symm}$, the spectral projections P_λ are also symmetric, but no longer binary, $P_\lambda \in M_N(\mathbb{R})^{symm}$. Moreover, these spectral projections P_λ can have 0 on the diagonal, pushing us into allowing self-edges in our colored graph formalism. We are led in this way to the following statement:

THEOREM 5.23. *Having a group action on a colored graph $G \curvearrowright X$ is the same as saying that the action of G leaves invariant the adjacency matrix d:*

$$d_{ij} = d_{g(i)g(j)} \quad , \quad \forall g \in G$$

Equivalently, the action must preserve the spectral projections of d, as follows:

$$d = \sum_\lambda \lambda P_\lambda \implies (P_\lambda)_{ij} = (P_\lambda)_{g(i)g(j)}$$

Moreover, when allowing self-edges, each P_λ will correspond to a colored graph X_λ.

PROOF. This follows indeed from the above discussion, and with some extra discussion regarding the precise colors that we use, as follows:

(1) When using real colors, the result follows from the linear algebra result regarding the diagonalization of real symmetric matrices, which tells us that the spectral projections of any such matrix $d \in M_N(\mathbb{R})^{symm}$ are also real and symmetric, $P_\lambda \in M_N(\mathbb{R})^{symm}$.

(2) When using complex colors, the result follows from the linear algebra result regarding the diagonalization of complex self-adjoint matrices, which tells us that the spectral projections of any such matrix $d \in M_N(\mathbb{C})^{sa}$ are also self-adjoint, $P_\lambda \in M_N(\mathbb{C})^{sa}$. □

The point with the perspective brought by the above results is that, when using permutation group tools for the study of the groups $G \subset S_N$ acting on our graph, $G \curvearrowright X$, what will eventually happen is that these tools, once sufficiently advanced, will become very close to the regular tools for the study of d, namely the same sort of mixture of linear algebra, calculus and probability, so in the end we will have a unified theory.

But probably too much talking, just trust me, we won't be doing groups and algebra here just because we are scared by analysis, and by the true graph problems. Quite the opposite. And we will see illustrations for this harmony and unity later on.

Leaving now the oriented or colored graphs aside, as per our general graph policy explained above, as a second general result about $X \to G(X)$, we have:

THEOREM 5.24. *The construction $X \to G(X)$ has the property*

$$G(X) = G(X^c)$$

where $X \to X^c$ is the complementation operation.

PROOF. This is clear from the construction of $G(X)$ from Theorem 5.18, and follows as well from the interpretation in Theorem 5.22, because the adjacency matrices of X, X^c are related by the following formula, where $\mathbb{I}_N$ is the all-one matrix:

$$d_X + d_{X^c} = \mathbb{I}_N - 1_N$$

Indeed, since on the right we have the adjacency matrix of the simplex, which commutes with everything, commutation with d_X is equivalent to commutation with d_{X^c}, and this gives the result, via the interpretation of $G(X)$ coming from Theorem 5.22. $\square$

In order to reach now to more advanced results, it is convenient to enlarge the attention to the colored graphs. Indeed, for the colored graphs, we can formulate:

THEOREM 5.25. *Having an action on a colored graph $G \curvearrowright X$ is the same as saying that the action leaves invariant the color components of X. Equivalently, with*

$$d = \sum_{c \in C} c d_c$$

being the color decomposition of the adjacency matrix, with color components

$$(d_c)_{ij} = \begin{cases} 1 & \text{if } d_{ij} = c \\ 0 & \text{otherwise} \end{cases}$$

the action must leave invariant all these color components d_c. Thus, the symmetry group $G(X) \subset S_N$ is the subgroup which preserves all these matrices d_c.

PROOF. As before with our other statements here, in the present section of this book, a lot of talking in the statement, with everything there being trivial. $\square$

I have this feeling that you might get to sleep, on the occasion of the present section, which is overly theoretical, this is how things are, we have to have some theory started, right. But, in the case it is so, I have something interesting for you, in relation with the above. Indeed, by combining Theorem 5.23 with Theorem 5.25, both trivialities, we are led to the following enigmatic statement, which all of the sudden wakes us up:

THEOREM 5.26. *Given an adjacency matrix of a graph X, which can be taken in a colored graph sense, $d \in M_N(\mathbb{C})$, or even binary as usual,*

$$d \in M_N(0,1)$$

a group action $G \curvearrowright X$ must preserve all "spectral-color" components of this matrix, obtained by succesively applying the spectral decomposition, and color decomposition.

PROOF. This is clear indeed by combining Theorem 5.23 and Theorem 5.25, and with the remark that, indeed, even for a usual binary matrix $d \in M_N(0,1)$ this leads to something non-trivial, because the spectral components of this matrix are no longer binary, and so all of the sudden, we are into colors and everything. $\square$

With the above result in hand, which is something quite unexpected, we are led into a quite interesting linear algebra question, which is surely new for you, namely:

QUESTION 5.27. *What are the spectral-color components of a matrix $d \in M_N(\mathbb{C})$, or even of a usual binary matrix $d \in M_N(0,1)$?*

This question is something non-trivial, and we will be back to this on several occasions, and notably in chapter 12 below when talking planar algebras in the sense of Jones [**56**], which provide the good framework for the study of such questions.

To be more precise, coming a bit in advance, we will see there that computing the spectral-color components of a matrix $d \in M_N(\mathbb{C})$ is the same as computing the planar algebra generated by d, viewed as 2-box inside the tensor planar algebra.

5c. Reflection groups

We already know a number of things about the circulant graphs from chapters 2-3, and we also know that these usually generalize to the case where we have a transitive action of an abelian group on X. Both the group theory and the linear algebra aspects here can be quite subtle, involving the classification of finite abelian groups, and generalized Fourier matrices, and that discussion from chapters 2-3 can be briefly summarized as follows:

FACT 5.28. *The transitive abelian group actions on graphs, $G \curvearrowright X$ with*

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_s}$$

can be investigated via discrete Fourier analysis, over the group G.

This is of course something very compact, and we refer to chapters 2-3 for details. In order to advance now on all this, we have the following result, which is standard in discrete Fourier analysis, extending what we previously knew in the circulant case:

THEOREM 5.29. *For a matrix $A \in M_N(\mathbb{C})$, the following are equivalent,*

(1) *A is G-invariant, $A_{ij} = \xi_{j-i}$, for a certain vector $\xi \in \mathbb{C}^N$,*
(2) *A is Fourier-diagonal, $A = F_G Q F_G^*$, for a certain diagonal matrix Q,*

and if so, $\xi = F_G^* q$, where $q \in \mathbb{C}^N$ is the vector formed by the diagonal entries of Q.

PROOF. This is something that we know from chapter 3 in the cyclic case, $G = \mathbb{Z}_N$, and the proof in general is similar, by using matrix indices as follows:

$$i, j \in G$$

To be more precise, in order to get started, with our generalization, let us decompose our finite abelian group G as a product of cyclic groups, as follows:

$$G=\mathbb{Z}_{N_1}\times\ldots\times\mathbb{Z}_{N_s}$$

The corresponding Fourier matrix decomposes then as well, as follows:

$$F_G=F_{N_1}\otimes\ldots\otimes F_{N_s}$$

Now if we set $w_i=e^{2\pi i/N_i}$, this means that we have the following formula:

$$(F_G)_{ij}=w_1^{i_1j_1}\ldots w_s^{i_sj_s}$$

We can now prove the equivalence in the statement, as follows:

(1) $\implies$ (2) Assuming $A_{ij}=\xi_{j-i}$, the matrix $Q=F_G^*AF_G$ is diagonal, as shown by the following computation, with all indices being group elements:

$$\begin{aligned}
Q_{ij}&=\sum_{kl}\overline{(F_G)}_{ki}A_{kl}(F_G)_{lj}\\
&=\sum_{kl}w_1^{-k_1i_1}\ldots w_s^{-k_si_s}\cdot\xi_{l-k}\cdot w_1^{l_1j_1}\ldots w_s^{l_sj_s}\\
&=\sum_{kl}w_1^{l_1j_1-k_1i_1}\ldots w_s^{l_sj_s-k_si_s}\xi_{l-k}\\
&=\sum_{kr}w_1^{(k_1+r_1)j_1-k_1i_1}\ldots w_s^{(k_s+r_s)j_s-k_si_s}\xi_r\\
&=\sum_r w_1^{r_1j_1}\ldots w_s^{r_sj_s}\xi_r\sum_k w_1^{k_1(j_1-i_1)}\ldots w_s^{k_s(j_s-i_s)}\\
&=\sum_r w_1^{r_1j_1}\ldots w_s^{r_sj_s}\xi_r\cdot N_1\delta_{i_1j_1}\ldots N_s\delta_{i_sj_s}\\
&=N\delta_{ij}\sum_r(F_G)_{jr}\xi_r
\end{aligned}$$

(2) $\implies$ (1) Assuming $Q=diag(q_1,\ldots,q_N)$, the matrix $A=F_GQF_G^*$ is G-invariant, as shown by the following computation, again with all indices being group elements:

$$\begin{aligned}
A_{ij}&=\sum_{kl}(F_G)_{ik}Q_{kk}\overline{(F_G)}_{kj}\\
&=\sum_k w_1^{i_1k_1}\ldots w_s^{i_sk_s}\cdot q_k\cdot w_1^{-j_1k_1}\ldots w_s^{-j_sk_s}\\
&=\sum_k w_1^{(i_1-j_1)k_1}\ldots w_s^{(i_s-j_s)k_s}q_k
\end{aligned}$$

To be more precise, in this formula the last term depends only on $j-i$, and so shows that we have $A_{ij}=\xi_{j-i}$, with ξ being the following vector:

$$\begin{aligned}\xi_i &= \sum_k w_1^{-i_1k_1}\ldots w_s^{-i_sk_s}q_k\\ &= \sum_k (F_G^*)_{ik}q_k\\ &= (F_G^*q)_i\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As another generalization of what we did in chapter 3, in relation now with the dihedral groups, and then with the hyperoctahedral groups, we can investigate the complex reflection groups H_N^s that we introduced in the above. Let us recall indeed that $H_N^s\subset U_N$ is the group of permutation-type matrices with s-th roots of unity as entries:

$$H_N^s=M_N(\mathbb Z_s\cup\{0\})\cap U_N$$

We know that at $s=1,2$, this group equals S_N,H_N, that we both know well. In analogy with what we know at $s=1,2$, we first have the following result:

Proposition 5.30. *The number of elements of H_N^s with $s\in\mathbb N$ is:*

$$|H_N^s|=s^NN!$$

At $s=\infty$, the group $K_N=H_N^\infty$ that we obtain is infinite.

Proof. This is indeed clear from our definition of H_N^s, as a matrix group as above, because there are $N!$ choices for a permutation-type matrix, and then s^N choices for the corresponding s-roots of unity, which must decorate the N nonzero entries. □

Once again in analogy with what we know at $s=1,2$, we have as well:

Theorem 5.31. *We have a wreath product decomposition as follows,*

$$H_N^s=\mathbb Z_s\wr S_N$$

which means by definition that we have a crossed product decomposition

$$H_N^s=\mathbb Z_s^N\rtimes S_N$$

with the permutations $\sigma\in S_N$ acting via $\sigma(e_1,\ldots,e_k)=(e_{\sigma(1)},\ldots,e_{\sigma(k)})$.

Proof. As explained in the proof of Proposition 5.30, the elements of H_N^s can be identified with the pairs $g=(e,\sigma)$ consisting of a permutation $\sigma\in S_N$, and a decorating vector $e\in\mathbb Z_s^N$, so that at the level of the cardinalities, we have:

$$|H_N|=|\mathbb Z_s^N\times S_N|$$

Now observe that the product formula for two such pairs $g = (e, \sigma)$ is as follows, with the permutations $\sigma \in S_N$ acting on the elements $f \in \mathbb{Z}_s^N$ as in the statement:

$$(e, \sigma)(f, \tau) = (ef^\sigma, \sigma\tau)$$

Thus, we are in the framework of crossed products, and we obtain $H_N^s = \mathbb{Z}_s^N \rtimes S_N$. But this can be written, by definition, as $H_N^s = \mathbb{Z}_s \wr S_N$, and we are done. □

In relation with graph symmetries, the above groups appear as follows:

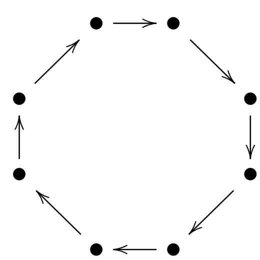
THEOREM 5.32. *The complex reflection group H_N^s appears as symmetry group,*

$$H_N^s = G(NC_s)$$

with NC_s consisting of N disjoint copies of the oriented cycle C_s.

PROOF. This is something elementary, the idea being as follows:

(1) Consider first the oriented cycle C_s, which looks as follows:



It is then clear that the symmetry group of this graph is the cyclic group $\mathbb{Z}_s$.

(2) In the general case now, where we have $N \in \mathbb{N}$ disjoint copies of the above cycle C_s, we must suitably combine the corresponding N copies of the cyclic group $\mathbb{Z}_s$. But this leads to the wreath product group $H_N^s = \mathbb{Z}_s \wr S_N$, as stated. □

Normally, with the theory that we have so far, we can investigate the graphs having small number of vertices. But for more here, going beyond what we have, we need more product results, and we will develop the needed theory in the next chapter.

5d. Partial symmetries

As a final topic for this theoretical chapter, let us discuss as well the partial symmetries of graphs. To be more precise, we will associate to any graph X having N vertices its semigroup of partial permutations $\widetilde{G}(X) \subset \widetilde{S}_N$, which is a quite interesting object.

In order to discuss all this, let us start with something well-known, namely:

Definition 5.33. *$\widetilde{S}_N$ is the semigroup of partial permutations of $\{1\ldots,N\}$,*

$$\widetilde{S}_N=\left\{\sigma:I\simeq J\Big|I,J\subset\{1,\ldots,N\}\right\}$$

with the usual composition operation for such partial permutations, namely

$$\sigma'\sigma:\sigma^{-1}(I'\cap J)\simeq\sigma'(I'\cap J)$$

being the composition of $\sigma':I'\simeq J'$ and $\sigma:I\simeq J$.

Observe that $\widetilde{S}_N$ is not simplifiable, because the null permutation $\emptyset\in\widetilde{S}_N$, having the empty set as domain/range, satisfies the following formula, for any $\sigma\in\widetilde{S}_N$:

$$\emptyset\sigma=\sigma\emptyset=\emptyset$$

Observe also that our semigroup $\widetilde{S}_N$ has a kind of "subinverse" map, which is not a true inverse in the semigroup sense, sending a partial permutation $\sigma:I\to J$ to its usual inverse $\sigma^{-1}:J\to I$. Many other algebraic things can be said, along these lines.

As a first interesting result now about $\widetilde{S}_N$, which shows that we are dealing here with some non-trivial combinatorics, not really present in the S_N context, we have:

Theorem 5.34. *The number of partial permutations is given by*

$$|\widetilde{S}_N|=\sum_{k=0}^N k!\binom{N}{k}^2$$

that is, $1,2,7,34,209,\ldots,$ *and we have the formula*

$$|\widetilde{S}_N|\simeq N!\sqrt{\frac{\exp(4\sqrt{N}-1)}{4\pi\sqrt{N}}}$$

in the $N\to\infty$ limit.

Proof. This is a mixture of trivial and non-trivial facts, as follows:

(1) The first assertion is clear, because in order to construct a partial permutation $\sigma:I\to J$ we must choose an integer $k=|I|=|J|$, then we must pick two subsets $I,J\subset\{1,\ldots,N\}$ having cardinality k, and there are $\binom{N}{k}$ choices for each, and finally we must construct a bijection $\sigma:I\to J$, and there are $k!$ choices here.

(2) As for the estimate, which is non-trivial, this is something standard, and well-known, and exercise for you here to look that up, and read the proof. $\square$

Another result, which is trivial, but is quite fundamental, is as follows:

Proposition 5.35. *We have a semigroup embedding $u:\widetilde{S}_N\subset M_N(0,1)$, given by*

$$u_{ij}(\sigma)=\begin{cases}1&\text{if }\sigma(j)=i\\0&\text{otherwise}\end{cases}$$

whose image are the matrices having at most one nonzero entry, on each row and column.

Proof. This is something which is trivial from definitions, with the above embedding $u:\widetilde{S}_N\subset M_N(0,1)$ extending the standard embedding $u:S_N\subset M_N(0,1)$, given by the permutation matrices, that we have been heavily using, so far. $\square$

Also at the algebraic level, we have the following simple and useful fact:

Proposition 5.36. *Any partial permutation $\sigma:I\simeq J$ can be factorized as*

$$\begin{tikzcd}
I \arrow[r, "\sigma"] \arrow[d, "\gamma"'] & J \\
\{1,\ldots,k\} \arrow[r, "\beta"'] & \{1,\ldots,k\} \arrow[u, "\alpha"']
\end{tikzcd}$$

with $\alpha,\beta,\gamma\in S_k$ being certain non-unique permutations, where $k=|Dom(\sigma)|$.

Proof. We can choose indeed any two bijections $I\simeq\{1,\ldots,k\}$ and $\{1,\ldots,k\}\simeq J$, and then complete them up to permutations $\gamma,\alpha\in S_N$. The remaining permutation $\beta\in S_k$ is then uniquely determined by the formula $\sigma=\alpha\beta\gamma$. $\square$

The above result is quite interesting, theoretically speaking, allowing us to reduce in principle questions about $\widetilde{S}_N$ to questions about the usual symmetric groups S_k, via some sort of homogeneous space procedure. We will be back to this, later in this book.

In relation now with the graphs, we have the following notion:

Definition 5.37. *Associated to any graph X is its partial symmetry semigroup*

$$\widetilde{G}(X)\subset\widetilde{S}_N$$

consisting of the partial permutations $\sigma\in\widetilde{S}_N$ whose action preserves the edges.

As a first observation, we have the following result, which provides an alternative to the above definition of the partial symmetry semigroup $\widetilde{G}(X)$:

Proposition 5.38. *Given a graph X with N vertices, we have*

$$\widetilde{G}(X)=\left\{\sigma\in\widetilde{S}_N\Big|d_{ij}=d_{\sigma(i)\sigma(j)},\ \forall i,j\in Dom(\sigma)\right\}$$

with $d\in M_N(0,1)$ being as usual the adjacency matrix of X.

PROOF. The construction of $\widetilde{G}(X)$ from Definition 5.37 reformulates as follows, in terms of the usual adjacency relation $i - j$ for the vertices:

$$\widetilde{G}(X) = \left\{\sigma \in \widetilde{S}_N \middle| i - j, \exists\, \sigma(i), \exists\, \sigma(j) \implies \sigma(i) - \sigma(j)\right\}$$

But this leads to the formula in the statement, in terms of the adjacency matrix d. □

In order to discuss now some examples, let us make the following convention:

DEFINITION 5.39. *In the context of the partial permutations $\sigma : I \to J$, with $I, J \subset \{1, \ldots, N\}$, we decompose the domain set I as a disjoint union*

$$I = I_1 \sqcup \ldots \sqcup I_p$$

with each I_r being an interval consisting of consecutive numbers, and being maximal with this property, and with everything being taken cyclically.

In other words, we represent the domain set $I \subset \{1, \ldots, N\}$ on a circle, with 1 following $1, \ldots, N$, and then we decompose it into intervals, in the obvious way. With this convention made, in the case of the oriented cycle, we have the following result:

PROPOSITION 5.40. *For the oriented cycle C_N^o we have*

$$\widetilde{G}(C_N^o) = \widetilde{\mathbb{Z}}_N$$

with the semigroup on the right consisting of the partial permutations

$$\sigma : I_1 \sqcup \ldots \sqcup I_p \to J$$

which are cyclic on any I_r, given there by $i \to i + k_r$, for a certain $k_r \in \{1, \ldots, N\}$.

PROOF. According to the definition of $\widetilde{G}(X)$, we have the following formula:

$$\widetilde{G}(C_N^o) = \left\{\sigma \in \widetilde{S}_N \middle| d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i, j \in Dom(\sigma)\right\}$$

On the other hand, the adjacency matrix of C_N^o is given by:

$$d_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the condition defining $\widetilde{G}(C_N^o)$ is as follows:

$$j = i + 1 \iff \sigma(j) = \sigma(i) + 1,\ \forall i, j \in Dom(\sigma)$$

But this leads to the conclusion in the statement. □

In the case of the unoriented cycle, the result is as follows:

PROPOSITION 5.41. *For the unoriented cycle C_N we have*

$$\widetilde{G}(C_N)=\widetilde{D}_N$$

with the semigroup on the right consisting of the partial permutations

$$\sigma:I_1\sqcup\ldots\sqcup I_p\to J$$

which are dihedral on any I_r, given there by $i\to\pm_r i+k_r$, for a certain $k_r\in\{1,\ldots,N\}$, and a certain choice of the sign $\pm_r\in\{-1,1\}$.

PROOF. The proof here is similar to the proof of Proposition 5.40. Indeed, the adjacency matrix of C_N is given by:

$$d_{ij}=\begin{cases}1&\text{if }j=i\pm 1\\0&\text{otherwise}\end{cases}$$

Thus, the condition defining $\widetilde{G}(C_N)$ is as follows:

$$j=i\pm 1\iff\sigma(j)=\sigma(i)\pm 1,\ \forall i,j\in Dom(\sigma)$$

But this leads to the conclusion in the statement. $\square$

An interesting question is whether the semigroups $\widetilde{\mathbb{Z}}_N,\widetilde{D}_N$ are related by a formula similar to $D_N=\mathbb{Z}_N\rtimes\mathbb{Z}_2$. This is not exactly the case, at least with the obvious definition for the $\rtimes$ operation, because at the level of cardinalities we have:

THEOREM 5.42. *The cardinalities of $\widetilde{\mathbb{Z}}_N,\widetilde{D}_N$ are given by the formulae*

$$|\widetilde{\mathbb{Z}}_N|=1+NK_1(N)+\sum_{p=2}^{[N/2]}N^pK_p(N)$$

$$|\widetilde{D}_N|=1+NK_1(N)+\sum_{p=2}^{[N/2]}(2N)^pK_p(N)$$

where $K_p(N)$ counts the sets having p cyclic components, $I=I_1\sqcup\ldots\sqcup I_p$.

PROOF. The first formula is clear from the description of $\widetilde{\mathbb{Z}}_N$ from Proposition 5.40, because for any domain set $I=I_1\sqcup\ldots\sqcup I_p$, we have N choices for each scalar k_r, producing a cyclic partial permutation $i\to i+k_r$ on the interval I_r. Thus we have, as claimed:

$$|\widetilde{\mathbb{Z}}_N|=\sum_{p=0}^{[N/2]}N^pK_p(N)$$

In the case of $\widetilde{D}_N$ the situation is similar, with Proposition 5.41 telling us that the N choices at the level of each interval I_r must be now replaced by $2N$ choices, as to have a dihedral permutation $i\to\pm_r i+k_r$ there. However, this is true only up to a subtlety, coming from the fact that at $p = 1$ the choice of the ± 1 sign is irrelevant. Thus, we are led to the formula in the statement, with $2N$ factors everywhere, except at $p = 1$. $\square$

Summarizing, the partial symmetry group problematics leads to some interesting questions, even for simple graphs like C_N^o and C_N. We will be back to this.

5e. Exercises

Welcome to pure mathematics, we certainly had a pure mathematics chapter here, and of pure mathematics nature will be as well our exercises, as follows:

Exercise 5.43. *Read more about Lie groups, and notably about Sp_N.*

Exercise 5.44. *Read more about finite groups, notably about Sylow theorems.*

Exercise 5.45. *And even more about finite groups, with complex reflection groups.*

Exercise 5.46. *Read about discrete groups too, and random walks on them.*

Exercise 5.47. *Meditate at the spectral-color decomposition, and its planar aspects.*

Exercise 5.48. *Meditate at our various graph formalisms, their pros and cons.*

Exercise 5.49. *Meditate at what else can be done, with discrete Fourier analysis.*

Exercise 5.50. *Find a proof for the asymptotic estimate for $|\widetilde{S}_N|$.*

Exercise 5.51. *Work out more examples of semigroups $\widetilde{G}(X)$.*

As bonus exercise, learn some representation theory for finite groups. We will discuss this soon in this book, but the more you know a bit in advance, the better.

CHAPTER 6

Graph products

6a. Small graphs

We have seen in the previous chapter how to associate to any finite graph X its symmetry group $G(X)$, and we have seen as well some basic properties of the correspondence $X \to G(X)$. Motivated by this, our goal in this chapter will be that of systematically computing the symmetry groups $G(X)$, for as many graphs X that we can.

Easy task, you would say, because we already have $G(X)$ for all the N-gons, and since $\infty + n = \infty$, no need for new computations, in order to improve our results. Jokes left aside, we certainly need here some precise objectives and strategy, so let us formulate:

Goal 6.1. *Take the integers one by one, $N = 2, 3, 4, 5, 6, \ldots$ and look at all graphs of order $|X| = N$. For each such graph, find a decomposition of type*

$$X = Y \times Z$$

with $\times$ being a certain graph product, and $|Y|, |Z| < N$, then work out a formula of type

$$G(X) = G(Y) \times G(Z)$$

with $\times$ being a certain group product, adapted to the above graph product, as to solve the problem by recurrence. When stuck, find some ad-hoc methods for dealing with X.

This plan sounds quite reasonable, so let us see how this works. At very small values of $N = 2, 3, 4, 5, 6, \ldots$ things are quite clear, and in fact no even need here for a plan, because the symmetry group is just obvious. However, since we will deal with higher N afterwards, it is useful to resist the temptation of simply recording the value of $G(X)$, and stick to the above plan, or at least record the various graph products $\times$ that we meet on the way, and the corresponding adapted group products $\times$ too.

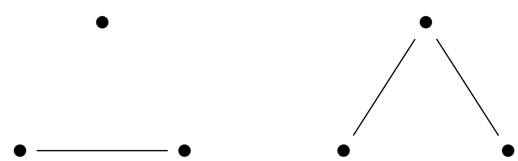
Getting started now, at $N = 2, 3$ everything is trivial, but let us record this:

Proposition 6.2. *At $N = 2, 3$ we have, with no product operations involved:*

1. *Two points $\bullet\ \bullet$ and the segment $\bullet - \bullet$, with symmetry group $\mathbb{Z}_2$.*
2. *Three points $\bullet\ \ \bullet\ \ \bullet$ and the triangle $\triangle$, with symmetry group S_3.*
3. *The graphs $\bullet - \bullet\ \ \ \bullet$ and $\bullet - \bullet - \bullet$, with symmetry group $\mathbb{Z}_2$.*

Proof. All this is self-explanatory, but us record however a few observations:

(1) All pairs of graphs in the above appear from each other via complementation, with this being obvious in all cases, save perhaps for (3), where the graphs are as follows:



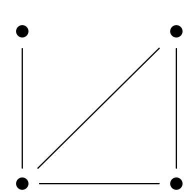
(2) Now since we have $G(X) = G(X^c)$, this suggests listing our graphs up to complementation. However, this is a quite bad idea, in practice, because believe me, you will end up sleeping bad at night, with thoughts of type damn, I forgot in my list this or that beautiful graph, only to realize later, after some computations in your head, that the graph in question was in fact complementary to a less beautiful graph, from your list.

(3) Another comment, subjective too, concerns the labeling of the groups that we found. For instance $\mathbb{Z}_1 = D_1 = S_1$, and $\mathbb{Z}_2 = D_2 = S_2$, and $D_3 = S_3$. Our policy will be that of regarding $\mathbb{Z}_N$ as the simplest group, because this is what this group is, and using it preferentially. Followed by S_N, because at least in questions regarding permutation groups, this is the second simplest group. As for D_N and other more specialized groups, we will only use them when needed, and with D_N being third on our list.

(4) As a mathematical comment now, however trivial, the fact that we have no product operations involved conceptually comes from the fact that $2, 3$ are both prime.

(5) However, in relation with this, observe that we have $S_3 = D_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, but this does not correspond to anything, at the level of the graphs $\bullet\ \bullet\ \bullet$ and $\triangle$. So, good to know, the world of graphs is somehow more rigid than that of the groups. $\square$

At $N = 4$ now, which is both big enough, and composite, several new phenomena appear. First, we have the presence of "ugly" graphs, of the following type, that no one will be ever interested in, and which do not decompose as products either:



So, leaving aside now this ugly graph, and its symmetry group $\mathbb{Z}_2$, and all sorts of other similar graphs, functioning on the same principle "too many vertices, for not much symmetry for the buck", we are left, quite obviously, with the transitive graphs, which do have lots of symmetry. So, let us update our Goal 6.1, as follows:

UPDATE 6.3. *In relation with our original program, we will restrict the attention to the transitive graphs. Moreover, we will list these transitive graphs following a*

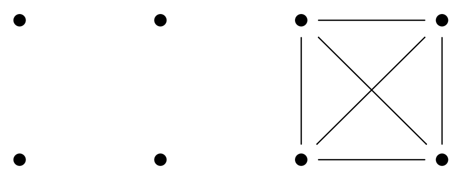
$$\{0, N-1\} \quad , \quad \{1, N-2\} \quad , \quad \{2, N-3\} \quad , \quad \ldots$$

scheme, according to their valence, and coupled via complementation.

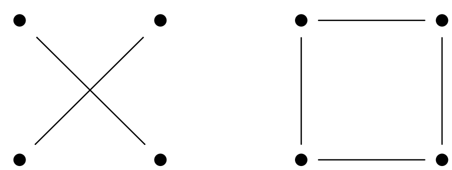
Here the scheme mentioned at the end is what comes out from Proposition 6.2 (1,2,3), and this is something quite self-explanatory, that will become clear as we work out more examples. Going back now to $N = 4$, with the above update made, we have:

PROPOSITION 6.4. *The transitive graphs at $N = 4$ are at follows:*

(1) *Valence $0, 3$: the empty graph and the tetrahedron, with symmetry group S_4,*



(2) *Valence $1, 4$: the two segments and the square, namely*



both products of a segment with itself, with symmetry group $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$.

PROOF. As before with Proposition 6.2, all this is trivial and self-explanatory, but some comments are in order, in regards with the last assertions, namely:

(1) Regarding the "products of a segment with itself" assertion, this is of course something informal, and very intuitive, which remains of course to be clarified. Thus, to be added on our to-do list, two different product operations for graphs $\times$ to be constructed, as to make this work. And no worries for this, we will be back to it, very soon.

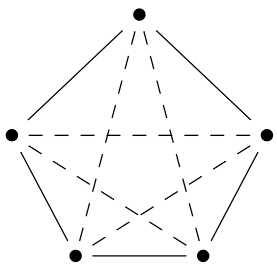
(2) As another comment, even when assuming that we managed to solve (1), with some suitable product operations for graphs $\times$ constructed, it is quite unclear how we can get, via these operations, the formula $G(X) = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$. Thus, a potential source of worries, and again, this is just a comment, and we will be back to this, very soon. $\square$

At $N = 5$ now, a small prime, things a bit similar to $N = 2, 3$, and we have:

PROPOSITION 6.5. *The transitive graphs at $N = 5$ are as follows:*

(1) *Valence $0, 4$: the empty graph and the simplex, with $G(X) = S_5$.*
(2) *Valence $1, 3$: none, since $N = 5$ is odd.*
(3) *Valence 2: the pentagon, self-complementary, with $G(X) = D_5$.*

PROOF. As before, everything is self-explanatory, with however the fact that the pentagon is self-dual being remarkable, definitely to be remembered, the picture being:



Thus, we are led to the conclusions in the statement. □

At $N = 6$ now, composite number, things get interesting again, and we have:

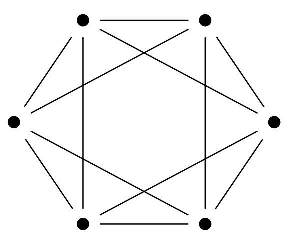
THEOREM 6.6. *The transitive graphs at $N = 6$ are as follows:*

1. *Valence $0, 5$: the empty graph and the simplex, with $G(X) = S_6$.*
2. *Valence $1, 4$: the 3 segments and the star graph, with $G(X) = H_3$.*
3. *Valence $2, 3$: the hexagon and the prism, with $G(X) = D_6$.*
4. *Valence $2, 3$ too: the 2 triangles and the wheel/utility graph, products.*

PROOF. As before, everything here is self-explanatory, the idea being as follows:

(1) Nothing special to be said about the empty graph and the simplex.

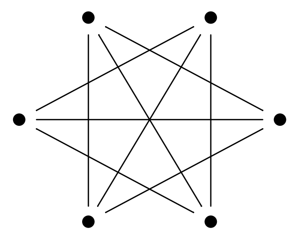
(2) At valence 1, we obviously have as only solution the 3 segments. Regarding now the complementary graph, this looks as follows, like a star:



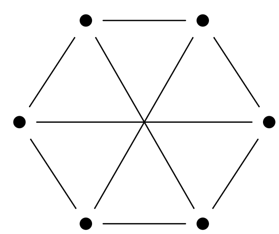
As for the symmetry group assertion, this follows from what we know about the hyperoctahedral groups. Indeed, the hyperoctahedral group H_3 appears by definition as symmetry group of the centered hypercube $\square_3 \subset \mathbb{R}^3$. But this group is also, obviously, the symmetry group of the space formed by the segments $[-1, 1]$ along each coordinate axis. We conclude that the symmetry group of the 3 segments is indeed H_3.

(3) At valence 2 now, which is the case left, along with the complementary case of valence 3, we have two solutions, namely the hexagon and the 2 triangles. Regarding the hexagon, have we have $G(X) = D_6$, and the only issue left is that of identifying the

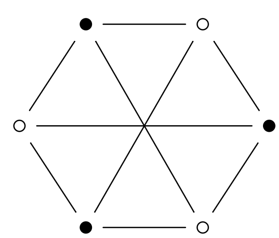
complementary graph. But this complementary graph is as follows, and when thinking a bit, by pulling one triangle, and then rotating it, you will see a prism here:



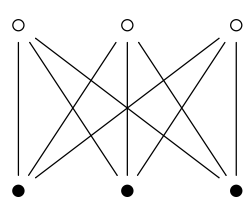
(4) Still at valence 2, and complementary valence 3, it remains to discuss the 2 triangles, and its complement. Nothing much to be said about the 2 triangles, but in what regards the complementary graph, there is something tricky here. Indeed, we can draw this complementary graph as follows, making it clear that we have a wheel:



But now, that we have this wheel, let us color the vertices as follows:



We can see here a bipartite graph, and by pulling apart the black and white vertices, we conclude that our graph is the utility graph, that we met in chapter 4:

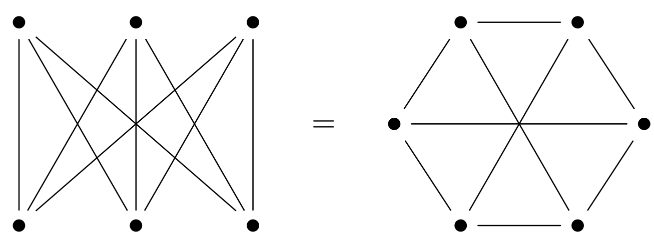


Finally, in what regards the computation of the symmetry group, the best here is to go back to the original 2 triangles, and draw them as follows:

But this is obviously a product graph, namely a product between a segment and a triangle, so the symmetry group must appear as some sort of product of $\mathbb{Z}_2$ and S_3. We will leave the details and computations here for a bit later, when discussing more in detail product operations, and the computation of the corresponding symmetry groups. $\square$

As an interesting conclusion coming from the above discussion, which is something useful and practice, that you always forget, let us record the following fact:

Conclusion 6.7. *The utility graph is the wheel,*



with this being best seen via the 2 triangles missing, in each case.

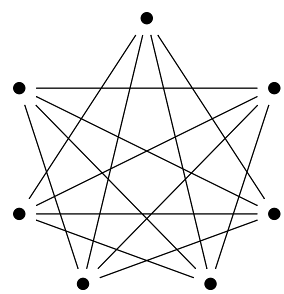
Here the first assertion looks a bit like a joke, because what can be more useful to mankind, than a wheel. However, it is not from here that the name "utility graph" comes from. The story here involves 3 companies, selling gas, water and electricity to 3 customers, and looking for a way to arrange their underground tubes and wires as not to cross. Thus, they are looking to implement their "utillity graph", which is the above one, in a planar way, and as we know well from chapter 4, this is not possible, and even leads to some deep theorems about planar graphs, those of Kuratowski and Wagner.

Back to work now, and to our graph enumeration program, at $N = 7$, which a small prime, things are a bit similar to those at $N = 2, 3, 5$, as follows:

Proposition 6.8. *The transitive graphs at $N = 7$ are as follows:*

1. *Valence $0, 6$: the empty graph and the simplex, with $G(X) = S_7$.*
2. *Valence $1, 5$: none, since $N = 7$ is odd.*
3. *Valence $2, 4$: the heptagon and its complement, with $G(X) = D_7$.*
4. *Valence 3: none, since $N = 7$ is odd.*

PROOF. As before, everything here is self-explanatory, coming from definitions, and from facts that we know well, with perhaps the only thing to be worked out being the picture of the complement of the heptagon, which looks as follows:



Thus, we are led to the conclusions in the statement. $\square$

6b. Medium graphs

Let us discuss now the graphs with bigger number of vertices, $N \geq 8$. In what regards the graphs with 8 vertices, to start with, with $N = 8$ being a multiply composite number, things get more complicated, with several new phenomena appearing, as follows:

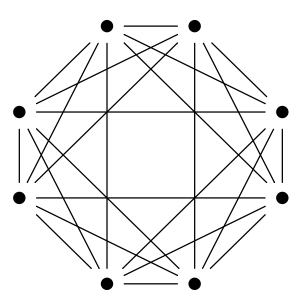
THEOREM 6.9. *The transitive graphs at $N = 8$ are as follows:*

1. *Valence $0, 7$: the empty graph and the simplex, with $G(X) = S_8$.*
2. *Valence $1, 6$: the 4 segments and the thick tyre, products.*
3. *Valence $2, 5$: the octagon and the big globe, with $G(X) = D_8$.*
4. *Valence $2, 5$ too: the 2 squares and the holy cross, products.*
5. *Valence $3, 4$: the 2 tetrahedra and the stop sign, products.*
6. *Valence $3, 4$ too: the cube and the metal cross, with $G(X) = H_3$.*
7. *Valence $3, 4$ too: the wheel and the big tent, with $G(X) = S_8$.*

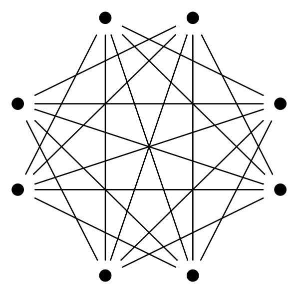
PROOF. As before, everything here is self-explanatory, the idea being as follows:

(1) Nothing much to be said about the empty graph, and its complement.

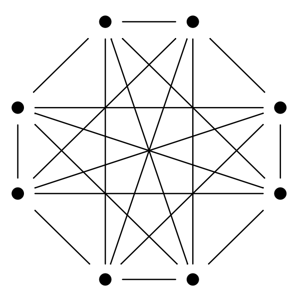
(2) At valence $1, 6$ we have the 4 segments, obviously a product, whose symmetry group we will compute later, and its complement, the thick tyre, which is as follows:



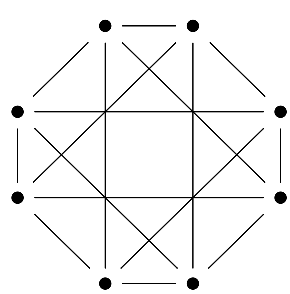
(3) At valence $2,5$ we have the octagon, whose symmetry group is $G(X)=D_8$, and its complement the big globe, which looks crowded too, as follows:



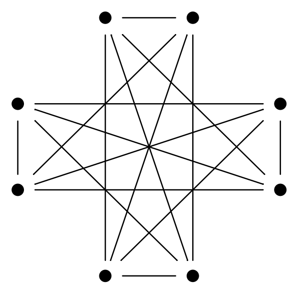
(4) At valence $2,5$ too we have the 2 squares, obviously a product, whose symmetry group we will compute later, and its complement the holy cross, as follows:



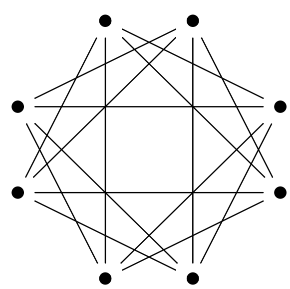
(5) At valence $3,4$ now, we first have the 2 tetrahedra, obviously a product, whose symmetry group we will compute later, and its complement, the stop sign:



(6) Still at valence $3,4$, we have as well the cube, whose symmetry group is $G(X)=H_3$, as we know well, and its complement the metal cross, which is as follows:



(7) Finally, still at valence $3,4$, we have as well the wheel, whose symmetry group is $G(X)=S_8$, and its complement the big tent, which is as follows:



Thus, we are led to the conclusions in the statement. $\square$

At $N=9$ now, things get again easier, but still non-trivial, as follows:

THEOREM 6.10. *The transitive graphs at $N=9$ are as follows:*

(1) *Valence $0,8$: the empty graph and the simplex, with $G(X)=S_9$.*
(2) *Valence $2,6$: the nonagon and its complement, with $G(X)=D_9$.*
(3) *Valence $2,6$ too: the 3 triangles and its complement, products.*
(4) *Valence 4: the torus graph and its complement, products.*
(5) *Valence 4 too: the wheel and its complement, with $G(X)=D_9$.*

PROOF. As before, everything here is self-explanatory, the comments being:

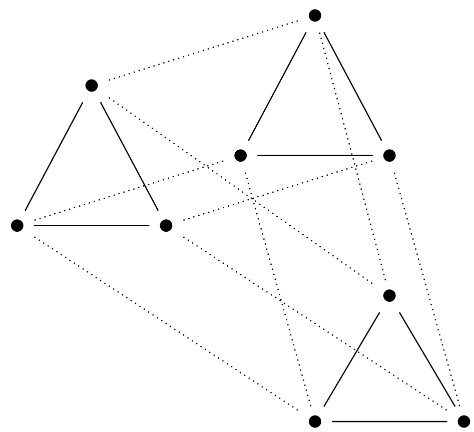
(1) Nothing much to be said about the empty graph, and its complement.

(2) Nothing much to be said either about the nonagon, and its complement.

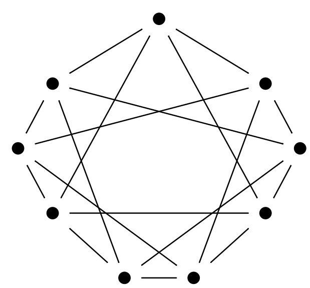
(3) Regarding the 3 triangles, and its complement, we will compute here $G(X)$ later.

(4) At valence 4 we have an interesting graph which appears, namely the discrete torus, which is the simplest discretization of a torus, as follows, and whose symmetry

group we will compute later, when discussing in detail the product operations:



(5) Also at valence 4, we have the wheel and its complement, with the wheel being drawn as follows, forced by the fact that 9 is odd, with two spokes at each vertex:



Thus, we are led to the conclusions in the statement. □

At $N = 10$ now, things get really tough, and we have:

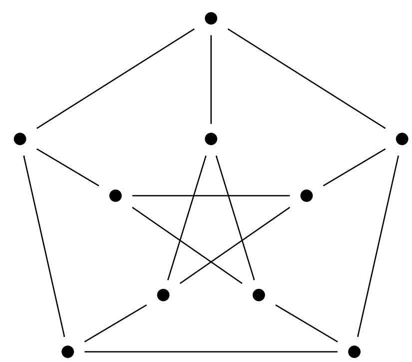
THEOREM 6.11. *The transitive graphs at $N = 10$ are as follows:*

(1) *Valence $0, 9$: the empty graph and the simplex, with $G(X) = S_{10}$.*
(2) *Valence $1, 8$: the 5 segments and its complement, products.*
(3) *Valence $2, 7$: the decagon and its complement, with $G(X) = D_{10}$.*
(4) *Valence $2, 7$ too: the 2 pentagons and its complement, product.*
(5) *Valence $3, 6$: the wheel and its complement, with $G(X) = D_{10}$.*
(6) *Valence $3, 6$ too: the pentagon prism and its complement, products.*
(7) *Valence $3, 6$ too: the Petersen graph and its complement.*
(8) *Valence $4, 5$: the two 5-simplices, and its complement, products.*
(9) *Valence $4, 5$ too: the 5-simplex prism and the torch, products.*
(10) *Valence $4, 5$ too: the reinforced wheel and its complement, with $G(X) = D_{10}$.*
(11) *Valence $4, 5$ too: the alternative reinforced wheel and its complement.*

PROOF. As before, everything here is self-explanatory, the idea being as follows:

(1-6) Nothing much to be said about these graphs, and their complements.

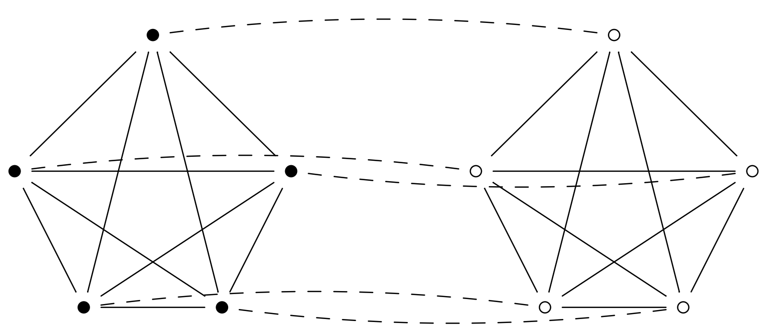
(7) At valence 3 we have as well the Petersen graph, which is as follows:



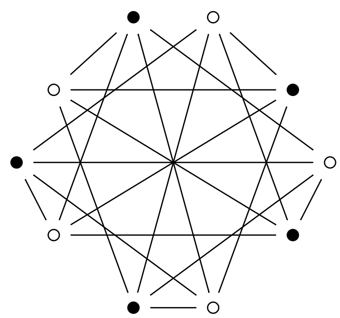
This graph seems to appear as some sort of product of the cycle C_5 with the segment C_2, but via a quite complicated procedure. We will be back to this, a bit later.

(8) Nothing much to be said about the two 5-simplices, and its complement, with the computation of the symmetry group here being left for later.

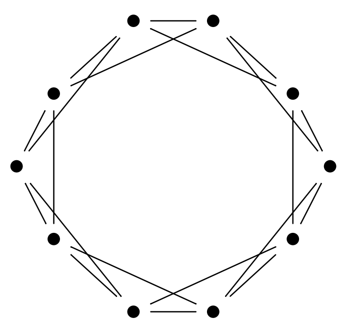
(9) Here there is some discussion to be made, because, contrary to our policy so far, listing graphs with smaller valence than their complements first, here we did the opposite. To start with, the 5-simplex prism, having valence 5, is as follows:



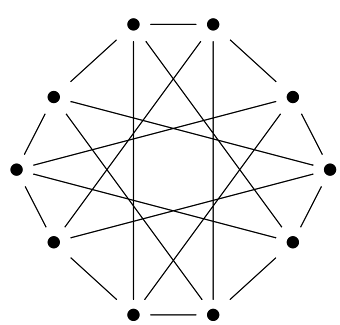
As for the complementary graph to this 5-simplex prism, this is the torch graph, having valence 4, which looks as follows:



(10-11) There is some discussion to be made here, concerning the two possible reinforced wheels on 10 vertices, having 2 spokes at each vertex. First we have a reinforced wheel as follows, whose symmetry group is D_{10}, as said in the statement:



But then we have as well the following alternative reinforced wheel, whose symmetry group is a bit more complicated to compute, and we will discuss this later:



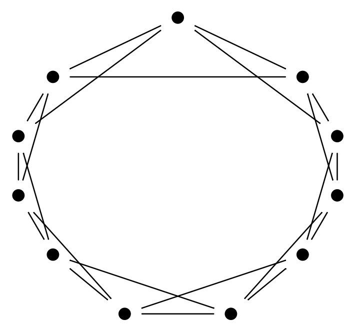
Thus, we are led to the conclusions in the statement. □

All the above is quite concerning, especially in what concerns the Petersen graph, which looks quite enigmatic. However, before leaving the subject, let us record as well the result at $N = 11$. Here the order being a small prime, things get again easier:

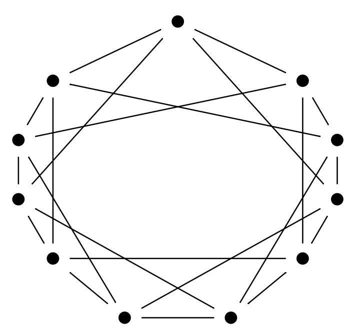
Proposition 6.12. *The transitive graphs at $N = 11$ are as follows:*

1. *Valence $0, 10$: the empty graph and the simplex, with $G(X) = S_{11}$.*
2. *Valence $2, 8$: the hendecagon and its complement, with $G(X) = D_{11}$.*
3. *Valence $4, 6$: the reinforced wheel and its complement, with $G(X) = D_{11}$.*
4. *Valence $4, 6$: the other reinforced wheel and complement, with $G(X) = D_{11}$.*

Proof. Here again business as usual, with the only discussion concerning the two possible reinforced wheels, with 2 spokes at each vertex. A first such wheel is follows:



But then, we have as well a second reinforced wheel, as follows:



Thus, we are led to the conclusions in the statement. □

And we will stop here our small N study, for several reasons:

(1) First of all, because we did some good work in the above, and found many types of graph products, that we have now to investigate in detail.

(2) Second, because of the Petersen graph at $N = 10$, which obviously brings us into uncharted territory, and needs some serious study, before going further.

(3) And third, because at $N = 12$, which is a dazzlingly composite number, things explode, with the number of transitive graphs here being a mighty 64.

6c. Standard products

For the transitive graphs, that we are mostly interested in, the point is that, according to the above, at very small values of the order, $N = 2, \ldots, 9$, these always decompose as products, via three main types of graph products, constructed as follows:

DEFINITION 6.13. *Let X, Y be two finite graphs.*

(1) *The direct product $X \times Y$ has vertex set $X \times Y$, and edges:*
$$(i, \alpha) - (j, \beta) \iff i - j,\ \alpha - \beta$$

(2) *The Cartesian product $X \,\square\, Y$ has vertex set $X \times Y$, and edges:*
$$(i, \alpha) - (j, \beta) \iff i = j,\ \alpha - \beta \text{ or } i - j, \alpha = \beta$$

(3) *The lexicographic product $X \circ Y$ has vertex set $X \times Y$, and edges:*
$$(i, \alpha) - (j, \beta) \iff \alpha - \beta \text{ or } \alpha = \beta,\ i - j$$

Several comments can be made here. First, the direct product $X \times Y$ is the usual one in a categorical sense, and we will leave clarifying this observation as an exercise. The Cartesian product $X \,\square\, Y$ is quite natural too from a geometric perspective, for instance because a product by a segment gives a prism. As for the lexicographic product $X \circ Y$, this is something interesting too, obtained by putting a copy of X at each vertex of Y.

At the level of symmetry groups, several things can be said, and we first have:

THEOREM 6.14. *We have group embeddings as follows, for any graphs X, Y,*
$$G(X) \times G(Y) \subset G(X \times Y)$$
$$G(X) \times G(Y) \subset G(X \,\square\, Y)$$
$$G(X) \wr G(Y) \subset G(X \circ Y)$$
but these embeddings are not always isomorphisms.

PROOF. The fact that we have indeed embeddings as above is clear from definitions. As for the counterexamples, in each case, these are easy to construct as well, provided by our study of small graphs, at $N = 2, \ldots, 11$, and we will leave this as an exercise. $\square$

The problem now is that of deciding when the embeddings in Theorem 6.14 are isomorphisms. And this is something non-trivial, because there are both examples and counterexamples for these isomorphisms, coming from the various computations of symmetry groups that we did in the above, at $N = 2, \ldots, 11$. We will see, however, that the problem can be solved, via a technical study, of spectral theory flavor.

Now speaking technical algebra and spectral theory, it is good time to go back to the permutation groups $G \subset S_N$, as introduced and studied in chapter 5, and study them a bit more, from an algebraic perspective. We first have the following basic fact:

THEOREM 6.15. *Given a subgroup $G \subset S_N$, regarded as matrix group via*

$$G \subset S_N \subset O_N$$

the standard coordinates of the group elements, $u_{ij}(g) = g_{ij}$, are given by:

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

Moreover, these functions $u_{ij} : G \to \mathbb{C}$ generate the algebra $C(G)$.

PROOF. Here the first assertion comes from the fact that the entries of the permutation matrices $\sigma \in S_N \subset O_N$, acting as $\sigma(e_i) = e_{\sigma(i)}$, are given by the following formula:

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

As for the second assertion, this comes from the Stone-Weierstrass theorem, because the coordinate functions $u_{ij} : G \to \mathbb{C}$ obviously separate the group elements $\sigma \in G$. $\square$

We are led in this way to the following definition:

DEFINITION 6.16. *The magic matrix associated to a permutation group $G \subset S_N$ is the $N \times N$ matrix of characteristic functions*

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

with the name "magic" coming from the fact that, on each row and each column, these characteristic functions sum up to 1.

The interest in this notion comes from the fact, that we know from Theorem 6.15, that the entries of the magic matrix generate the algebra of functions on our group:

$$C(G) = < u_{ij} >$$

We will talk more in detail later about such matrices, and their correspondence with the subgroups $G \subset S_N$, and what can be done with it, in the general framework of representation theory. However, for making our point, here is the general principle:

PRINCIPLE 6.17. *Everything that you can do with your group $G \subset S_N$ can be expressed in terms of the magic matrix $u = (u_{ij})$, quite often with good results.*

This principle comes from the above Stone-Weierstrass result, $C(G) = < u_{ij} >$. Indeed, when coupled with some basic spectral theory, and more specifically with the Gelfand theorem from operator algebras, this result tells us that our group G appears as the spectrum of the algebra $< u_{ij} >$, therefore leading to the above principle. But more on this later in this book, when discussing spectral theory, and the Gelfand theorem.

For the moment, we will just take Definition 6.16 as it is, something technical of group theory, of rather functional analysis flavor, that we can use in our proofs when needed. And we will take Principle 6.17 also as it is, namely a claim that this is useful indeed.

As an illustration for all this, in relation with the graphs, we have:

THEOREM 6.18. *Given a subgroup $G \subset S_N$, the transpose of its action map $X \times G \to X$ on the set $X = \{1, \ldots, N\}$, given by $(i, \sigma) \to \sigma(i)$, is given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Also, in the case where we have a graph with N vertices, the action of G on the vertex set X leaves invariant the edges precisely when we have

$$du = ud$$

with d being as usual the adjacency matrix of the graph.

PROOF. There are several things going on here, the idea being as follows:

(1) Given a subgroup $G \subset S_N$, if we set $X = \{1, \ldots, N\}$, we have indeed an action map as follows, and with the reasons of using $X \times G$ instead of the perhaps more familiar $G \times X$ being dictated by some quantum algebra, that we will do later in this book:

$$a : X \times G \to X \quad , \quad a(i, \sigma) = \sigma(i)$$

(2) Now by transposing this map, we obtain a morphism of algebras, as follows:

$$\Phi : C(X) \to C(X) \otimes C(G) \quad , \quad \Phi(f)(i, \sigma) = f(\sigma(i))$$

When evaluated on the Dirac masses, this map Φ is then given by:

$$\Phi(e_i)(j, \sigma) = e_i(\sigma(j)) = \delta_{\sigma(j)i}$$

Thus, in tensor product notation, we have the following formula, as desired:

$$\Phi(e_i)(j, \sigma) = \left(\sum_j e_j \otimes u_{ji} \right)(j, \sigma)$$

(3) Regarding now the second assertion, observe first that we have:

$$(du)_{ij}(\sigma) = \sum_k d_{ik} u_{kj}(\sigma) = \sum_k d_{ik} \delta_{\sigma(j)k} = d_{i\sigma(j)}$$

On the other hand, we have as well the following formula:

$$(ud)_{ij}(\sigma) = \sum_k u_{ik}(\sigma) d_{kj} = \sum_k \delta_{\sigma(k)i} d_{kj} = d_{\sigma^{-1}(i)j}$$

Thus $du = ud$ reformulates as $d_{ij} = d_{\sigma(i)\sigma(j)}$, which gives the result. $\square$

So long for magic unitaries, and their basic properties, and we will be back to this, on several occasions, in what follows. In fact, the magic matrices will get increasingly important, as the present book develops, because not far away from now, when starting to talk about quantum permutation groups G, and their actions on the graphs X, these beasts will not really exist, as concrete objects G, but their associated magic matrices $u = (u_{ij})$ will exist, and we will base our whole study on them. More on this later.

Back to graphs now, we want to know when the embeddings in Theorem 6.14 are isomorphisms. In what regards the first two products, we have here the following result, coming with a proof from [8], obtained by using the magic matrix technology:

THEOREM 6.19. *Let X and Y be finite connected regular graphs. If their spectra $\{\lambda\}$ and $\{\mu\}$ do not contain 0 and satisfy*

$$\left\{\lambda_i/\lambda_j\right\} \cap \left\{\mu_k/\mu_l\right\} = \{1\}$$

then $G(X \times Y) = G(X) \times G(Y)$. Also, if their spectra satisfy

$$\left\{\lambda_i - \lambda_j\right\} \cap \left\{\mu_k - \mu_l\right\} = \{0\}$$

then $G(X \,\square\, Y) = G(X) \times G(Y)$.

PROOF. This is something quite standard, the idea being as follows:

(1) First, we know from Theorem 6.14 that we have embeddings as follows, valid for any two graphs X, Y, and coming from definitions:

$$G(X) \times G(Y) \subset G(X \times Y)$$

$$G(X) \times G(Y) \subset G(X \,\square\, Y)$$

(2) Now let λ_1 be the valence of X. Since X is regular we have $\lambda_1 \in Sp(X)$, with 1 as eigenvector, and since X is connected λ_1 has multiplicity 1. Thus if P_1 is the orthogonal projection onto $\mathbb{C}1$, the spectral decomposition of d_X is of the following form:

$$d_X = \lambda_1 P_1 + \sum_{i \neq 1} \lambda_i P_i$$

We have a similar formula for the adjacency matrix d_Y, namely:

$$d_Y = \mu_1 Q_1 + \sum_{j \neq 1} \mu_j Q_j$$

(3) But this gives the following formulae for the graph products:

$$d_{X \times Y} = \sum_{ij} (\lambda_i \mu_j) P_i \otimes Q_j$$

$$d_{X \,\square\, Y} = \sum_{ij} (\lambda_i + \mu_i) P_i \otimes Q_j$$

Here the projections form partitions of unity, and the scalar are distinct, so these are spectral decompositions. The coactions will commute with any of the spectral projections, and so with both $P_1 \otimes 1$, $1 \otimes Q_1$. In both cases the universal coaction v is the tensor product of its restrictions to the images of $P_1 \otimes 1$, $1 \otimes Q_1$, which gives the result. □

Regarding now the lexicographic product, things here are more tricky. Let us first recall that the lexicographic product of two graphs $X \circ Y$ is obtained by putting a copy of X at each vertex of Y, the formula for the edges being as follows:

$$(i,\alpha) - (j,\beta) \iff \alpha - \beta \text{ or } \alpha = \beta,\ i - j$$

In what regards now the computation of the symmetry group, as before we must do here some spectral theory, and we are led in this way to the following result:

Theorem 6.20. *Let X,Y be regular graphs, with X connected. If their spectra $\{\lambda_i\}$ and $\{\mu_j\}$ satisfy the condition*

$$\left\{\lambda_1 - \lambda_i \Big| i \neq 1\right\} \cap \left\{ -n\mu_j\right\} = \emptyset$$

where n and λ_1 are the order and valence of X, then $G(X \circ Y) = G(X) \wr G(Y)$.

Proof. This is something quite tricky, the idea being as follows:

(1) First, we know from Theorem 6.14 that we have an embedding as follows, valid for any two graphs X,Y, and coming from definitions:

$$G(X) \wr G(Y) \subset G(X \circ Y)$$

(2) We denote by P_i, Q_j the spectral projections corresponding to λ_i, μ_j. Since X is connected we have $P_1 = \mathbb{I}/n$, and we obtain:

$$\begin{aligned} d_{X\circ Y} &= d_X \otimes 1 + \mathbb{I} \otimes d_Y \\ &= \left(\sum_i \lambda_i P_i\right) \otimes \left(\sum_j Q_j\right) + (nP_1) \otimes \left(\sum_i \mu_j Q_j\right) \\ &= \sum_j (\lambda_1 + n\mu_j)(P_1 \otimes Q_j) + \sum_{i \neq 1} \lambda_i (P_i \otimes 1) \end{aligned}$$

In this formula the projections form a partition of unity and the scalars are distinct, so this is the spectral decomposition of $d_{X\circ Y}$.

(3) Now let W be the universal magic matrix for $X \circ Y$. Then W must commute with all spectral projections, and in particular:

$$[W, P_1 \otimes Q_j] = 0$$

Summing over j gives $[W, P_1 \otimes 1] = 0$, so $1 \otimes C(Y)$ is invariant under the coaction. So, consider the restriction of W, which gives a coaction of $G(X \circ Y)$ on $1 \otimes C(Y)$, that we can denote as follows, with y being a certain magic unitary:

$$W(1\otimes e_a)=\sum_b 1\otimes e_b\otimes y_{ba}$$

(4) On the other hand, according to our definition of W, we can write:

$$W(e_i\otimes 1)=\sum_{jb} e_j\otimes e_b\otimes x_{ji}^b$$

By multiplying by the previous relation, found in (3), we obtain:

$$\begin{aligned}W(e_i\otimes e_a)&=\sum_{jb} e_j\otimes e_b\otimes y_{ba}x_{ji}^b\\&=\sum_{jb} e_j\otimes e_b\otimes x_{ji}^b y_{ba}\end{aligned}$$

But this shows that the coefficients of W are of the following form:

$$W_{jb,ia}=y_{ba}x_{ji}^b=x_{ji}^b y_{ba}$$

(5) In order to advance, consider now the following matrix:

$$x^b=(x_{ij}^b)$$

Since the map W above is a morphism of algebras, each row of x^b is a partition of unity. Also, by using the antipode map S, which is transpose to $g\to g^{-1}$, we have:

$$\begin{aligned}S\left(\sum_j x_{ji}^b\right)&=S\left(\sum_{ja} x_{ji}^b y_{ba}\right)\\&=S\left(\sum_{ja} W_{jb,ia}\right)\\&=\sum_{ja} W_{ia,jb}\\&=\sum_{ja} x_{ij}^a y_{ab}\\&=\sum_a y_{ab}\\&=1\end{aligned}$$

As a conclusion to this, the matrix x^b constructed above is magic.

(6) We check now that both x^a, y commute with d_X, d_Y. We have:

$$(d_{X\circ Y})_{ia,jb}=(d_X)_{ij}\delta_{ab}+(d_Y)_{ab}$$

Thus the two products between W and $d_{X\circ Y}$ are given by:

$$(Wd_{X\circ Y})_{ia,kc}=\sum_j W_{ia,jc}(d_X)_{jk}+\sum_{jb}W_{ia,jb}(d_Y)_{bc}$$

$$(d_{X\circ Y}W)_{ia,kc}=\sum_j(d_X)_{ij}W_{ja,kc}+\sum_{jb}(d_Y)_{ab}W_{jb,kc}$$

(7) Now since the magic matrix W commutes by definition with $d_{X\circ Y}$, the terms on the right in the above equations are equal, and by summing over c we get:

$$\sum_j x_{ij}^a(d_X)_{jk}+\sum_{cb}y_{ab}(d_Y)_{bc}=\sum_j(d_X)_{ij}x_{jk}^a+\sum_{cb}(d_Y)_{ab}y_{bc}$$

The second sums in both terms are equal to the valence of Y, so we get:

$$[x^a,d_X]=0$$

Now once again from the formula coming from $[W,d_{X\circ Y}]=0$, we get:

$$[y,d_Y]=0$$

(8) Summing up, the coefficients of W are of the following form, where x^b are magic unitaries commuting with d_X, and y is a magic unitary commuting with d_Y:

$$W_{jb,ia}=x_{ji}^b y_{ba}$$

But this gives a morphism $C(G(X)\wr G(Y))\to G(X\circ Y)$ mapping $u_{ji}^{(b)}\to x_{ji}^b$ and $v_{ba}\to y_{ba}$, which is inverse to the morphism in (1), as desired. $\square$

As before with the other graph products, there is some further spectral theory to be done here, plus working out examples. There are as well a number of finer results, for instance the theorem of Sabidussi, to be discussed. We will be back to this.

6d. Kneser graphs

Good news, we have now enough tools in our bag for getting back to the transitive graphs of order $N=2,\ldots,11$, enumerated and partially studied in the beginning of this chapter, compute the missing symmetry groups there, and end up with a nice table.

Let us begin with some notations, as follows:

Definition 6.21. *We use the following notations for graphs:*
1. *K_N is the complete graph on N vertices.*
2. *C_N is the cycle on N vertices.*
3. *$P(X)$ the prism over a graph X.*
4. *C_N^k is the cycle with chords of length k.*

In what regards the transitive graphs of order $N=2,\ldots,9$, here we have all the symmetry groups already computed, except for 6 of them, which are as follows:

PROPOSITION 6.22. *The missing symmetry groups at $N \leq 9$ are as follows:*

(1) *For the two triangles $2K_3$ we obtain $S_3 \wr \mathbb{Z}_2$.*
(2) *For the four segments $4K_2$ we obtain H_4.*
(3) *For the two squares $2C_4$ we obtain $H_2 \wr \mathbb{Z}_2$.*
(4) *For the two tetrahedra $2K_4$ we obtain $S_4 \wr \mathbb{Z}_2$.*
(5) *For the three triangles $3K_3$ we obtain $S_3 \wr S_3$.*
(6) *For the torus graph $K_3 \times K_3$ we obtain $S_3 \wr \mathbb{Z}_2$.*

PROOF. This follows indeed by applying the various product results from the previous section, and we will leave the details here as an instructive exercise. □

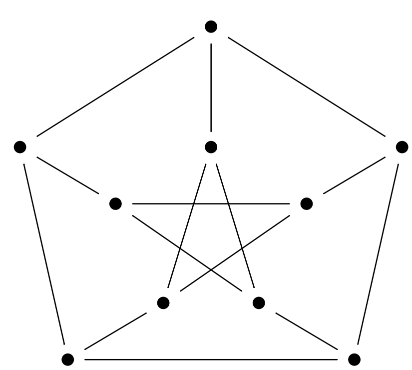
Regarding the graphs of order $N = 10, 11$, things fine at $N = 11$, but at $N = 10$ we have 7 symmetry groups left. The first 6 of them can be computed as follows:

PROPOSITION 6.23. *The missing symmetry groups at $N = 10$ are as follows:*

(1) *For the five segments $5K_2$ we obtain H_5.*
(2) *For the two pentagons $2C_5$ we obtain $D_5 \wr \mathbb{Z}_2$.*
(3) *For the pentagon prism $P(C_5)$ we obtain D_{10}.*
(4) *For the two 5-simplices $2K_5$ we obtain $S_5 \wr \mathbb{Z}_2$.*
(5) *For the 5-simplex prism $P(K_5)$ we obtain $S_5 \times \mathbb{Z}_2$.*
(6) *For the second reinforced wheel C_{10}^4 we obtain $\mathbb{Z}_2 \wr D_5$.*

PROOF. This follows too by applying the various product results from the previous section, and we will leave the details here as an instructive exercise. □

We have kept the best for the end. Time now to face the only graph left, which is the famous Petersen graph, that we know well since chapter 4, which looks as follows:



Intuition suggests that this graph should appear as a product of the cycle C_5 with the segment C_2. However, there are many bugs with this idea, which does not work.

In order to view the Petersen graph as part of a larger family, and have some general theory going, we have to proceed in a quite unexpected way, as follows:

Proposition 6.24. *The Petersen graph is part of the Kneser graph family,*

$$P_{10}=K(5,2)$$

with $K(n,s)$ having as vertices the s-element subsets of $\{1,\ldots,n\}$, and with the edges being drawn between distinct subsets.

Proof. Consider indeed the Kneser graph $K(n,s)$, as constructed above. At $n=5$, $s=2$ the vertices are the $\binom{5}{2}=10$ subsets of $\{1,\ldots,5\}$ having 2 elements, and since each such subset $\{p,q\}$ is disjoint from exactly 3 other such subsets $\{u,v\}$, our graph is trivalent, and when drawing the picture, we obtain indeed the Petersen graph. $\square$

Many things can be said about the Kneser graphs, and among others, we have:

Theorem 6.25. *The Kneser graphs $K(n,s)$ have the following properties:*

(1) *$K(n,1)$ is the complete graph K_n.*
(2) *$K(n,2)$ is the complement of the line graph $L(K_n)$.*
(3) *$K(2m-1,m-1)$ is the so-called odd graph O_m.*
(4) *The symmetry group of $K(n,s)$ is the symmetric group S_n.*

In particular, $P_{10}=K(5,2)=L(K_5)^c=O_3$, having symmetry group S_5.

Proof. All this is quite self-explanatory, the idea being as follows:

(1) This is something trivial, coming from definitions.

(2) This is clear too from definitions, with the line graph $L(X)$ of a given graph X being by definition the incidence graph of the edges of X.

(3) This stands for the definition of O_m, as being the graph $K(2m-1,m-1)$.

(4) This is clear too, coming from the definition of the Kneser graphs. $\square$

All this is very nice, and time to conclude. We have the following table, based on our various results above, containing all transitive graphs up to $N=11$ vertices, up to complementation, and their symmetry groups, written by using the conventions from Definition 6.21, and with the extra convention that C_n^+ with n even denote the wheels:

Order	Graph	Symmetry group
2	K_2	$\mathbb{Z}_2$
3	K_3	S_3
4	$2K_2$	H_2
4	K_4	S_4
5	C_5	D_5
5	K_5	S_5
6	C_6	D_6
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$
6	$3K_2$	H_3
6	K_6	S_6
7	C_7	D_7
7	K_7	S_7
8	C_8, C_8^+	D_8
8	$P(C_4)$	H_3
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$
8	$2C_4$	$H_2 \wr \mathbb{Z}_2$
8	$4K_2$	H_4
8	K_8	S_8
9	C_9, C_9^3	D_9
9	$K_3 \times K_3$	$S_3 \wr \mathbb{Z}_2$
9	$3K_3$	$S_3 \wr S_3$
9	K_9	S_9
10	C_{10}, C_{10}^2, C_{10}^+, $P(C_5)$	D_{10}
10	$P(K_5)$	$S_5 \times \mathbb{Z}_2$
10	C_{10}^4	$\mathbb{Z}_2 \wr D_5$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$
10	$5K_2$	H_5
10	K_{10}	S_{10}
10	P_{10}	S_5
11	C_{11}, C_{11}^2, C_{11}^3	D_{11}
11	K_{11}	S_{11}

Afterwards, at $N = 12$ and higher, the study becomes quite complicated, but we have results for several key series of graphs. We will be back to this, later in this book.

6e. Exercises

We had a very exciting chapter here, with plenty of explicit graphs, nice pictures, and beautiful symmetry groups. As exercises on all this, we have:

Exercise 6.26. *Draw all graphs with $N = 9$ vertices, on a torus.*

Exercise 6.27. *Clarify which graphs at $N \leq 11$ equal their complement.*

Exercise 6.28. *Enumerate all graphs on $N = 12$ vertices, and good luck here.*

Exercise 6.29. *Try saying something nice, about the non-transitive graphs.*

Exercise 6.30. *Meditate on the magic matrices, and what can be done with them.*

Exercise 6.31. *Find more direct proofs for all our general product results.*

Exercise 6.32. *Work out all the concrete applications of our product results.*

Exercise 6.33. *Learn more about the Kneser graphs, and their various properties.*

As bonus exercise, further meditate on the Petersen graph, and what can be done with it. This is indeed the typical counterexample to all sorts of questions, in graph theory.

CHAPTER 7

Spectral theory

7a. Function algebras

It is possible to do many other things, based on the techniques developed above, by further building on that material. But, my proposal now would be to slow down, temporarily forget about the graphs X, and focus instead on the finite groups G. These are our main tools, and before going to war, you have to sharpen your blades.

In order to discuss this, let us first have a second look at the magic unitaries, introduced in chapter 6. We have the following result, summarizing our knowledge from there:

THEOREM 7.1. *Given a subgroup $G \subset S_N \subset O_N$, the standard coordinates $u_{ij} \in C(G)$ generate $C(G)$, are given by the following formula, and form a magic matrix:*

$$u_{ij} = \chi\left(\sigma \in G \Big| \sigma(j) = i\right)$$

These coordinates appear as well as the coefficients of the transpose of the action map $a : X \times G \to X$ on the set $X = \{1, \ldots, N\}$, given by $(i, \sigma) \to \sigma(i)$, which is given by:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Also, in the case where we have a graph with N vertices, the action of G on the vertex set X leaves invariant the edges precisely when $du = ud$.

PROOF. This is something that we know from chapter 6, the idea being as follows:

(1) Since the action of the group elements $\sigma \in G \subset S_N \subset O_N$ on the standard basis of $\mathbb{R}^N$ is given by $\sigma(e_i) = e_{\sigma(i)}$, we have $\sigma_{ij} = \delta_{\sigma(j)i}$, which gives the formula of u_{ij}.

(2) The fact that our matrix $u = (u_{ij})$ is indeed magic, in the sense that its entries sum up to 1, on each row and each column, follows from our formula of u_{ij}.

(3) By Stone-Weierstrass we get $C(G) = < u_{ij} >$, since the coordinate functions u_{ij} separate the points of G, in the sense that $\sigma \neq \pi$ needs $\sigma_{ij} \neq \pi_{ij}$, for some i, j.

(4) Regarding the action $a : X \times G \to X$ and the coaction $\Phi : C(X) \to C(X) \otimes C(G)$, all this looks scary, but is in fact a triviality, as explained in chapter 6.

(5) Finally, in what regards the last assertion, concerning $du = ud$, this again looks a bit abstract and scary, but is again a triviality, as explained in chapter 6. $\square$

We have the following result, further building on the above:

THEOREM 7.2. *The symmetry group $G(X)$ of a graph X having N vertices is given by the following formula, at the level of the corresponding algebra of functions,*

$$C(G(X))=C(S_N)\Big/\Big\langle du=ud\Big\rangle$$

with $d\in M_N(0,1)$ being as usual the adjacency matrix of X.

PROOF. This follows indeed from Theorem 7.1, and more specifically, is just an abstract reformulation of the last assertion there. $\square$

In order to further build on all this, the idea will be that of getting rid of S_N, or rather of the corresponding algebra of functions $C(S_N)$, and formulating everything in terms of magic matrices. To be more precise, leaving aside X, we have the following question:

QUESTION 7.3. *With a suitable algebra formalism, do we have*

$$C(S_N)=A\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

with A standing for "universal algebra generated by"?

At the first glance, this question might seem overly theoretical and abstract, but the claim is that such things can be useful, in order to deal with graphs. Indeed, assuming a positive answer to this question, by Theorem 7.2 we would conclude that $C(G(X))$ is the universal algebra generated by the entries of a magic matrix commuting with d.

Which is something very nice, and potentially useful, among others putting under a more conceptual light the various product computations from the previous chapter, done with magic matrices. In a word, we have seen in the previous chapter that the magic matrices can be very useful objects, so let us go now for it, and reformulate everything in terms of them, along the lines of Question 7.3, and of the comments afterwards.

Getting to work now, the algebra $C(S_N)$ from Question 7.3, that we want to axiomatize via magic matrices, is something very particular, and we have:

FACT 7.4. *The function algebra $C(S_N)$ has the following properties:*

(1) *It is a complex algebra.*
(2) *It is finite dimensional.*
(3) *It is commutative, $fg=gf$.*
(4) *It has an involution, given by $f^*(x)=\overline{f(x)}$.*
(5) *It has a norm, given by $||f||=\sup_x|f(x)|$.*

So, which of these properties shall we choose, for our axiomatization? Definitely (1), and then common sense would suggest to try the combination (2+3). However, since we will be soon in need, in this book, of algebras which are infinite dimensional, or not commutative, or both, let us go instead with the combination (4+5), for most of our axiomatization work, and then with (2) or (3) added, if needed, towards the end.

In short, trust me here, we need to do some algebra and this is what we will do, we will learn useful things in what follows, and here are some axioms, to start with:

DEFINITION 7.5. *A C^*-algebra is a complex algebra A, given with an involution $a \to a^*$ and a norm $a \to ||a||$, such that:*

1. *The norm and involution are related by $||aa^*|| = ||a||^2$.*
2. *A is complete, as metric space, with respect to the norm.*

As a first basic class of examples, which are of interest for us, in relation with Question 7.3, we have the algebras of type $A = C(X)$, with X being a finite, or more generally compact space, with the usual involution and norm of functions, namely:

$$f^*(x) = \overline{f(x)} \quad , \quad ||f|| = \sup_x |f(x)|$$

Observe that such algebras are commutative, $fg = gf$, and also that both the conditions in Definition 7.5 are satisfied, with (1) being something trivial, and with (2) coming from the well-known fact that a uniform limit of continuous function is continuous.

Interestingly, and of guaranteed interest for many considerations to follow, in this book, as a second basic class of examples we have the matrix algebras $A = M_N(\mathbb{C})$, with the usual involution and norm of the complex matrices, namely:

$$(M^*)_{ij} = \overline{M}_{ij} \quad , \quad ||M|| = \sup_{||x||=1} ||Mx||$$

Observe that such algebras are finite dimensional, and also that the two conditions in Definition 7.5 are satisfied, with (1) being a good linear algebra exercise for you, via double inequality, and with (2) being trivial, our algebra being finite dimensional.

Summarizing, good definition that we have, so let us develop now some theory, for the C^*-algebras. Inspired by the matrix algebra examples, we first have:

THEOREM 7.6. *Given an element $a \in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \Big| a - \lambda \notin A^{-1} \right\}$$

The following spectral theory results hold, exactly as in the $A = M_N(\mathbb{C})$ case:

1. *We have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*
2. *We have $\sigma(f(a)) = f(\sigma(a))$, for any $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.*
3. *The spectrum $\sigma(a)$ is compact, non-empty, and contained in $D_0(||a||)$.*
4. *The spectra of unitaries $(u^* = u^{-1})$ and self-adjoints $(a = a^*)$ are on $\mathbb{T}, \mathbb{R}$.*
5. *The spectral radius of normal elements $(aa^* = a^*a)$ is given by $\rho(a) = ||a||$.*

In addition, assuming $a \in A \subset B$, the spectra of a with respect to A and to B coincide.

PROOF. Here the assertions (1-5), which are of course formulated a bit informally, are well-known for the matrix algebra $A = M_N(\mathbb{C})$, and the proof in general is similar:

(1) Assuming that $1 - ab$ is invertible, with inverse c, we have $abc = cab = c - 1$, and it follows that $1 - ba$ is invertible too, with inverse $1 + bca$. Thus $\sigma(ab), \sigma(ba)$ agree on $1 \in \mathbb{C}$, and by linearity, it follows that $\sigma(ab), \sigma(ba)$ agree on any point $\lambda \in \mathbb{C}^*$.

(2) The formula $\sigma(f(a)) = f(\sigma(a))$ is clear for polynomials, $f \in \mathbb{C}[X]$, by factorizing $f - \lambda$, with $\lambda \in \mathbb{C}$. Then, the extension to the rational functions is straightforward, because $P(a)/Q(a) - \lambda$ is invertible precisely when $P(a) - \lambda Q(a)$ is.

(3) By using $1/(1-b) = 1 + b + b^2 + \ldots$ for $||b|| < 1$ we obtain that $a - \lambda$ is invertible for $|\lambda| > ||a||$, and so $\sigma(a) \subset D_0(||a||)$. It is also clear that $\sigma(a)$ is closed, so what we have is a compact set. Finally, assuming $\sigma(a) = \emptyset$ the function $f(\lambda) = \varphi((a-\lambda)^{-1})$ is well-defined, for any $\varphi \in A^*$, and by Liouville we get $f = 0$, contradiction.

(4) Assuming $u^* = u^{-1}$ we have $||u|| = 1$, and so $\sigma(u) \subset D_0(1)$. But with $f(z) = z^{-1}$ we obtain via (2) that we have as well $\sigma(u) \subset f(D_0(1))$, and this gives $\sigma(u) \subset \mathbb{T}$. As for the result regarding the self-adjoints, this can be obtained from the result for the unitaries, by using (2) with functions of type $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$.

(5) It is routine to check, by integrating quantities of type $z^n/(z - a)$ over circles centered at the origin, and estimating, that the spectral radius is given by $\rho(a) = \lim ||a^n||^{1/n}$. But in the self-adjoint case, $a = a^*$, this gives $\rho(a) = ||a||$, by using exponents of type $n = 2^k$, and then the extension to the general normal case is straightforward.

(6) Regarding now the last assertion, the inclusion $\sigma_B(a) \subset \sigma_A(a)$ is clear. For the converse, assume $a - \lambda \in B^{-1}$, and set $b = (a-\lambda)^*(a-\lambda)$. We have then:

$$\sigma_A(b) - \sigma_B(b) = \left\{\mu \in \mathbb{C} - \sigma_B(b) \Big| (b-\mu)^{-1} \in B - A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a - \lambda \in A^{-1}$, as desired. □

With these ingredients, we can now a prove a key result, as follows:

THEOREM 7.7 (Gelfand). *Any commutative C^*-algebra is of the form $A = C(X)$, with*

$$X = \left\{\chi : A \to \mathbb{C}\,,\ \text{normed algebra character}\right\}$$

with topology making continuous the evaluation maps $ev_a : \chi \to \chi(a)$.

PROOF. This is something quite tricky, the idea being as follows:

(1) Given a commutative C^*-algebra A, let us define a space X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(2) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ decomposition formula for usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove $ev_{a^*} = ev_a^*$ for the self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, by Theorem 7.6 (4), because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(3) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The above result is something truly remarkable, and we can now formulate:

Definition 7.8. *Given an arbitrary C^*-algebra A, we can write it as*

$$A = C(X)$$

with X compact quantum space. When A is commutative, X is a usual compact space.

Which is very nice, but obviously, a bit off-topic. More on quantum spaces a bit later in this book, and for the moment, just take this as it came, namely math professor supposed to lecture on something, and ending up in lecturing on something else.

More seriously now, the Gelfand theorem that we just learned is something very useful, and getting back to our original Question 7.3, we can answer it, as follows:

Theorem 7.9. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

Proof. The universal algebra A in the statement being commutative, by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$. Thus we have proved the first assertion, and the second assertion is clear as well. □

In relation now with graphs, we have the following result:

THEOREM 7.10. *The symmetry group of a graph X having N vertices is given by the following formula, at the level of the corresponding algebra of functions,*

$$C(G(X)) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic},\ du = ud\right)$$

with $d \in M_N(0,1)$ being as usual the adjacency matrix of X, and the multiplication, unit and inversion map of $G(X)$ appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on $G(X)$, by transposing.

PROOF. This follows indeed from Theorem 7.9, by combining it with Theorem 7.2, which tells us that when dealing with a graph X, with adjacency matrix $d \in M_N(0,1)$, we simply must add the condition $du = ud$, on the corresponding magic matrix u. $\square$

Let us discuss as well what happens in relation with the partial permutations, introduced at the end of chapter 5. We first have the following result:

THEOREM 7.11. *The algebra of functions on $\widetilde{S}_N$ has the following presentation, with submagic meaning formed of projections, pairwise orthogonal on rows and columns,*

$$C(\widetilde{S}_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{submagic}\right)$$

and the multiplication and unit of $\widetilde{S}_N$ appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij}$$

defined at the algebraic level, of functions on $\widetilde{S}_N$, by transposing.

PROOF. This is very similar to the proof of Theorem 7.9, with the result again coming from the Gelfand theorem, applied to the universal algebra in the statement. $\square$

In relation now with graphs, the result, which is quite tricky, is as follows:

THEOREM 7.12. *Given a graph X with N vertices, and adjacency matrix $d \in M_N(0,1)$, consider its partial automorphism semigroup, given by:*

$$\widetilde{G}(X) = \left\{\sigma \in \widetilde{S}_N\Big|d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i,j \in Dom(\sigma)\right\}$$

We have then the following formula, with $R = diag(R_i)$, $C = diag(C_j)$, with R_i, C_j being the row and column sums of the associated submagic matrix u:

$$C(\widetilde{G}(X)) = C(\widetilde{S}_N)\Big/\Big\langle R(du - ud)C = 0\Big\rangle$$

Moreover, when using the relation $du = ud$ instead of the above one, we obtain a certain semigroup $\bar{G}(X) \subset \widetilde{G}(X)$, which can be strictly smaller.

Proof. This requires a bit of abstract thinking, the idea being as follows:

(1) To start with, we will use the formula from Theorem 7.11, namely:

$$C(\widetilde{S}_N)=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{submagic}\right)$$

(2) Getting now to graphs, the definition of $\widetilde{G}(X)$ in the statement reformulates as follows, in terms of the usual adjacency relation $i-j$ for the vertices:

$$\widetilde{G}(X)=\left\{\sigma\in\widetilde{S}_N\Big|i-j,\exists\,\sigma(i),\exists\,\sigma(j)\implies\sigma(i)-\sigma(j)\right\}$$

Indeed, this reformulation is something which is clear from definitions.

(3) In view of this, we have the following product computation:

$$\begin{aligned}(du)_{ij}(\sigma)&=\sum_k d_{ik}u_{kj}(\sigma)\\&=\sum_{k\sim i}u_{kj}(\sigma)\\&=\begin{cases}1&\text{if }\sigma(j)-i\\0&\text{otherwise}\end{cases}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}(ud)_{ij}(\sigma)&=\sum_k u_{ik}d_{kj}(\sigma)\\&=\sum_{k\sim j}u_{ik}(\sigma)\\&=\begin{cases}1&\text{if }\sigma^{-1}(i)-j\\0&\text{otherwise}\end{cases}\end{aligned}$$

To be more precise, in the above two formulae the "otherwise" cases include by definition the cases where $\sigma(j)$, respectively $\sigma^{-1}(i)$, is undefined.

(4) On the other hand, we have as well the following formulae:

$$R_i(\sigma)=\sum_j u_{ij}(\sigma)=\begin{cases}1&\text{if }\exists\,\sigma^{-1}(i)\\0&\text{otherwise}\end{cases}$$

$$C_j(\sigma)=\sum_i u_{ij}(\sigma)=\begin{cases}1&\text{if }\exists\,\sigma(j)\\0&\text{otherwise}\end{cases}$$

(5) Now by multiplying the above formulae, we obtain the following formulae:

$$(R_i(du)_{ij}C_j)(\sigma)=\begin{cases}1&\text{if }\sigma(j)-i\text{ and }\exists\,\sigma^{-1}(i)\text{ and }\exists\,\sigma(j)\\0&\text{otherwise}\end{cases}$$

$$(R_i(ud)_{ij}C_j)(\sigma)=\begin{cases}1&\text{if }\sigma^{-1}(i)-j\text{ and }\exists\,\sigma^{-1}(i)\text{ and }\exists\,\sigma(j)\\0&\text{otherwise}\end{cases}$$

(6) We conclude that the relations in the statement, $R_i(du)_{ij}C_j=R_i(ud)_{ij}C_j$, when applied to a given element $\sigma\in\widetilde{S}_N$, correspond to the following condition:

$$\exists\,\sigma^{-1}(i),\ \exists\,\sigma(j)\implies[\sigma(j)-i\iff\sigma^{-1}(i)-j]$$

But with $i=\sigma(k)$, this latter condition reformulates as follows:

$$\exists\,\sigma(k),\ \exists\,\sigma(j)\implies[\sigma(j)-\sigma(k)\iff k-j]$$

Thus we must have $\sigma\in\widetilde{G}(X)$, and we obtain the presentation result for $\widetilde{G}(X)$.

(7) Regarding now the second assertion, the simplest counterexample here is simplex X_N, having N vertices and edges everywhere. Indeed, the adjacency matrix of this simplex is $d=\mathbb{I}_N-1_N$, with $\mathbb{I}_N$ being the all-1 matrix, and so the commutation of this matrix with u corresponds to the fact that u must be bistochastic. Thus, u must be in fact magic, and we obtain $\bar{G}(X_N)=S_N$, which is smaller than $\widetilde{G}(X_N)=\widetilde{S}_N$. □

Many interesting things can be said here, and we have of course many explicit examples, for the graphs having small number of vertices. We will be back to this.

7b. Representation theory

Let us discuss now another advanced algebraic topic, namely the Peter-Weyl theory for the finite groups, and in particular for the permutation groups. The idea here will be that there are several non-trivial things that can be said about the group actions on graphs, $G\curvearrowright X$, by using the Peter-Weyl theory for finite groups, according to:

PRINCIPLE 7.13. *Any finite group action on a finite graph $G\curvearrowright X$, with $|X|=N$, produces a unitary representation of G, obtained as*

$$G\to S_N\subset U_N$$

that we can decompose and study by using the Peter-Weyl theory for G. And with this study leading to non-trivial results about the action $G\curvearrowright X$, and about X itself.

Getting started now, with this program, we first have to forget about the finite graphs X, and develop the Peter-Weyl theory for the finite groups G. We first have:

DEFINITION 7.14. *A representation of a finite group G is a morphism as follows:*

$$u : G \to U_N$$

The character of such a representation is the function $\chi : G \to \mathbb{C}$ given by

$$g \to Tr(u_g)$$

where Tr is the usual trace of the $N \times N$ matrices, $Tr(M) = \sum_i M_{ii}$.

As a basic example here, for any finite group we always have available the trivial 1-dimensional representation, which is by definition as follows:

$$u : G \to U_1 \quad , \quad g \to (1)$$

At the level of non-trivial examples now, most of the groups that we met so far, in chapter 5, naturally appear as subgroups $G \subset U_N$. In this case, the embedding $G \subset U_N$ is of course a representation, called fundamental representation:

$$u : G \subset U_N \quad , \quad g \to g$$

In this situation, there are many other representations of G, which are equally interesting. For instance, we can define the representation conjugate to u, as being:

$$\bar{u} : G \subset U_N \quad , \quad g \to \bar{g}$$

In order to clarify all this, and see which representations are available, let us first discuss the various operations on the representations. The result here is as follows:

PROPOSITION 7.15. *The representations of a finite group G are subject to:*

(1) *Making sums. Given representations u, v, having dimensions N, M, their sum is the $N + M$-dimensional representation $u + v = diag(u, v)$.*
(2) *Making products. Given representations u, v, having dimensions N, M, their tensor product is the NM-dimensional representation $(u \otimes v)_{ia,jb} = u_{ij}v_{ab}$.*
(3) *Taking conjugates. Given a representation u, having dimension N, its complex conjugate is the N-dimensional representation $(\bar{u})_{ij} = \bar{u}_{ij}$.*
(4) *Spinning by unitaries. Given a representation u, having dimension N, and a unitary $V \in U_N$, we can spin u by this unitary, $u \to VuV^*$.*

PROOF. The fact that the operations in the statement are indeed well-defined, among maps from G to unitary groups, is indeed routine, and this gives the result. □

In relation now with characters, we have the following result:

PROPOSITION 7.16. *We have the following formulae, regarding characters*

$$\chi_{u+v} = \chi_u + \chi_v \quad , \quad \chi_{u\otimes v} = \chi_u \chi_v \quad , \quad \chi_{\bar{u}} = \bar{\chi}_u \quad , \quad \chi_{VuV^*} = \chi_u$$

in relation with the basic operations for the representations.

PROOF. All these assertions are elementary, by using the following well-known trace formulae, valid for any two square matrices g,h, and any unitary V:

$$Tr(diag(g,h)) = Tr(g) + Tr(h) \quad , \quad Tr(g \otimes h) = Tr(g)Tr(h)$$

$$Tr(\bar{g}) = \overline{Tr(g)} \quad , \quad Tr(VgV^*) = Tr(g)$$

Thus, we are led to the conclusions in the statement. □

Assume now that we are given a finite group $G \subset U_N$. By using the above operations, we can construct a whole family of representations of G, as follows:

DEFINITION 7.17. *Given a finite group $G \subset U_N$, its Peter-Weyl representations are the tensor products between the fundamental representation and its conjugate:*

$$u : G \subset U_N \quad , \quad \bar{u} : G \subset U_N$$

We denote these tensor products $u^{\otimes k}$, with $k = \circ\bullet\bullet\circ\ldots$ being a colored integer, with the colored tensor powers being defined according to the rules

$$u^{\otimes\circ} = u \quad , \quad u^{\otimes\bullet} = \bar{u} \quad , \quad u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$$

and with the convention that $u^{\otimes\emptyset}$ is the trivial representation $1 : G \to U_1$.

Here are a few examples of such Peter-Weyl representations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes\circ\circ} = u \otimes u \quad , \quad u^{\otimes\circ\bullet} = u \otimes \bar{u}$$

$$u^{\otimes\bullet\circ} = \bar{u} \otimes u \quad , \quad u^{\otimes\bullet\bullet} = \bar{u} \otimes \bar{u}$$

Observe also that the characters of Peter-Weyl representations are given by the following formula, with the powers χ being given by $\chi^\circ = \chi$, $\chi^\bullet = \bar{\chi}$ and multiplicativity:

$$\chi_{u^{\otimes k}} = (\chi_u)^k$$

In order now to advance, let us formulate the following key definition:

DEFINITION 7.18. *Given a finite group G, and two of its representations,*

$$u : G \to U_N \quad , \quad v : G \to U_M$$

we define the linear space of intertwiners between these representations as being

$$Hom(u,v) = \left\{ T \in M_{M\times N}(\mathbb{C}) \Big| Tu_g = v_g T, \forall g \in G \right\}$$

and we use the following conventions:

(1) *We use the notations $Fix(u) = Hom(1,u)$, and $End(u) = Hom(u,u)$.*
(2) *We write $u \sim v$ when $Hom(u,v)$ contains an invertible element.*
(3) *We say that u is irreducible, and write $u \in Irr(G)$, when $End(u) = \mathbb{C}1$.*

The terminology here is very standard, with Hom and End standing for "homomorphisms" and "endomorphisms", and with Fix standing for "fixed points". We have:

THEOREM 7.19. *The following happen:*

(1) *The intertwiners are stable under composition:*
$$T\in Hom(u,v)\ ,\ S\in Hom(v,w)\implies ST\in Hom(u,w)$$

(2) *The intertwiners are stable under taking tensor products:*
$$S\in Hom(u,v)\ ,\ T\in Hom(w,t)\implies S\otimes T\in Hom(u\otimes w,v\otimes t)$$

(3) *The intertwiners are stable under taking adjoints:*
$$T\in Hom(u,v)\implies T^*\in Hom(v,u)$$

(4) *Thus, the Hom spaces form a tensor $*$-category.*

PROOF. All this is clear from definitions, the verifications being as follows:

(1) This follows indeed from the following computation, valid for any $g\in G$:
$$STu_g=Sv_gT=w_gST$$

(2) Again, this is clear, because we have the following computation:
$$\begin{aligned}(S\otimes T)(u_g\otimes w_g)&=Su_g\otimes Tw_g\\&=v_gS\otimes t_gT\\&=(v_g\otimes t_g)(S\otimes T)\end{aligned}$$

(3) This follows from the following computation, valid for any $g\in G$:
$$\begin{aligned}Tu_g=v_gT&\implies u_g^*T^*=T^*v_g^*\\&\implies T^*v_g=u_gT^*\end{aligned}$$

(4) This is just an abstract conclusion of (1,2,3), with a tensor $*$-category being by definition an abstract beast satisfying these conditions (1,2,3). We will be back to tensor categories later on in this book, with more details on all this. □

As a main consequence of Theorem 7.19, we have:

THEOREM 7.20. *Given a representation $u:G\to U_N$, the linear space*
$$End(u)\subset M_N(\mathbb C)$$
is a $*$-algebra, with respect to the usual involution of the matrices.

PROOF. We know from Theorem 7.19 (1) that $End(u)$ is a subalgebra of $M_N(\mathbb C)$, and we know as well from Theorem 7.19 (3) that this subalgebra is stable under the involution $*$. Thus, what we have here is a $*$-subalgebra of $M_N(\mathbb C)$, as claimed. □

Our claim now is that Theorem 7.20 gives us everything that we need, in order to have some advanced representation theory started, for our finite groups G. Indeed, we can combine this result with the following standard fact, from matrix algebra:

THEOREM 7.21. *Let $A\subset M_N(\mathbb C)$ be a $*$-algebra.*

(1) *We can write $1=p_1+\ldots+p_k$, with $p_i\in A$ being central minimal projections.*
(2) *The linear spaces $A_i=p_iAp_i$ are non-unital $*$-subalgebras of A.*
(3) *We have a non-unital $*$-algebra sum decomposition $A=A_1\oplus\ldots\oplus A_k$.*
(4) *We have unital $*$-algebra isomorphisms $A_i\simeq M_{n_i}(\mathbb C)$, with $n_i=rank(p_i)$.*
(5) *Thus, we have a $*$-algebra isomorphism $A\simeq M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$.*

PROOF. This is something standard, whose proof is however quite long, as follows:

(1) Consider an arbitrary $*$-algebra of the $N\times N$ matrices, $A\subset M_N(\mathbb C)$, as in the statement. Let us first look at the center of this algebra, $Z(A)=A\cap A'$. It is elementary to prove that this center, as an algebra, is of the following form:

$$Z(A)\simeq\mathbb C^k$$

Consider now the standard basis $e_1,\ldots,e_k\in\mathbb C^k$, and let $p_1,\ldots,p_k\in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1,\ldots,p_k\in A$ are central minimal projections, summing up to 1:

$$p_1+\ldots+p_k=1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

(2) We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i=p_iAp_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2=p_i=p_i^*$.

(3) We prove now that the above algebras $A_i\subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A=A_1\oplus\ldots\oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1+\ldots+p_k=1$ and the projection equations $p_i^2=p_i=p_i^*$, we conclude that we have the needed generation property, namely:

$$A_1+\ldots+A_k=A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1+\ldots+p_k=1$, and from the projection equations $p_i^2=p_i=p_i^*$.

(4) Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i=p_iAp_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i=rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i\simeq M_{n_i}(\mathbb C)$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

(5) We can now obtain the result, by putting together what we have. Indeed, by using the results from (3) and (4), we obtain an isomorphism as follows:

$$A \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

In addition to this, a careful look at the isomorphisms established in (4) shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, step (2) above, by a certain unitary matrix $U \in U_N$:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. $\square$

We can now formulate our first Peter-Weyl theorem, as follows:

THEOREM 7.22 (PW1). *Let $u : G \to U_N$ be a representation, consider the algebra $A = End(u)$, and write its unit as above, with p_i being central minimal projections:*

$$1 = p_1 + \ldots + p_k$$

The representation u decomposes then as a direct sum, as follows,

$$u = u_1 + \ldots + u_k$$

with each u_i being an irreducible representation, obtained by restricting u to $Im(p_i)$.

PROOF. This follows from Theorem 7.20 and Theorem 7.21, as follows:

(1) As a first observation, by replacing G with its image $u(G) \subset U_N$, we can assume if we want that our representation u is faithful, $G \subset_u U_N$. However, this replacement will not be really needed, and we will keep using $u : G \to U_N$, as above.

(2) In order to prove the result, we will need some preliminaries. We first associate to our representation $u : G \to U_N$ the corresponding action map on $\mathbb{C}^N$. If a linear subspace $V \subset \mathbb{C}^N$ is invariant, the restriction of the action map to V is an action map too, which must come from a subrepresentation $v \subset u$. This is clear indeed from definitions, and with the remark that the unitaries, being isometries, restrict indeed into unitaries.

(3) Consider now a projection $p \in End(u)$. From $pu = up$ we obtain that the linear space $V = Im(p)$ is invariant under u, and so this space must come from a subrepresentation $v \subset u$. It is routine to check that the operation $p \to v$ maps subprojections to subrepresentations, and minimal projections to irreducible representations.

(4) With these preliminaries in hand, let us decompose the algebra $End(u)$ as in Theorem 7.21, by using the decomposition $1 = p_1 + \ldots + p_k$ into minimal projections. If we denote by $u_i \subset u$ the subrepresentation coming from the vector space $V_i = Im(p_i)$, then we obtain in this way a decomposition $u = u_1 + \ldots + u_k$, as in the statement. $\square$

Here is now our second Peter-Weyl theorem, complementing Theorem 7.22:

THEOREM 7.23 (PW2). *Given a subgroup $G \subset_u U_N$, any irreducible representation*

$$v : G \to U_M$$

appears inside a tensor product of the fundamental representation u and its adjoint $\bar{u}$.

PROOF. We define the space of coefficients a representation $v : G \to U_M$ to be:

$$C_v = span\Big[g \to (v_g)_{ij}\Big]$$

The construction $v \to C_v$ is then functorial, in the sense that it maps subrepresentations into linear subspaces. Also, we have an inclusion of linear spaces as follows:

$$C_v \subset < g_{ij} >$$

On the other hand, by definition of the Peter-Weyl representations, we have:

$$< g_{ij} >= \sum_k C_{u^{\otimes k}}$$

Thus, we must have an inclusion as follows, for certain exponents $k_1, \ldots, k_p$:

$$C_v \subset C_{u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}}$$

We conclude that we have an inclusion of representations, as follows:

$$v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$$

Together with Theorem 7.22, this leads to the conclusion in the statement. $\square$

As a third Peter-Weyl theorem, which is something more advanced, we have:

THEOREM 7.24 (PW3). *We have a direct sum decomposition of linear spaces*

$$C(G) = \bigoplus_{v \in Irr(G)} M_{\dim(v)}(\mathbb{C})$$

with the summands being pairwise orthogonal with respect to the scalar product

$$< a, b >= \int_G ab^*$$

where $\int_G$ is the averaging over G.

PROOF. This is something more tricky, the idea being as follows:

(1) By combining the previous two Peter-Weyl results, from Theorem 7.22 and Theorem 7.23, we deduce that we have a linear space decomposition as follows:

$$C(G)=\sum_{v\in Irr(G)}C_v=\sum_{v\in Irr(G)}M_{\dim(v)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C_v\perp C_w$$

(2) We will need the basic fact, whose proof is elementary, that for any representation v we have the following formula, where P is the orthogonal projection on $Fix(v)$:

$$\left(id\otimes\int_G\right)v=P$$

(3) We will also need the basic fact, whose proof is elementary too, that for any two representations v,w we have an isomorphism as follows, called Frobenius isomorphism:

$$Hom(v,w)\simeq Fix(v\otimes\bar{w})$$

(4) Now back to our orthogonality question from (1), let us set indeed:

$$P_{ia,jb}=\int_G v_{ij}w_{ab}^*$$

Then P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(v,w)=\{0\}$$

Thus we have $P=0$, and this gives the result. $\square$

Finally, we have the following result, completing the Peter-Weyl theory:

THEOREM 7.25 (PW4). *The characters of irreducible representations belong to*

$$C(G)_{central}=\left\{f\in C(G)\Big|f(gh)=f(hg),\forall g,h\in G\right\}$$

called algebra of central functions on G, and form an orthonormal basis of it.

PROOF. We have several things to be proved, the idea being as follows:

(1) Observe first that $C(G)_{central}$ is indeed an algebra, which contains all the characters. Conversely, consider a function $f\in C(G)$, written as follows:

$$f=\sum_{v\in Irr(G)}f_v$$

The condition $f\in C(G)_{central}$ states then that for any $v\in Irr(G)$, we must have:

$$f_v\in C(G)_{central}$$

But this means precisely that the coefficient f_v must be a scalar multiple of χ_v, and so the characters form a basis of $C(G)_{central}$, as stated.

(2) The fact that we have an orthogonal basis follows from Theorem 7.24.

(3) As for the fact that the characters have norm 1, this follows from:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the above integrals $\int_G v_{ij} v_{kl}^*$ form the orthogonal projection onto the following vector space:

$$Fix(v \otimes \bar{v}) \simeq End(v) = \mathbb{C}1$$

Thus, the proof of our theorem is now complete. □

So long for Peter-Weyl theory. As a comment, our approach here, which was rather functional analytic, was motivated by what we will be doing later in this book, in relation with quantum groups. For a more standard presentation of the Peter-Weyl theory for finite groups, there are many good books available, such as the book of Serre [**83**].

7c. Transitive actions

With the above understood, let us get back to graphs. As already mentioned in the beginning of the previous section, the general principle is that representation theory and the Peter-Weyl theorems can help in understanding the group actions on graphs, $G \curvearrowright X$, by regarding these actions as unitary representations of G, as follows:

$$u : G \to S_N \subset U_N$$

To be more precise, from a finite group and representation theory perspective, the whole graph symmetry problematics can be reformulated as follows:

Problem 7.26. *Given a finite group representation $u : G \to U_N$:*

1. *Is this representation magic, in the sense that it factorizes through S_N?*
2. *If so, what are the $0-1$ graph adjacency matrices $d \in End(u)$?*
3. *Among these latter matrices, which ones generate $End(u)$?*

Generally speaking, these questions are quite difficult, ultimately leading into planar algebras in the sense of Jones [**56**], and despite having learned in this chapter some serious C^*-algebra theory, and Peter-Weyl theory, we are not ready yet for such things. All this will have to wait for chapter 12 below, when we will discuss planar algebras.

In the meantime we can, however, discuss some elementary applications of Peter-Weyl theory to the study of graphs. As starting point, we have the following result:

THEOREM 7.27. *Given a transitive action $G \curvearrowright X$, any group character*

$$\chi : G \to \mathbb{T}$$

canonically produces an eigenfunction and eigenvalue of X, given by

$$f(g(0)) = \chi(g) \quad , \quad \lambda = \sum_{g(0)-0} \chi(g)$$

with $0 \in X$ being a chosen vertex of the graph.

PROOF. This is something elementary, coming from definitions, as follows:

(1) Let us fix a vertex $0 \in X$, and consider the following set:

$$S = \left\{ g \in G \middle| g(0) - 0 \right\}$$

Observe that this set $S \subset G$ satisfies $1 \notin S$, and also $S = S^{-1}$, due to:

$$\begin{aligned} g(0) - 0 &\implies g^{-1}(g(0)) - g^{-1}(0) \\ &\implies 0 - g^{-1}(0) \\ &\implies g^{-1}(0) - 0 \end{aligned}$$

(2) As a comment here, the condition $1 \notin S = S^{-1}$ is something quite familiar, namely the Cayley graph assumption from chapter 4. More about this in chapter 8 below, where we will systematically discuss the Cayley graphs, as a continuation of that material.

(3) Now given a character $\chi : G \to \mathbb{T}$ as in the statement, we can construct a function on the vertex set of our graph, $f : X \to \mathbb{T}$, according to the following formula:

$$f(g(0)) = \chi(g)$$

Observe that this function is indeed well-defined, everywhere on the graph, thanks to our assumption that the group action $G \curvearrowright X$ is transitive.

(4) Our claim now is that this function $f : X \to \mathbb{T}$ is an eigenfunction of the adjacency matrix of the graph. Indeed, we have the following computation, for any $g \in G$:

$$\begin{aligned}(df)(g(0)) &= \sum_{g(0)-j} f(j) \\ &= \sum_{g(0)-h(0)} f(h(0)) \\ &= \sum_{g(0)-h(0)} \chi(h) \\ &= \sum_{s\in S} \chi(sg) \\ &= \sum_{s\in S} \chi(s)\chi(g) \\ &= \sum_{s\in S} \chi(s) f(g(0))\end{aligned}$$

Thus, we are led to the conclusion in the statement. $\square$

The above result is quite interesting, and as a continuation of the story, we have:

THEOREM 7.28. *In the context of the eigenfunctions and eigenvalues constructed above, coming from transitive actions $G \curvearrowright X$ and characters $\chi : G \to \mathbb{T}$:*

(1) *For the trivial character, $\chi = 1$, we obtain the trivial eigenfunction, $f = 1$.*
(2) *When G is assumed to be abelian, this gives the diagonalization of d.*

Moreover, it is possible to generalize this construction, by using arbitrary irreducible representations instead of characters, and this gives again the diagonalization of d.

PROOF. There are several things going on here, the idea being as follows:

(1) This is clear from the construction from the proof of Theorem 7.27, which for trivial character, $\chi = 1$, gives $f = 1$, and $\lambda = |S|$, with the notations there.

(2) For an abelian group G, the characters form the dual group $\widehat{G} \simeq G$, and so we get a collection of $|\widehat{G}| = |G| = |X|$ eigenfunctions, as needed for diagonalizing d.

(3) In what regards the last assertion, the construction there is straightforward, with the final conclusion coming from the following formula, coming itself from Peter-Weyl:

$$\sum_{r\in Irr(G)} (\dim r)^2 = |G|$$

We will leave clarifying the details here as an instructive exercise, and we will come back to this in chapter 8, when discussing more systematically the Cayley graphs. $\square$

7d. Asymptotic aspects

We would like to end this chapter with something still advanced, in relation with representations, but more analytic and refreshing, featuring some probability.

It is about formal graphs X_N and their formal symmetry groups $G_N \subset S_N$ that we want to talk about, in the $N \to \infty$ limit, a bit as the physicists do. But, how to do this? Not clear at all, because while it is easy to talk about series of graphs $X = (X_N)$, in an intuitive way, the corresponding symmetry groups $G_N \subset S_N$ are not necessarily compatible with each other, as to form an axiomatizable object $G = (G_N)$.

Long story short, we are into potentially difficult mathematics here, and as a more concrete question that we can attempt to solve, we have:

PROBLEM 7.29. *What families of groups $G_N \subset S_N$ have compatible representation theory invariants, in the $N \to \infty$ limit? And then, can we use this in order to talk about "good" families of graphs X_N, whose symmetry groups $G_N = G(X_N)$ are of this type?*

But probably too much talking, let us get to work. The simplest graphs are undoubtedly the empty graphs and the simplices, with symmetry group S_N, so as a first question, we would like to know if the symmetric groups S_N themselves have compatible representation theory invariants, with some nice $N \to \infty$ asymptotics to work out.

And here, surprise, or miracle, the answer is indeed yes, with the result, which is something very classical, remarkable, and beautiful, being as follows:

THEOREM 7.30. *The probability for a random $\sigma \in S_N$ to have no fixed points is*

$$P \simeq \frac{1}{e}$$

in the $N \to \infty$ limit, where $e = 2.7182\ldots$ is the usual constant from analysis. More generally, the main character of S_N, which counts these permutations, given by

$$\chi = \sum_i \sigma_{ii}$$

via the standard embedding $S_N \subset O_N$, follows the Poisson law p_1, in the $N \to \infty$ limit. Even more generally, the truncated characters of S_N, given by

$$\chi = \sum_{i=1}^{[tN]} \sigma_{ii}$$

with $t > 0$, follow the Poisson laws p_t, in the $N \to \infty$ limit.

Proof. Obviously, many things going on here. The idea is as follows:

(1) In order to prove the first assertion, which is the key one, and probably the most puzzling one too, we will use the inclusion-exclusion principle. Let us set:

$$S_N^k=\left\{\sigma\in S_N\Big|\sigma(k)=k\right\}$$

The set of permutations having no fixed points, called derangements, is then:

$$X_N=\left(\bigcup_k S_N^k\right)^c$$

Now the inclusion-exclusion principle tells us that we have:

$$\begin{aligned}
|X_N| &= \left|\left(\bigcup_k S_N^k\right)^c\right|\\
&= |S_N|-\sum_k|S_N^k|+\sum_{k<l}|S_N^k\cap S_N^l|-\ldots+(-1)^N\sum_{k_1<\ldots<k_N}|S_N^{k_1}\cup\ldots\cup S_N^{k_N}|\\
&= N!-N(N-1)!+\binom{N}{2}(N-2)!-\ldots+(-1)^N\binom{N}{N}(N-N)!\\
&= \sum_{r=0}^N(-1)^r\binom{N}{r}(N-r)!
\end{aligned}$$

Thus, the probability that we are interested in, for a random permutation $\sigma\in S_N$ to have no fixed points, is given by the following formula:

$$P=\frac{|X_N|}{N!}=\sum_{r=0}^N\frac{(-1)^r}{r!}$$

Since on the right we have the expansion of $1/e$, this gives the result.

(2) Let us construct now the main character of S_N, as in the statement. The permutation matrices being given by $\sigma_{ij}=\delta_{i\sigma(j)}$, we have the following formula:

$$\chi(\sigma)=\sum_i\delta_{\sigma(i)i}=\#\left\{i\in\{1,\ldots,N\}\Big|\sigma(i)=i\right\}$$

In order to establish now the asymptotic result in the statement, regarding these characters, we must prove the following formula, for any $r\in\mathbb{N}$, in the $N\to\infty$ limit:

$$P(\chi=r)\simeq\frac{1}{r!e}$$

We already know, from (1), that this formula holds at $r=0$. In the general case now, we have to count the permutations $\sigma\in S_N$ having exactly r points. Now since having such a permutation amounts in choosing r points among $1,\ldots,N$, and then permuting the $N-r$ points left, without fixed points allowed, we have:

$$\begin{aligned}
\#\left\{\sigma\in S_N\Big|\chi(\sigma)=r\right\} &= \binom{N}{r}\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}\\
&= \frac{N!}{r!(N-r)!}\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}\\
&= N!\times\frac{1}{r!}\times\frac{\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}}{(N-r)!}
\end{aligned}$$

By dividing everything by $N!$, we obtain from this the following formula:

$$\frac{\#\left\{\sigma\in S_N\Big|\chi(\sigma)=r\right\}}{N!}=\frac{1}{r!}\times\frac{\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}}{(N-r)!}$$

Now by using the computation at $r=0$, that we already have, from (1), it follows that with $N\to\infty$ we have the following estimate:

$$P(\chi=r)\simeq\frac{1}{r!}\cdot P(\chi=0)\simeq\frac{1}{r!}\cdot\frac{1}{e}$$

Thus, we obtain as limiting measure the Poisson law of parameter 1, as stated.

(3) Finally, let us construct the truncated characters of S_N, as in the statement. As before in the case $t=1$, we have the following computation, coming from definitions:

$$\chi_t(\sigma)=\sum_{i=1}^{[tN]}\delta_{\sigma(i)i}=\#\left\{i\in\{1,\ldots,[tN]\}\Big|\sigma(i)=i\right\}$$

Also before in the proofs of (1) and (2), we obtain by inclusion-exclusion that:

$$\begin{aligned}
P(\chi_t=0) &= \frac{1}{N!}\sum_{r=0}^{[tN]}(-1)^r\sum_{k_1<\ldots<k_r<[tN]}|S_N^{k_1}\cap\ldots\cap S_N^{k_r}|\\
&= \frac{1}{N!}\sum_{r=0}^{[tN]}(-1)^r\binom{[tN]}{r}(N-r)!\\
&= \sum_{r=0}^{[tN]}\frac{(-1)^r}{r!}\cdot\frac{[tN]!(N-r)!}{N!([tN]-r)!}
\end{aligned}$$

Now with $N\to\infty$, we obtain from this the following estimate:

$$P(\chi_t=0)\simeq\sum_{r=0}^{[tN]}\frac{(-1)^r}{r!}\cdot t^r\simeq e^{-t}$$

More generally, by counting the permutations $\sigma \in S_N$ having exactly r fixed points among $1, \ldots, [tN]$, as in the proof of (2), we obtain:

$$P(\chi_t = r) \simeq \frac{t^r}{r! e^t}$$

Thus, we obtain in the limit a Poisson law of parameter t, as stated. □

Escalating difficulties, let us discuss now the hyperoctahedral group H_N. Here the result is more technical, getting us into more advanced probability, as follows:

THEOREM 7.31. *For the hyperoctahedral group $H_N \subset O_N$, the law of the truncated character $\chi = g_{11} + \ldots + g_{ss}$ with $s = [tN]$ is, in the $N \to \infty$ limit, the measure*

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r \sum_{p=0}^{\infty} \frac{(t/2)^{|r|+2p}}{(|r|+p)!p!}$$

called Bessel law of parameter $t > 0$.

PROOF. We regard H_N as being the symmetry group of the graph $I_N = \{I^1, \ldots, I^N\}$ formed by n segments. The diagonal coefficients are then given by:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ 1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let us denote by F_g, R_g the number of segments among $\{I^1, \ldots, I^s\}$ which are fixed, respectively returned by an element $g \in H_N$. With this notation, we have:

$$u_{11} + \ldots + u_{ss} = F_g - R_g$$

Let us denote by P_N the probabilities computed over H_N. The density of the law of the variable $u_{11} + \ldots + u_{ss}$ at a point $r \geq 0$ is then given by the following formula:

$$D(r) = P_N(F_g - R_g = r) = \sum_{p=0}^{\infty} P_N(F_g = r + p, R_g = p)$$

Assume first that we are in the case $t = 1$. We have the following computation:

$$\begin{aligned} \lim_{N\to\infty} D(r) &= \lim_{N\to\infty} \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} P_N(F_g + R_g = r + 2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} \frac{1}{e(r+2p)!} \\ &= \frac{1}{e} \sum_{p=0}^{\infty} \frac{(1/2)^{r+2p}}{(r+p)!p!} \end{aligned}$$

The general case $0 < t \leq 1$ follows by performing some modifications in the above computation. Indeed, the asymptotic density can be computed as follows:

$$\begin{aligned}
\lim_{N\to\infty} D(r) &= \lim_{N\to\infty}\sum_{p=0}^{\infty}(1/2)^{r+2p}\binom{r+2p}{r+p}P_N(F_g+R_g=r+2p)\\
&= \sum_{p=0}^{\infty}(1/2)^{r+2p}\binom{r+2p}{r+p}\frac{t^{r+2p}}{e^t(r+2p)!}\\
&= e^{-t}\sum_{p=0}^{\infty}\frac{(t/2)^{r+2p}}{(r+p)!p!}
\end{aligned}$$

Together with $D(-r) = D(r)$, this gives the formula in the statement. □

In order to further discuss now all this, and extend the above results, we will need a number of standard probabilistic preliminaries. We have the following notion:

Definition 7.32. *Associated to any compactly supported positive measure ν on $\mathbb{C}$, not necessarily of mass 1, is the probability measure*

$$p_\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{1}{n}\nu\right)^{*n}$$

where $t = mass(\nu)$, called compound Poisson law.

In what follows we will be mainly interested in the case where the measure ν is discrete, as is for instance the case for $\nu = t\delta_1$ with $t > 0$, which produces the Poisson laws. The following standard result allows one to detect compound Poisson laws:

Proposition 7.33. *For $\nu = \sum_{i=1}^s t_i\delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{C}$, we have*

$$F_{p_\nu}(y) = \exp\left(\sum_{i=1}^s t_i(e^{iyz_i}-1)\right)$$

where F denotes the Fourier transform.

Proof. Let μ_n be the Bernoulli measure appearing in Definition 7.32, under the convolution sign. We have then the following Fourier transform computation:

$$\begin{aligned}
F_{\mu_n}(y) = \left(1-\frac{t}{n}\right)+\frac{1}{n}\sum_{i=1}^s t_ie^{iyz_i} \quad&\implies\quad F_{\mu_n^{*n}}(y) = \left(\left(1-\frac{t}{n}\right)+\frac{1}{n}\sum_{i=1}^s t_ie^{iyz_i}\right)^n\\
&\implies\quad F_{p_\nu}(y) = \exp\left(\sum_{i=1}^s t_i(e^{iyz_i}-1)\right)
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

Getting back now to the Poisson and Bessel laws, we have:

THEOREM 7.34. *The Poisson and Bessel laws are compound Poisson laws,*

$$p_t = p_{t\delta_1} \quad , \quad b_t = p_{t\varepsilon}$$

where δ_1 is the Dirac mass at 1, and ε is the centered Bernoulli law, $\varepsilon = (\delta_1 + \delta_{-1})/2$.

PROOF. We have two assertions here, the idea being as follows:

(1) The first assertion, regarding the Poisson law p_t, is clear from Definition 7.22, which for $\nu = t\delta_1$ takes the following form:

$$p_\nu = \lim_{n\to\infty}\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\,\delta_1\right)^{*n}$$

But this is a usual Poisson limit, producing the Poisson law p_t, as desired.

(2) Regarding the second assertion, concerning the Bessel law b_t, by further building on the various formulae from Theorem 7.31 and its proof, we have:

$$F_{b_t}(y) = \exp\left(t\left(\frac{e^{iy} + e^{-iy}}{2} - 1\right)\right)$$

On the other hand, the formula in Proposition 7.33 gives, for $\nu = t\varepsilon$, the same Fourier transform formula. Thus, with $\nu = t\varepsilon$ we have $p_\nu = b_t$, as claimed. □

More generally now, and in answer to Problem 7.29, for the groups H_N^s from chapter 5 the asymptotic truncated characters follow the law $b_t^s = p_{t\varepsilon_s}$, with ε_s being the uniform measure on the s-roots of unity, and if you are really picky about what happens with $N \to \infty$, these are the only solutions. For details on this, you can check my book [5].

7e. Exercises

We had an exciting chapter here, mixing advanced algebra techniques with advanced analysis and probability. As exercises on all this, we have:

EXERCISE 7.35. *Fine-tune your Peter-Weyl learning, with some character tables.*

EXERCISE 7.36. *Clarify what else can be done with magics, in the graph context.*

EXERCISE 7.37. *Learn more about planar algebras, and their relation with graphs.*

EXERCISE 7.38. *Learn about the representations of S_N, at fixed $N \in \mathbb{N}$.*

EXERCISE 7.39. *Have a look at the representations of H_N too, at fixed $N \in \mathbb{N}$.*

EXERCISE 7.40. *Revise your probability knowledge, notably with the CLT and PLT.*

EXERCISE 7.41. *Learn more about compound Poisson laws, and their properties.*

EXERCISE 7.42. *Learn more about the Bessel laws, and their properties.*

As bonus exercise, learn some advanced probability. Among others, because we will soon get into quantum, and there is no quantum without probability.

CHAPTER 8

Cayley graphs

8a. Cayley graphs

We discuss here the Cayley graphs of finite groups, and what can be done with them. We have already met these graphs in chapter 3, their definition being as follows:

DEFINITION 8.1. *Associated to any finite group $G =< S >$, with the generating set S assumed to satisfy $1 \notin S = S^{-1}$, is its Cayley graph, constructed as follows:*

(1) *The vertices are the group elements $g \in G$.*
(2) *Edges $g - h$ are drawn when $h = gs$, with $s \in S$.*

As a first observation, the Cayley graph is indeed a graph, because our assumption $S = S^{-1}$ on the generating set shows that we have $g - h \implies h - g$, as we should, and also because our assumption $1 \notin S$ excludes the self-edges, $g \not- g$.

Observe also that the Cayley graph depends, in a crucial way, on the generating set S satisfying $1 \notin S = S^{-1}$. Indeed, if we choose for instance $S = G - \{1\}$, we obtain the complete graph with $N = |G|$ vertices, and this regardless of what our group G is. Thus, the Cayley graph as constructed above is not exactly associated to the group G, but rather to the group G viewed as finitely generated group, $G =< S >$.

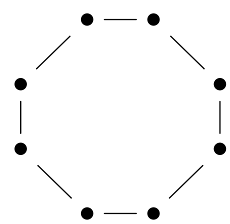
In view of this, we will usually look for generating sets S which are minimal, in order to perform our Cayley graph construction, and get non-trivial graphs. Here are now some examples, with minimal generating sets, that we know well from chapter 3:

PROPOSITION 8.2. *We have the following examples of Cayley graphs, each time with respect to the standard, minimal generating set satisfying $S = S^{-1}$:*

(1) *For $\mathbb{Z}_N$ we obtain the cycle graph C_N.*
(2) *For $\mathbb{Z}_2 \times \mathbb{Z}_3$ we obtain the prism $P(C_3)$.*
(3) *For $\mathbb{Z}_2^N$ we obtain the hypercube graph $\square_N$.*

PROOF. This is something elementary, that we know well from chapter 3, with the generating set in question being $S = \{-1, 1\}$ for the cyclic group $\mathbb{Z}_N$, written additively, unless we are in the case $N = 2$, where this set simply becomes $S = \{1\}$, again written additively, and with the generating sets for products being obtained in the obvious minimal way, by taking the union of the generating sets of the components:

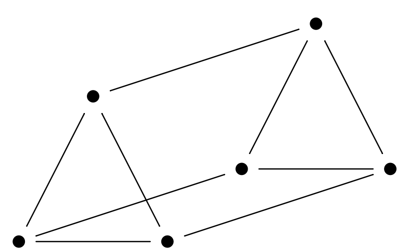
(1) For the group $\mathbb{Z}_N =< -1, 1 >$, written additively, our condition for the edges $g - h$ reads $g = h \pm 1$, so we are led to the cycle graph C_N, namely:



(2) For the group $\mathbb{Z}_2 \times \mathbb{Z}_3 =< (1,0),(0,1),(0,2) >$, again written additively, our condition for the edges takes the following form:

$$(g,a) - (h,b) \iff g = h,\ a = b \pm 1 \text{ or } g = h+1, a = b$$

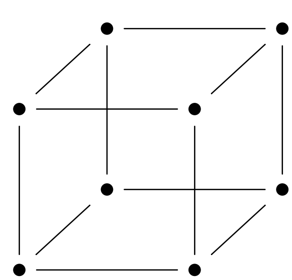
But this leads to the prism graph $P(C_3)$, which is as follows:



(3) Finally, for the group $\mathbb{Z}_2^N =< 1_1, \ldots, 1_N >$, with 1_i standing for the standard generators of the components, our condition for the edges takes the following form:

$$g - h \iff \exists!\, i, g_i \neq h_i$$

Now if we represent the elements of $\mathbb{Z}_2^N = (0,1)^N$ as points in $\mathbb{R}^N$, in the obvious way, we are led to the hypercube graph $\square_N$, which is as follows:



Thus, we are led to the conclusions in the statement. $\square$

The above result dates back to chapter 3, and time now to improve it, by using the product operations for graphs, that we learned in chapter 6. We have:

THEOREM 8.3. *Given two groups $G=<S>$ and $H=<T>$, we have*

$$G\times H=<S\times 1,1\times T>$$

and at the level of the corresponding Cayley graphs, we have the formula

$$X_{G\times H}=X_G\,\square\,X_H$$

involving the Cartesian product operation for graphs $\square$.

PROOF. We have indeed a generating set, which satisfies the condition $1\notin S=S^{-1}$. Now observe that when constructing the Cayley graph, the edges are as follows:

$$(g,a)-(h,b)\iff g=h,\ a-b \text{ or } g-h, a=b$$

Thus, we obtain indeed a Cartesian product $X_{G\times H}=X_G\,\square\,X_H$, as claimed. $\square$

Along the same lines, let us record as well the following result, which is however rather anecdotical, first because the subgroup $K\subset G\times H$ appearing below might not equal the group $G\times H$ itself, and also because, even when it does, the generating set $S\times T$ that we use is much bigger than the generating set $S\times 1,1\times T$ from Theorem 8.3:

PROPOSITION 8.4. *Given two groups $G=<S>$ and $H=<T>$, if we set*

$$K=<S\times T>\subset G\times H$$

then at the level of the corresponding Cayley graphs, we have the formula

$$X_K=X_G\times X_H$$

involving the direct product operation for graphs $\times$.

PROOF. As before, we have a set satisfying the condition $1\notin S=S^{-1}$. Now observe that when constructing the Cayley graph, the edges are as follows:

$$(g,a)-(h,b)\iff g-h,\ a-b$$

Thus, we obtain in this case a direct product $X_K=X_G\times X_H$, as claimed. $\square$

As mentioned above, this latter result is something anecdotical, and we will not use it, in what follows. Thus, our convention for products will be the one in Theorem 8.3. We can now generalize the various computations from Proposition 8.2, as follows:

THEOREM 8.5. *For a finite abelian group, written as*

$$G=\mathbb{Z}_{N_1}\times\ldots\times\mathbb{Z}_{N_k}$$

and with standard generating set, the corresponding Cayley graph is:

$$X_G=C_{N_1}\,\square\,\ldots\,\square\,C_{N_k}$$

That is, we obtain a Cartesian product of cycle graphs.

PROOF. This follows indeed from Proposition 8.2 (1), and from Theorem 8.3. $\square$

Getting now to symmetry groups, as a first observation, we have:

PROPOSITION 8.6. *Given a finite group, $G =< S >$ with $1 \notin S = S^{-1}$, we have an action of this group on its associated Cayley graph,*

$$G \curvearrowright X_G$$

and so $G \subset G(X_G)$. However, this inclusion is not an isomorphism, in general.

PROOF. We have several assertions here, the idea being as follows:

(1) Consider indeed the Cayley action of G on itself, which is given by:

$$G \subset S_G \quad , \quad g \to [h \to gh]$$

(2) Thus G acts on the vertices of its Cayley graph X_G, and our claim is that the edges are preserved by this action. Indeed, given an edge, $h - hs$ with $s \in S$, a group element $g \in G$ will transform it into $gh - ghs$, which is an edge too.

(3) Thus, the first assertion holds indeed. As for the second assertion, this holds too, and in a very convincing way, for instance because for the groups in Proposition 8.2, the corresponding Cayley graphs and their symmetry groups are as follows:

$$\mathbb{Z}_N \to C_N \to D_N$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \to P(C_3) \to D_6$$

$$\mathbb{Z}_2^N \to \square_N \to H_N$$

Thus, we are led to the conclusions in the statement. $\square$

In relation with the above, a first question would be that of suitably fine-tuning the construction of the Cayley graph X_G, as to have as end result $G = G(X_G)$, which would be something nice. We will discuss this later, the idea being that this can be done indeed, by adding orientation, or colors, or both, to the construction of the Cayley graph X_G.

As a second question now, we can try to understand the operation $G \to G(X_G)$, with our present definition for the Cayley graph X_G. Many things can be said here, and to start with, we have the following result, generalizing our computation for $\mathbb{Z}_2^N$:

THEOREM 8.7. *The symmetry group of the Cayley graph of $G = \mathbb{Z}_s^N$, which is*

$$X_G = C_s^{\square N}$$

according to our previous results, is the complex reflection group H_N^s:

$$G(X_G) = H_N^s$$

Thus, in this case, the operation $G \to G(X_G)$ is given by $\mathbb{Z}_s^N \to \mathbb{Z}_s^N \rtimes S_N$.

PROOF. This is something that we discussed in chapter 3 at $s = 2$, with the graph involved there being the hypercube $X_G = \square_N$. In general the proof is similar, by using the theory of the complex reflection groups H_N^s, that we developed in chapter 5. $\square$

Many other things can be said, along these lines, for instance with a generalization of the computation $\mathbb{Z}_2 \times \mathbb{Z}_3 \to P(C_3) \to D_6$ too, and with a number of other results regarding the operation $G \to X_G \to G(X_G)$, for the finite abelian groups G. We will leave doing some study here as an instructive exercise.

8b. Orientation, colors

We discuss now various versions of the Cayley graph construction $G \to X_G$, which are all technically useful, obtained by adding orientation, or colors, or both. Let us start with a straightforward oriented version of Definition 8.1, as follows:

Definition 8.8. *Associated to any finite group $G =< S >$, with the generating set S assumed to satisfy $1 \notin S$, is its oriented Cayley graph, constructed as follows:*

1. *The vertices are the group elements $g \in G$.*
2. *Edges $g \to h$ are drawn when $h = gs$, with $s \in S$.*

Observe that the oriented Cayley graph is indeed a graph, because our assumption $1 \notin S$ excludes the self-edges, $g \not\to g$. Observe also that in the case $S = S^{-1}$ each edge $g \to h$ has as companion edge $h \to g$, so in this case, up to replacing the pairs of edges $g \leftrightarrow h$ by usual edges $g - h$, we obtain the previous, unoriented Cayley graph.

As in the case of unoriented Cayley graphs, our graph depends a lot on the chosen writing $G =< S >$, and the point is to look for generating sets S which are minimal, in order to perform our Cayley graph construction, and get non-trivial graphs.

In view of this, what we basically have to do, in order to get started with some theory, is to review the material from the previous section, by dropping the assumption $S = S^{-1}$ there, which in practice means to remove around $1/2$ of the generators.

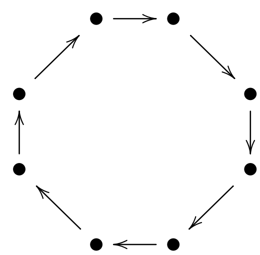
Getting started now, as a first result, in relation with Proposition 8.2, we have:

Proposition 8.9. *We have the following examples of oriented Cayley graphs, each time with respect to the standard, minimal generating set:*

1. *For $\mathbb{Z}_N$ we obtain the oriented cycle C_N^o.*
2. *For $\mathbb{Z}_2 \times \mathbb{Z}_3$ we obtain the oriented prism $P(C_3^o)$.*
3. *For $\mathbb{Z}_2^N$ we obtain the hypercube graph $\square_N$.*

Proof. This is something elementary, with the convention that the generating set is $S = \{1\}$ for the cyclic group $\mathbb{Z}_N$, written additively, and with the generating sets for products being obtained by taking the union of the generating sets for components:

(1) For the group $\mathbb{Z}_N =< 1 >$, written additively, our condition for the edges $g \to h$ reads $h = g + 1$, so we are led to the oriented cycle graph C_N^o, namely:

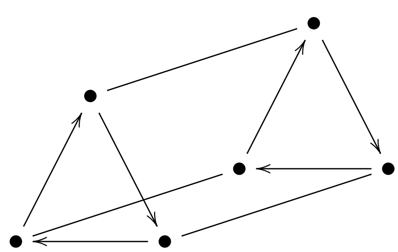


As an observation here, that will be of importance later, in the particular case $N = 2$ what we get is the graph $\bullet \longleftrightarrow \bullet$, which is the same as the usual segment, $\bullet - \bullet$.

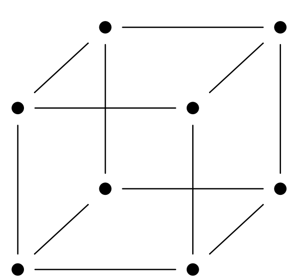
(2) For the group $\mathbb{Z}_2 \times \mathbb{Z}_3 =< (1,0), (0,1) >$, again written additively, our condition for the edges takes the following form:

$$(g,a) \to (h,b) \Longleftrightarrow g = h,\ b = a + 1 \text{ or } h = g + 1, a = b$$

But this leads to the oriented prism graph $P(C_3^o)$, which is as follows, with the convention that the usual segments $\bullet - \bullet$ stand for double edges $\bullet \longleftrightarrow \bullet$:



(3) Finally, for the group $\mathbb{Z}_2^N =< 1_1, \ldots, 1_N >$, with 1_i standing for the standard generators of the components, written additively, there is no true orientation, due to the reason explained in (1), and we are led to the old hypercube graph $\square_N$, namely:



Thus, we are led to the conclusions in the statement. $\square$

As before in the unoriented case, we can generalize these computations, as follows:

THEOREM 8.10. *Given two groups $G =< S >$ and $H =< T >$, we have*

$$G \times H =< S \times 1, 1 \times T >$$

and at the level of the corresponding oriented Cayley graphs, we have the formula

$$X_{G\times H} = X_G \,\square\, X_H$$

involving the Cartesian product operation for oriented graphs $\square$.

PROOF. We have indeed a generating set, which satisfies the condition $1 \notin S$. Now observe that when constructing the Cayley graph, the edges are as follows:

$$(g,a) \to (h,b) \iff g = h,\ a \to b \text{ or } g \to h, a = b$$

Thus, we obtain indeed a Cartesian product $X_{G\times H} = X_G \,\square\, X_H$, as claimed. $\square$

At the level of symmetry groups now, of these Cayley graphs, things get more interesting, in the present oriented graph setting. Let us recall indeed that in the unoriented setting, the basic computations of type $G \to X_G \to G(X_G)$ were as follows:

$$\mathbb{Z}_N \to C_N \to D_N$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \to P(C_3) \to D_6$$

$$\mathbb{Z}_2^N \to \square_N \to H_N$$

When adding orientation, according to Proposition 8.9, these computations become:

$$\mathbb{Z}_N \to C_N^o \to \mathbb{Z}_N$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \to P(C_3^o) \to \mathbb{Z}_2 \times \mathbb{Z}_3$$

$$\mathbb{Z}_2^N \to \square_N \to H_N$$

Thus, we are getting closer to a formula of type $G = G(X_G)$. In order to discuss this, let us start with a straightforward analogue of Proposition 8.6, as follows:

PROPOSITION 8.11. *Given a finite group, $G =< S >$ with $1 \notin S$, we have an action of this group on its associated oriented Cayley graph,*

$$G \curvearrowright X_G$$

and so $G \subset G(X_G)$. However, this inclusion is not an isomorphism, in general.

PROOF. We have several assertions here, the idea being as follows:

(1) Consider indeed the Cayley action of G on itself, given by:

$$G \subset S_G \quad , \quad g \to [h \to gh]$$

(2) Thus G acts on the vertices of its Cayley graph X_G, and our claim is that the edges are preserved by this action. Indeed, given an edge, $h \to hs$ with $s \in S$, a group element $g \in G$ will transform it into $gh \to ghs$, which is an edge too.

(3) Thus, the first assertion holds indeed. As for the second assertion, this holds too, the counterexample coming from the computation $\mathbb{Z}_2^N \to \square_N \to H_N$, discussed above. $\square$

Summarizing, as already said, we are certainly getting closer to a formula of type $G = G(X_G)$, but we are still not there, with some counterexamples still persisting. We will come back to this question in a moment, with a solution using colors.

Before leaving the subject, let us record the following result, due to Sabidussi:

THEOREM 8.12. *An oriented graph X is the oriented Cayley graph of a given group G precisely when it admits a simply transitive action of G.*

PROOF. This is something elementary, which is clear in one sense, from the proof of Proposition 8.11, and which in the other sense can be established as follows:

– Given a graph X as in the statement, pick an arbitrary vertex, and label it 1.

– Then, label each vertex $v \in X$ by the unique $g \in G$ mapping $1 \to v$.

– Finally, define $S \subset G$ as being the set of labels i, such that $1 \to i$.

Indeed, with these operations performed, it follows from definitions that what we get is a generating set for our group, $G =< S >$, satisfying the condition $1 \notin S$, and the Cayley graph of $G =< S >$ follows to be the original graph X itself, as desired. □

Moving ahead, let us attempt now to further modify our Cayley graph formalism, as to have $G = G(X_G)$. As a first observation, this is certainly possible, due to:

THEOREM 8.13. *Any finite group G appears as $G = G(X_G)$, with X_G being the complete oriented graph having G as set of vertices, and with the edges being colored by*

$$d_{hk} = h^{-1}k$$

according to the usual colored graph conventions, with color set $C = G$.

PROOF. Consider indeed the Cayley action of G on itself, which is given by:

$$G \subset S_G \quad , \quad g \to [h \to gh]$$

We have $d_{gh,gk} = d_{hk}$, which gives an action $G \curvearrowright X_G$, and so an inclusion $G \subset G(X_G)$. Conversely now, pick an arbitrary permutation $\sigma \in S_G$. We have then:

$$\begin{aligned}
\sigma \in G(X_G) \quad & \implies \quad d_{\sigma(h)\sigma(k)} = d_{hk} \\
& \implies \quad \sigma(h)^{-1}\sigma(k) = h^{-1}k \\
& \implies \quad \sigma(1)^{-1}\sigma(k) = k \\
& \implies \quad \sigma(k) = \sigma(1)k \\
& \implies \quad \sigma \in G
\end{aligned}$$

Thus, the inclusion $G \subset G(X_G)$ is an equality, as desired. □

Summarizing, we must add colors to our Cayley graph formalism, as follows:

Definition 8.14. *Associated to any finite group $G =< S >$, with the generating set S satisfying $1 \notin S$, is its oriented colored Cayley graph, constructed as follows:*

(1) *The vertices are the group elements $g \in G$.*
(2) *Edges $g \to h$ are drawn when $h = gs$, with $s \in S$.*
(3) *Such an edge $g \to h$ is colored by the element $s = g^{-1}h \in S$.*

As before, this oriented colored Cayley graph is indeed a graph, because our assumption $1 \notin S$ excludes the self-edges, $g \not\to g$. Observe also that in the case $S = S^{-1}$ each edge $g \to h$ has as companion edge $h \to g$, so in this case, up to replacing the pairs of edges $g \leftrightarrow h$ by usual edges $g = h$, we obtain an unoriented colored graph.

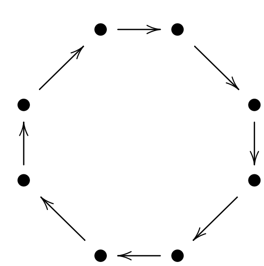
At the level of examples, the graph in Theorem 8.13 appears as in Definition 8.14, with generating set $S = G - \{1\}$. Also, in relation with Proposition 8.2 and Proposition 8.9, and with the associated symmetry groups too, we have the following result:

Proposition 8.15. *We have the following examples of oriented colored Cayley graphs, each time with respect to the standard, minimal generating set:*

(1) *For $\mathbb{Z}_N$ we obtain C_N^o, having symmetry group $\mathbb{Z}_N$.*
(2) *For $\mathbb{Z}_2 \times \mathbb{Z}_3$ we obtain the bicolored $P(C_3^o)$, with symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_3$.*
(3) *For $\mathbb{Z}_2^N$ we obtain the N-colored cube $\square_N$, having symmetry group $\mathbb{Z}_2^N$.*

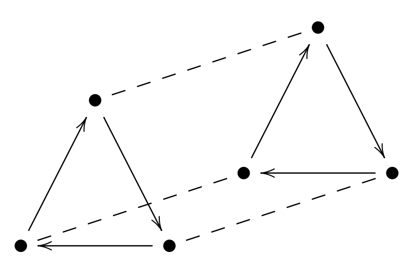
Proof. This basically comes from our previous computations, as follows:

(1) For the group $\mathbb{Z}_N =< 1 >$, written as usual additively, the number of colors is $|\{1\}| = 1$, and so no colors, and the graph is the usual oriented cycle C_N^o, namely:



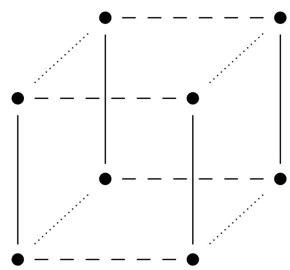
But, as we know well from the above, the symmetry group of this graph is $\mathbb{Z}_N$.

(2) For the group $\mathbb{Z}_2 \times \mathbb{Z}_3 =< (1,0), (0,1) >$, again written additively, we have 2 colors, and we obtain the bicolored version of the oriented prism $P(C_3^o)$, which is as follows:



But the symmetry group of this bicolored oriented prism is $\mathbb{Z}_2 \times \mathbb{Z}_3$, as claimed.

(3) Finally, for the group $\mathbb{Z}_2^N =< 1_1, \ldots, 1_N >$, with 1_i standing for the standard generators of the components, written as usual additively, here there is no true orientation, due to $1 = -1$, but we have however N colors, those of the generators. We are led in this way to the N-colored version of the old hypercube graph $\square_N$, namely:



But the symmetry group of this latter graph is $\mathbb{Z}_2^N$, as claimed. $\square$

All this is quite interesting, so let us generalize now the above results. In what regards the product operations, the result here, which is very standard, is as follows:

THEOREM 8.16. *Given two groups $G =< S >$ and $H =< T >$, we have*

$$G \times H =< S \times 1, 1 \times T >$$

and at the level of the corresponding oriented colored Cayley graphs, we have the formula

$$X_{G \times H} = X_G \,\square\, X_H$$

involving the Cartesian product operation for oriented colored graphs $\square$.

PROOF. We have indeed a generating set, which satisfies the condition $1 \notin S$. Now observe that when constructing the Cayley graph, the edges are as follows:

$$(g, a) \to (h, b) \iff g = h,\ a \to b \text{ or } g \to h, a = b$$

Thus, we obtain indeed a Cartesian product $X_{G \times H} = X_G \,\square\, X_H$, as claimed. $\square$

As for the generalization of our symmetry group computations, this is as follows:

THEOREM 8.17. *Given a group $G =< S >$ with $1 \notin S$, we have the formula*

$$G = G(X_G)$$

with X_G being the corresponding oriented colored Cayley graph.

PROOF. We use the same method as for Theorem 8.13, which corresponds to the case $S = G - \{1\}$. The adjacency matrix of the graph X_G in the statement is given by:

$$d_{hk} = \begin{cases} h^{-1}k & \text{if } h^{-1}k \in S \\ 0 & \text{otherwise} \end{cases}$$

Thus we have an action $G \curvearrowright X_G$, and so on inclusion $G \subset G(X_G)$. Conversely now, pick an arbitrary permutation $\sigma \in S_G$. We know that σ must preserve all the color components of d, which are the following matrices, depending on a color $c \in S$:

$$d^c_{hk} = \begin{cases} 1 & \text{if } h^{-1}k = c \\ 0 & \text{otherwise} \end{cases}$$

In other words, we have the following equivalences:

$$\begin{aligned} \sigma \in G(X) &\iff d^c_{\sigma(h)\sigma(k)} = d_{hk}, \forall c \in S \\ &\iff \sigma(h)^{-1}\sigma(k) = h^{-1}k, \forall h^{-1}k \in S \end{aligned}$$

Now observe that with $h = 1$ we obtain from this that we have:

$$\begin{aligned} k \in S &\implies \sigma(1)^{-1}\sigma(k) = k \\ &\implies \sigma(k) = \sigma(1)k \end{aligned}$$

Next, by taking $h \in S$, we obtain from the above formula that we have:

$$\begin{aligned} k \in hS &\implies \sigma(h)^{-1}\sigma(k) = h^{-1}k \\ &\implies \sigma(k) = \sigma(h)h^{-1}k \\ &\implies \sigma(k) = (\sigma(1)h)h^{-1}k \\ &\implies \sigma(k) = \sigma(1)k \end{aligned}$$

Thus with $g = \sigma(1)$ we have the following formula, for any $k \in S$:

$$\sigma(k) = gk$$

But the same method shows that this formula holds as well for any $k \in S^2$, then for any $k \in S^3$, any $k \in S^4$, and so on. Thus the above formula $\sigma(k) = gk$ holds for any $k \in G$, and so the inclusion $G \subset G(X_G)$ is an equality, as desired. □

The above results are not the end of the story, but rather the beginning of it. Indeed, at a more advanced level, we have the following classical result, due to Frucht:

THEOREM 8.18 (Frucht). *Any finite group G appears as the symmetry group*

$$G = G(X)$$

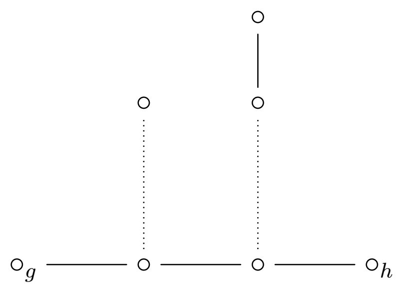
of a certain uncolored, unoriented graph X.

PROOF. This is something quite tricky, the idea being as follows:

(1) We can start with the graph in Definition 8.8, namely the associated oriented Cayley graph X_G, with the convention that the edges $g \to h$ exist when:

$$g^{-1}h \in S$$

(2) The point now is that we can make a suitable unoriented graph out of this graph, by replacing each edge with a copy of the following graph, with the height being in a chosen bijection with the corresponding element $g^{-1}h \in S$:



(3) With these replacements made, and under suitable assumptions on the generating set S, namely the usual $1 \notin S$ assumption, plus the fact that $S \cap S^{-1}$ must consist only of involutions, one can prove that G appears indeed of the symmetry group of this graph X. We will leave checking the details here as an instructive exercise. $\square$

8c. Brauer theorems

The above results are certainly interesting from a graph theory perspective, but from a group theory perspective, they remain a bit anecdotical. We would like to present now a series of alternative results, going in the other sense, that is, featuring less graphs, or rather featuring some combinatorial objects which are more complicated and abstract than graphs, but which can be extremely useful for the study of finite groups.

The idea will be a bit the same as for the Frucht theorem, namely that of viewing an arbitrary finite group G as symmetry group of a combinatorial object, $G = G(X)$. What will change, however, is the nature of X, the general principle being as follows:

PRINCIPLE 8.19. *Any finite, or even general compact group G appears as the symmetry group of its corresponding Tannakian category C_G,*

$$G = G(C_G)$$

and by suitably delinearizing C_G, say via a Brauer theorem of type $C_G = span(D_G)$, we can view G as symmetry group of a certain combinatorial object D_G.

Excited about this? Does not look easy, all this material, with both Tannaka and Brauer being quite scary names, in the context of group theory. But, believe me, all this is worth learning, and there is nothing better, when doing graphs or any kind of other algebraic discipline, to have in your bag some cutting-edge technology regarding the groups, such as the results of Tannaka and Brauer. So, we will go for this.

Getting started now, we will develop our theory as a continuation of the Peter-Weyl theory developed in chapter 7. That theory was developed for the finite groups, but with some minimal changes, and we will leave clarifying the details here to you, everything works in fact for a compact group G. So, our starting point will be:

THEOREM 8.20. *We have the following Peter-Weyl results, valid for any compact group $G \subset_u U_N$, with the orthogonality being with respect to the Haar integration:*

1. *Any representation decomposes as a sum of irreducible representations.*
2. *Each irreducible representation appears inside a certain tensor power $u^{\otimes k}$.*
3. *$C(G) = \overline{\bigoplus}_{v \in Irr(G)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible representations form an orthonormal system.*

PROOF. As explained above, this is something that we know well from chapter 7, in the finite group case, and in the general compact group case the proof is similar, with the only technical point being that of proving, somewhere between (1,2) and (3,4), the existence of the Haar measure. But this can be proved by using the arguments from chapter 7, with that chapter being written precisely with this idea in mind, namely that of having a quite straightforward extension to the compact group case, whenever needed. □

Going now towards Tannakian duality, let us start with:

DEFINITION 8.21. *A tensor category over $H = \mathbb{C}^N$ is a collection $C = (C_{kl})$ of linear spaces $C_{kl} \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$ satisfying the following conditions:*

1. *$S, T \in C$ implies $S \otimes T \in C$.*
2. *If $S, T \in C$ are composable, then $ST \in C$.*
3. *$T \in C$ implies $T^* \in C$.*
4. *Each C_{kk} contains the identity operator.*
5. *$C_{\emptyset k}$ with $k = \circ\bullet, \bullet\circ$ contain the operator $R : 1 \to \sum_i e_i \otimes e_i$.*
6. *$C_{kl,lk}$ with $k, l = \circ, \bullet$ contain the flip operator $\Sigma : a \otimes b \to b \otimes a$.*

Here, as usual, the tensor powers $H^{\otimes k}$, which are Hilbert spaces depending on a colored integer $k = \circ\bullet\bullet\circ\ldots$, are defined by the following formulae, and multiplicativity:

$$H^{\otimes\emptyset} = \mathbb{C} \quad , \quad H^{\otimes\circ} = H \quad , \quad H^{\otimes\bullet} = \bar{H} \simeq H$$

We have already met such categories, when dealing with the Tannakian categories of the closed subgroups $G \subset U_N$, and our knowledge can be summarized as follows:

PROPOSITION 8.22. *Given a closed subgroup $G \subset U_N$, its Tannakian category*

$$C_{kl} = \left\{ T \in \mathcal{L}(H^{\otimes k}, H^{\otimes l}) \Big| T g^{\otimes k} = g^{\otimes l} T, \forall g \in G \right\}$$

is a tensor category over $H = \mathbb{C}^N$. Conversely, given a tensor category C over $\mathbb{C}^N$,

$$G = \left\{ g \in U_N \Big| T g^{\otimes k} = g^{\otimes l} T, \forall k, l, \forall T \in C_{kl} \right\}$$

is a closed subgroup of U_N.

PROOF. This is something that we basically know, the idea being as follows:

(1) Regarding the first assertion, we have to check here the axioms (1-6) in Definition 8.21. The axioms (1-4) being all clear from definitions, let us establish (5). But this follows from the fact that each element $g \in G$ is a unitary, which can be reformulated as follows, with $R : 1 \to \sum_i e_i \otimes e_i$ being the map in Definition 8.21:

$$R \in Hom(1, g \otimes \bar{g}) \quad , \quad R \in Hom(1, \bar{g} \otimes g)$$

Regarding now the condition in Definition 8.21 (6), this comes from the fact that the matrix coefficients $g \to g_{ij}$ and their conjugates $g \to \bar{g}_{ij}$ commute with each other.

(2) Regarding the second assertion, we have to check that the subset $G \subset U_N$ constructed in the statement is a closed subgroup. But, assuming $g, h \in G$, we have $gh \in G$, due to the following computation, valid for any k, l and any $T \in C_{kl}$:

$$\begin{aligned} T(gh)^{\otimes k} &= Tg^{\otimes k}h^{\otimes k} \\ &= g^{\otimes l}Th^{\otimes k} \\ &= g^{\otimes l}h^{\otimes l}T \\ &= (gh)^{\otimes l}T \end{aligned}$$

Also, we have $1 \in G$, trivially. And also, assuming $g \in G$, we have $g^{-1} \in G$, due to:

$$\begin{aligned} T(g^{-1})^{\otimes k} &= (g^{-1})^{\otimes l}[g^{\otimes l}T](g^{-1})^{\otimes k} \\ &= (g^{-1})^{\otimes l}[Tg^{\otimes k}](g^{-1})^{\otimes k} \\ &= (g^{-1})^{\otimes l}T \end{aligned}$$

Finally, the fact that our subgroup $G \subset U_N$ is closed is clear from definitions. □

Summarizing, we have so far precise axioms for the tensor categories $C = (C_{kl})$, given in Definition 8.21, as well as correspondences as follows:

$$G \to C_G \quad , \quad C \to G_C$$

We will prove in what follows that these correspondences are inverse to each other. In order to get started, we first have the following technical result:

PROPOSITION 8.23. *Consider the following conditions:*

(1) $C = C_{G_C}$*, for any tensor category* C*.*
(2) $G = G_{C_G}$*, for any closed subgroup* $G \subset U_N$*.*

We have then (1) $\implies$ (2)*. Also,* $C \subset C_{G_C}$ *is automatic.*

PROOF. Given $G \subset U_N$, we have $G \subset G_{C_G}$. On the other hand, by using (1) we have $C_G = C_{G_{C_G}}$. Thus, we have an inclusion of closed subgroups of U_N, which becomes an isomorphism at the level of the associated Tannakian categories, so $G = G_{C_G}$. Finally, the fact that we have an inclusion $C \subset C_{G_C}$ is clear from definitions. □

The point now is that it is possible to prove that we have $C_{G_C}\subset C$, by doing some abstract algebra, and we are led in this way to the following conclusion:

THEOREM 8.24. *The Tannakian duality constructions*

$$C\to G_C\quad,\quad G\to C_G$$

are inverse to each other.

PROOF. This is something quite tricky, the idea being as follows:

(1) According to Proposition 8.23, we must prove $C_{G_C}\subset C$. For this purpose, given a tensor category $C=(C_{kl})$ over a Hilbert space H, consider the following $*$-algebra:

$$E_C=\bigoplus_{k,l}C_{kl}\subset\bigoplus_{k,l}B(H^{\otimes k},H^{\otimes l})\subset B\left(\bigoplus_k H^{\otimes k}\right)$$

Consider also, inside this $*$-algebra, the following $*$-subalgebra:

$$E_C^{(s)}=\bigoplus_{|k|,|l|\leq s}C_{kl}\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

(2) It is then routine to check that we have equivalences as follows:

$$\begin{aligned}C_{G_C}\subset C&\iff E_{C_{G_C}}\subset E_C\\&\iff E_{C_{G_C}}^{(s)}\subset E_C^{(s)},\forall s\\&\iff E_{C_{G_C}}^{(s)'}\supset E_C^{(s)'},\forall s\end{aligned}$$

(3) Summarizing, we would like to prove that we have $E_C^{(s)'}\subset E_{C_{G_C}}^{(s)'}$. But this can be done by doing some abstract algebra, and we refer here to Malacarne [**67**], or to the paper of Woronowicz [**100**]. For more on all this, you have as well my book [**5**]. □

With this piece of general theory in hand, let us go back to Principle 8.19, and develop the second idea there, namely delinearization and Brauer theorems. We have:

DEFINITION 8.25. *A category of crossing partitions is a collection $D=\bigsqcup_{k,l}D(k,l)$ of subsets $D(k,l)\subset P(k,l)$, having the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*
6. *The sets $P(k,\bar{k})$ with $|k|=2$ contain the crossing partition $\slashed{\cap}$.*

Observe the similarity with Definition 8.21, and more on this in a moment. In order now to construct a Tannakian category out of such a category, we will need:

PROPOSITION 8.26. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. This is something elementary, the computations being as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\ =&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\ =&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\ =&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}&T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\ =&\sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\ =&\sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\ =&\ N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p} < T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}), e_{i_1}\otimes\ldots\otimes e_{i_p} > e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix} e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

We can now formulate a key theoretical result, as follows:

THEOREM 8.27. *Any category of crossing partitions $D\subset P$ produces a series of compact groups $G=(G_N)$, with $G_N\subset U_N$ for any $N\in\mathbb N$, via the formula*

$$C_{kl}=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

for any k,l, and Tannakian duality. We call such groups easy.

PROOF. Indeed, once we fix an integer $N\in\mathbb N$, the various axioms in Definition 8.25 show, via Proposition 8.26, that the following spaces form a Tannakian category:

$$span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

Thus, Tannakian duality applies, and provides us with a closed subgroup $G_N\subset U_N$ such that the following equalities are satisfied, for any colored integers k,l:

$$C_{kl}=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

Thus, we are led to the conclusion in the statement. □

At the level of basic examples of easy groups, these are the real and complex rotation groups, coming from the following key theorem of Brauer:

THEOREM 8.28. *We have the following results:*

(1) *U_N is easy, coming from the category of all matching pairings $\mathcal P_2$.*
(2) *O_N is easy too, coming from the category of all pairings P_2.*

PROOF. This can be deduced from Tannakian duality, the idea being as follows:

(1) The group U_N being defined via the relations $u^*=u^{-1}$, $u^t=\bar u^{-1}$, the associated Tannakian category is $C=span(T_\pi|\pi\in D)$, with:

$$D=<\,\overset{\sqcap}{\circ\bullet}\,,\,\overset{\sqcap}{\bullet\circ}\,>=\mathcal P_2$$

(2) The group $O_N\subset U_N$ being defined by imposing the relations $u_{ij}=\bar{u}_{ij}$, the associated Tannakian category is $C=span(T_\pi|\pi\in D)$, with:

$$D=<\mathcal{P}_2,\,{}^{\circ}_{\bullet},\,{}^{\bullet}_{\circ}>=P_2$$

Thus, we are led to the conclusions in the statement. $\square$

Moving now towards finite groups, we first have the following result:

THEOREM 8.29. *The symmetric group S_N, regarded as group of unitary matrices,*

$$S_N\subset O_N\subset U_N$$

via the permutation matrices, is easy, coming from the category of all partitions P.

PROOF. Consider indeed the group S_N, regarded as a group of unitary matrices, with each permutation $\sigma\in S_N$ corresponding to the associated permutation matrix:

$$\sigma(e_i)=e_{\sigma(i)}$$

Consider as well the easy group $G\subset O_N$ coming from the category of all partitions P. Since P is generated by the one-block "fork" partition $Y\in P(2,1)$, we have:

$$C(G)=C(O_N)\Big/\Big\langle T_Y\in Hom(u^{\otimes 2},u)\Big\rangle$$

The linear map associated to Y is given by the following formula:

$$T_Y(e_i\otimes e_j)=\delta_{ij}e_i$$

In order to do the computations, we use the following formulae:

$$u=(u_{ij})_{ij}\quad,\quad u^{\otimes 2}=(u_{ij}u_{kl})_{ik,jl}\quad,\quad T_Y=(\delta_{ijk})_{i,jk}$$

We therefore obtain the following formula:

$$(T_Yu^{\otimes 2})_{i,jk}=\sum_{lm}(T_Y)_{i,lm}(u^{\otimes 2})_{lm,jk}=u_{ij}u_{ik}$$

On the other hand, we have as well the following formula:

$$(uT_Y)_{i,jk}=\sum_l u_{il}(T_Y)_{l,jk}=\delta_{jk}u_{ij}$$

Thus, the relation defining $G\subset O_N$ reformulates as follows:

$$T_Y\in Hom(u^{\otimes 2},u)\iff u_{ij}u_{ik}=\delta_{jk}u_{ij},\forall i,j,k$$

In other words, the elements u_{ij} must be projections, which must be pairwise orthogonal on the rows of $u=(u_{ij})$. We conclude that $G\subset O_N$ is the subgroup of matrices $g\in O_N$ having the property $g_{ij}\in\{0,1\}$. Thus we have $G=S_N$, as desired. $\square$

The hyperoctahedral group H_N is easy as well, the result here being as follows:

THEOREM 8.30. *The hyperoctahedral group H_N, regarded as group of matrices,*
$$S_N \subset H_N \subset O_N$$
is easy, coming from the category of partitions with even blocks P_{even}.

PROOF. This follows as usual from Tannakian duality. To be more precise, consider the following one-block partition, which, as the name indicates, looks like a H letter:
$$H \in P(2,2)$$
The linear map associated to this partition is then given by:
$$T_H(e_i \otimes e_j) = \delta_{ij} e_i \otimes e_i$$
By using this formula, we have the following computation:
$$\begin{aligned}
(T_H \otimes id)u^{\otimes 2}(e_a \otimes e_b) &= (T_H \otimes id)\left(\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes u_{ij}u_{kl}\right)(e_a \otimes e_b)\\
&= (T_H \otimes id)\left(\sum_{ik} e_i \otimes e_k \otimes u_{ia}u_{kb}\right)\\
&= \sum_i e_i \otimes e_i \otimes u_{ia}u_{ib}
\end{aligned}$$
On the other hand, we have as well the following computation:
$$\begin{aligned}
u^{\otimes 2}(T_H \otimes id)(e_a \otimes e_b) &= \delta_{ab}\left(\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes u_{ij}u_{kl}\right)(e_a \otimes e_a)\\
&= \delta_{ab}\sum_{ij} e_i \otimes e_k \otimes u_{ia}u_{ka}
\end{aligned}$$
We conclude from this that we have the following equivalence:
$$T_H \in End(u^{\otimes 2}) \iff \delta_{ik}u_{ia}u_{ib} = \delta_{ab}u_{ia}u_{ka}, \forall i,k,a,b$$
But the relations on the right tell us that the entries of $u = (u_{ij})$ must satisfy $\alpha\beta = 0$ on each row and column of u, and so that the corresponding closed subgroup $G \subset O_N$ consists of the matrices $g \in O_N$ which are permutation-like, with ± 1 nonzero entries. Thus, the corresponding group is $G = H_N$, and as a conclusion to this, we have:
$$C(H_N) = C(O_N)\Big/\Big\langle T_H \in End(u^{\otimes 2})\Big\rangle$$
But this means that the hyperoctahedral group H_N is easy, coming from the category of partitions $D =< H >= P_{even}$. Thus, we are led to the conclusion in the statement. $\square$

More generally now, we have in fact the following result, regarding the series of complex reflection groups H_N^s, which covers both the groups S_N, H_N:

THEOREM 8.31. *The complex reflection group $H_N^s = \mathbb{Z}_s \wr S_N$ is easy, the corresponding category P^s consisting of the partitions satisfying the condition*

$$\#\circ = \# \bullet (s)$$

as a weighted sum, in each block. In particular, we have the following results:

(1) *S_N is easy, coming from the category P.*
(2) *$H_N = \mathbb{Z}_2 \wr S_N$ is easy, coming from the category P_{even}.*
(3) *$K_N = \mathbb{T} \wr S_N$ is easy, coming from the category $\mathcal{P}_{even}$.*

PROOF. This is something that we already know at $s = 1, 2$, from Theorems 8.29 and 8.30. In general, the proof is similar, based on Tannakian duality. To be more precise, in what regards the main assertion, the idea here is that the one-block partition $\pi \in P(s)$, which generates the category of partitions P^s in the statement, implements the relations producing the subgroup $H_N^s \subset S_N$. As for the last assertions, these are all elementary:

(1) At $s = 1$ we know that we have $H_N^1 = S_N$. Regarding now the corresponding category, here the condition $\#\circ = \# \bullet (1)$ is automatic, and so $P^1 = P$.

(2) At $s = 2$ we know that we have $H_N^2 = H_N$. Regarding now the corresponding category, here the condition $\#\circ = \# \bullet (2)$ reformulates as follows:

$$\# \circ + \#\bullet = 0(2)$$

Thus each block must have even size, and we obtain, as claimed, $P^2 = P_{even}$.

(3) At $s = \infty$ we know that we have $H_N^\infty = K_N$. Regarding now the corresponding category, here the condition $\#\circ = \# \bullet (\infty)$ reads:

$$\#\circ = \#\bullet$$

But this is the condition defining $\mathcal{P}_{even}$, and so $P^\infty = \mathcal{P}_{even}$, as claimed. $\square$

Summarizing, we have many examples. In fact, our list of easy groups has currently become quite big, and here is a selection of the main results that we have so far:

THEOREM 8.32. *We have a diagram of compact groups as follows,*

$$\begin{tikzcd} K_N \arrow[r] & U_N \\ H_N \arrow[u] \arrow[r] & O_N \arrow[u] \end{tikzcd}$$

where $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$, and all these groups are easy.

PROOF. This follows from the above results. To be more precise, we know that the above groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{tikzcd}
\mathcal{P}_{even} \arrow[d] & \mathcal{P}_2 \arrow[l] \arrow[d] \\
P_{even} & P_2 \arrow[l]
\end{tikzcd}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have reached to the conclusions formulated in Principle 8.19. All this remains of course a bit away from graph theory, but we will make a good use of what we learned here, later on in this book, especially when talking quantum groups.

8d. Infinite graphs

We would like to end this chapter, and this first half of the present book, with a discussion regarding the infinite graphs. Generally speaking, the subject is heavily analytic, and it is beyond our purposes here to really get into this, at least at this stage of things. However, we have already seen some interesting examples of infinite graphs in the above, and also some of our basic definitions in the above extend in a quite straightforward way to the infinite graph setting, and all this is certainly worth an informal discussion.

To start with, an infinite graph X is the same thing as a finite graph, except for the fact that the set of vertices is infinite, $|X| = \infty$, assumed countable. As in the finite group case, many interesting examples appear as Cayley graphs, as follows:

DEFINITION 8.33. *Associated to any discrete group $G =< S >$, with the generating set S assumed to satisfy $1 \notin S = S^{-1}$, is its Cayley graph, constructed as follows:*

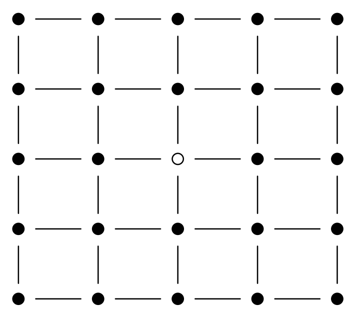
1. *The vertices are the group elements $g \in G$.*
2. *Edges $g - h$ are drawn when $h = gs$, with $s \in S$.*

Of course, we can talk as well about oriented Cayley graphs, and about colorings too, exactly as in the finite group case, but this discussion being quite informal anyway, we will focus on the most standard types of Cayley graphs, which are those above.

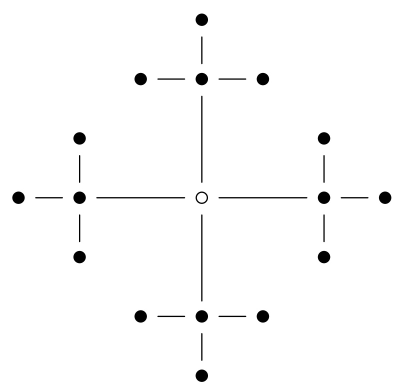
At the level of basic examples, we have two of them, as follows:

(1) Consider the group $\mathbb{Z}^N$, that is, the free abelian group on N generators. We can represent the group elements as vectors in $\mathbb{R}^N$, in the obvious way, by using the embedding $\mathbb{Z}^N \subset \mathbb{R}^N$, and if we endow our group with its standard generating set, namely

$S = \{\pm 1_1, \ldots, \pm 1_N\}$, written additively, the edges will appear precisely at the edges of the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$. Thus, the Cayley graph that we get is precisely this lattice:



(2) Consider now the free group $F_N = \mathbb{Z}^{*N}$ on N generators, with its standard generating set, formed by the N generators and their inverses, $S = \{g_1^{\pm 1}, \ldots, g_N^{\pm 1}\}$. As explained in chapter 4, when discussing trees, at $N = 2$ the corresponding Cayley graph is as follows, and in general the picture is similar, with valence 4 being replaced by valence $4N$:



So, these will be our basic examples of infinite graphs. And with the comment that these are both excellent graphs, not only aesthetically, but mathematically too.

Regarding now the general theory for the infinite graphs X, we can certainly talk about the corresponding symmetry groups $G(X) \subset S_\infty$, and other algebraic aspects, but the main questions remain those of analytic nature, notably in relation with:

QUESTION 8.34. *Given an infinite rooted graph X:*

(1) *What is the number L_k of length k loops based at the root?*
(2) *What about the probability measure μ having the numbers L_k as moments?*

We refer to chapter 1 for the computation for $\mathbb{Z}$, that is, for the Cayley graph of $\mathbb{Z} = F_1$, and to chapter 2 for some further computations, for the graphs $\mathbb{N}$ and D_∞. There are many interesting questions here, notably with the computation for the Cayley graphs of $\mathbb{Z}^N$ and F_N at arbitrary $N \in \mathbb{N}$, leading to a lot of interesting mathematics, and for more on all this, we refer to any advanced probability book.

Finally, another set of interesting questions appears in relation with the Laplace operator introduced in chapter 4. Indeed, as explained there, for the Cayley graph of $\mathbb{Z}^N$, all this is related to the discretization of the basic equations of physics, via the finite element method. For more on all this, we refer to any good PDE book.

As a last topic of discussion, let us get back to the $N \to \infty$ considerations from the end of chapter 7. It is pretty much clear, from the discussion there, that, although this might seem related to the infinite graphs, the world of infinite graphs is in fact too narrow, for discussing such things. However, as good news, we have something new to say on that subject, by using the Brauer theorems from the previous section, namely:

Theorem 8.35. *Given a family of easy groups $G = (G_N)$, coming from a category of crossing partitions $D \subset P$, we have the following formula, with $D_k = D(0,k)$:*

$$\lim_{N\to\infty}\int_{G_N}\chi^k = |D_k|$$

More generally, we have the following formula, with $|.|$ being the number of blocks:

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k = \sum_{\pi\in D_k} t^{|\pi|}$$

In the case of the groups $S = (S_N)$, $H = (H_N)$, and more generally $H^s = (H_N^s)$, we recover in this way, more conceptually, our previous probabilistic results.

Proof. There are several things going on here, the idea being as follows:

(1) To start with, we have the following formula, coming from Peter-Weyl, and then from easiness, with u standing as usual for the fundamental representation:

$$\begin{aligned}\int_{G_N}\chi^k &= \int_{G_N}\chi_{u^{\otimes k}}\\ &= \dim(Fix(u^{\otimes k}))\\ &= \dim\big(span(T_\pi|\pi\in D_k)\big)\end{aligned}$$

(2) The point now is that, with $N \to \infty$, the above vectors T_π become linearly independent. This is something not exactly trivial, the standard argument here being that it is enough to check this for the biggest possible category, $D = P$, and that for this category, the determinant of the Gram matrix of the vectors T_π can be explicitely computed, thanks to a result of Lindstöm, and follows to be nonzero, with $N \to \infty$.

(3) Long story short, we have the first formula in the statement, at $t = 1$. As for the general $t > 0$ formula, this can be deduced as well, via more technical integration, called Weingarten formula, again by using Peter-Weyl and easiness.

(4) Finally, in what regards the examples as the end, for $S = (S_N)$, where $D = P$, and at $t = 1$, it is notorious that the measure having as moments the Bell numbers $B_k = |P_k|$ is the Poisson law p_1. Thus we have the general H_N^s result at $s = 1, t = 1$, and the extension to the case of arbitrary parameters $s \in \mathbb{N}$ and $t > 0$ is straightforward. $\square$

And good news, that is all. In the hope that you liked the first half of this book, and of course with some apologies for going a bit off-topic on a number of occassions, as for instance with this Theorem 8.35 that we just proved, featuring no graphs. The point, however, with all this, is that all this learning will be very useful, for what comes next.

8e. Exercises

We had a quite varied chapter here, but at the level of exercises, we would like to insist on the infinite graphs, which are certainly worth a more detailed look:

Exercise 8.36. *Draw the Cayley graphs of other infinite groups G.*

Exercise 8.37. *Clarify the formalism of symmetry groups $G(X) \subset S_\infty$.*

Exercise 8.38. *Clarify too the partial symmetry groups $\widetilde{G}(X) \subset \widetilde{S}_\infty$.*

Exercise 8.39. *Do the loop computations for the Cayley graph of $\mathbb{Z}^N$.*

Exercise 8.40. *Do the loop computations for the Cayley graph of F_N too.*

Exercise 8.41. *Do the F_N computations appear as "liberations" of those for $\mathbb{Z}^N$?*

Exercise 8.42. *Study the wave equation on various infinite graphs.*

Exercise 8.43. *Study the heat equation too, on various infinite graphs.*

As bonus exercise, learn some functional analysis. This would be useful in connection with the above, and also in connection with what we will be doing next, in this book.

Part III

Quantum symmetries

Here's my key
Philosophy
A freak like me
Just needs infinity

CHAPTER 9

Quantum symmetry

9a. Quantum spaces

Welcome to quantum symmetry. Our purpose in what follows will be to look for hidden, quantum symmetries of graphs, according to the following principle:

PRINCIPLE 9.1. *The following happen, in the quantum world:*

(1) *S_N has a free analogue S_N^+, which is a compact quantum group.*
(2) *This quantum group S_N^+ is infinite, and reminding SO_3, at $N\geq 4$.*
(3) *$S_N\to S_N^+$ can be however understood, using algebra and probability.*
(4) *S_N^+ is the quantum symmetry group $G^+(K_N)$ of the complete graph K_N.*
(5) *In fact, any graph $X\subset K_N$ has a quantum symmetry group $G^+(X)\subset S_N^+$.*
(6) *$G(X)\subset G^+(X)$ can be an isomorphism or not, depending on X.*
(7) *$G(X)\to G^+(X)$ can be understood, via algebra and probability.*

Excited about this? We will learn this technology, in this chapter, and in the next one. To be more precise, in this chapter we will talk about Hilbert spaces, operator algebras, quantum spaces, quantum groups, and then about (1), with a look into (2,3) too. And then, in the next chapter, we will talk about (4,5), with a look into (6,7) too.

Getting started now, we already know a bit about operator algebras and quantum spaces from chapter 7, but that material was explained in a hurry, time now to do this the right way. At the gates of the quantum world are the Hilbert spaces:

DEFINITION 9.2. *A Hilbert space is a complex vector space H with a scalar product $<x,y>$, which will be linear at left and antilinear at right,*

$$<\lambda x,y>=\lambda<x,y>\quad,\quad<x,\lambda y>=\bar{\lambda}<x,y>$$

and which is complete with respect to corresponding norm

$$||x||=\sqrt{<x,x>}$$

in the sense that any sequence $\{x_n\}$ which is a Cauchy sequence, having the property $||x_n-x_m||\to 0$ with $n,m\to\infty$, has a limit, $x_n\to x$.

Here our convention for the scalar products, written $<x,y>$ and being linear at left, is one among others, often used by mathematicians, and also by certain professional quantum physicists, like myself. As further comments now on Definition 9.2, there is some mathematics encapsulated there, needing some discussion. First, we have:

THEOREM 9.3. *Given an index set I, which can be finite or not, the space*

$$l^2(I)=\left\{(x_i)_{i\in I}\Big|\sum_i|x_i|^2<\infty\right\}$$

is a Hilbert space, with scalar product as follows:

$$<x,y>=\sum_i x_i\bar{y}_i$$

When I is finite, $I=\{1,\ldots,N\}$, we obtain in this way the usual space $H=\mathbb{C}^N$.

PROOF. All this is well-known and routine, the idea being as follows:

(1) We know that $l^2(I)\subset\mathbb{C}^I$ is the space of vectors satisfying $||x||<\infty$. We want to prove that $l^2(I)$ is a vector space, that $<x,y>$ is a scalar product on it, that $l^2(I)$ is complete with respect to $||.||$, and finally that for $|I|<\infty$ we have $l^2(I)=\mathbb{C}^{|I|}$.

(2) The last assertion, $l^2(I)=\mathbb{C}^{|I|}$ for $|I|<\infty$, is clear, because in this case the sums are finite, so the condition $||x||<\infty$ is automatic. So, we know at least one thing.

(3) Next, we can use the Cauchy-Schwarz inequality, which is as follows, coming from the positivity of the degree 2 quantity $f(t)=||twx+y||^2$, with $t\in\mathbb{R}$ and $w\in\mathbb{T}$:

$$|<x,y>|\leq||x||\cdot||y||$$

(4) Indeed, with Cauchy-Schwarz in hand, everything is straightforward. We first obtain, by raising to the square and expanding, that for any $x,y\in l^2(I)$ we have:

$$||x+y||\leq||x||+||y||$$

(5) Thus $l^2(I)$ is indeed a vector space, and $<x,y>$ is surely a scalar product on it, because all the conditions for a scalar product are trivially satisfied.

(6) Finally, the completness with respect to $||.||$ follows in the obvious way, the limit of a Cauchy sequence $\{x^n\}$ being the vector $y=(y_i)$ given by $y_i=\lim_{n\to\infty}x_i^n$. $\square$

Going now a bit abstract, we have, more generally, the following result:

THEOREM 9.4. *Given a space X with a positive measure μ on it, the space*

$$L^2(X)=\left\{f:X\to\mathbb{C}\Big|\int_X|f(x)|^2\,d\mu(x)<\infty\right\}$$

with the convention $f=g$ when $||f-g||=0$, is a Hilbert space, with scalar product:

$$<f,g>=\int_X f(x)\overline{g(x)}\,d\mu(x)$$

When $X=I$ is discrete, $\mu(\{x\})=1$ for any $x\in X$, we obtain the previous space $l^2(I)$.

PROOF. This is something routine, remake of Theorem 9.3, as follows:

(1) The proof of the first, and main assertion is something perfectly similar to the proof of Theorem 9.3, by replacing everywhere the sums by integrals.

(2) As for the last assertion, when μ is the counting measure all our integrals here become usual sums, and so we recover in this way Theorem 9.3. □

As a third and last theorem about Hilbert spaces, that we will need, we have:

THEOREM 9.5. *Any Hilbert space H has an orthonormal basis $\{e_i\}_{i\in I}$, which is by definition a set of vectors whose span is dense in H, and which satisfy*

$$< e_i, e_j >= \delta_{ij}$$

with δ being a Kronecker symbol. The cardinality $|I|$ of the index set, which can be finite, countable, or worse, depends only on H, and is called dimension of H. We have

$$H \simeq l^2(I)$$

in the obvious way, mapping $\sum \lambda_i e_i \to (\lambda_i)$. The Hilbert spaces with $\dim H = |I|$ being countable, including $l^2(\mathbb{N})$ and $L^2(\mathbb{R})$, are all isomorphic, and are called separable.

PROOF. We have many assertions here, the idea being as follows:

(1) In finite dimensions an orthonormal basis $\{e_i\}_{i\in I}$ can be constructed by starting with any vector space basis $\{x_i\}_{i\in I}$, and using the Gram-Schmidt procedure. But the same method works in arbitrary dimensions, by using the Zorn lemma.

(2) Regarding $L^2(\mathbb{R})$, here we can argue that, since functions can be approximated by polynomials, we have a countable algebraic basis, namely $\{x^n\}_{n\in\mathbb{N}}$, called the Weierstrass basis, that we can orthogonalize afterwards by using Gram-Schmidt. □

Moving ahead, now that we know what our vector spaces are, we can talk about infinite matrices with respect to them. And the situation here is as follows:

THEOREM 9.6. *Given a Hilbert space H, consider the linear operators $T : H \to H$, and for each such operator define its norm by the following formula:*

$$||T|| = \sup_{||x||=1} ||Tx||$$

The operators which are bounded, $||T|| < \infty$, form then a complex algebra $B(H)$, which is complete with respect to $||.||$. When H comes with a basis $\{e_i\}_{i\in I}$, we have

$$B(H) \subset \mathcal{L}(H) \subset M_I(\mathbb{C})$$

where $\mathcal{L}(H)$ is the algebra of all linear operators $T : H \to H$, and $\mathcal{L}(H) \subset M_I(\mathbb{C})$ is the correspondence $T \to M$ obtained via the usual linear algebra formulae, namely:

$$T(x) = Mx \quad , \quad M_{ij} =< Te_j, e_i >$$

In infinite dimensions, none of the above two inclusions is an equality.

Proof. This is something straightforward, the idea being as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S+T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the completness assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x) + T_n(y) \\ &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \\ &= T(x) + T(y) \end{aligned}$$

Similarly, we have $T(\lambda x) = \lambda T(x)$, and we conclude that $T \in \mathcal{L}(H)$.

(3) With this done, it remains to prove now that we have $T \in B(H)$, and that $T_n \to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned} ||T_n - T_m|| \leq \varepsilon \ , \ \forall n,m \geq N &\implies ||T_n x - T_m x|| \leq \varepsilon \ , \ \forall ||x|| = 1 \ , \ \forall n,m \geq N \\ &\implies ||T_n x - Tx|| \leq \varepsilon \ , \ \forall ||x|| = 1 \ , \ \forall n \geq N \\ &\implies ||T_N x - Tx|| \leq \varepsilon \ , \ \forall ||x|| = 1 \\ &\implies ||T_N - T|| \leq \varepsilon \end{aligned}$$

But this gives both $T \in B(H)$, and $T_N \to T$ in norm, and we are done.

(4) Regarding the embeddings, the correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$, so we have indeed an embedding as follows, as claimed:

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

In finite dimensions we have an isomorphism, because any $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, given by $< Te_j, e_i >= M_{ij}$. However, in infinite dimensions, we have matrices not producing operators, as for instance the all-one matrix.

(5) As for the examples of linear operators which are not bounded, these are more complicated, coming from logic, and we will not need them in what follows. □

Finally, as a second and last result regarding the operators, we will need:

THEOREM 9.7. *Each operator $T\in B(H)$ has an adjoint $T^*\in B(H)$, given by:*

$$<Tx,y>=<x,T^*y>$$

The operation $T\to T^*$ is antilinear, antimultiplicative, involutive, and satisfies:

$$||T||=||T^*||\quad,\quad ||TT^*||=||T||^2$$

When H comes with a basis $\{e_i\}_{i\in I}$, the operation $T\to T^*$ corresponds to

$$(M^*)_{ij}=\overline{M}_{ji}$$

at the level of the associated matrices $M\in M_I(\mathbb C)$.

PROOF. This is standard too, and can be proved in 3 steps, as follows:

(1) The existence of the adjoint operator T^*, given by the formula in the statement, comes from the fact that the function $\varphi(x)=<Tx,y>$ being a linear map $H\to\mathbb C$, we must have a formula as follows, for a certain vector $T^*y\in H$:

$$\varphi(x)=<x,T^*y>$$

Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^*=S^*+T^*\quad,\quad(\lambda T)^*=\bar\lambda T^*\quad,\quad(ST)^*=T^*S^*\quad,\quad(T^*)^*=T$$

Observe also that we have indeed $T^*\in B(H)$, because:

$$\begin{aligned}||T||&=\sup_{||x||=1}\sup_{||y||=1}<Tx,y>\\&=\sup_{||y||=1}\sup_{||x||=1}<x,T^*y>\\&=||T^*||\end{aligned}$$

(2) Regarding now $||TT^*||=||T||^2$, which is a key formula, observe that we have:

$$||TT^*||\leq||T||\cdot||T^*||=||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned}||T||^2&=\sup_{||x||=1}|<Tx,Tx>|\\&=\sup_{||x||=1}|<x,T^*Tx>|\\&\leq||T^*T||\end{aligned}$$

By replacing $T\to T^*$ we obtain from this $||T||^2\leq||TT^*||$, as desired.

(3) Finally, when H comes with a basis, the formula $<Tx,y>=<x,T^*y>$ applied with $x=e_i$, $y=e_j$ translates into the formula $(M^*)_{ij}=\overline{M}_{ji}$, as desired. □

Generally speaking, the theory of bounded operators can be developed in analogy with the theory of the usual matrices, and the main results can be summarized as follows:

Fact 9.8. *The following happen, extending the spectral theorem for matrices:*

(1) *Any self-adjoint operator, $T = T^*$, is diagonalizable.*
(2) *More generally, any normal operator, $TT^* = T^*T$, is diagonalizable.*
(3) *In fact, any family $\{T_i\}$ of commuting normal operators is diagonalizable.*

You might wonder here, why calling this Fact instead of Theorem. In answer, this is something which is quite hard to prove, and in fact not only we will not prove this, but we will also find a way of short-circuiting all this. But more on this in a moment, for now, let us enjoy this. As a consequence of all this, we can formulate as well:

Fact 9.9. *The following happen, regarding the closed $*$-algebras $A \subset B(H)$:*

(1) *For $A =< T >$ with T normal, we have $A = C(X)$, with $X = \sigma(T)$.*
(2) *In fact, all commutative algebras are of the form $A = C(X)$, with X compact.*
(3) *In general, we can write $A = C(X)$, with X being a compact quantum space.*

To be more precise here, the first assertion is more or less part of the spectral theorems from Fact 9.8, with the spectrum of an operator $T \in B(H)$ being defined as follows:

$$\sigma(T) = \left\{\lambda \in \mathbb{C}\Big| T - \lambda \notin B(H)^{-1}\right\}$$

Regarding the second assertion, if we write $A = span(T_i)$, then the family $\{T_i\}$ consists of commuting normal operators, and this leads to the above conclusion, with X being a certain compact space associated to the family $\{T_i\}$. As for the third assertion, which is something important to us, this is rather a philosophical conclusion, to all this.

Very good all this, so we have quantum spaces, and you would say, it remains to understand the proofs of all the above, and then we are all set, ready to go ahead with quantum groups, and the rest of our program. However, there is a bug with all this:

Bug 9.10. *Besides the spectral theorem in infinite dimensions being something tough, the resulting notion of compact quantum spaces is not very satisfactory, because we cannot define operator algebras $A \subset B(H)$ with generators and relations, as we would love to.*

In short, nice try with the above, but time now to forget all this, and invent something better. And, fortunately, the solution to our problem exists, due to Gelfand, with the starting definition here, that we already met in chapter 7, being as follows:

Definition 9.11. *A C^*-algebra is a complex algebra A, having a norm $||.||$ making it a Banach algebra, and an involution $*$, related to the norm by the formula*

$$||aa^*|| = ||a||^2$$

which must hold for any $a \in A$.

As a basic example, the full operator algebra $B(H)$ is a C^*-algebra, and so is any norm closed $*$-subalgebra $A\subset B(H)$. We will see in a moment that a converse of this holds, in the sense that any C^*-algebra appears as an operator algebra, $A\subset B(H)$.

But, let us start with finite dimensions. We know that the matrix algebra $M_N(\mathbb{C})$ is a C^*-algebra, with the usual matrix norm and involution of matrices, namely:

$$||M||=\sup_{||x||=1}||Mx||\quad,\quad (M^*)_{ij}=\bar{M}_{ji}$$

More generally, any $*$-subalgebra $A\subset M_N(\mathbb{C})$ is automatically closed, and so is a C^*-algebra. In fact, in finite dimensions, the situation is as follows:

Theorem 9.12. *The finite dimensional C^*-algebras are exactly the algebras*

$$A=M_{N_1}(\mathbb{C})\oplus\ldots\oplus M_{N_k}(\mathbb{C})$$

with norm $||(a_1,\ldots,a_k)||=\sup_i||a_i||$, and involution $(a_1,\ldots,a_k)^*=(a_1^*,\ldots,a_k^*)$.

Proof. This is something that we discussed in chapter 7, the idea being that this comes by splitting the unit of our algebra A as a sum of central minimal projections, $1=p_1+\ldots+p_k$. Indeed, when doing so, each of the $*$-algebras $A_i=p_iAp_i$ follows to be a matrix algebra, $A_i\simeq M_{N_i}(\mathbb{C})$, and this gives the decomposition in the statement. $\square$

In order to develop more theory, we will need a technical result, as follows:

Theorem 9.13. *Given an element $a\in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a)=\left\{\lambda\in\mathbb{C}\Big|a-\lambda\notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A=B(H)$ case:

(1) *We have $\sigma(ab)\cup\{0\}=\sigma(ba)\cup\{0\}$.*
(2) *We have $\sigma(f(a))=f(\sigma(a))$, for any $f\in\mathbb{C}(X)$ having poles outside $\sigma(a)$.*
(3) *The spectrum $\sigma(a)$ is compact, non-empty, and contained in $D_0(||a||)$.*
(4) *The spectra of unitaries $(u^*=u^{-1})$ and self-adjoints $(a=a^*)$ are on $\mathbb{T},\mathbb{R}$.*
(5) *The spectral radius of normal elements $(aa^*=a^*a)$ is given by $\rho(a)=||a||$.*

In addition, assuming $a\in A\subset B$, the spectra of a with respect to A and to B coincide.

Proof. All the above assertions, which are of course formulated a bit informally, are well-known to hold for the full operator algebra $A=B(H)$, and the proof in general is similar. We refer here to chapter 7, where all this was already discussed. $\square$

With these ingredients, we can now a prove a key result of Gelfand, as follows:

Theorem 9.14. *Any commutative C^*-algebra A is of the form*

$$A=C(X)$$

with $X=Spec(A)$ being the space of Banach algebra characters $\chi:A\to\mathbb{C}$.

PROOF. This is something that we know too from chapter 7, the idea being that with X as in the statement, we have a morphism of algebras as follows:

$$ev : A \to C(X) \quad , \quad a \to ev_a = [\chi \to \chi(a)]$$

(1) Quite suprisingly, the fact that ev is involutive is not trivial. But here we can argue that it is enough to prove that we have $ev_{a^*} = ev_a^*$ for the self-adjoint elements a, which in turn follows from Theorem 9.13 (4), which shows that we have:

$$ev_a(\chi) = \chi(a) \in \sigma(a) \subset \mathbb{R}$$

(2) Next, since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. $\square$

In view of Theorem 9.14, we can formulate the following definition:

DEFINITION 9.15. *Given an arbitrary C^*-algebra A, we can write*

$$A = C(X)$$

and call the abstract space X a compact quantum space.

In other words, we can define the category of compact quantum spaces X as being the category of the C^*-algebras A, with the arrows reversed. A morphism $f : X \to Y$ corresponds by definition to a morphism $\Phi : C(Y) \to C(X)$, a product of spaces $X \times Y$ corresponds by definition to a product of algebras $C(X) \otimes C(Y)$, and so on.

All this is of course a bit speculative, and as a first true result, we have:

THEOREM 9.16. *The finite quantum spaces are exactly the disjoint unions of type*

$$X = M_{N_1} \sqcup \ldots \sqcup M_{N_k}$$

where M_N is the finite quantum space given by $C(M_N) = M_N(\mathbb{C})$.

PROOF. For a compact quantum space X, coming from a C^*-algebra A via the formula $A = C(X)$, being finite can only mean that the following number is finite:

$$|X| = \dim_{\mathbb{C}} A < \infty$$

Thus, by using Theorem 9.12, we are led to the conclusion that we must have:

$$C(X) = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

But since direct sums of algebras A correspond to disjoint unions of quantum spaces X, via the correspondence $A = C(X)$, this leads to the conclusion in the statement. $\square$

Finally, at the general level, we have as well the following key result:

THEOREM 9.17. *Any C^*-algebra appears as an operator algebra:*

$$A \subset B(H)$$

Moreover, when A is separable, which is usually the case, H can be taken separable.

PROOF. This result, called GNS representation theorem after Gelfand-Naimark-Segal, comes as a continuation of the Gelfand theorem, the idea being as follows:

(1) Let us first prove that the result holds in the commutative case, $A = C(X)$. Here, we can pick a positive measure on X, and construct our embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to [g \to fg]$$

(2) In general the proof is similar, the idea being that given a C^*-algebra A we can construct a Hilbert space $H = L^2(A)$, and then an embedding as above:

$$A \subset B(L^2(A)) \quad , \quad a \to [b \to ab]$$

(3) Finally, the last assertion is clear, because when A is separable, meaning that it has a countable algebraic basis, so does the associated Hilbert space $H = L^2(A)$. ☐

Many other things can be said about the C^*-algebras, and we recommend here any operator algebra book. But for our purposes here, the above will do.

9b. Quantum groups

We are ready now to introduce the quantum groups. The axioms here, due to Woronowicz [99], and slightly modified for our purposes, are as follows:

DEFINITION 9.18. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

define morphisms of C^*-algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$ and $S : A \to A^{opp}$, called comultiplication, counit and antipode.

Here the tensor product needed for Δ can be any C^*-algebra tensor product, and more on this later. In order to get rid of redundancies, coming from this and from amenability issues, we will divide everything by an equivalence relation, as follows:

DEFINITION 9.19. *We agree to identify two Woronowicz algebras, $(A, u) = (B, v)$, when we have an isomorphism of $*$-algebras*

$$< u_{ij} > \simeq < v_{ij} >$$

mapping standard coordinates to standard coordinates, $u_{ij} \to v_{ij}$.

We say that A is cocommutative when $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have then the following key result, from [99], providing us with examples:

THEOREM 9.20. *The following are Woronowicz algebras, which are commutative, respectively cocommutative:*

(1) $C(G)$, *with* $G \subset U_N$ *compact Lie group. Here the structural maps are:*
$$\Delta(\varphi) = \left[(g,h) \to \varphi(gh)\right] \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = \left[g \to \varphi(g^{-1})\right]$$
(2) $C^*(\Gamma)$, *with* $F_N \to \Gamma$ *finitely generated group. Here the structural maps are:*
$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$
Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we first have to exhibit a certain matrix u, and then prove that we have indeed a Woronowicz algebra. The constructions are as follows:

(1) For the first assertion, we can use the matrix $u = (u_{ij})$ formed by the standard matrix coordinates of G, which is by definition given by:
$$g = \begin{pmatrix} u_{11}(g) & \ldots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \ldots & u_{NN}(g) \end{pmatrix}$$

(2) For the second assertion, we can use the diagonal matrix formed by generators:
$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, regarding the last assertion, in the commutative case this follows from the Gelfand theorem, and in the cocommutative case, we will be back to this. □

In order to get now to quantum groups, we will need as well:

PROPOSITION 9.21. *Assuming that* $G \subset U_N$ *is abelian, we have an identification of Woronowicz algebras* $C(G) = C^*(\Gamma)$, *with* Γ *being the Pontrjagin dual of* G:
$$\Gamma = \left\{\chi : G \to \mathbb{T}\right\}$$
Conversely, assuming that $F_N \to \Gamma$ *is abelian, we have an identification of Woronowicz algebras* $C^*(\Gamma) = C(G)$, *with* G *being the Pontrjagin dual of* Γ:
$$G = \left\{\chi : \Gamma \to \mathbb{T}\right\}$$
Thus, the Woronowicz algebras which are both commutative and cocommutative are exactly those of type $A = C(G) = C^*(\Gamma)$, *with* G, Γ *being abelian, in Pontrjagin duality.*

PROOF. This follows from the Gelfand theorem applied to $C^*(\Gamma)$, and from the fact that the characters of a group algebra come from the characters of the group. □

In view of this result, and of the findings from Theorem 9.20 too, we have the following definition, complementing Definition 9.18 and Definition 9.19:

Definition 9.22. *Given a Woronowicz algebra, we write it as follows, and call G a compact quantum Lie group, and Γ a finitely generated discrete quantum group:*

$$A = C(G) = C^*(\Gamma)$$

Also, we say that G, Γ are dual to each other, and write $G = \widehat{\Gamma}, \Gamma = \widehat{G}$.

Let us discuss now some tools for studying the Woronowicz algebras, and the underlying quantum groups. First, we have the following result:

Proposition 9.23. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

(2) *S satisfies the antipode axiom, on the $*$-algebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

Proof. As a first observation, the result holds in the commutative case, $A = C(G)$ with $G \subset U_N$. Indeed, here we know from Theorem 9.20 that Δ, ε, S appear as functional analytic transposes of the multiplication, unit and inverse maps m, u, i:

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^t$$

Thus, in this case, the various conditions in the statement on Δ, ε, S simply come by transposition from the group axioms satisfied by m, u, i, namely:

$$m(m \times id) = m(id \times m)$$

$$m(u \times id) = m(id \times u) = id$$

$$m(i \times id)\delta = m(id \times i)\delta = 1$$

Here $\delta(g) = (g, g)$. Observe also that the result holds as well in the cocommutative case, $A = C^*(\Gamma)$ with $F_N \to \Gamma$, trivially. In general now, the first axiom follows from:

$$(\Delta \otimes id)\Delta(u_{ij}) = (id \otimes \Delta)\Delta(u_{ij}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

As for the other axioms, the verifications here are similar. □

In order to reach now to more advanced results, the idea will be that of doing representation theory. Following Woronowicz [99], let us start with the following definition:

DEFINITION 9.24. *Given (A,u), we call corepresentation of it any unitary matrix $v\in M_n(\mathcal{A})$, with $\mathcal{A}=<u_{ij}>$, satisfying the same conditions as u, namely:*

$$\Delta(v_{ij})=\sum_k v_{ik}\otimes v_{kj}\quad,\quad\varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

We also say that v is a representation of the underlying compact quantum group G.

In the commutative case, $A=C(G)$ with $G\subset U_N$, we obtain in this way the finite dimensional unitary smooth representations $v:G\to U_n$, via the following formula:

$$v(g)=\begin{pmatrix}v_{11}(g)&\ldots&v_{1n}(g)\\ \vdots&&\vdots\\ v_{n1}(g)&\ldots&v_{nn}(g)\end{pmatrix}$$

With this convention, we have the following fundamental result, from [99]:

THEOREM 9.25. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, by linearity we can assume that $a\in A$ is the coefficient of certain corepresentation, $a=(\tau\otimes id)v$. But in this case, an elementary computation gives the following formula, with P_φ being the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi = \xi$ implies $[(id \otimes \varphi)v]\xi = \xi$, we have $P_\varphi \geq P$, where P is the orthogonal projection onto the fixed point space in the statement, namely:

$$Fix(v) = \left\{\xi \in \mathbb{C}^n \Big| v\xi = \xi\right\}$$

The point now is that when $\varphi \in A^*$ is faithful, by using a standard positivity trick, we can prove that we have $P_\varphi = P$, exactly as in the classical case.

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

We can now develop, again following [99], the Peter-Weyl theory for the corepresentations of A. Consider the dense subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

With this convention, we have the following result, also from [99]:

THEOREM 9.26. *We have the following Peter-Weyl type results:*
1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. *$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. This is something that we met in chapters 7-8, in the case where $G \subset U_N$ is a finite group, or more generally a compact group. In general, when G is a compact quantum group, the proof is quite similar, using Theorem 9.25. □

Finally, no discussion about compact and discrete quantum groups would be complete without a word on amenability. The result here, again from [99], is as follows:

THEOREM 9.27. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*
1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $\Phi : A_{red} \to A_{full}$, by slicing the map $\widetilde{\Delta} : A_{red} \to A_{red} \otimes A_{full}$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some functional analysis. See [99]. □

This was for the basic theory of the quantum groups in the sense of Woronowicz, quickly explained. For more on all this, we have for instance my book [6].

9c. Quantum permutations

Following Wang [91], let us discuss now the construction and basic properties of the quantum permutation group S_N^+. Let us first look at S_N. We have here:

THEOREM 9.28. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

PROOF. This is something that we know from chapter 7, coming from the Gelfand theorem, applied to the universal algebra in the statement. Indeed, that algebra follows to be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$, as desired. □

Following now Wang [91], we can liberate S_N, as follows:

THEOREM 9.29. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Thus the space S_N^+ is a compact quantum group, called quantum permutation group.

PROOF. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$(u_{ij}^\Delta)^2 = \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} = \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} = u_{ij}^\Delta$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u_{ij}^\Delta)^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^\Delta$$

The row and column sums for the matrix u^Δ can be computed as follows:

$$\sum_j u_{ij}^\Delta = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

$$\sum_i u_{ij}^\Delta = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$. □

Our first task now is to make sure that Theorem 9.29 produces indeed a new quantum group, which does not collapse to S_N. Following Wang [91], we have:

THEOREM 9.30. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The fact that we have indeed an embedding as above follows from Theorem 9.28. Observe that in fact more is true, because Theorems 9.28 and 9.29 give:

$$C(S_N) = C(S_N^+) \Big/ \left\langle ab = ba \right\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N=2$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Thus $C(S_2^+)$ is commutative, and equals its biggest commutative quotient, $C(S_2)$.

Case $N=3$. It is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}u_{11}u_{22}&=u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\&=u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\&=u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\&=u_{11}u_{22}u_{11}\end{aligned}$$

Indeed, by conjugating, $u_{22}u_{11}=u_{11}u_{22}u_{11}$, so $u_{11}u_{22}=u_{22}u_{11}$, as desired.

Case $N=4$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose $p,q\in B(H)$ as for the algebra $<p,q>$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. $\square$

As a first observation, as a matter of doublechecking our findings, we are not wrong with our formalism, because as explained once again in [91], we have as well:

THEOREM 9.31. *The quantum permutation group S_N^+ acts on the set $X=\{1,\ldots,N\}$, the corresponding coaction map $\Phi:C(X)\to C(X)\otimes C(S_N^+)$ being given by:*

$$\Phi(e_i)=\sum_j e_j\otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(tr\otimes id)\Phi=tr(.)1$, where $tr(e_i)=\frac{1}{N},\forall i$.

PROOF. Our claim is that given a compact matrix quantum group G, the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u = (u_{ij})$ is a magic corepresentation of $C(G)$:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(e_i)\Phi(e_k) = \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} = \sum_j e_j \otimes u_{ji} u_{jk}$$

$$\Phi(e_i e_k) = \delta_{ik}\Phi(e_i) = \delta_{ik} \sum_j e_j \otimes u_{ji}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji} u_{jk} = \delta_{ik} u_{ji} \quad , \quad \forall i,j,k$$

Similarly, Φ is unital when $\sum_i u_{ji} = 1$, $\forall j$. Finally, the fact that Φ is a $*$-morphism translates into $u_{ij} = u_{ij}^*$, $\forall i,j$. Summing up, in order for $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u. Regarding now the preservation of the trace, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N} \sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u. We conclude from this that $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and this gives the result. □

9d. Liberation theory

In order to study S_N^+, and better understand the liberation operation $S_N \to S_N^+$, we can use representation theory. We have the following version of Tannakian duality:

THEOREM 9.32. *The following operations are inverse to each other:*

(1) *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to A$, which associates to a tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, going back to Woronowicz's paper [100] in a slightly different form, with the idea being as follows:

– We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

– We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Some elementary algebra shows then that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving $C_{A_C} \subset C$, and this can be done by doing some algebra, and using von Neumann's bicommutant theorem. See [6]. $\square$

We will need as well, following the classical work of Weyl, Brauer and many others, the notion of "easiness". Let us start with the following definition:

Definition 9.33. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A set $D = \bigsqcup_{k,l} D(k,l)$ with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.*
2. *Stability under the vertical concatenation, $(\pi,\sigma) \to [^\sigma_\pi]$.*
3. *Stability under the upside-down turning, $\pi \to \pi^*$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The set $P(0,2)$ contains the semicircle partition $\cap$.*

Observe that this is precisely the definition that we used in chapter 8, with the condition there on the basic crossing $\slashed{\chi}$, which produces commutativity via Tannakian duality, removed. In relation with the quantum groups, we have the following notion:

Definition 9.34. *A compact quantum matrix group G is called easy when*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for any colored integers k,l, for certain sets of partitions $D(k,l) \subset P(k,l)$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Again, this is something coming as a continuation of the material from chapter 8. Many things can be said here, but getting now straight to the point, we have:

Theorem 9.35. *We have the following results:*

1. *S_N is easy, coming from the category of all partitions P.*
2. *S_N^+ is easy, coming from the category of all noncrossing partitions NC.*

Proof. This is something quite fundamental, with the proof, using the above Tannakian results and subsequent easiness theory, being as follows:

(1) S_N^+. We know that this quantum group comes from the magic condition. In order to interpret this magic condition, consider the fork partition:

$$Y \in P(2,1)$$

By arguing as in chapter 8, we conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij}u_{ik} = \delta_{jk}u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we conclude that S_N^+ is indeed easy, the corresponding category of partitions being, as desired:

$$D =< Y >= NC$$

(2) S_N. Here there is no need for new computations, because we have:

$$S_N = S_N^+ \cap O_N$$

At the categorical level means that S_N is easy, coming from:

$$< NC, \slash\!\!\!\backslash >= P$$

Thus, we are led to the conclusions in the statement. $\square$

Summarizing, we have now a good understanding of the liberation operation $S_N \to S_N^+$, the idea being that this comes, via Tannakian duality, from $P \to NC$.

In order to go further in this direction, we will need the following result, with $*$ being the classical convolution, and $\boxplus$ being Voiculescu's free convolution operation [90]:

Theorem 9.36. *The following Poisson type limits converge, for any $t > 0$,*

$$p_t = \lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\delta_0 + \frac{1}{n}\delta_t\right)^{*n}$$

$$\pi_t = \lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\delta_0 + \frac{1}{n}\delta_t\right)^{\boxplus n}$$

the limiting measures being the Poisson law p_t, and the Marchenko-Pastur law π_t,

$$p_t = \frac{1}{e^t}\sum_{k=0}^{\infty}\frac{t^k\delta_k}{k!}$$

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

whose moments are given by the following formulae:

$$M_k(p_t) = \sum_{\pi\in P(k)} t^{|\pi|} \quad , \quad M_k(\pi_t) = \sum_{\pi\in NC(k)} t^{|\pi|}$$

The Marchenko-Pastur measure π_t is also called free Poisson law.

PROOF. This is something quite advanced, related to probability theory, free probability theory, and random matrices, the idea being as follows:

(1) The first step is that of finding suitable functional transforms, which linearize the convolution operations in the statement. In the classical case this is the logarithm of the Fourier transform $\log F$, and in the free case this is Voiculescu's R-transform.

(2) With these tools in hand, the above limiting theorems can be proved in a standard way, a bit as when proving the Central Limit Theorem. The computations give the moment formulae in the statement, and the density computations are standard as well.

(3) Finally, in order for the discussion to be complete, what still remains to be explained is the precise nature of the "liberation" operation $p_t \to \pi_t$, as well as the random matrix occurrence of π_t. This is more technical, and we refer here to [16], [69], [90]. □

Getting back now to quantum permutations, the results here are as follows:

THEOREM 9.37. *The law of the main character, given by*

$$\chi = \sum_i u_{ii}$$

for S_N/S_N^+ becomes p_1/π_1 with $N \to \infty$. As for the truncated character

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

for S_N/S_N^+, with $t \in (0,1]$, this becomes p_t/π_t with $N \to \infty$.

PROOF. This is again something quite technical, the idea being as follows:

(1) In the classical case this is well-known, and follows by using the inclusion-exclusion principle, and then letting $N \to \infty$, as explained in chapter 7.

(2) In the free case there is no such simple argument, and we must use what we know about S_N^+, namely its easiness property. We know from easiness that we have:

$$Fix(u^{\otimes k}) = span(NC(k))$$

On the other hand, a direct computation shows that the partitions in $P(k)$, and in particular those in $NC(k)$, implemented as linear maps via the operation $\pi \to T_\pi$ from

Definition 9.34, become linearly independent with $N\geq k$. Thus we have, as desired:

$$\begin{aligned}\int_{S_N^+}\chi^k&=\dim\left(Fix(u^{\otimes k})\right)\\&=\dim\left(span\left(T_\pi\Big|\pi\in NC(k)\right)\right)\\&\simeq|NC(k)|\\&=\sum_{\pi\in NC(k)}1^{|\pi|}\end{aligned}$$

(3) In the general case now, where our parameter is an arbitrary number $t\in(0,1]$, the above computation does not apply, but we can still get away with Peter-Weyl theory. Indeed, we know from Theorem 9.25 how to compute the Haar integration of S_N^+, out of the knowledge of the fixed point spaces $Fix(u^{\otimes k})$, and in practice, by using easiness, this leads to the following formula, called Weingarten integration formula:

$$\int_{S_N^+}u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in NC(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

Here the δ symbols are Kronecker type symbols, checking whether the indices fit or not with the partitions, and $W_{kN}=G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$, where $|.|$ is the number of blocks. Now by using this formula for computing the moments of χ_t, we obtain:

$$\begin{aligned}\int_{S_N^+}\chi_t^k&=\sum_{i_1=1}^{[tN]}\ldots\sum_{i_k=1}^{[tN]}\int u_{i_1i_1}\ldots u_{i_ki_k}\\&=\sum_{\pi,\sigma\in NC(k)}W_{kN}(\pi,\sigma)\sum_{i_1=1}^{[tN]}\ldots\sum_{i_k=1}^{[tN]}\delta_\pi(i)\delta_\sigma(i)\\&=\sum_{\pi,\sigma\in NC(k)}W_{kN}(\pi,\sigma)G_{k[tN]}(\sigma,\pi)\\&=Tr(W_{kN}G_{k[tN]})\end{aligned}$$

(4) The point now is that with $N\to\infty$ the Gram matrix G_{kN}, and so the Weingarten matrix W_{kN} too, becomes asymptotically diagonal. We therefore obtain:

$$\int_{S_N^+}\chi_t^k\simeq\sum_{\pi\in NC(k)}t^{|\pi|}$$

Thus, we are led to the conclusion in the statement. For details, see [6]. □

9e. Exercises

This was a pleasant chapter to write for me, because I've been doing such things for long, but probably quite hard to read, for you. As exercises, we have:

Exercise 9.38. *Do Gram-Schmidt for spaces $L^2(X)$ of your choice.*

Exercise 9.39. *Clarify the details in the proof in the spectral radius formula.*

Exercise 9.40. *Learn more about the GNS embedding theorem, and its proof.*

Exercise 9.41. *Learn the details of Woronowicz's Peter-Weyl theory.*

Exercise 9.42. *Can we compute the Haar measure via Peter-Weyl, and how.*

Exercise 9.43. *Find a new, very simple proof for $S_3^+ = S_3$.*

Exercise 9.44. *Prove that the quantum group S_4^+ is coamenable.*

Exercise 9.45. *Compute the representations of S_N^+, at $N \geq 4$.*

As bonus exercise, learn some quantum mechanics, from Feynman [**40**], or Griffiths [**45**], or Weinberg [**93**]. There ain't no quantum without quantum mechanics.

CHAPTER 10

Graph symmetries

10a. Graph symmetries

We can get back now to graphs. By using the quantum permutation group S_N^+ constructed in the previous chapter, we can perform the following construction:

THEOREM 10.1. *Given a finite graph X, with adjacency matrix $d \in M_N(0,1)$, the following construction produces a quantum permutation group,*

$$C(G^+(X)) = C(S_N^+)\Big/\Big\langle du = ud\Big\rangle$$

whose classical version $G(X)$ is the usual automorphism group of X.

PROOF. The fact that we have indeed a quantum group comes from the fact that $du = ud$ reformulates as $d \in End(u)$, which makes it clear that Δ, ε, S factorize. Regarding now the second assertion, we must establish here the following equality:

$$C(G(X)) = C(S_N)\Big/\Big\langle du = ud\Big\rangle$$

For this purpose, recall that we have $u_{ij}(\sigma) = \delta_{\sigma(j)i}$. We therefore obtain:

$$(du)_{ij}(\sigma) = \sum_k d_{ik} u_{kj}(\sigma) = \sum_k d_{ik}\delta_{\sigma(j)k} = d_{i\sigma(j)}$$

On the other hand, we have as well the following formula:

$$(ud)_{ij}(\sigma) = \sum_k u_{ik}(\sigma) d_{kj} = \sum_k \delta_{\sigma(k)i} d_{kj} = d_{\sigma^{-1}(i)j}$$

Thus $du = ud$ reformulates as $d_{ij} = d_{\sigma(i)\sigma(j)}$, which gives the result. □

Let us work out some examples. With the convention that $\hat{*}$ is the dual free product, obtained by diagonally concatenating the magic unitaries, we have:

PROPOSITION 10.2. *The construction $X \to G^+(X)$ has the following properties:*

(1) *For the N-point graph, having no edges at all, we obtain S_N^+.*
(2) *For the N-simplex, having edges everywhere, we obtain as well S_N^+.*
(3) *We have $G^+(X) = G^+(X^c)$, where X^c is the complementary graph.*
(4) *For a disconnected union, we have $G^+(X)\,\hat{*}\,G^+(Y) \subset G^+(X \sqcup Y)$.*
(5) *For the square, we obtain a non-classical, proper subgroup of S_4^+.*

PROOF. All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have $d = 0$.

(2) Here $d = \mathbb{I} - 1$, where $\mathbb{I}$ is the all-one matrix, and the magic condition gives $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$. We conclude that $du = ud$ is automatic, and so $G^+(X) = S_N^+$.

(3) The adjacency matrices of X, X^c being related by the following formula:

$$d_X + d_{X^c} = \mathbb{I} - 1$$

By using now the above formula $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$, we conclude that $d_X u = ud_X$ is equivalent to $d_{X^c}u = ud_{X^c}$. Thus, we obtain, as claimed, $G^+(X) = G^+(X^c)$.

(4) The adjacency matrix of a disconnected union is given by:

$$d_{X \sqcup Y} = diag(d_X, d_Y)$$

Now let $w = diag(u, v)$ be the fundamental corepresentation of $G^+(X) \,\hat{*}\, G^+(Y)$. Then $d_X u = ud_X$ and $d_Y v = vd_Y$, and we obtain, as desired, $d_{X \sqcup Y} w = wd_{X \sqcup Y}$.

(5) We know from (3) that we have $G^+(\square) = G^+(|\ |)$. We know as well from (4) that we have $\mathbb{Z}_2 \,\hat{*}\, \mathbb{Z}_2 \subset G^+(|\ |)$. It follows that $G^+(\square)$ is non-classical. Finally, the inclusion $G^+(\square) \subset S_4^+$ is indeed proper, because $S_4 \subset S_4^+$ does not act on the square. □

In order to further advance, and to explicitely compute various quantum automorphism groups, we can use the spectral decomposition of d, as follows:

THEOREM 10.3. *A closed subgroup $G \subset S_N^+$ acts on a graph X precisely when*

$$P_\lambda u = uP_\lambda \quad , \quad \forall \lambda \in \mathbb{R}$$

where $d = \sum_\lambda \lambda \cdot P_\lambda$ is the spectral decomposition of the adjacency matrix of X.

PROOF. Since $d \in M_N(0, 1)$ is a symmetric matrix, we can consider indeed its spectral decomposition, $d = \sum_\lambda \lambda \cdot P_\lambda$, and we have the following formula:

$$< d >= span\left\{P_\lambda \Big| \lambda \in \mathbb{R}\right\}$$

Thus $d \in End(u)$ when $P_\lambda \in End(u)$ for all $\lambda \in \mathbb{R}$, which gives the result. □

In order to exploit Theorem 10.3, we will often combine it with the following fact:

PROPOSITION 10.4. *Given a closed subgroup $G \subset S_N^+$, with associated coaction*

$$\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G) \quad , \quad e_i \to \sum_j e_j \otimes u_{ji}$$

and a linear subspace $V \subset \mathbb{C}^N$, the following are equivalent:

1. *The magic matrix $u = (u_{ij})$ commutes with P_V.*
2. *V is invariant, in the sense that $\Phi(V) \subset V \otimes C(G)$.*

PROOF. Let $P = P_V$. For any $i \in \{1, \ldots, N\}$ we have the following formula:

$$\Phi(P(e_i)) = \Phi\left(\sum_k P_{ki} e_k\right) = \sum_{jk} P_{ki} e_j \otimes u_{jk} = \sum_j e_j \otimes (uP)_{ji}$$

On the other hand the linear map $(P \otimes id)\Phi$ is given by a similar formula:

$$(P \otimes id)(\Phi(e_i)) = \sum_k P(e_k) \otimes u_{ki} = \sum_{jk} P_{jk} e_j \otimes u_{ki} = \sum_j e_j \otimes (Pu)_{ji}$$

Thus $uP = Pu$ is equivalent to $\Phi P = (P \otimes id)\Phi$, and the conclusion follows. $\square$

As an application of the above results, we have the following computation:

THEOREM 10.5. *The quantum automorphism group of the N-cycle is, at $N \neq 4$:*

$$G^+(X) = D_N$$

However, at $N = 4$ we have $D_4 \subset G^+(X) \subset S_4^+$, with proper inclusions.

PROOF. We know from Proposition 10.2, and from $S_N = S_N^+$ at $N \leq 3$, that the various assertions hold indeed at $N \leq 4$. So, assume $N \geq 5$. Given a N-th root of unity, $w^N = 1$, the vector $\xi = (w^i)$ is an eigenvector of d, with eigenvalue as follows:

$$\lambda = w + w^{N-1}$$

Now by taking $w = e^{2\pi i/N}$, it follows that the are eigenvectors of d are:

$$1, f, f^2, \ldots, f^{N-1}$$

More precisely, the invariant subspaces of d are as follows, with the last subspace having dimension 1 or 2 depending on the parity of N:

$$\mathbb{C}1,\ \mathbb{C}f \oplus \mathbb{C}f^{N-1},\ \mathbb{C}f^2 \oplus \mathbb{C}f^{N-2}, \ldots$$

Assuming $G \subset G^+(X)$, consider the coaction $\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G)$, and write:

$$\Phi(f) = f \otimes a + f^{N-1} \otimes b$$

By taking the square of this equality we obtain the following formula:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2 + 1 \otimes (ab + ba)$$

It follows that $ab = -ba$, and that $\Phi(f^2)$ is given by the following formula:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2$$

By multiplying this with $\Phi(f)$ we obtain the following formula:

$$\Phi(f^3) = f^3 \otimes a^3 + f^{N-3} \otimes b^3 + f^{N-1} \otimes ab^2 + f \otimes ba^2$$

Now since $N \geq 5$ implies that $1, N-1$ are different from $3, N-3$, we must have $ab^2 = ba^2 = 0$. By using this and $ab = -ba$, we obtain by recurrence on k that:

$$\Phi(f^k) = f^k \otimes a^k + f^{N-k} \otimes b^k$$

In particular at $k=N-1$ we obtain the following formula:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^{N-1}+f\otimes b^{N-1}$$

On the other hand we have $f^*=f^{N-1}$, so by applying $*$ to $\Phi(f)$ we get:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^*+f\otimes b^*$$

Thus $a^*=a^{N-1}$ and $b^*=b^{N-1}$. Together with $ab^2=0$ this gives:

$$(ab)(ab)^*=abb^*a^*=ab^Na^{N-1}=(ab^2)b^{N-2}a^{N-1}=0$$

From positivity we get from this $ab=0$, and together with $ab=-ba$, this shows that a,b commute. On the other hand $C(G)$ is generated by the coefficients of Φ, which are powers of a,b, and so $C(G)$ must be commutative, and we obtain the result. $\square$

The above result is quite suprising, but we will be back to this, with a more conceptual explanation for the fact that the square $\square$ has quantum symmetry. Back to theory now, we have the following useful result, complementary to Theorem 10.3:

THEOREM 10.6. *Given a matrix $p\in M_N(\mathbb{C})$, consider its "color" decomposition*

$$p=\sum_{c\in\mathbb{C}}c\cdot p_c$$

with the color components $p_c\in M_N(0,1)$ with $c\in\mathbb{C}$ being constructed as follows:

$$(p_c)_{ij}=\begin{cases}1&\text{if }p_{ij}=c\\0&\text{otherwise}\end{cases}$$

Then a magic matrix $u=(u_{ij})$ commutes with p iff it commutes with all matrices p_c.

PROOF. Consider the multiplication and counit maps of the algebra $\mathbb{C}^N$:

$$M:e_i\otimes e_j\to e_ie_j\quad,\quad C:e_i\to e_i\otimes e_i$$

Since M,C intertwine $u,u^{\otimes 2}$, their iterations $M^{(k)},C^{(k)}$ intertwine $u,u^{\otimes k}$, and so:

$$M^{(k)}p^{\otimes k}C^{(k)}=\sum_{c\in\mathbb{C}}c^kp_c\in End(u)$$

Now since this formula holds for any $k\in\mathbb{N}$, we obtain the result. $\square$

The above results can be combined, and we are led to the following statement:

THEOREM 10.7. *A closed subgroup $G\subset S_N^+$ acts on a graph X precisely when*

$$u=(u_{ij})$$

commutes with all the matrices coming from the color-spectral decomposition of d.

PROOF. This follows by combining Theorem 10.3 and Theorem 10.6, with the "color-spectral" decomposition in the statement referring to what comes out by succesively doing the color and spectral decomposition, until the process stabilizes. $\square$

This latter statement is quite interesting, with the color-spectral decomposition there being something quite intriguing. We will be back to this in chapter 12, when discussing planar algebras, which is the good framework for discussing such things.

10b. Product operations

We would like to understand how the operation $X \to G^+(X)$ behaves under taking various products of graphs, in analogy with what we know about $X \to G(X)$, from chapter 6. As a first observation, things can be quite tricky here, as shown by:

FACT 10.8. *Although the graph formed by two points $\bullet\bullet$ has no quantum symmetry, the graph formed by two copies of it, namely $\bullet\,\bullet\;\bullet\,\bullet$, does have quantum symmetry.*

Which looks quite scary, but no worries, we will manage to reach to a better understanding of this. Getting to work now, let us recall from chapter 6 that we have:

DEFINITION 10.9. *Let X, Y be two finite graphs.*

(1) *The direct product $X \times Y$ has vertex set $X \times Y$, and edges:*

$$(i,\alpha) - (j,\beta) \Longleftrightarrow i - j,\ \alpha - \beta$$

(2) *The Cartesian product $X \,\square\, Y$ has vertex set $X \times Y$, and edges:*

$$(i,\alpha) - (j,\beta) \Longleftrightarrow i = j,\ \alpha - \beta \text{ or } i - j, \alpha = \beta$$

(3) *The lexicographic product $X \circ Y$ has vertex set $X \times Y$, and edges:*

$$(i,\alpha) - (j,\beta) \Longleftrightarrow \alpha - \beta \text{ or } \alpha = \beta,\ i - j$$

The above products are all well-known in graph theory, and we have already studied them in chapter 6, in relation with symmetry groups, with the following conclusion:

THEOREM 10.10. *We have standard embeddings, as follows,*

$$G(X) \times G(Y) \subset G(X \times Y)$$

$$G(X) \times G(Y) \subset G(X \,\square\, Y)$$

$$G(X) \wr G(Y) \subset G(X \circ Y)$$

and under suitable spectral assumptions, these embeddings are isomorphisms.

PROOF. This is something that we know from chapter 6, and we refer to the discussion there for both the precise statement of the last assertion, and for the proofs. $\square$

In order to discuss the quantum analogues of these embeddings, we need to introduce first a number of operations on the compact quantum groups, similar to the above operations for the finite groups. Following Wang [91], we first have:

Proposition 10.11. *Given two compact quantum groups G,H, so is their product $G\times H$, constructed according to the following formula:*

$$C(G\times H)=C(G)\otimes C(H)$$

Equivalently, at the level of the associated discrete duals Γ,Λ, we can set

$$C^*(\Gamma\times\Lambda)=C^*(\Gamma)\otimes C^*(\Lambda)$$

and we obtain the same equality of Woronowicz algebras as above.

Proof. Assume indeed that we have two Woronowicz algebras, (A,u) and (B,v). Our claim is that the following construction produces a Woronowicz algebra:

$$C=A\otimes B\quad,\quad w=diag(u,v)$$

Indeed, the matrix w is unitary, and its coefficients generate C. As for the existence of the maps Δ,ε,S, this follows from the functoriality properties of $\otimes$. But with this claim in hand, the first assertion is clear. As for the second assertion, let us recall that when G,H are classical and abelian, we have the following formula:

$$\widehat{G\times H}=\widehat{G}\times\widehat{H}$$

Thus, our second assertion is simply a reformulation of the first assertion, with the $\times$ symbol used there being justified by this well-known group theory formula. $\square$

Another standard operation, again from [91], is that of taking subgroups:

Proposition 10.12. *Let G be compact quantum group, and let $I\subset C(G)$ be a closed $*$-ideal satisfying the following condition:*

$$\Delta(I)\subset C(G)\otimes I+I\otimes C(G)$$

We have then a closed quantum subgroup $H\subset G$, constructed as follows:

$$C(H)=C(G)/I$$

At the dual level we obtain a quotient of discrete quantum groups, $\widehat{\Gamma}\to\widehat{\Lambda}$.

Proof. This follows indeed from the above conditions on I, which are designed precisely as for Δ,ε,S to factorize through the quotient. As for the last assertion, this is just a reformulation, coming from the functoriality properties of the Pontrjagin duality. $\square$

Regarding now taking quotients, the result here, again from [91], is as follows:

Proposition 10.13. *Let G be a compact quantum group, and $v=(v_{ij})$ be a corepresentation of $C(G)$. We have then a quotient quantum group $G\to H$, given by:*

$$C(H)=<v_{ij}>$$

At the dual level we obtain a discrete quantum subgroup, $\widehat{\Lambda}\subset\widehat{\Gamma}$.

PROOF. Here the first assertion follows from definitions, and the second assertion is a reformulation of it, using the basic properties of Pontrjagin duality. □

Finally, we will need the notion of free wreath product, from [17], as follows:

PROPOSITION 10.14. *Given closed subgroups $G\subset S_N^+$, $H\subset S_k^+$, with magic corepresentations u,v, the following construction produces a closed subgroup of S_{Nk}^+:*

$$C(G\wr_* H)=(C(G)^{*k}*C(H))/<[u_{ij}^{(a)},v_{ab}]=0>$$

When G,H are classical, the classical version of $G\wr_* H$ is the wreath product $G\wr H$.

PROOF. Consider indeed the matrix $w_{ia,jb}=u_{ij}^{(a)}v_{ab}$, over the quotient algebra in the statement. Our claim is that this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row sums are:

$$\sum_{jb}w_{ia,jb}=\sum_{jb}u_{ij}^{(a)}v_{ab}=\sum_b v_{ab}\sum_j u_{ij}^{(a)}=1$$

As for the column sums, these are as follows:

$$\sum_{ia}w_{ia,jb}=\sum_{ia}u_{ij}^{(a)}v_{ab}=\sum_a v_{ab}\sum_i u_{ij}^{(a)}=1$$

With this in hand, it is routine to check that $G\wr_* H$ is indeed a quantum permutation group. Finally, the last assertion is standard as well. See [17]. □

With the above discussed, we can now go back to graphs. Following [8], the standard embeddings from Theorem 10.10 have the following quantum analogues:

THEOREM 10.15. *We have embeddings as follows,*

$$G^+(X)\times G^+(Y)\subset G^+(X\times Y)$$
$$G^+(X)\times G^+(Y)\subset G^+(X\,\square\, Y)$$
$$G^+(X)\wr_* G^+(Y)\subset G^+(X\circ Y)$$

with the operation $\wr_*$ being a free wreath product.

PROOF. We use the following identification, given by $\delta_{(i,\alpha)}=\delta_i\otimes\delta_\alpha$:

$$C(X\times Y)=C(X)\otimes C(Y)$$

(1) The adjacency matrix of the direct product is given by:

$$d_{X\times Y}=d_X\otimes d_Y$$

Thus if u commutes with d_X and v commutes with d_Y, then $u\otimes v=(u_{ij}v_{\alpha\beta})_{(i\alpha,j\beta)}$ is a magic unitary that commutes with $d_{X\times Y}$. But this gives a morphism as follows:

$$C(G^+(X\times Y))\to C(G^+(X)\times G^+(Y))$$

Finally, the surjectivity of this morphism follows by summing over i and β.

(2) The adjacency matrix of the Cartesian product is given by:

$$d_{X\,\square\,Y}=d_X\otimes 1+1\otimes d_Y$$

Thus if u commutes with d_X and v commutes with d_Y, then $u\otimes v=(u_{ij}v_{\alpha\beta})_{(i\alpha,j\beta)}$ is a magic unitary that commutes with $d_{X\,\square\,Y}$, and this gives the result.

(3) The adjacency matrix of the lexicographic product $X\circ Y$ is given by:

$$d_{X\circ Y}=d_X\otimes 1+\mathbb{I}\otimes d_Y$$

Now let u,v be the magic unitary matrices of $G^+(X),G^+(Y)$. The magic unitary matrix of $G^+(X)\wr_* G^+(Y)$ is then given by the following formula:

$$w_{ia,jb}=u_{ij}^{(a)}v_{ab}$$

Since u,v commute with d_X,d_Y, we get that w commutes with $d_{X\circ Y}$. But this gives the desired morphism, and the surjectivity follows by summing over i and b. $\square$

The problem now is that of deciding when the embeddings in Theorem 10.15 are isomorphisms. Following [8], we first have the following result:

THEOREM 10.16. *Let X and Y be finite connected regular graphs. If their spectra $\{\lambda\}$ and $\{\mu\}$ do not contain 0 and satisfy*

$$\left\{\lambda_i/\lambda_j\right\}\cap\left\{\mu_k/\mu_l\right\}=\{1\}$$

then $G^+(X\times Y)=G^+(X)\times G^+(Y)$. Also, if their spectra satisfy

$$\left\{\lambda_i-\lambda_j\right\}\cap\left\{\mu_k-\mu_l\right\}=\{0\}$$

then $G^+(X\,\square\,Y)=G^+(X)\times G^+(Y)$.

PROOF. This is something quite standard, the idea being as follows:

(1) First, we know from Theorem 10.15 that we have embeddings as follows, valid for any two graphs X,Y, and coming from definitions:

$$G^+(X)\times G^+(Y)\subset G^+(X\times Y)$$

$$G^+(X)\times G^+(Y)\subset G^+(X\,\square\,Y)$$

(2) Now let λ_1 be the valence of X. Since X is regular we have $\lambda_1\in Sp(X)$, with 1 as eigenvector, and since X is connected λ_1 has multiplicity 1. Hence if P_1 is the orthogonal projection onto $\mathbb{C}1$, the spectral decomposition of d_X is of the following form:

$$d_X=\lambda_1P_1+\sum_{i\neq 1}\lambda_iP_i$$

We have a similar formula for the adjacency matrix d_Y, namely:

$$d_Y=\mu_1Q_1+\sum_{j\neq 1}\mu_jQ_j$$

(3) But this gives the following formulae for products:

$$d_{X\times Y}=\sum_{ij}(\lambda_i\mu_j)P_i\otimes Q_j$$

$$d_{X\,\square\,Y}=\sum_{ij}(\lambda_i+\mu_i)P_i\otimes Q_j$$

Here the projections form partitions of unity, and the scalar are distinct, so these are spectral decompositions. The coactions will commute with any of the spectral projections, and hence with both $P_1\otimes 1$, $1\otimes Q_1$. In both cases the universal coaction v is the tensor product of its restrictions to the images of $P_1\otimes 1$, $1\otimes Q_1$, which gives the result. □

Regarding now the lexicographic products, the result here is as follows:

THEOREM 10.17. *Let X,Y be regular graphs, with X connected. If their spectra $\{\lambda_i\}$ and $\{\mu_j\}$ satisfy the condition*

$$\left\{\lambda_1-\lambda_i\Big|i\neq 1\right\}\cap\left\{-n\mu_j\right\}=\emptyset$$

where n and λ_1 are the order and valence of X, then $G^+(X\circ Y)=G^+(X)\wr_* G^+(Y)$.

PROOF. This is something quite tricky, the idea being as follows:

(1) First, we know from Theorem 10.15 that we have an embedding as follows, valid for any two graphs X,Y, and coming from definitions:

$$G^+(X)\wr_* G^+(Y)\subset G^+(X\circ Y)$$

(2) We denote by P_i,Q_j the spectral projections corresponding to λ_i,μ_j. Since X is connected we have $P_1=\mathbb{I}/n$, and we obtain:

$$\begin{aligned}
d_{X\circ Y} &= d_X\otimes 1+\mathbb{I}\otimes d_Y\\
&= \left(\sum_i\lambda_iP_i\right)\otimes\left(\sum_j Q_j\right)+(nP_1)\otimes\left(\sum_i\mu_jQ_j\right)\\
&= \sum_j(\lambda_1+n\mu_j)(P_1\otimes Q_j)+\sum_{i\neq 1}\lambda_i(P_i\otimes 1)
\end{aligned}$$

Since in this formula the projections form a partition of unity, and the scalars are distinct, we conclude that this is the spectral decomposition of $d_{X\circ Y}$.

(3) Now let W be the universal magic matrix for $X\circ Y$. Then W must commute with all spectral projections, and in particular, we have:

$$[W,P_1\otimes Q_j]=0$$

Summing over j gives $[W, P_1 \otimes 1] = 0$, so $1 \otimes C(Y)$ is invariant under the coaction. So, consider the restriction of W, which gives a coaction of $G^+(X \circ Y)$ on $1 \otimes C(Y)$, that we can denote as follows, with y being a certain magic unitary:

$$W(1 \otimes e_a) = \sum_b 1 \otimes e_b \otimes y_{ba}$$

(4) On the other hand, according to our definition of W, we can write:

$$W(e_i \otimes 1) = \sum_{jb} e_j \otimes e_b \otimes x_{ji}^b$$

By multiplying by the previous relation, found in (3), we obtain:

$$W(e_i \otimes e_a) = \sum_{jb} e_j \otimes e_b \otimes y_{ba} x_{ji}^b = \sum_{jb} e_j \otimes e_b \otimes x_{ji}^b y_{ba}$$

But this shows that the coefficients of W are of the following form:

$$W_{jb,ia} = y_{ba} x_{ji}^b = x_{ji}^b y_{ba}$$

(5) Consider now the matrix $x^b = (x_{ij}^b)$. Since W is a morphism of algebras, each row of x^b is a partition of unity. Also, by using the antipode, we have:

$$S\left(\sum_j x_{ji}^b\right) = S\left(\sum_{ja} W_{jb,ia}\right) = \sum_{ja} W_{ia,jb} = 1$$

As a conclusion to this, the matrix x^b constructed above is magic.

(6) We check now that both x^a, y commute with d_X, d_Y. We have:

$$(d_{X \circ Y})_{ia,jb} = (d_X)_{ij} \delta_{ab} + (d_Y)_{ab}$$

Thus the two products between W and $d_{X \circ Y}$ are given by:

$$(W d_{X \circ Y})_{ia,kc} = \sum_j W_{ia,jc} (d_X)_{jk} + \sum_{jb} W_{ia,jb} (d_Y)_{bc}$$

$$(d_{X \circ Y} W)_{ia,kc} = \sum_j (d_X)_{ij} W_{ja,kc} + \sum_{jb} (d_Y)_{ab} W_{jb,kc}$$

(7) Now since the magic matrix W commutes by definition with $d_{X \circ Y}$, the terms on the right in the above equations are equal, and by summing over c we get:

$$\sum_j x_{ij}^a (d_X)_{jk} + \sum_{cb} y_{ab} (d_Y)_{bc} = \sum_j (d_X)_{ij} x_{jk}^a + \sum_{cb} (d_Y)_{ab} y_{bc}$$

The second sums in both terms are equal to the valence of Y, so we get:

$$[x^a, d_X] = 0$$

Now once again from the formula coming from $[W,d_{X\circ Y}]=0$, we get:

$$[y,d_Y]=0$$

(8) Summing up, the coefficients of W are of the following form, where x^b are magic unitaries commuting with d_X, and y is a magic unitary commuting with d_Y:

$$W_{jb,ia}=x_{ji}^b y_{ba}$$

But this gives a morphism $C(G^+(X)\wr_* G^+(Y))\to G^+(X\circ Y)$ mapping $u_{ji}^{(b)}\to x_{ji}^b$ and $v_{ba}\to y_{ba}$, which is inverse to the morphism in (1), as desired. $\square$

As a main application of the above result, we have:

THEOREM 10.18. *Given a connected graph X, and $k\in\mathbb N$, we have the formulae*

$$G(kX)=G(X)\wr S_k\quad,\quad G^+(kX)=G^+(X)\wr_* S_k^+$$

where $kX=X\sqcup\ldots\sqcup X$ is the k-fold disjoint union of X with itself.

PROOF. The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is elementary to check that we have an inclusion as follows, for any finite graph X:

$$G^+(X)\wr_* S_k^+\subset G^+(kX)$$

Regarding now the reverse inclusion, which requires X to be connected, this follows by doing some matrix analysis, by using the commutation with u. To be more precise, let us denote by w the fundamental corepresentation of $G^+(kX)$, and set:

$$u_{ij}^{(a)}=\sum_b w_{ia,jb}\quad,\quad v_{ab}=\sum_i v_{ab}$$

It is then routine to check, by using the fact that X is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus, we obtain:

$$G^+(kX)\subset G^+(X)\wr_* S_k^+$$

But this gives the result, as a consequence of Theorem 10.17. See [8]. $\square$

We refer to [8] and related papers for further results, along these lines.

10c. Reflection groups

We know that we have results involving free wreath products, which replace the usual wreath products from the classical case. In particular, the quantum symmetry group of the graph formed by N segments is the hyperoctahedral quantum group $H_N^+=\mathbb Z_2\wr_* S_N^+$, appearing as a free analogue of the usual hyperoctahedral group $H_N=\mathbb Z_2\wr S_N$.

The free analogues of the reflection groups H_N^s can be constructed as follows:

Definition 10.19. *The algebra $C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements u_{ij}, subject to the following relations,*

1. *$u = (u_{ij})$ is unitary,*
2. *$u^t = (u_{ji})$ is unitary,*
3. *$p_{ij} = u_{ij}u_{ij}^*$ is a projection,*
4. *$u_{ij}^s = p_{ij}$,*

with Woronowicz algebra maps Δ, ε, S constructed by universality.

Here we allow the value $s = \infty$, with the convention that the last axiom simply disappears in this case. Observe that at $s < \infty$ the normality condition is actually redundant, because a partial isometry a subject to the relation $aa^* = a^s$ is normal.

Observe also that we have an inclusion of quantum groups $H_N^s \subset H_N^{s+}$ which is a liberation, in the sense that the classical version of H_N^{s+}, obtained by dividing by the commutator ideal, is the group H_N^s. Indeed, this follows exactly as for $S_N \subset S_N^+$.

In analogy with the results from the real case, we have the following result:

Proposition 10.20. *The algebras $C(H_N^{s+})$ with $s = 1, 2, \infty$, and their presentation relations in terms of the entries of the matrix $u = (u_{ij})$, are as follows:*

1. *For $C(H_N^{1+}) = C(S_N^+)$, the matrix u is magic: all its entries are projections, summing up to 1 on each row and column.*
2. *For $C(H_N^{2+}) = C(H_N^+)$ the matrix u is cubic: it is orthogonal, and the products of pairs of distinct entries on the same row or the same column vanish.*
3. *For $C(H_N^{\infty+}) = C(K_N^+)$ the matrix u is unitary, its transpose is unitary, and all its entries are normal partial isometries.*

Proof. This is something elementary, the idea being as follows:

(1) This follows from definitions, and from some standard operator algebra tricks.

(2) This follows again from definitions, and standard operator algebra tricks.

(3) This is just a translation of the definition of $C(H_N^{s+})$, at $s = \infty$. □

Let us prove now that H_N^{s+} with $s < \infty$ is a quantum permutation group. For this purpose, we must change the fundamental representation. Let us start with:

Definition 10.21. *A (s, N)-sudoku matrix is a magic unitary of size sN, of the form*

$$m = \begin{pmatrix} a^0 & a^1 & \dots & a^{s-1} \\ a^{s-1} & a^0 & \dots & a^{s-2} \\ \vdots & \vdots & & \vdots \\ a^1 & a^2 & \dots & a^0 \end{pmatrix}$$

where $a^0, \dots, a^{s-1}$ are $N \times N$ matrices.

The basic examples of such matrices come from the group H_n^s. Indeed, with $w = e^{2\pi i/s}$, each of the N^2 matrix coordinates $u_{ij} : H_N^s \to \mathbb{C}$ decomposes as follows:

$$u_{ij} = \sum_{r=0}^{s-1} w^r a_{ij}^r$$

Here each a_{ij}^r is a function taking values in $\{0,1\}$, and so a projection in the C^*-algebra sense, and it follows from definitions that these projections form a sudoku matrix. Now with this sudoku matrix notion in hand, we have the following result:

Theorem 10.22. *The following happen:*

1. *The algebra $C(H_N^s)$ is isomorphic to the universal commutative C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*
2. *The algebra $C(H_N^{s+})$ is isomorphic to the universal C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*

Proof. The first assertion follows from the second one. In order to prove now the second assertion, consider the universal algebra in the statement:

$$A = C^*\left(a_{ij}^p \,\Big|\, \left(a_{ij}^{q-p}\right)_{pi,qj} = (s,N) - \text{sudoku}\right)$$

Consider also the algebra $C(H_N^{s+})$. According to Definition 10.19, this algebra is presented by certain relations R, that we will call here level s cubic conditions:

$$C(H_N^{s+}) = C^*\left(u_{ij} \,\Big|\, u = N \times N \text{ level } s \text{ cubic}\right)$$

We will construct now a pair of inverse morphisms between these algebras.

(1) Our first claim is that $U_{ij} = \sum_p w^{-p} a_{ij}^p$ is a level s cubic unitary. Indeed, by using the sudoku condition, the verification of (1-4) in Definition 10.19 is routine.

(2) Our second claim is that the elements $A_{ij}^p = \frac{1}{s}\sum_r w^{rp} u_{ij}^r$, with the convention $u_{ij}^0 = p_{ij}$, form a level s sudoku unitary. Once again, the proof here is routine.

(3) According to the above, we can define a morphism $\Phi : C(H_N^{s+}) \to A$ by the formula $\Phi(u_{ij}) = U_{ij}$, and a morphism $\Psi : A \to C(H_N^{s+})$ by the formula $\Psi(a_{ij}^p) = A_{ij}^p$.

(4) It is then routine to check that Φ, Ψ are inverse morphisms, by a direct computation of their compositions. Thus, we have an isomorphism $C(H_N^{s+}) = A$, as claimed. $\square$

In order to further advance, we will need the following simple fact:

PROPOSITION 10.23. *A $sN \times sN$ magic unitary commutes with the matrix*

$$\Sigma = \begin{pmatrix} 0 & I_N & 0 & \ldots & 0 \\ 0 & 0 & I_N & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & I_N \\ I_N & 0 & 0 & \ldots & 0 \end{pmatrix}$$

precisely when it is a sudoku matrix in the sense of Definition 10.21.

PROOF. This follows from the fact that commutation with Σ means that the matrix is circulant. Thus, we obtain the sudoku relations from Definition 10.21. □

Now let Z_s be the oriented cycle with s vertices, and consider the graph NZ_s consisting of N disjoint copies of it. Observe that, with a suitable labeling of the vertices, the adjacency matrix of this graph is the above matrix Σ. We obtain from this:

THEOREM 10.24. *We have the following results:*
(1) *H_N^s is the symmetry group of NZ_s.*
(2) *H_N^{s+} is the quantum symmetry group of NZ_s.*

PROOF. This is something elementary, the idea being as follows:

(1) This follows indeed from definitions.

(2) This follows from Theorem 10.18 and Proposition 10.23, because the algebra $C(H_N^{s+})$ is the quotient of the algebra $C(S_{sN}^+)$ by the relations making the fundamental corepresentation commute with the adjacency matrix of NZ_s. □

Next in line, we must talk about wreath products. We have here:

THEOREM 10.25. *We have isomorphisms as follows,*

$$H_N^s = \mathbb{Z}_s \wr S_N \quad , \quad H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

with $\wr$ being a wreath product, and $\wr_*$ being a free wreath product.

PROOF. This follows from the following formulae, valid for any connected graph X, and explained before in this chapter, applied to the graph Z_s:

$$G(NX) = G(X) \wr S_N \quad , \quad G^+(NX) = G^+(X) \wr_* S_N^+$$

Alternatively, (1) follows from definitions, and (2) can be proved directly, by constructing a pair of inverse morphisms. For details here, we refer to the literature. □

Regarding now the easiness property of H_N^s, H_N^{s+}, we have here:

THEOREM 10.26. *The quantum groups H_N^s, H_N^{s+} are easy, the corresponding categories*

$$P^s \subset P \quad , \quad NC^s \subset NC$$

consisting of partitions satisfying $\#\circ = \#\bullet\,(s)$, as a weighted sum, in each block.

PROOF. This is something quite routine, the idea being as follows:

(1) We already know this for the reflection group H_N^s, from chapter 8, and the idea is that the computation there works for H_N^{s+} too, with minimal changes. Indeed, at $s=1$, to start with, this is something that we already know, from chapter 9.

(2) At $s=2$ now, we know that $H_N^+ \subset O_N^+$ appears via the cubic relations, namely:

$$u_{ij}u_{ik} = u_{ji}u_{ki} = 0 \quad , \quad \forall j \neq k$$

We conclude, exactly as in chapter 8, that H_N^+ is indeed easy, coming from:

$$D = < H > = NC_{even}$$

(3) Regarding now that case $s = \infty$, for the quantum group K_N^+, the proof here is similar, leading this time to the category $\mathcal{NC}_{even}$ of noncrossing matching partitions.

(4) Summarizing, we have the result at $s = 1, 2, \infty$. But the passage to the general case $s \in \mathbb{N}$ is then routine, by using functoriality, and the result at $s = \infty$. $\square$

All the above is very nice, and at the first glance, it looks like a complete theory of quantum reflection groups. However, there is a skeleton in the closet, coming from:

FACT 10.27. *The symmetry groups of the hypercube $\square_N \subset \mathbb{R}^N$ are*

$$G(\square_N) = H_N \quad , \quad G^+(\square_N) \neq H_N^+$$

with the problem coming from the fact that H_N^+ does not act on $\square_N$.

Excited about this? We are here at the heart of quantum algebra, with all sorts of new phenomena, having no classical counterpart, waiting to be explored. In answer, we will prove that we have an equality as follows, with O_N^{-1} being a certain twist of O_N:

$$G^+(\square_N) = O_N^{-1}$$

In order to introduce this new quantum group O_N^{-1}, we will need:

PROPOSITION 10.28. *There is a signature map $\varepsilon : P_{even} \to \{-1, 1\}$, given by*

$$\varepsilon(\tau) = (-1)^c$$

where c is the number of switches needed to make τ noncrossing. In addition:
1. *For $\tau \in S_k$, this is the usual signature.*
2. *For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.*
3. *For $\tau \leq \pi \in NC_{even}$, the signature is 1.*

PROOF. The fact that ε is indeed well-defined comes from the fact that the number c in the statement is well-defined modulo 2, which is standard combinatorics. In order to prove now the remaining assertion, observe that any partition $\tau \in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. With this convention:

(1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau \in P_2$, the standard form is of type $\tau' = |\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau \to \tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{even}$ comes from $\pi \in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

With the above result in hand, we can now formulate:

DEFINITION 10.29. *Associated to a partition $\pi \in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where the signed Kronecker symbols

$$\bar{\delta}_\pi \in \{-1, 0, 1\}$$

are given by $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with $\tau = \ker\binom{i}{j}$.

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j:\ker\binom{i}{j} \geq \pi} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon : P_{even} \to \{\pm 1\}$, we are led to the following formula, which coincides with the one above:

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{j:\ker\binom{i}{j} = \tau} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Getting now to quantum groups, we have the following construction:

THEOREM 10.30. *Given a category of partitions $D \subset P_{even}$, the construction*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\bar{T}_\pi \Big| \pi \in D(k,l)\right)$$

produces via Tannakian duality a quantum group G_N^{-1}, for any $N \in \mathbb{N}$.

PROOF. It is routine to check that the assignement $\pi \to \bar{T}_\pi$ is categorical, in the sense that we have the following formulae, where $c(\pi, \sigma)$ are certain positive integers:

$$\bar{T}_\pi \otimes \bar{T}_\sigma = \bar{T}_{[\pi\sigma]} \quad , \quad \bar{T}_\pi \bar{T}_\sigma = N^{c(\pi,\sigma)} \bar{T}_{[^\sigma_\pi]} \quad , \quad \bar{T}_\pi^* = \bar{T}_{\pi^*}$$

But with this, the result follows from the Tannakian results from chapter 9. $\square$

We can unify the easy quantum groups, or at least the examples coming from categories $D \subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 10.31. *A quantum group G is called q-easy, or quizzy, with deformation parameter $q = \pm 1$, when its tensor category appears as*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\dot{T}_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P_{even}$, where, for $q = -1, 1$:

$$\dot{T} = \bar{T}, T$$

The Schur-Weyl twist of G is the quizzy quantum group G^{-1} obtained via $q \to -q$.

As an illustration for all this, which might seem quite abstract, we can now twist the orthogonal group O_N, and the unitary group U_N too. The result here is as follows:

Theorem 10.32. *The twists of O_N, U_N are obtained by replacing the commutation relations $ab = ba$ between the coordinates u_{ij} and their adjoints u_{ij}^* with the relations*

$$ab = \pm ba$$

with anticommutation on rows and columns, and commutation otherwise.

Proof. The basic crossing, $\ker\binom{ij}{ji}$ with $i \neq j$, comes from the transposition $\tau \in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\chi}(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from $\bar{T}_{\chi}$ correspond to the relations defining O_N^{-1}, U_N^{-1}, we obtain the result. $\square$

Getting back now to graphs, we have the following result, from [12]:

Theorem 10.33. *The quantum symmetry group of the N-hypercube is*

$$G^+(\square_N) = O_N^{-1}$$

with the corresponding coaction map on the vertex set being given by

$$\Phi : C^*(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N) \otimes C(O_N^{-1}) \quad , \quad g_i \to \sum_j g_j \otimes u_{ji}$$

via the standard identification $\square_N = \widehat{\mathbb{Z}_2^N}$. In particular we have $G^+(\square) = O_2^{-1}$.

PROOF. This follows from a standard algebraic study, done in [12], as follows:

(1) Our first claim is that $\square_N$ is the Cayley graph of $\mathbb{Z}_2^N =< \tau_1, \ldots, \tau_N >$. Indeed, the vertices of this latter Cayley graph are the products of the following form:

$$g = \tau_1^{i_1} \ldots \tau_N^{i_N}$$

The sequence of exponents defining such an element determines a point of $\mathbb{R}^N$, which is a vertex of the cube. Thus the vertices of the Cayley graph are the vertices of the cube, and in what regards the edges, this is something that we know too, from chapter 3.

(2) Our second claim now, which is something routine, coming from an elementary computation, is that when identifying the vector space spanned by the vertices of $\square_N$ with the algebra $C^*(\mathbb{Z}_2^N)$, the eigenvectors and eigenvalues of $\square_N$ are given by:

$$v_{i_1 \ldots i_N} = \sum_{j_1 \ldots j_N} (-1)^{i_1 j_1 + \ldots + i_N j_N} \tau_1^{j_1} \ldots \tau_N^{j_N}$$

$$\lambda_{i_1 \ldots i_N} = (-1)^{i_1} + \ldots + (-1)^{i_N}$$

(3) We prove now that the quantum group O_N^{-1} acts on the cube $\square_N$. For this purpose, observe first that we have a map as follows:

$$\Phi : C^*(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N) \otimes C(O_N^{-1}) \quad , \quad \tau_i \to \sum_j \tau_j \otimes u_{ji}$$

It is routine to check that for $i_1 \neq i_2 \neq \ldots \neq i_l$ we have:

$$\Phi(\tau_{i_1} \ldots \tau_{i_l}) = \sum_{j_1 \neq \ldots \neq j_l} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l}$$

In terms of eigenspaces E_s of the adjacency matrix, this gives, as desired:

$$\Phi(E_s) \subset E_s \otimes C(O_N^{-1})$$

(4) Conversely now, consider the universal coaction on the cube:

$$\Psi : C^*(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N) \otimes C(G) \quad , \quad \tau_i \to \sum_j \tau_j \otimes u_{ji}$$

By applying Ψ to the relation $\tau_i \tau_j = \tau_j \tau_i$ we get $u^t u = 1$, so the matrix $u = (u_{ij})$ is orthogonal. By applying Ψ to the relation $\tau_i^2 = 1$ we get:

$$1 \otimes \sum_k u_{ki}^2 + \sum_{k<l} \tau_k \tau_l \otimes (u_{ki} u_{li} + u_{li} u_{ki}) = 1 \otimes 1$$

This gives $u_{ki} u_{li} = -u_{li} u_{ki}$ for $i \neq j$, $k \neq l$, and by using the antipode we get $u_{ik} u_{il} = -u_{il} u_{ik}$ for $k \neq l$. Also, by applying Ψ to $\tau_i \tau_j = \tau_j \tau_i$ with $i \neq j$ we get:

$$\sum_{k<l} \tau_k \tau_l \otimes (u_{ki} u_{lj} + u_{li} u_{kj}) = \sum_{k<l} \tau_k \tau_l \otimes (u_{kj} u_{li} + u_{lj} u_{ki})$$

Identifying coefficients, it follows that for $i \neq j$ and $k \neq l$, we have:

$$u_{ki}u_{lj} + u_{li}u_{kj} = u_{kj}u_{li} + u_{lj}u_{ki}$$

In other words, we have $[u_{ki}, u_{lj}] = [u_{kj}, u_{li}]$. By using the antipode we get:

$$[u_{jl}, u_{ik}] = [u_{il}, u_{jk}]$$

Now by combining these relations we get:

$$[u_{il}, u_{jk}] = [u_{ik}, u_{jl}] = [u_{jk}, u_{il}] = -[u_{il}, u_{jk}]$$

Thus $[u_{il}, u_{jk}] = 0$, so the elements u_{ij} satisfy the relations for $C(O_N^{-1})$, as desired. □

Many other things can be said about twists, reflection groups, and their actions on graphs, and for an introduction to this, we recommend [**12**], followed by [**81**].

10d. Small graphs

Generally speaking, the graphs having small number of vertices can be investigated by using product operations plus complementation. The first graph which is resistent to such a study is the torus graph $K_3 \times K_3$, but we have here, following [**8**]:

THEOREM 10.34. *The torus graph, obtained as a product of a triangle with itself,*

$$X = K_3 \times K_3$$

has no quantum symmetry, $G^+(X) = G(X) = S_3 \wr \mathbb{Z}_2$.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, we have $Sp(X) = \{-2, 1, 4\}$, the corresponding eigenspaces being as follows, with $\xi_{ij} = \xi^i \otimes \xi^j$, where $\xi = (1, w, w^2)$, with $w = e^{2\pi i/3}$:

$$E_{-2} = \mathbb{C}\xi_{10} \oplus \mathbb{C}\xi_{01} \oplus \mathbb{C}\xi_{20} \oplus \mathbb{C}\xi_{02}$$

$$E_1 = \mathbb{C}\xi_{11} \oplus \mathbb{C}\xi_{12} \oplus \mathbb{C}\xi_{21} \oplus \mathbb{C}\xi_{22}$$

$$E_4 = \mathbb{C}\xi_{00}$$

(2) Since the universal coaction $v : C(X) \to C(X) \otimes A$ preserves the eigenspaces, we can write formulae as follows, for some $a, b, c, d, \alpha, \beta, \gamma, \delta \in A$:

$$v(\xi_{10}) = \xi_{10} \otimes a + \xi_{01} \otimes b + \xi_{20} \otimes c + \xi_{02} \otimes d$$

$$v(\xi_{01}) = \xi_{10} \otimes \alpha + \xi_{01} \otimes \beta + \xi_{20} \otimes \gamma + \xi_{02} \otimes \delta$$

Taking the square of $v(\xi_{10})$ gives the following formula:

$$v(\xi_{20}) = \xi_{20} \otimes a^2 + \xi_{02} \otimes b^2 + \xi_{10} \otimes c^2 + \xi_{01} \otimes d^2$$

Also, from eigenspace preservation, we have the following relations:

$$ab = -ba \ , \ ad = -da \ , \ bc = -cb \ , \ cd = -dc$$

$$ac + ca = -(bd + db)$$

(3) Now since a,b anticommute, their squares have to commute. On the other hand, by applying v to the equality $\xi_{10}^*=\xi_{20}$, we get the following formulae for adjoints:

$$a^*=a^2\ ,\ b^*=b^2\ ,\ c^*=c^2\ ,\ d^*=d^2$$

The commutation relation $a^2b^2=b^2a^2$ reads now $a^*b^*=b^*a^*$, and by taking adjoints we get $ba=ab$. Together with $ab=-ba$ this gives:

$$ab=ba=0$$

The same method applies to ad,bc,cd, and we end up with:

$$ab=ba=0\ ,\ ad=da=0\ ,\ bc=cb=0\ ,\ cd=dc=0$$

(4) We apply now v to the equality $1=\xi_{10}\xi_{20}$. We get that 1 is the sum of 16 terms, all of them of the form $\xi_{ij}\otimes P$, where P are products between a,b,c,d and their squares. Due to the above formulae 8 terms vanish, and the 8 remaining ones give:

$$1=a^3+b^3+c^3+d^3$$

We have as well the relations coming from eigenspace preservation, namely:

$$ac^2=ca^2=bd^2=db^2=0$$

(5) Now from $ac^2=0$ we get $a^2c^2=0$, and by taking adjoints this gives $ca=0$. The same method applies to ac,bd,db, and we end up with:

$$ac=ca=0\ ,\ bd=db=0$$

In the same way we can show that $\alpha,\beta,\gamma,\delta$ pairwise commute:

$$\alpha\beta=\beta\alpha=\ldots=\gamma\delta=\delta\gamma=0$$

(6) In order to finish the proof, it remains to show that a,b,c,d commute with $\alpha,\beta,\gamma,\delta$. For this purpose, we apply v to the following equality:

$$\xi_{10}\xi_{01}=\xi_{01}\xi_{10}$$

We obtain in this way an equality between two sums having 16 terms each, and by using again the eigenspace preservation condition we get the following formulae relating the corresponding 32 products $a\alpha,\alpha a$, and so on:

$$\begin{aligned}
a\alpha=\alpha a=0 &\quad,\quad b\beta=\beta b=0\\
c\gamma=\gamma c=0 &\quad,\quad d\delta=\delta d=0\\
a\gamma+c\alpha+b\delta+d\beta=0 &\quad,\quad \alpha c+\gamma a+\beta d+\delta b=0\\
a\beta+b\alpha=\alpha b+\beta a &\quad,\quad b\gamma+c\beta=\beta c+\gamma b\\
c\delta+d\gamma=\gamma d+\delta c &\quad,\quad a\delta+d\alpha=\alpha d+\delta a
\end{aligned}$$

(7) Now observe that multiplying the first equality in the third row on the left by a and on the right by γ gives $a^2\gamma^2=0$, and by taking adjoints we get $\gamma a=0$. The

same method applies to the other 7 products involved in the third row, so all 8 products involved in the third row vanish. That is, we have the following formulae:

$$a\gamma = c\alpha = b\delta = d\beta = \alpha c = \gamma a = \beta d = \delta b = 0$$

(8) We use now the first equality in the fourth row. Multiplying it on the left by a gives $a^2\beta = a\beta a$, and multiplying it on the right by a gives $a\beta a = \beta a^2$. Thus we get $a^2\beta = \beta a^2$. On the other hand from $a^3 + b^3 + c^3 + d^3 = 1$ we get $a^4 = a$, so:

$$a\beta = a^4\beta = a^2a^2\beta = \beta a^2a^2 = \beta a$$

Finally, one can show in a similar manner that the missing commutation formulae $a\delta = \delta a$ and so on, hold as well. Thus the algebra A is commutative, as desired. □

As a second graph which is resistent to a routine product study, we have the Petersen graph P_{10}. In order to explain the computation here, done by Schmidt in [80], we will need a number of preliminaries. Let us start with the following notion, from [17]:

DEFINITION 10.35. *The reduced quantum automorphism group of X is given by*

$$C(G^*(X)) = C(G^+(X))\Big/\Big\langle u_{ij}u_{kl} = u_{kl}u_{ij}\Big|\forall i-k, j-l\Big\rangle$$

with $i - j$ standing as usual for the fact that i, j are connected by an edge.

As explained by Bichon in [17], the above construction produces indeed a quantum group $G^*(X)$, which sits as an intermediate subgroup, as follows:

$$G(X) \subset G^*(X) \subset G^+(X)$$

There are many things that can be said about this construction, but in what concerns us, we will rather use it as a technical tool. Following Schmidt [80], we have:

PROPOSITION 10.36. *Assume that a regular graph X is strongly regular, with parameters $\lambda = 0$ and $\mu = 1$, in the sense that:*

1. *$i - j$ implies that i, j have λ common neighbors.*
2. *$i \not- j$ implies that i, j have μ common neighbors.*

The quantum group inclusion $G^*(X) \subset G^+(X)$ is then an isomorphism.

PROOF. This is something quite tricky, the idea being as follows:

(1) First of all, regarding the statement, a graph is called regular, with valence k, when each vertex has exactly k neighbors. Then we have the notion of strong regularity, given by the conditions (1,2) in the statement. And finally we have the notion of strong regularity with parameters $\lambda = 0, \mu = 1$, that the statement is about, and with as main example here P_{10}, which is 3-regular, and strongly regular with $\lambda = 0, \mu = 1$.

(2) Regarding now the proof, we must prove that the following commutation relation holds, with u being the magic unitary of the quantum group $G^+(X)$:

$$u_{ij}u_{kl} = u_{kl}u_{ij} \ , \ \forall i-k, j-l$$

(3) But for this purpose, we can use the $\lambda = 0, \mu = 1$ strong regularity of our graph, by inserting some neighbors into our computation. To be more precise, we have:

$$\begin{aligned}
u_{ij}u_{kl} &= u_{ij}u_{kl}\sum_{s-l} u_{is}\\
&= u_{ij}u_{kl}u_{ij} + \sum_{s-l,s\neq j} u_{ij}u_{kl}u_{is}\\
&= u_{ij}u_{kl}u_{ij} + \sum_{s-l,s\neq j} u_{ij}\left(\sum_a u_{ka}\right)u_{is}\\
&= u_{ij}u_{kl}u_{ij} + \sum_{s-l,s\neq j} u_{ij}u_{is}\\
&= u_{ij}u_{kl}u_{ij}
\end{aligned}$$

(4) But this gives the result. Indeed, we conclude from this that $u_{ij}u_{kl}$ is self-adjoint, and so, by conjugating, that we have $u_{ij}u_{kl} = u_{kl}u_{ij}$, as desired. □

In the particular case of the Petersen graph P_{10}, which in addition is 3-regular, we can further build on the above result, and still following Schmidt [80], we have:

THEOREM 10.37. *The Petersen graph has no quantum symmetry,*

$$G^+(P_{10}) = G(P_{10}) = S_5$$

with S_5 acting in the obvious way.

PROOF. In view of Proposition 10.36, we must prove that the following commutation relation holds, with u being the magic unitary of the quantum group $G^+(P_{10})$:

$$u_{ij}u_{kl} = u_{kl}u_{ij} \ , \ \forall i \not- k, j \not- l$$

We can assume $i \neq k$, $j \neq l$. Now if we denote by s,t the unique vertices having the property $i-s, k-s$ and $j-t, l-t$, a routine study shows that we have:

$$u_{ij}u_{kl} = u_{ij}u_{st}u_{kl}$$

With this in hand, if we denote by q the third neighbor of t, we obtain:

$$\begin{aligned}
u_{ij}u_{kl} &= u_{ij}u_{st}u_{kl}(u_{ij} + u_{il} + u_{iq})\\
&= u_{ij}u_{st}u_{kl}u_{ij} + 0 + 0\\
&= u_{ij}u_{st}u_{kl}u_{ij}\\
&= u_{ij}u_{kl}u_{ij}
\end{aligned}$$

Thus the element $u_{ij}u_{kl}$ is self-adjoint, and we obtain, as desired:

$$u_{ij}u_{kl} = u_{kl}u_{ij}$$

As for the fact that the usual symmetry group is S_5, this is something that we know well from chapter 6, coming from the Kneser graph picture of P_{10}. □

As an application of this, we have the following classification table from [8], improved by using [80], containing all the vertex-transitive graphs of order ≤ 11 modulo complementation, with their classical and quantum symmetry groups:

Order	Graph	Classical group	Quantum group
2	K_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	K_3	S_3	S_3
4	$2K_2$	H_2	H_2^+
4	K_4	S_4	S_4^+
5	C_5	D_5	D_5
5	K_5	S_5	S_5^+
6	C_6	D_6	D_6
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$
6	$3K_2$	H_3	H_3^+
6	K_6	S_6	S_6^+
7	C_7	D_7	D_7
7	K_7	S_7	S_7^+
8	C_8, C_8^+	D_8	D_8
8	$P(C_4)$	H_3	$S_4^+ \times \mathbb{Z}_2$
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$	$S_4^+ \wr_* \mathbb{Z}_2$
8	$2C_4$	$H_2 \wr \mathbb{Z}_2$	$H_2^+ \wr_* \mathbb{Z}_2$
8	$4K_2$	H_4	H_4^+
8	K_8	S_8	S_8^+
9	C_9, C_9^3	D_9	D_9
9	$K_3 \times K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$
9	$3K_3$	$S_3 \wr S_3$	$S_3 \wr_* S_3$
9	K_9	S_9	S_9^+
10	C_{10}, C_{10}^2, C_{10}^+, $P(C_5)$	D_{10}	D_{10}
10	$P(K_5)$	$S_5 \times \mathbb{Z}_2$	$S_5^+ \times \mathbb{Z}_2$
10	C_{10}^4	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$	$D_5 \wr_* \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$	$S_5^+ \wr_* \mathbb{Z}_2$
10	$5K_2$	H_5	H_5^+
10	K_{10}	S_{10}	S_{10}^+
10	P_{10}	S_5	S_5
11	C_{11}, C_{11}^2, C_{11}^3	D_{11}	D_{11}
11	K_{11}	S_{11}	S_{11}^+

Here K denote the complete graphs, C the cycles with chords, and P stands for prisms. Moreover, by using more advanced techniques, the above table can be considerably extended. For more on all this, we refer to Schmidt's papers [**80**], [**81**], [**82**].

10e. Exercises

We had a quite technical algebraic chapter here, in the hope that you survived all this, and here are a few exercises, in relation with the above:

Exercise 10.38. *Clarify all the details in relation with free wreath products.*

Exercise 10.39. *Work out some basic applications of our usual product results.*

Exercise 10.40. *Work out basic applications of our lexicographic product results.*

Exercise 10.41. *Work out colored versions of our lexicographic product results.*

Exercise 10.42. *Clarify the easiness property of the complex reflection groups.*

Exercise 10.43. *Discuss the representations of the complex reflection groups.*

Exercise 10.44. *Discuss the character laws for the complex reflection groups.*

Exercise 10.45. *Come up with new twisting results, inspired by our O_N^{-1} result.*

As a bonus exercise, learn more about the usual complex reflection groups, and their classification into series and exceptional groups, from Shephard and Todd [**84**].

CHAPTER 11

Advanced results

11a. Orbits, orbitals

We have seen in chapter 10 how to develop the basic theory of quantum symmetry groups of graphs $G^+(X)$, with this corresponding more or less to the knowledge of the subject from the mid to the end 00s. Many things have happened since, and in the remainder of this book we will attempt to navigate the subject, with more on graphs in the remainder of this Part III, and with a look into quantum graphs too, in Part IV.

In the present chapter we would like to discuss a number of more advanced techniques for the computation of $G^+(X)$. More specifically, we would like to talk about:

(1) The orbits and orbitals of subgroups $G \subset S_N^+$, the point here being that an action $G \curvearrowright X$ is possible when the adjacency matrix of X is constant on the orbitals of G. This is a key observation of Lupini, Mančinska and Roberson [**66**], heavily used ever since.

(2) The study of dual quantum groups $\widehat{\Gamma} \subset S_N^+$, and their actions on graphs $\widehat{\Gamma} \curvearrowright X$, when Γ is finite, or classical, or arbitrary. Here the main ideas, which are actually related to the orbit and orbital problematics, are due to Bichon [**18**] and McCarthy [**70**].

(3) The quantum group actions $G \curvearrowright X$ on graphs which are circulant, $\mathbb{Z}_N \curvearrowright X$, or more generally which admit an action of a finite abelian group, $A \curvearrowright X$. There is a lot of Fourier magic here, first discussed in my paper with Bichon and Chenevier [**11**].

(4) The construction and main properties of the quantum semigroup of quantum partial isometries $\widetilde{G}^+(X) \subset \widetilde{S}_N^+$, in analogy with what we know about $\widetilde{G}(X) \subset \widetilde{S}_N$. This is something more recent, not yet truly developed, and believed to be of interest.

So, this will be the plan for the present chapter, obviously many interesting things to be discussed, and I can only imagine that you are quite excited about this. Unfortunately, there is no easy way of compacting 20 years of math into one chapter, and we have:

Disclaimer 11.1. *Contrary to the previous chapters, where new theory was accompanied by decent theorems using it, here we will be rather theoretical, explaining what is needed to know, in order to read the recent papers on the subject, and their theorems.*

And as a second disclaimer, this will be just half of the story, because we still have to talk afterwards about planar algebra methods, in chapter 12. Or perhaps one quarter, or one sixth of the story, because we have quantum graphs to talk about too, in Part IV.

But probably too much talking, leaving aside all this arithmetics of knowledge, let us get to work, according to our (1,2,3,4) plan above, for the present chapter.

To start with, a useful tool for the study of the permutation groups $G\subset S_N$ are the orbits of the action $G\curvearrowright\{1,\ldots,N\}$. In the quantum permutation group case, $G\subset S_N^+$, following Bichon [18], the orbits can be introduced as follows:

Theorem 11.2. *Given a closed subgroup $G\subset S_N^+$, with standard coordinates denoted $u_{ij}\in C(G)$, the following defines an equivalence relation on $\{1,\ldots,N\}$,*

$$i\sim j\iff u_{ij}\neq 0$$

that we call orbit decomposition associated to the action $G\curvearrowright\{1,\ldots,N\}$. In the classical case, $G\subset S_N$, this is the usual orbit equivalence.

Proof. We first check the fact that we have indeed an equivalence relation. The reflexivity axiom $i\sim i$ follows by using the counit, as follows:

$$\varepsilon(u_{ii})=1\implies u_{ii}\neq 0$$

The symmetry axiom $i\sim j\implies j\sim i$ follows by using the antipode:

$$S(u_{ji})=u_{ij}\implies [u_{ij}\neq 0\implies u_{ji}\neq 0]$$

As for the transitivity axiom $i\sim k,k\sim j\implies i\sim j$, this follows by using the comultiplication, and positivity. Consider indeed the following formula:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$$

On the right we have a sum of projections, and we obtain from this, as desired:

$$\begin{aligned}u_{ik}\neq 0,u_{kj}\neq 0&\implies u_{ik}\otimes u_{kj}>0\\&\implies \Delta(u_{ij})>0\\&\implies u_{ij}\neq 0\end{aligned}$$

Finally, in the classical case, where $G\subset S_N$, the standard coordinates are:

$$u_{ij}=\chi\left(\sigma\in G\Big|\sigma(j)=i\right)$$

Thus $u_{ij}\neq 0$ means that i,j must be in the same orbit, as claimed. □

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, also from [18]:

THEOREM 11.3. *Given a closed subgroup $G \subset S_N^+$, with magic matrix $u = (u_{ij})$, consider the associated coaction map, on the space $X = \{1, \ldots, N\}$:*

$$\Phi : C(X) \to C(X) \otimes C(G) \quad , \quad e_i \to \sum_j e_j \otimes u_{ji}$$

The following three subalgebras of $C(X)$ are then equal,

$$Fix(u) = \left\{\xi \in C(X)\Big| u\xi = \xi\right\}$$

$$Fix(\Phi) = \left\{\xi \in C(X)\Big|\Phi(\xi) = \xi \otimes 1\right\}$$

$$Fix(\sim) = \left\{\xi \in C(X)\Big| i \sim j \implies \xi_i = \xi_j\right\}$$

where $\sim$ is the orbit equivalence relation constructed in Theorem 11.2.

PROOF. The fact that we have $Fix(u) = Fix(\Phi)$ is standard, with this being valid for any corepresentation $u = (u_{ij})$. Indeed, we first have the following computation:

$$\begin{aligned}
\xi \in Fix(u) &\iff u\xi = \xi \\
&\iff (u\xi)_j = \xi_j, \forall j \\
&\iff \sum_i u_{ji}\xi_i = \xi_j, \forall j
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}
\xi \in Fix(\Phi) &\iff \Phi(\xi) = \xi \otimes 1 \\
&\iff \sum_i \Phi(e_i)\xi_i = \xi \otimes 1 \\
&\iff \sum_{ij} e_j \otimes u_{ji}\xi_i = \sum_j e_j \otimes \xi_j \\
&\iff \sum_i u_{ji}\xi_i = \xi_j, \forall j
\end{aligned}$$

Thus we have $Fix(u) = Fix(\Phi)$, as claimed. Regarding now the equality of this algebra with $Fix(\sim)$, observe first that given a vector $\xi \in Fix(\sim)$, we have:

$$\begin{aligned}
\sum_i u_{ji}\xi_i &= \sum_{i \sim j} u_{ji}\xi_i \\
&= \sum_{i \sim j} u_{ji}\xi_j \\
&= \sum_i u_{ji}\xi_j \\
&= \xi_j
\end{aligned}$$

Thus $\xi \in Fix(u) = Fix(\Phi)$. Finally, for the reverse inclusion, we know from Theorem 11.2 that the magic unitary $u = (u_{ij})$ is block-diagonal, with respect to the orbit decomposition there. But this shows that the algebra $Fix(u) = Fix(\Phi)$ decomposes as well with respect to the orbit decomposition, so in order to prove the result, we are left with a study in the transitive case. More specifically we must prove that if the action is transitive, then u is irreducible, and this being clear, we obtain the result. See [**18**]. $\square$

We have as well a useful analytic result, as follows:

THEOREM 11.4. *Given a closed subgroup $G \subset S_N^+$, the matrix*

$$P_{ij} = \int_G u_{ij}$$

is the orthogonal projection onto $Fix(\sim)$, and determines the orbits of $G \curvearrowright \{1, \ldots, N\}$.

PROOF. This follows from Theorem 11.3, and from the standard fact, coming from Peter-Weyl theory, that P is the orthogonal projection onto $Fix(u)$. $\square$

As an application of the above, let us discuss now the notion of transitivity. We have here the following result, once again coming from [**18**]:

THEOREM 11.5. *For a closed subgroup $G \subset S_N^+$, the following are equivalent:*

(1) *G is transitive, in the sense that $i \sim j$, for any i, j.*
(2) *$Fix(u) = \mathbb{C}\xi$, where ξ is the all-one vector.*
(3) *$\int_G u_{ij} = \frac{1}{N}$, for any i, j.*

In the classical case, $G \subset S_N$, this is the usual notion of transitivity.

PROOF. This is well-known in the classical case. In general, the proof is as follows:

(1) $\iff$ (2) We use the standard fact that the fixed point space of a corepresentation coincides with the fixed point space of the associated coaction:

$$Fix(u) = Fix(\Phi)$$

As explained in the beginning of this chapter, the fixed point space of the magic corepresentation $u = (u_{ij})$ has the following interpretation, in terms of orbits:

$$Fix(u) = \left\{ \xi \in C(X) \Big| i \sim j \implies \xi(i) = \xi(j) \right\}$$

In particular, the transitivity condition corresponds to $Fix(u) = \mathbb{C}\xi$, as stated.

(2) $\iff$ (3) This is clear from the general properties of the Haar integration, and more precisely from the fact that $(\int_G u_{ij})_{ij}$ is the projection onto $Fix(u)$. $\square$

Following Lupini, Mančinska and Roberson [**66**], let us discuss now the higher orbitals. Things are quite tricky here, and we have the following result, to start with:

THEOREM 11.6. *For a closed aubgroup $G\subset S_N^+$, with magic unitary $u=(u_{ij})$, and $k\in\mathbb N$, the relation*

$$(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)\iff u_{i_1j_1}\ldots u_{i_kj_k}\neq0$$

is reflexive and symmetric, and is transitive at $k=1,2$. In the classical case, $G\subset S_N$, this relation is transitive at any $k\in\mathbb N$, and is the usual k-orbital equivalence.

PROOF. This is known from [66], the proof being as follows:

(1) The reflexivity of $\sim$ follows by using the counit, as follows:

$$\begin{aligned}\varepsilon(u_{i_ri_r})=1,\forall r&\implies\varepsilon(u_{i_1i_1}\ldots u_{i_ki_k})=1\\&\implies u_{i_1i_1}\ldots u_{i_ki_k}\neq0\\&\implies(i_1,\ldots,i_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(2) The symmetry follows by applying the antipode, and then the involution:

$$\begin{aligned}(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)&\implies u_{i_1j_1}\ldots u_{i_kj_k}\neq0\\&\implies u_{j_ki_k}\ldots u_{j_1i_1}\neq0\\&\implies u_{j_1i_1}\ldots u_{j_ki_k}\neq0\\&\implies(j_1,\ldots,j_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(3) The transitivity at $k=1,2$ is more tricky. Here we need to prove that:

$$u_{i_1j_1}\ldots u_{i_kj_k}\neq0\ ,\ u_{j_1l_1}\ldots u_{j_kl_k}\neq0\implies u_{i_1l_1}\ldots u_{i_kl_k}\neq0$$

In order to do so, we use the following formula:

$$\Delta(u_{i_1l_1}\ldots u_{i_kl_k})=\sum_{s_1\ldots s_k}u_{i_1s_1}\ldots u_{i_ks_k}\otimes u_{s_1l_1}\ldots u_{s_kl_k}$$

At $k=1$, we already know this. At $k=2$ now, we can use the following trick:

$$\begin{aligned}(u_{i_1j_1}\otimes u_{j_1l_1})\Delta(u_{i_1l_1}u_{i_2l_2})(u_{i_2j_2}\otimes u_{j_2l_2})&=\sum_{s_1s_2}u_{i_1j_1}u_{i_1s_1}u_{i_2s_2}u_{i_2j_2}\otimes u_{j_1l_1}u_{s_1l_1}u_{s_2l_2}u_{j_2l_2}\\&=u_{i_1j_1}u_{i_2j_2}\otimes u_{j_1l_1}u_{j_2l_2}\end{aligned}$$

Indeed, we obtain from this the following implication, as desired:

$$u_{i_1j_1}u_{i_2j_2}\neq0,u_{j_1l_1}u_{j_2l_2}\neq0\implies u_{i_1l_1}u_{i_2l_2}\neq0$$

(4) Finally, assume that we are in the classical case, $G\subset S_N$. We have:

$$u_{ij}=\chi\left(\sigma\in G\Big|\sigma(j)=i\right)$$

But this formula shows that we have the following equivalence:

$$u_{i_1j_1}\ldots u_{i_kj_k}\neq0\iff\exists\sigma\in G,\ \sigma(i_1)=j_1,\ldots,\sigma(i_k)=j_k$$

In other words, $(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)$ happens precisely when $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ are in the same k-orbital of G, and this gives the last assertion. ☐

The above result raises the question about what exactly happens at $k=3$, in relation with transitivity, and the answer here is negative in general. To be more precise, as explained by McCarthy in [70], there are closed subgroups $G\subset S_N^+$, as for instance the Kac-Paljutkin quantum group $G\subset S_4^+$, for which $\sim$ is not transitive at $k=3$.

In view of all this, we can only formulate a modest definition, as follows:

DEFINITION 11.7. *Given a closed subgroup $G\subset S_N^+$, consider the relation defined by:*

$$(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)\iff u_{i_1j_1}\ldots u_{i_kj_k}\neq0$$

1. *At $k=1$, the equivalence classes of $\sim$ are called orbits of G.*
2. *At $k=2$, the equivalence classes of $\sim$ are called orbitals of G.*
3. *At $k\geq3$, if $\sim$ is transitive, we call its equivalence classes k-orbitals of G.*

It is possible to say more things here, but generally speaking, the good theory remains at $k=1,2$. In what follows we will focus on the case $k=2$, where $\sim$ is given by:

$$(i,k)\sim(j,l)\iff u_{ij}u_{kl}\neq0$$

As a key theoretical result on the subject, again from [66], we have the following key analogue of Theorem 11.3, which makes a connection with the graph problematics:

THEOREM 11.8. *Given a closed subgroup $G\subset S_N^+$, with magic matrix $u=(u_{ij})$, consider the following vector space coaction map, where $X=\{1,\ldots,N\}$:*

$$\Phi:C(X\times X)\to C(X\times X)\otimes C(G)\quad,\quad e_{ik}\to\sum_{jl}e_{jl}\otimes u_{ji}u_{lk}$$

The following three algebras are then isomorphic,

$$End(u)=\left\{d\in M_N(\mathbb C)\Big|du=ud\right\}$$

$$Fix(\Phi)=\left\{\xi\in C(X\times X)\Big|\Phi(\xi)=\xi\otimes1\right\}$$

$$Fix(\sim)=\left\{\xi\in C(X\times X)\Big|(i,k)\sim(j,l)\implies\xi_{ik}=\xi_{jl}\right\}$$

where $\sim$ is the orbital equivalence relation from Definition 11.7 (2).

PROOF. This follows by doing some computations which are quite similar to those from the proof of Theorem 11.3, and we refer here to [66], for the details. $\square$

As already mentioned, the above result makes a quite obvious connection with the graph problematics, the precise statement here being as follows:

THEOREM 11.9. *In order for a quantum permutation group $G\subset S_N^+$ to act on a graph X, having N vertices, the adjacency matrix $d\in M_N(0,1)$ of the graph must be, when regarded as function on the set $\{1,\ldots,N\}^2$, constant on the orbitals of G.*

PROOF. This follows indeed from the following isomorphism, from Theorem 11.8:

$$End(u) \simeq Fix(\sim)$$

For more on all this, details, examples, and applications too, we refer to [66]. □

Finally, as a theoretical application of the theory of orbitals, as developed above, let us discuss now the notion of double transitivity. Following [66], we have:

DEFINITION 11.10. *Let $G \subset S_N^+$ be a closed subgroup, with magic unitary $u = (u_{ij})$, and consider as before the equivalence relation on $\{1, \ldots, N\}^2$ given by:*

$$(i, k) \sim (j, l) \iff u_{ij} u_{kl} \neq 0$$

(1) *The equivalence classes under $\sim$ are called orbitals of G.*
(2) *G is called doubly transitive when the action has two orbitals.*

In other words, we call $G \subset S_N^+$ doubly transitive when $u_{ij} u_{kl} \neq 0$, for any $i \neq k, j \neq l$.

To be more precise, it is clear from definitions that the diagonal $D \subset \{1, \ldots, N\}^2$ is an orbital, and that its complement D^c must be a union of orbitals. With this remark in hand, the meaning of (2) is that the orbitals must be D, D^c.

In more analytic terms, we have the following result, also from [66]:

THEOREM 11.11. *For a doubly transitive subgroup $G \subset S_N^+$, we have:*

$$\int_G u_{ij} u_{kl} = \begin{cases} \frac{1}{N} & \text{if } i = k, j = l \\ 0 & \text{if } i = k, j \neq l \text{ or } i \neq k, j = l \\ \frac{1}{N(N-1)} & \text{if } i \neq k, j \neq l \end{cases}$$

Moreover, this formula characterizes the double transitivity.

PROOF. We use the standard fact, from [99], that the integrals in the statement form the projection onto $Fix(u^{\otimes 2})$. Now if we assume that G is doubly transitive, $Fix(u^{\otimes 2})$ has dimension 2, and therefore coincides with $Fix(u^{\otimes 2})$ for the usual symmetric group S_N. Thus the integrals in the statement coincide with those for the symmetric group S_N, which are given by the above formula. Finally, the converse is clear as well. □

So long for orbits, orbitals, and transitivity. As already mentioned, Theorem 11.9, which makes the connection with the actions on graphs $G \curvearrowright X$, is something quite far-reaching, and for the continuation of this, we refer to [66] and subsequent papers.

11b. Dual formalism

As another application of the orbit theory developed above, following Bichon [18], let us discuss now the group duals $\widehat{\Gamma} \subset S_N^+$. We first have the following result:

THEOREM 11.12. *Given a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, we have an embedding $\widehat{\Gamma} \subset S_N^+$, with $N = N_1 + \ldots + N_k$, having the following properties:*

1. *This embedding appears by diagonally joining the embeddings $\widehat{\mathbb{Z}_{N_k}} \subset S_{N_k}^+$, and the corresponding magic matrix has blocks of sizes $N_1, \ldots, N_k$.*
2. *The equivalence relation on $X = \{1, \ldots, N\}$ coming from the orbits of the action $\widehat{\Gamma} \curvearrowright X$ appears by refining the partition $N = N_1 + \ldots + N_k$.*

PROOF. This is something elementary, the idea being as follows:

(1) Given a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, we have indeed a standard embedding as follows, with $N = N_1 + \ldots + N_k$, that we actually know well since chapter 9:

$$\begin{aligned}
\widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}} = \widehat{\mathbb{Z}_{N_1}} \,\hat{*}\, \ldots \,\hat{*}\, \widehat{\mathbb{Z}_{N_k}} \\
&\simeq \mathbb{Z}_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{N_k} \subset S_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k} \\
&\subset S_{N_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k}^+ \subset S_N^+
\end{aligned}$$

(2) Regarding the magic matrix, our claim is that this is as follows, $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$ with $w_N = e^{2\pi i/N}$ being Fourier matrices, and g_l being the standard generator of $\mathbb{Z}_{N_l}$:

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* & & \\ & \ddots & \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix} \quad , \quad I_l = \begin{pmatrix} 1 & & & \\ & g_l & & \\ & & \ddots & \\ & & & g_l^{N_l - 1} \end{pmatrix}$$

(3) Indeed, let us recall that the magic matrix for $\mathbb{Z}_N \subset S_N \subset S_N^+$ is given by:

$$v_{ij} = \chi\left(\sigma \in \mathbb{Z}_N \middle| \sigma(j) = i\right) = \delta_{i-j}$$

Let us apply now the Fourier transform. According to our usual Pontrjagin duality conventions, we have a pair of inverse isomorphisms, as follows:

$$\Phi : C(\mathbb{Z}_N) \to C^*(\mathbb{Z}_N) \quad , \quad \delta_i \to \frac{1}{N} \sum_k w^{ik} g^k$$

$$\Psi : C^*(\mathbb{Z}_N) \to C(\mathbb{Z}_N) \quad , \quad g^i \to \sum_k w^{-ik} \delta_k$$

Here $w=e^{2\pi i/N}$, and we use the standard Fourier analysis convention that the indices are $0,1,\ldots,N-1$. With $F=\frac{1}{\sqrt{N}}(w^{ij})$ and $I=diag(g^i)$ as above, we have:

$$\begin{aligned} u_{ij} &= \Phi(v_{ij}) \\ &= \frac{1}{N}\sum_k w^{(i-j)k}g^k \\ &= \frac{1}{N}\sum_k w^{ik}g^kw^{-jk} \\ &= (FIF^*)_{ij} \end{aligned}$$

Thus, the magic matrix that we are looking for is $u=FIF^*$, as claimed.

(4) Finally, the second assertion in the statement is clear from the fact that u is block-diagonal, with blocks corresponding to the partition $N=N_1+\ldots+N_k$. $\square$

As a first comment on the above result, not all group dual subgroups $\widehat{\Gamma}\subset S_N^+$ appear as above, a well-known counterexample here being the Klein group:

$$K=\mathbb{Z}_2\times\mathbb{Z}_2\subset S_4\subset S_4^+$$

Indeed, with $K=\{1,a,b,c\}$, where $c=ab$, consider the embedding $K\subset S_4$ given by $a=(12)(34)$, $b=(13)(24)$, $c=(14)(23)$. The corresponding magic matrix is:

$$u=\begin{pmatrix}\delta_1&\delta_a&\delta_b&\delta_c\\ \delta_a&\delta_1&\delta_c&\delta_b\\ \delta_b&\delta_c&\delta_1&\delta_a\\ \delta_c&\delta_b&\delta_a&\delta_1\end{pmatrix}\in M_4(C(K))$$

Now since this matrix is not block-diagonal, the only choice for $K=\widehat{K}$ to appear as in Theorem 11.12 would be via a quotient map $\mathbb{Z}_4\to K$, which is impossible. As a second comment now on Theorem 11.12, in the second assertion there we really have a possible refining operation, as shown by the example provided by the trivial group, namely:

$$\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}\to\{1\}$$

In order to further discuss all this, let us first enlarge the attention to the group dual subgroups $\widehat{\Gamma}\subset G$ of an arbitrary closed subgroup $G\subset U_N^+$. We have here:

THEOREM 11.13. *Given a closed subgroup $G\subset U_N^+$ and a matrix $Q\in U_N$, we let $T_Q\subset G$ be the diagonal torus of G, with fundamental representation spinned by Q:*

$$C(T_Q)=C(G)\Big/\left\langle(QuQ^*)_{ij}=0\Big|\forall i\neq j\right\rangle$$

This torus is then a group dual, $T_Q=\widehat{\Lambda}_Q$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=(QuQ^*)_{ii}$, which are unitaries inside $C(T_Q)$.

PROOF. Since $v = QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i = v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q) = C^*(\Lambda_Q)$, where $\Lambda_Q =< g_1, \ldots, g_N >$ is the group in the statement, and this gives the result. $\square$

Generally speaking, the above family $\{T_Q | Q \in U_N\}$ can be thought of as being a kind of "maximal torus" for $G \subset U_N^+$. Now back to quantum permutations, we have:

THEOREM 11.14. *For the quantum permutation group S_N^+, the discrete group quotient $F_N \to \Lambda_Q$ with $Q \in U_N$ comes from the following relations:*

$$\begin{cases} g_i = 1 & \text{if } \sum_l Q_{il} \neq 0 \\ g_i g_j = 1 & \text{if } \sum_l Q_{il} Q_{jl} \neq 0 \\ g_i g_j g_k = 1 & \text{if } \sum_l Q_{il} Q_{jl} Q_{kl} \neq 0 \end{cases}$$

Also, given a decomposition $N = N_1 + \ldots + N_k$, for the matrix $Q = diag(F_{N_1}, \ldots, F_{N_k})$, where $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$ with $\xi = e^{2\pi i/N}$ is the Fourier matrix, we obtain

$$\Lambda_Q = \mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$$

with dual embedded into S_N^+ in a standard way, as in Theorem 11.12.

PROOF. This can be proved by a direct computation, as follows:

(1) Fix a unitary matrix $Q \in U_N$, and consider the following quantities:

$$\begin{cases} c_i = \sum_l Q_{il} \\ c_{ij} = \sum_l Q_{il} Q_{jl} \\ d_{ijk} = \sum_l \bar{Q}_{il} \bar{Q}_{jl} Q_{kl} \end{cases}$$

We write $w = QvQ^*$, where v is the fundamental corepresentation of $C(S_N^+)$. Assume $X \simeq \{1, \ldots, N\}$, and let α be the coaction of $C(S_N^+)$ on $C(X)$. Let us set:

$$\varphi_i = \sum_l \bar{Q}_{il} \delta_l \in C(X)$$

Also, let $g_i = (QvQ^*)_{ii} \in C^*(\Lambda_Q)$. If β is the restriction of α to $C^*(\Lambda_Q)$, then:

$$\beta(\varphi_i) = \varphi_i \otimes g_i$$

(2) Now recall that $C(X)$ is the universal C^*-algebra generated by elements $\delta_1, \ldots, \delta_N$ which are pairwise orthogonal projections. Writing these conditions in terms of the linearly independent elements φ_i by means of the formulae $\delta_i = \sum_l Q_{il} \varphi_l$, we find that the

universal relations for $C(X)$ in terms of the elements φ_i are as follows:

$$\begin{cases}\sum_i c_i\varphi_i = 1\\ \varphi_i^* = \sum_j c_{ij}\varphi_j\\ \varphi_i\varphi_j = \sum_k d_{ijk}\varphi_k\end{cases}$$

(3) Let $\widetilde{\Lambda}_Q$ be the group in the statement. Since β preserves these relations, we get:

$$\begin{cases}c_i(g_i-1)=0\\ c_{ij}(g_ig_j-1)=0\\ d_{ijk}(g_ig_j-g_k)=0\end{cases}$$

We conclude from this that Λ_Q is a quotient of $\widetilde{\Lambda}_Q$. On the other hand, it is immediate that we have a coaction map as follows:

$$C(X)\to C(X)\otimes C^*(\widetilde{\Lambda}_Q)$$

Thus $C(\widetilde{\Lambda}_Q)$ is a quotient of $C(S_N^+)$. Since w is the fundamental corepresentation of S_N^+ with respect to the basis $\{\varphi_i\}$, it follows that the generator w_{ii} is sent to $\widetilde{g}_i\in\widetilde{\Lambda}_Q$, while w_{ij} is sent to zero. We conclude that $\widetilde{\Lambda}_Q$ is a quotient of Λ_Q. Since the above quotient maps send generators on generators, we conclude that $\Lambda_Q=\widetilde{\Lambda}_Q$, as desired.

(4) We apply the result found in (3), with the N-element set X there being:

$$X=\mathbb{Z}_{N_1}\sqcup\ldots\sqcup\mathbb{Z}_{N_k}$$

With this choice, we have $c_i=\delta_{i0}$ for any i. Also, we have $c_{ij}=0$, unless i,j,k belong to the same block to Q, in which case $c_{ij}=\delta_{i+j,0}$, and also $d_{ijk}=0$, unless i,j,k belong to the same block of Q, in which case $d_{ijk}=\delta_{i+j,k}$. We conclude from this that Λ_Q is the free product of k groups which have generating relations as follows:

$$g_ig_j=g_{i+j}\quad,\quad g_i^{-1}=g_{-i}$$

But this shows that our group is $\Lambda_Q=\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}$, as stated. □

In relation now with actions on graphs, let us start with the following simple fact:

Proposition 11.15. *In order for a closed subgroup $G\subset U_K^+$ to appear as $G=G^+(X)$, for a certain graph X having N vertices, the following must happen:*

1. *We must have a representation $G\subset U_N^+$.*
2. *This representation must be magic, $G\subset S_N^+$.*
3. *We must have a graph X having N vertices, such that $d\in End(u)$.*
4. *X must be in fact such that the Tannakian category of G is precisely $<d>$.*

Proof. This is more of an empty statement, coming from the definition of the quantum automorphism group $G^+(X)$, as formulated in chapter 10. □

The above result remains something quite philosophical. In the group dual case, that we will be interested in now, we can combine it with the following result:

PROPOSITION 11.16. *Given a discrete group $\Gamma =< g_1,\ldots,g_N >$, embed diagonally $\widehat{\Gamma}\subset U_N^+$, via the unitary matrix $u = diag(g_1,\ldots,g_N)$. We have then the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=\left\{T=(T_{j_1\ldots j_l,i_1\ldots i_k})\Big|g_{i_1}\ldots g_{i_k}\neq g_{j_1}\ldots g_{j_l}\implies T_{j_1\ldots j_l,i_1\ldots i_k}=0\right\}$$

and in particular, with $k = l = 1$, we have the formula

$$End(u)=\left\{T=(T_{ji})\Big|g_i\neq g_j\implies T_{ji}=0\right\}$$

with the linear maps being identified with the corresponding scalar matrices.

PROOF. This is indeed elementary, with the first assertion coming from:

$$\begin{aligned}
T\in Hom(u^{\otimes k},u^{\otimes l})&\iff Tu^{\otimes k}=u^{\otimes l}T\\
&\iff (Tu^{\otimes k})_{j_1\ldots j_l,i_1\ldots i_k}=(u^{\otimes l}T)_{j_1\ldots j_l,i_1\ldots i_k}\\
&\iff T_{j_1\ldots j_l,i_1\ldots i_k}g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}T_{j_1\ldots j_l,i_1\ldots i_k}\\
&\iff T_{j_1\ldots j_l,i_1\ldots i_k}(g_{i_1}\ldots g_{i_k}-g_{j_1}\ldots g_{j_l})=0
\end{aligned}$$

As for the second assertion, this follows from the first one. □

Still in the group dual setting, we have now, refining Proposition 11.15:

THEOREM 11.17. *In order for a group dual $\widehat{\Gamma}$ as above to appear as $G = G^+(X)$, for a certain graph X having N vertices, the following must happen:*

1. *First, we need a quotient map $\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}\to\Gamma$.*
2. *Let $u = diag(I_1,\ldots,I_k)$, with $I_l = diag(\mathbb{Z}_{N_l})$, for any l.*
3. *Consider also the matrix $F = diag(F_{N_1},\ldots,F_{N_k})$.*
4. *We must then have a graph X having N vertices.*
5. *This graph must be such that $F^*dF\neq 0\implies I_i = I_j$.*
6. *In fact, $< F^*dF >$ must be the category in Proposition 11.16.*

PROOF. This is something rather informal, which follows from the above. □

Going ahead, in connection with the Fourier tori from Theorem 11.14, we have:

PROPOSITION 11.18. *The Fourier tori of $G^+(X)$ are the biggest quotients*

$$\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}\to\Gamma$$

whose duals act on the graph, $\widehat{\Gamma}\curvearrowright X$.

PROOF. We have indeed the following computation, at $F = 1$:

$$\begin{aligned} C(T_1(G^+(X))) &= C(G^+(X))/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < [d,u] = 0 >]/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < u_{ij} = 0, \forall i \neq j >]/ < [d,u] = 0 > \\ &= C(T_1(S_N^+))/ < [d,u] = 0 > \end{aligned}$$

Thus, we obtain the result, with the remark that the quotient that we are interested in appears via relations of type $d_{ij} = 1 \implies g_i = g_j$. The proof in general is similar. □

In order to formulate our main result, let us call $G \subset G^+$ a Fourier liberation when G^+ is generated by G, and its Fourier tori. With this convention, we have:

THEOREM 11.19. *Consider the following conditions:*

(1) *We have $G(X) = G^+(X)$.*
(2) *$G(X) \subset G^+(X)$ is a Fourier liberation.*
(3) *$\widehat{\Gamma} \curvearrowright X$ implies that Γ is abelian.*

We have then the equivalence $(1) \iff (2) + (3)$.

PROOF. This is something elementary. Indeed, the implications $(1) \implies (2,3)$ are trivial. As for $(2,3) \implies (1)$, assuming $G(X) \neq G^+(X)$, from (2) we know that $G^+(X)$ has at least one non-classical Fourier torus, and this contradicts (3), as desired. □

As an application of this, we have the following elementary result, which is a particular case of more general and advanced results regarding the random graphs, from [66]:

THEOREM 11.20. *For a finite graph X, the probability for having an action*

$$\widehat{\Gamma} \curvearrowright X$$

with Γ being a non-abelian group goes to 0 with $|X| \to \infty$.

PROOF. Observe first that in the cyclic case, where $F = F_N$ is a usual Fourier matrix, associated to a cyclic group $\mathbb{Z}_N$, we have the following formula, with $w = e^{2\pi i/N}$:

$$(F^*dF)_{ij} = \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} = \sum_{kl} w^{lj-ik} d_{kl} = \sum_{k \sim l} w^{lj-ik}$$

In the general setting now, where we have a quotient map $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N_1 + \ldots + N_k = N$, the computation is similar, as follows, with $w_i = e^{2\pi i/N_i}$:

$$(F^*dF)_{ij} = \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} = \sum_{k \sim l} (F^*)_{ik} F_{lj} = \sum_{k:i, l:j, k \sim l} (w_{N_i})^{-ik} (w_{N_j})^{lj}$$

The point now is that with the partition $N_1 + \ldots + N_k = N$ fixed, and with $d \in M_N(0,1)$ being random, we have $(F^*dF)_{ij} \neq 0$ almost everywhere in the $N \to \infty$ limit, and so we obtain $I_i = I_j$ almost everywhere, and so abelianity of Γ, with $N \to \infty$. □

Many other things can be said, as a continuation of the above, and we refer here to [66], [70] for an introduction, and to the recent literature on the subject, for more.

11c. Circulant graphs

Changing topics, but still obsessed by Fourier analysis, let us discuss now, following [11], some sharp results in the circulant graph case. Let us start with:

DEFINITION 11.21. *Associated to any circulant graph X having N vertices are:*
1. *The set $S\subset\mathbb{Z}_N$ given by $i-_X j\iff j-i\in S$.*
2. *The group $E\subset\mathbb{Z}_N^*$ consisting of elements g such that $gS=S$.*
3. *The number $k=|E|$, called type of X.*

The interest in the type k is that this is the good parameter measuring the complexity of the spectral theory of X, as we will soon see. To start with, here are a few basic examples, and properties of the type, with φ being the Euler function:

(1) The type can be $2,4,6,8,\ldots$ This is because $\{\pm1\}\subset E$.

(2) C_N is of type 2. Indeed, we have $S=\{\pm1\}$, $E=\{\pm1\}$.

(3) K_N is of type $\varphi(N)$. Indeed, here $S=\emptyset$, $E=\mathbb{Z}_N^*$.

Let us first discuss the spectral theory of the circulant graphs. In what follows X will be a circulant graph having p vertices, with p prime. We denote by ξ the column vector $(1,w,w^2,\ldots,w^{p-1})$, where $w=e^{2\pi i/p}$. With this convention, we have:

THEOREM 11.22. *The eigenspaces of d are given by $V_0=\mathbb{C}1$ and*

$$V_x=\bigoplus_{a\in E}\mathbb{C}\,\xi^{xa}$$

with $x\in\mathbb{Z}_p^*$. Moreover, we have $V_x=V_y$ if and only if $xE=yE$.

PROOF. Since d is circulant, we have $d(\xi^x)=f(x)\xi^x$, with $f:\mathbb{Z}_p\to\mathbb{C}$ being:

$$f(x)=\sum_{t\in S}w^{xt}$$

Let $K=\mathbb{Q}(w)$ and let H be the Galois group of the Galois extension $\mathbb{Q}\subset K$. We have then a well-known group isomorphism, as follows:

$$\mathbb{Z}_p^*\simeq H\quad,\quad x\to s_x=[w\to w^x]$$

Also, we know from a standard theorem of Dedekind that the family $\{s_x\mid x\in\mathbb{Z}_p^*\}$ is free in $\mathrm{End}_{\mathbb{Q}}(K)$. Now for $x,y\in\mathbb{Z}_p^*$ consider the following operator:

$$L=\sum_{t\in S}s_{xt}-\sum_{t\in S}s_{yt}\in End_{\mathbb{Q}}(K)$$

We have $L(w) = f(x) - f(y)$, and since L commutes with the action of H, we have:

$$L = 0 \iff L(w) = 0 \iff f(x) = f(y)$$

By linear independence of the family $\{s_x \mid x \in \mathbb{Z}_p^*\}$ we get:

$$f(x) = f(y) \iff xS = yS \iff xE = yE$$

It follows that d has precisely $1 + (p-1)/k$ distinct eigenvalues, the corresponding eigenspaces being those in the statement. □

Consider now a commutative ring $(R, +, \cdot)$. We denote by R^* the group of invertibles, and we assume $2 \in R^*$. A subgroup $G \subset R^*$ is called even if $-1 \in G$. We have:

Definition 11.23. *An even subgroup $G \subset R^*$ is called 2-maximal if, inside G:*

$$a - b = 2(c - d) \implies a = \pm b$$

We call $a = b, c = d$ trivial solutions, and $a = -b = c - d$ hexagonal solutions.

To be more precise, in what regards our terminology, consider the group $G \subset \mathbb{C}$ formed by k-th roots of unity, with k even. An equation of the form $a - b = 2(c - d)$ with $a, b, c, d \in G$ says that the diagonals $a - b$ and $c - d$ must be parallel, and that the first one is twice as much as the second one. But this can happen only when a, c, d, b are consecutive vertices of a regular hexagon, and here we have $a + b = 0$.

The relation with our quantum symmetry considerations will come from:

Proposition 11.24. *Assume that R has the property $3 \neq 0$, and consider a 2-maximal subgroup $G \subset R^*$. Then, the following happen:*

1. *$2, 3 \notin G$.*
2. *$a + b = 2c$ with $a, b, c \in G$ implies $a = b = c$.*
3. *$a + 2b = 3c$ with $a, b, c \in G$ implies $a = b = c$.*

Proof. All these assertions are elementary, as follows:

(1) This follows from the following formulae, which cannot hold in G:

$$4 - 2 = 2(2 - 1) \quad , \quad 3 - (-1) = 2(3 - 1)$$

Indeed, the first one would imply $4 = \pm 2$, and the second one would imply $3 = \pm 1$. But from $2 \in R^*$ and $3 \neq 0$ we get $2, 4, 6 \neq 0$, contradiction.

(2) We have $a - b = 2(c - b)$. For a trivial solution we have $a = b = c$, and for a hexagonal solution we have $a + b = 0$, hence $c = 0$, hence $0 \in G$, contradiction.

(3) We have $a - c = 2(c - b)$. For a trivial solution we have $a = b = c$, and for a hexagonal solution we have $a + c = 0$, hence $b = -2a$, hence $2 \in G$, contradiction. □

As a first result now, coming from this study, we have:

THEOREM 11.25. *A circulant graph X, on $p \geq 5$ prime vertices, such that*

$$E \subset \mathbb{Z}_p$$

is 2-maximal, has no quantum symmetry, $G^+(X) = G(X)$.

PROOF. This comes from the above results, via a long algebraic study, as follows:

(1) We use Proposition 11.24, which ensures that V_1, V_2, V_3 are eigenspaces of d. By 2-maximality of E, these eigenspaces are different. From the eigenspace preservation in Theorem 11.22 we get formulae of the following type, with $r_a, r'_a, r''_a \in \mathcal{A}$:

$$\alpha(\xi) = \sum_{a \in E} \xi^a \otimes r_a \quad , \quad \alpha(\xi^2) = \sum_{a \in E} \xi^{2a} \otimes r'_a \quad , \quad \alpha(\xi^3) = \sum_{a \in E} \xi^{3a} \otimes r''_a$$

(2) We take the square of the first relation, we compare with the formula of $\alpha(\xi^2)$, and we use 2-maximality. We obtain in this way the following formula:

$$\alpha(\xi^2) = \sum_{c \in E} \xi^{2c} \otimes r_c^2$$

(3) We multiply now this relation by the formula of $\alpha(\xi)$, we compare with the formula of $\alpha(\xi^3)$, and we use 2-maximality. We obtain the following formula:

$$\alpha(\xi^3) = \sum_{b \in E} \xi^{3b} \otimes r_b^3$$

(4) As a conclusion, the three formulae in the beginning are in fact as follows:

$$\alpha(\xi) = \sum_{a \in E} \xi^a \otimes r_a \quad , \quad \alpha(\xi^2) = \sum_{a \in E} \xi^{2a} \otimes r_a^2 \quad , \quad \alpha(\xi^3) = \sum_{a \in E} \xi^{3a} \otimes r_a^3$$

(5) Our claim now is that for $a \neq b$, we have the following "key formula":

$$r_a r_b^3 = 0$$

Indeed, in order to prove this claim, consider the following equality:

$$\left(\sum_{a \in E} \xi^a \otimes r_a\right)\left(\sum_{b \in E} \xi^{2b} \otimes r_b^2\right) = \sum_{c \in E} \xi^{3c} \otimes r_c^3$$

By eliminating all $a = b$ terms, which produce the sum on the right, we get:

$$\sum \left\{ r_a r_b^2 \Big| a, b \in E,\ a \neq b,\ a + 2b = x \right\} = 0$$

(6) We fix now elements $a, b \in E$ satisfying $a \neq b$. We know from 2-maximality that the equation $a + 2b = a' + 2b'$ with $a', b' \in E$ has at most one non-trivial solution, namely the hexagonal one, given by $a' = -a$ and $b' = a + b$. Now with $x = a + 2b$, we get that the above equality is in fact one of the following two equalities:

$$r_a r_b^2 = 0 \quad , \quad r_a r_b^2 + r_{-a} r_{a+b}^2 = 0$$

(7) In the first case, we are done. In the second case, we know that $a_1 = b$ and $b_1 = a + b$ are distinct elements of E. So, consider the following equation, over E:

$$a_1 + 2b_1 = a_1' + 2b_1'$$

The hexagonal solution of this equation, given by $a_1' = -a_1$ and $b_1' = a_1 + b_1$, cannot appear, because $b_1' = a_1 + b_1$ can be written as $b_1' = a + 2b$, and by 2-maximality we get $b_1' = -a = b$, which contradicts $a + b \in E$. Thus the equation $a_1 + 2b_1 = a_1' + 2b_1'$ with $a_1', b_1' \in E$ has only trivial solutions, and with $x = a_1 + 2b_1$ in the above, we get:

$$r_{a_1} r_{b_1}^2 = 0$$

Now remember that this follows by identifying coefficients in $\alpha(\xi)\alpha(\xi^2) = \alpha(\xi^3)$. The same method applies to the formula $\alpha(\xi^2)\alpha(\xi) = \alpha(\xi^3)$, and we get as well:

$$r_{b_1}^2 r_{a_1} = 0$$

We have now all ingredients for finishing the proof of the key formula, as follows:

$$r_a r_b^3 = r_a r_b^2 r_b = -r_{-a} r_{a+b}^2 r_b = -r_{-a} r_{b_1}^2 r_{a_1} = 0$$

(8) We come back now to the following formula, proved for $s = 1, 2, 3$:

$$\alpha(\xi^s) = \sum_{a \in E} \xi^{sa} \otimes r_a^s$$

By using the key formula, we get by recurrence on s that this holds in general.

(9) In particular with $s = p - 1$ we get the following formula:

$$\alpha(\xi^{-1}) = \sum_{a \in E} \xi^{-a} \otimes r_a^{p-1}$$

On the other hand, from $\xi^* = \xi^{-1}$ we get the following formula:

$$\alpha(\xi^{-1}) = \sum_{a \in E} \xi^{-a} \otimes r_a^*$$

But this gives $r_a^* = r_a^{p-1}$ for any a. Now by using the key formula we get:

$$(r_a r_b)(r_a r_b)^* = r_a r_b r_b^* r_a^* = r_a r_b^p r_a^* = (r_a r_b^3)(r_b^{p-3} r_a^*) = 0$$

(10) But this gives $r_a r_b = 0$. Thus, we have the following equalities:

$$r_a r_b = r_b r_a = 0$$

On the other hand, $\mathcal{A}$ is generated by coefficients of α, which are in turn powers of elements r_a. It follows that $\mathcal{A}$ is commutative, and we are done. □

Still following [**11**], we can now formulate a main result, as follows:

THEOREM 11.26. *A type k circulant graph having $p >> k$ vertices, with p prime, has no quantum symmetry.*

PROOF. This follows from Theorem 11.25 and some arithmetics, as follows:

(1) Let k be an even number, and consider the group of k-th roots of unity $G = \{1, w, \ldots, w^{k-1}\}$, where $w = e^{2\pi i/k}$. By standard arithmetics, G is 2-maximal in $\mathbb{C}$.

(2) As a continuation of this, again by some standard arithmetics, for $p > 6^{\varphi(k)}$, with φ being the Euler function, any subgroup $E \subset \mathbb{Z}_p^*$ of order k is 2-maximal.

(3) But this proves our result. Indeed, by using (2), we can apply Theorem 11.25 provided that we have $p > 6^{\varphi(k)}$, and our graph has no quantum symmetry, as desired. □

We should mention that the above result, from [**11**], is quite old. The challenge is to go beyond this, with results for the graphs having an abelian group action, $A \curvearrowright X$.

11d. Partial symmetries

As a last topic for this chapter, let us discuss the construction and main properties of the quantum semigroup $\widetilde{G}^+(X) \subset \widetilde{S}_N^+$, in analogy with what we know about $\widetilde{G}(X) \subset \widetilde{S}_N$. This is something more recent, and potentially interesting too. Let us start with:

DEFINITION 11.27. *$C(\widetilde{S}_N^+)$ is the universal C^*-algebra generated by the entries of a $N \times N$ submagic matrix u, with comultiplication and counit maps given by*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij}$$

where "submagic" means formed of projections, which are pairwise orthogonal on rows and columns. We call $\widetilde{S}_N^+$ semigroup of quantum partial permutations of $\{1, \ldots, N\}$.

Observe that the morphisms Δ, ε constructed above satisfy the usual axioms for a comultiplication and an antipode, in the bialgebra setting, namely:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

As a conclusion to this, the basic properties of the quantum semigroup $\widetilde{S}_N^+$ that we constructed in Definition 11.27 can be summarized as follows:

PROPOSITION 11.28. *We have maps as follows,*

$$\begin{matrix} C(S_N^+) & \leftarrow & C(\widetilde{S}_N^+) \\ \downarrow & & \downarrow \\ C(S_N) & \leftarrow & C(\widetilde{S}_N) \end{matrix} \quad : \quad \begin{matrix} S_N^+ & \subset & \widetilde{S}_N^+ \\ \cup & & \cup \\ S_N & \subset & \widetilde{S}_N \end{matrix}$$

with the bialgebras at left corresponding to the quantum semigroups at right.

PROOF. This is clear from the above discussion, and from the well-known fact that projections which sum up to 1 are pairwise orthogonal. □

We recall from chapter 9 that we have $S_N^+ \neq S_N$, starting from $N = 4$. At the semigroup level things get interesting starting from $N = 2$, where we have:

PROPOSITION 11.29. *We have an isomorphism as follows,*

$$C(\widetilde{S}_2^+) \simeq \left\{(x,y) \in C^*(D_\infty) \oplus C^*(D_\infty)\Big|\varepsilon(x) = \varepsilon(y)\right\}$$

where $\varepsilon : C^*(D_\infty) \to \mathbb{C}1$ is the counit, given by the formula

$$u = \begin{pmatrix} p \oplus 0 & 0 \oplus r \\ 0 \oplus s & q \oplus 0 \end{pmatrix}$$

where p, q and r, s are the standard generators of the two copies of $C^*(D_\infty)$.

PROOF. Consider an arbitrary 2×2 matrix formed by projections:

$$u = \begin{pmatrix} P & R \\ S & Q \end{pmatrix}$$

This matrix is submagic when the following conditions are satisfied:

$$PR = PS = QR = QS = 0$$

Now observe that these conditions tell us that the non-unital algebras $X =< P, Q >$ and $Y =< R, S >$ must commute, and must satisfy $xy = 0$, for any $x \in X, y \in Y$. Thus, if we denote by Z the universal algebra generated by two projections, we have:

$$C(\widetilde{S}_2^+) \simeq \mathbb{C}1 \oplus Z \oplus Z$$

Now since we have $C^*(D_\infty) = \mathbb{C}1 \oplus Z$, we obtain an isomorphism as follows:

$$C(\widetilde{S}_2^+) \simeq \left\{(\lambda + a, \lambda + b)\Big|\lambda \in \mathbb{C}, a, b \in Z\right\}$$

Thus, we are led to the conclusion in the statement. □

We recall now from chapter 5 that given a graph X with N vertices, and adjacency matrix $d \in M_N(0,1)$, its partial automorphism semigroup is given by:

$$\widetilde{G}(X) = \left\{\sigma \in \widetilde{S}_N\Big|d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i, j \in Dom(\sigma)\right\}$$

We also have the following formula, from chapter 7, with $R = diag(R_i)$, $C = diag(C_j)$, with R_i, C_j being the row and column sums of the associated submagic matrix u:

$$C(\widetilde{G}(X)) = C(\widetilde{S}_N)\Big/\Big\langle R(du - ud)C = 0\Big\rangle$$

With these results in hand, we are led to the following statement:

THEOREM 11.30. *The following construction, with R, C being the diagonal matrices formed by the row and column sums of u, produces a subsemigroup $\widetilde{G}^+(X) \subset \widetilde{S}_N^+$,*

$$C(\widetilde{G}^+(X)) = C(\widetilde{S}_N^+) \Big/ \Big\langle R(du - ud)C = 0 \Big\rangle$$

called semigroup of quantum partial automorphisms of X, whose classical version is $\widetilde{G}(X)$. When using $du = ud$, we obtain a semigroup $\bar{G}^+(X) \subset \widetilde{G}^+(X)$ which can be smaller.

PROOF. All this is elementary, the idea being as follows:

(1) In order to construct the comultiplication Δ, consider the following elements:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

By using the fact that u is submagic, we deduce that we have:

$$R_i^U (dU)_{ij} C_j^U = \Delta(R_i(du)_{ij}C_j)$$

$$R_i^U (Ud)_{ij} C_j^U = \Delta(R_i(ud)_{ij}C_j)$$

Thus we can define Δ by mapping $u_{ij} \to U_{ij}$, as desired.

(2) Regarding now ε, the algebra in the statement has indeed a morphism ε defined by $u_{ij} \to \delta_{ij}$, because the following relations are trivially satisfied:

$$R_i(d1_N)_{ij}C_j = R_i(1_N d)_{ij}C_j$$

(3) Regarding now S, we must prove that we have a morphism S given by $u_{ij} \to u_{ji}$. For this purpose, we know that with $R = diag(R_i)$ and $C = diag(C_j)$, we have:

$$R(du - ud)C = 0$$

Now when transposing this formula, we obtain:

$$C^t(u^t d - du^t)R^t = 0$$

Since C^t, R^t are respectively the diagonal matrices formed by the row sums and column sums of u^t, we conclude that the relations $R(du - ud)C = 0$ are satisfied by the transpose matrix u^t, and this gives the existence of the subantipode map S.

(4) The fact that we have $\widetilde{G}^+(X)_{class} = \widetilde{G}(X)$ follows from $(S_N^+)_{class} = S_N$.

(5) Finally, the last assertion follows from our similar results from chapter 7, by taking classical versions, the simplest counterexample being the simplex. □

As a first result now regarding the correspondence $X \to \widetilde{G}^+(X)$, we have:

PROPOSITION 11.31. *For any finite graph X we have*

$$\widetilde{G}^+(X) = \widetilde{G}^+(X^c)$$

where X^c is the complementary graph.

PROOF. The adjacency matrices of a graph X and of its complement X^c are related by the following formula, where $\mathbb{I}_N$ is the all-1 matrix:

$$d_X + d_{X^c} = \mathbb{I}_N - 1_N$$

Thus, in order to establish the formula in the statement, we must prove that:

$$R_i(\mathbb{I}_N u)_{ij} C_j = R_i(u\mathbb{I}_N)_{ij} C_j$$

For this purpose, let us recall that, the matrix u being submagic, its row sums and column sums R_i, C_j are projections. By using this fact, we have:

$$R_i(\mathbb{I}_N u)_{ij} C_j = R_i C_j C_j = R_i C_j$$

$$R_i(u\mathbb{I}_N)_{ij} C_j = R_i R_i C_j = R_i C_j$$

Thus we have proved our equality, and the conclusion follows. $\square$

In order to discuss now various aspects of the correspondence $X \to \widetilde{G}^+(X)$, it is technically convenient to slightly enlarge our formalism, as follows:

DEFINITION 11.32. *Associated to any complex-colored oriented graph X, with adjacency matrix $d \in M_N(\mathbb{C})$, is its semigroup of partial automorphisms, given by*

$$\widetilde{G}(X) = \left\{ \sigma \in \widetilde{S}_N \middle| d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i,j \in Dom(\sigma) \right\}$$

as well as its quantum semigroup of quantum partial automorphisms, given by

$$C(\widetilde{G}^+(X)) = C(\widetilde{S}_N^+) \Big/ \Big\langle R(du - ud)C = 0 \Big\rangle$$

where $R = diag(R_i)$, $C = diag(C_j)$, with R_i, C_j being the row and column sums of u.

With this notion in hand, following the material in chapter 10, let us discuss now the color independence. Let m, γ be the multiplication and comultiplication of $\mathbb{C}^N$:

$$m(e_i \otimes e_j) = \delta_{ij} e_i \quad , \quad \gamma(e_i) = e_i \otimes e_i$$

We denote by $m^{(p)}, \gamma^{(p)}$ their iterations, given by the following formulae:

$$m^{(p)}(e_{i_1} \otimes \ldots \otimes e_{i_1}) = \delta_{i_1 \ldots i_p} e_{i_1} \quad , \quad \gamma^{(p)}(e_i) = e_i \otimes \ldots \otimes e_i$$

Our goal is to use these iterations in the semigroup case, exactly as we did in chapter 10, in the quantum group case. We will need some technical results. Let us start with:

PROPOSITION 11.33. *We have the following formulae,*

$$m^{(p)} u^{\otimes p} = u m^{(p)} \quad , \quad u^{\otimes p} \gamma^{(p)} = \gamma^{(p)} u$$

valid for any submagic matrix u.

PROOF. We have the following computations, which prove the first formula:

$$m^{(p)}u^{\otimes p}(e_{i_1}\otimes\ldots\otimes e_{i_p})=\sum_j e_j\otimes u_{ji_1}\ldots u_{ji_p}=\delta_{i_1\ldots i_p}\sum_j e_j\otimes u_{ji_1}$$

$$um^{(p)}(e_{i_1}\otimes\ldots\otimes e_{i_p})=\delta_{i_1\ldots i_p}u(e_{i_1})=\delta_{i_1\ldots i_p}\sum_j e_j\otimes u_{ji_1}$$

We have as well the following computations, which prove the second formula:

$$u^{\otimes p}\gamma^{(p)}(e_i)=u^{\otimes p}(e_i\otimes\ldots\otimes e_i)=\sum_j e_j\otimes\ldots\otimes e_j\otimes u_{ji}$$

$$\gamma^{(p)}u(e_i)=\gamma^{(p)}\left(\sum_j e_j\otimes u_{ji}\right)=\sum_j e_j\otimes\ldots\otimes e_j\otimes u_{ji}$$

Summarizing, we have proved both formulae in the statement. □

We will need as well a second technical result, as follows:

PROPOSITION 11.34. *We have the following formulae, with u,m,γ being as before,*

$$m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)}=Rd^{\times p}\quad,\quad m^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)}=d^{\times p}C$$

and with $\times$ being the componentwise, or Hadamard, product of matrices.

PROOF. We have the following computations, which prove the first formula:

$$m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)}(e_i)=m^{(p)}R^{\otimes p}d^{\otimes p}(e_i\otimes\ldots\otimes e_i)=\sum_j e_j\otimes R_jd_{ji}^p$$

$$Rd^{\times p}(e_i)=R\left(\sum_j e_j\otimes d_{ji}^p\right)=\sum_j e_j\otimes R_jd_{ji}^p$$

We have as well the following computations, which prove the second formula:

$$m^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)}(e_i)=m^{(p)}d^{\otimes p}(e_i\otimes\ldots\otimes e_i\otimes C_i)=\sum_j e_j\otimes d_{ji}^pC_i$$

$$d^{\times p}C(e_i)=d^{\times p}(e_i\otimes C_i)=\sum_j e_j\otimes d_{ji}^pC_i$$

Thus, we have proved both formulae in the statement. □

We can now prove a key result, as follows:

PROPOSITION 11.35. *We have the following formulae, with u,m,γ being as before,*

$$m^{(p)}(Rdu)^{\otimes p}\gamma^{(p)}=Rd^{\times p}u\quad,\quad m^{(p)}(udC)^{\otimes p}\gamma^{(p)}=ud^{\times p}C$$

and with $\times$ being the componentwise product of matrices.

PROOF. By using the formulae in Propositions 11.33 and 11.34, we get:

$$\begin{aligned} m^{(p)}(Rdu)^{\otimes p}\gamma^{(p)} &= m^{(p)}R^{\otimes p}d^{\otimes p}u^{\otimes p}\gamma^{(p)} \\ &= m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)}u \\ &= Rd^{\times p}u \end{aligned}$$

Once again by using Proposition 11.33 and Proposition 11.34, we have:

$$\begin{aligned} m^{(p)}(udC)^{\otimes p}\gamma^{(p)} &= m^{(p)}u^{\otimes p}d^{\otimes p}C^{\otimes p}\gamma^{(p)} \\ &= um^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)} \\ &= ud^{\times p}C \end{aligned}$$

Thus, we have proved both formulae in the statement. □

We can now prove the color independence result, as follows:

THEOREM 11.36. *The quantum semigroups of quantum partial isomorphisms of finite graphs are subject to the "independence on the colors" formula*

$$\left[d_{ij} = d_{kl} \iff d'_{ij} = d'_{kl}\right] \implies \widetilde{G}^+(X) = \widetilde{G}^+(X')$$

valid for any graphs X, X', having adjacency matrices d, d'.

PROOF. Given a matrix $d \in M_N(\mathbb{C})$, consider its color decomposition, which is as follows, with the color components d_c being by definition 0-1 matrices:

$$d = \sum_{c \in \mathbb{C}} c \cdot d_c$$

We want to prove that a given quantum semigroup G acts on (X, d) if and only if it acts on (X, d_c), for any $c \in \mathbb{C}$. For this purpose, consider the following linear space:

$$E_u = \left\{ f \in M_N(\mathbb{C}) \Big| Rfu = ufC \right\}$$

In terms of this space, we want to prove that we have:

$$d \in E_u \implies d_c \in E_u, \forall c \in \mathbb{C}$$

For this purpose, observe that we have the following implication, as a consequence of the formulae established in Proposition 11.35:

$$Rdu = udC \implies Rd^{\times p}u = ud^{\times p}C$$

We conclude that we have the following implication:

$$d \in E_u \implies d^{\times p} \in E_u, \forall p \in \mathbb{N}$$

But this gives the result, exactly as in chapter 10, via the standard fact that the color components d_c can be obtained from the componentwise powers $d^{\times p}$. □

In contrast with what happens for the groups or quantum groups, in the semigroup setting we do not have a spectral decomposition result as well. To be more precise, consider as before the following linear space, associated to a submagic matrix u:

$$E_u=\left\{d\in M_N(\mathbb C)\Big|Rdu=udC\right\}$$

It is clear that E_u is a linear space, containing 1, and stable under the adjoint operation $*$ too. We also know from Theorem 11.36 that E_u is stable under color decomposition. However, E_u is not stable under taking products, and so is not an algebra, in general.

In general, the computation of $\widetilde{G}^+(X)$ remains a very interesting question. Interesting as well is the question of generalizing all this to the infinite graph case, $|X|=\infty$, with the key remark that this might be simpler than talking about $G^+(X)$ with $|X|=\infty$.

11e. Exercises

We had a quite technical chapter, with a lot of tricky noncommutative algebra, and as exercises we have, and no surprise here, more noncommutative algebra, as follows:

Exercise 11.37. *Learn about the Kac-Paljutkin quantum group.*

Exercise 11.38. *Learn about exotic subgroups $S_N\subset G\subset S_N^+$, and their orbitals.*

Exercise 11.39. *Learn more about random graphs, in the classical case.*

Exercise 11.40. *Learn about quantum information aspects of all the above.*

Exercise 11.41. *Clarify the arithmetic input that we used for circulant graphs.*

Exercise 11.42. *Extend our results on circulant graphs, using abelian groups.*

Exercise 11.43. *Compute some further quantum semigroups of type $\widetilde{G}^+(X)$.*

Exercise 11.44. *Construct a quantum semigroup $\widetilde{G}^+(X)$, when X is infinite.*

As bonus exercise, read in detail the papers [66] and [82]. These will bring your knowledge to the state of the art of the subject, as of the late 2010s.

CHAPTER 12

Planar algebras

12a. Operator algebras

We discuss here a question that we met several times in this book, in relation with our graph theory investigations, be them classical of quantum, namely the computation of the algebra generated by the adjacency matrix $d \in M_N(0,1)$ of a graph X, under the operations consisting in taking the spectral decomposition, and the color decomposition. This question makes in fact sense for any complex matrix, as follows:

QUESTION 12.1. *What is the algebra generated by a matrix $d \in M_N(\mathbb{C})$,*

$$\lhd\, d \rhd = ?$$

with respect to the spectral decomposition, and the color decomposition?

Here we use the above symbols in the lack of something known and standard, regarding this seemingly alien operation. However, as we will soon discover, there is nothing that alien regarding that operation, which is in fact something very familiar in operator algebras, knot theory, quantum field theory, and many more, namely the operation which consists in computing the associated planar algebra, in the sense of Jones [56].

So, this will be our plan, and the whole discussion will bring us on a long trip into modern mathematics, featuring advanced operator algebras, advanced linear algebra, and other advanced things, quite often with a flavor of modern topology and modern physics too, and as a bonus, we will reach to an answer to the following question too:

QUESTION 12.2. *What is the general mathematical theory behing the advanced topics that we saw in Part I, namely walks on ADE graphs, and knot invariants?*

With the answer to this latter question being, and you guessed right, again planar algebras in the sense of Jones [56]. But probably enough advertisement, let us get to work. We will need some basic von Neumann algebra theory, coming as a complement to the basic C^*-algebra theory developed in chapter 9, and we have here:

THEOREM 12.3. *For a $*$-algebra of operators $A \subset B(H)$ the following conditions are equivalent, and if satisfied, we say that A is a von Neumann algebra:*

(1) *A is closed under the weak topology, making each $T \to Tx$ continuous.*
(2) *A equals its bicommutant, $A = A''$, computed inside $B(H)$.*

Proof. This is von Neumann's bicommutant theorem, that we actually invoked a few times already, when talking Tannakian duality, in its finite dimensional particular case, which is elementary, with the discussion here, and then the proof, being as follows:

(1) As a first comment, the weak topology on $B(H)$, making each $T \to Tx$ with $x \in H$ continuous, is indeed weaker than the norm topology, in the sense that we have:

$$||T_n - T|| \to 0 \implies ||T_n x - Tx|| \to 0, \forall x \in H$$

In particular, we see that a von Neumann algebra in the sense of (1), that is, closed under the weak topology, must be a C^*-algebra, that is, closed under the norm.

(2) Before getting further, let us see if the converse of this fact is true. This is certainly true in finite dimensions, $H = \mathbb{C}^N$, where we have $B(H) = M_N(\mathbb{C})$, and where the operator $*$-algebras $A \subset B(H)$ are as follows, automatically closed both for the norm topology, and the weak topology, and with these two topologies actually coinciding:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

(3) In infinite dimensions, however, things change. Indeed, let us first take a look at the most basic examples of commutative C^*-algebras that we know, the commutative ones. These naturally appear from compact measured spaces X, as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to [g \to fg]$$

(4) But, it is pretty much clear that such an algebra will not be weakly closed, unless X is discrete, with the details here being left to you. So, in infinite dimensions, there are far less von Neumann algebras than C^*-algebras, with this being good to know.

(5) Still talking about this, the following natural question appears, what happens if we take the weak closure of the algebra $C(X) \subset B(L^2(X))$ considered above? And the answer here, obtained via some basic measure theory and functional analysis, that we will leave as an exercise, is that we obtain the following algebra:

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \to [g \to fg]$$

(6) But this is quite interesting, because forgetting now about C^*-algebras, what we have here is a nice method of producing von Neumann algebras, in the weakly closed sense, and with the measured space X being no longer required to be compact.

(7) As a conclusion to all this, "von Neumann algebras have something to do with measure theory, in the same way as C^*-algebras have something to do with topology". Which sounds quite deep, so good, and time to stop here. More on this later.

(8) Hang on, we are not done yet with the preliminaries, because all the above was in relation with the condition (1) in the statement, and we still have the condition (2) in the statement to comment on. So, here we go again, with a basic exploration, of that

condition. To start with, given a subalgebra $A \subset B(H)$, or even a subset $A \subset B(H)$, we can talk about its commutant inside $B(H)$, constructed as follows:

$$A' = \left\{ T \in B(H) \middle| TS = ST, \forall S \in A \right\}$$

Now if we take the commutant A'' of this commutant A', it is obvious that the elements of the original algebra or set A will be inside. Thus, we have an inclusion as follows:

$$A \subset A''$$

(9) The question is now, why $A = A''$ should be equivalent to A being weakly closed, and why should we care about this? These are both good questions, so let us start with the first one. As a first observation, in finite dimensions the bicommutant condition is automatic, because with $A \subset M_N(\mathbb{C})$ being as in (2) above, its commutant is:

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

But now, by taking again the commutant, we obtain the original algebra A:

$$A'' = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

(10) Moving now to infinite dimensions, the first thought goes into taking the commutant of the basic examples of C^*-algebras, $C(X) \subset B(L^2(X))$. But here, up to some mesure theory and functional analysis work, that we will leave as an exercise, we are led to the following conclusion, which proves the bicommutant theorem in this case:

$$C(X)'' = L^\infty(X)$$

(11) Summarizing, we have some intuition on the condition $A = A''$ from the statement, and we can also say, based on the above, that the method for proving the bicommutant theorem would be that of establishing the following equality, for any $*$-subalgebra $A \subset B(H)$, with on the right being the closure with respect to the weak topology:

$$A'' = \overline{A}^w$$

(12) Before getting to work, however, we still have a question to be answered, namely, why should we care about all this? I mean, the condition (1) in the statement, weak closedness, looks very nice and mathematical, that would be a good axiom for the von Neumann algebras, so why bothering with commutants, and with the condition (2).

(13) In answer, at the elementary level, and with my apologies for calling these damn things "elementary", we have seen in chapters 8-9, when struggling with Tannakian duality, that the bicommutant operation and theorem can be something very useful.

(14) In answer too, at the advanced level now, in abstract quantum mechanics the vectors of the Hilbert space $x \in H$ are the states of the system, and the linear self-adjoint operators $T : H \to H$ are the observables, and taking the commutant of a set or algebra of observables is something extremely natural. And this is how von Neumann came upon such things, back in the 1930s, and looking now retrospectively, we can even say that his

bicommutant theorem is not only important in the context of quantum mechanics, but even "makes abstract quantum mechanics properly work". So, in short, trust me, with the present bicommutant theorem we are into first-class mathematics and physics.

(15) Time perhaps for the proof? We recall from (11) that we would like to prove the following equality, for any $*$-algebra of operators $A \subset B(H)$:

$$A'' = \overline{A}^w$$

(16) Let us first prove $\supset$. Since we have $A \subset A''$, we just have to prove that A'' is weakly closed. But, assuming $T_i \to T$ weakly, we have indeed:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS, \ \forall S \in A' \\ &\implies ST = TS, \ \forall S \in A' \\ &\implies T \in A \end{aligned}$$

(17) Let us prove now $\subset$. Here we must establish the following implication:

$$T \in A'' \implies T \in \overline{A}^w$$

For this purpose, we use an amplification trick. Consider indeed the Hilbert space K obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

(18) The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \Big| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Big| S_{ij} \in A' \right\}$$

(19) We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix}T&&\\&\ddots&\\&&T\end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

(20) Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any vector $x \in K$, we conclude that any operator $T \in A''$ belongs to the weak closure of A. Thus, we have $A'' \subset \overline{A}^w$, as desired. $\square$

Very nice all this, but as you can see, the von Neumann algebras are far more subtle objects than the C^*-algebras, and their proper understanding, even at the very basic level, is a far more complicated business than what we quickly did in chapter 9, for the C^*-algebras. Welcome to the real quantum, the quantum mechanics one.

Moving ahead, the continuation of the story involves an accumulation of non-trivial results, due to Murray and von Neumann, from the 1930s and 1940s, and then due to Connes, much later, in the 1970s, the conclusions being as follows:

Theorem 12.4. *The von Neumann algebras are as follows:*

(1) *In the commutative case, these are the algebras $A = L^\infty(X)$, with X measured space, represented on $H = L^2(X)$, up to a multiplicity.*
(2) *If we write the center as $Z(A) = L^\infty(X)$, then we have a decomposition of type $A = \int_X A_x\,dx$, with the fibers A_x having trivial center, $Z(A_x) = \mathbb{C}$.*
(3) *The factors, $Z(A) = \mathbb{C}$, can be fully classified in terms of II_1 factors, which are those satisfying $\dim A = \infty$, and having a faithful trace $tr : A \to \mathbb{C}$.*

Proof. This is something quite heavy, the idea being as follows:

(1) As already discussed above, it is clear that $L^\infty(X)$ is indeed a von Neumann algebra on $H = L^2(X)$. The converse can be proved as well, by using spectral theory, one way of viewing this being by saying that, given a commutative von Neumann algebra $A \subset B(H)$, its elements $T \in A$ are commuting normal operators, so the Spectral Theorem for such operators applies, and gives $A = L^\infty(X)$, for some measured space X.

(2) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis. To be more precise, in finite dimensions this is something that we know well, with the formula $A=\int_X A_x\,dx$ corresponding to our usual direct sum decomposition, namely:

$$A=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

In infinite dimensions, things are more complicated, but the idea remains the same, namely using (1) for the commutative von Neumann algebra $Z(A)$, as to get a measured space X, and then making your way towards a decomposition of type $A=\int_X A_x\,dx$.

(3) This is something fairly heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. To be more precise, the various type of factors can be classified as follows:

– Type I. These are the matrix algebras $M_N(\mathbb{C})$, called of type I_N, and their infinite generalization, $B(H)$ with H infinite dimensional, called of type I_∞. Although these factors are very interesting and difficult mathematical objects, from the perspective of the general von Neumann algebra classification work, they are dismissed as "trivial".

– Type II. These are the infinite dimensional factors having a trace, which is a usual trace $tr:A\to\mathbb{C}$ in the type II_1 case, and is something more technical, possibly infinite, in the remaining case, the type II_∞ one, with these latter factors being of the form $B(H)\otimes A$, with A being a II_1 factor, and with H being an infinite dimensional Hilbert space.

– Type III. These are the factors which are infinite dimensional, and do not have a trace $tr:A\to\mathbb{C}$. Murray and von Neumann struggled a lot with such beasts, with even giving an example being a non-trivial task, but later Connes came and classified them, basically showing that they appear from II_1 factors, via crossed product constructions. $\square$

So long for basic, or rather advanced but foundational, von Neumann algebra theory. In what follows we will focus on the II_1 factors, according to the following principle:

Principle 12.5. *The building blocks of the von Neumann algebra theory are the* II_1 *factors, which are the von Neumann algebras having the following properties:*

(1) *They are infinite dimensional,* $\dim A=\infty$.
(2) *They are factors, their center being* $Z(A)=\mathbb{C}$.
(3) *They have a faithful trace* $tr:A\to\mathbb{C}$.

But you might perhaps ask, is it even clear that such beasts exist? Good point, and in answer, given a discrete group Γ, you can talk about its von Neumann algebra, obtained by talking the weak closure of the usual group algebra, or group C^*-algebra:

$$L(\Gamma)\subset B(l^2(\Gamma))$$

This algebra is then infinite dimensional when Γ is infinite, and also has a trace, given on group elements by $tr(g)=\delta_{g1}$. As for the center, this consists of the functions on Γ

which are constant on the conjugacy classes, so when Γ has infinite conjugacy classes, called ICC property, what we have is a factor. Thus, as a conclusion, when Γ is infinite and has the ICC property, its von Neumann algebra $L(\Gamma)$ is a II_1 factor.

Let us summarize this finding, along with a bit more, as follows:

THEOREM 12.6. *We have the following examples of* II_1 *factors:*

(1) *The group von Neumann algebras $L(\Gamma)$, with Γ being an infinite group, having the infinite conjugacy class (ICC) property.*
(2) *The Murray-von Neumann hyperfinite factor $R = \overline{\cup_k M_{n_k}(\mathbb{C})}^{\,w}$, with the limit being independent on the summands, and on the inclusions between them.*
(3) *With the remark that when Γ as above is assumed to be amenable, its associated* II_1 *factor $L(\Gamma)$ is the Murray-von Neumann hyperfinite factor R.*

PROOF. Here the first assertion comes from the above discussion, and the rest, regarding the factor R, is due to Murray and von Neumann, using standard functional analysis. With the remark however that the notion of hyperfiniteness can be plugged into the general considerations from Theorem 12.4, and with the resulting questions, which are of remarkable difficulty, having been solved only relatively recently, basically by Connes in the 1970s, and with a last contribution by Haagerup in the 1980s, the general idea being that, in the end, everything hyperfinite can be reconstructed from R. $\square$

Many other things can be said, along these lines, and if truly interested in theoretical physics, be that quantum mechanics, or statistical mechanics, or other, have a look at all this, von Neumann algebras, this is first-class mathematical technology.

12b. Subfactor theory

In view of Principle 12.5, and its quantum mechanics ramifications, it looks reasonable to forget about the Hilbert space H, about operators $T \in B(H)$, about other von Neumann algebras and factors $A \subset B(H)$ that might appear, about other mathematics and physics too, why not about your friends, spouse and hobbies too, but please keep teaching some calculus, that is first class mathematics, and focus on the II_1 factors.

With this idea in mind, we have our objects, the II_1 factors, but what about morphisms. And here, a natural idea is that of looking at the inclusions of such factors:

DEFINITION 12.7. *A subfactor is an inclusion of* II_1 *factors $A_0 \subset A_1$.*

So, these will be the objects that we will be interested in, in what follows. With the comment that, while quantum mechanics and von Neumann algebras have been around for a while, since the 1920s, and Definition 12.7 is something very natural emerging from this, it took mathematics and physics a lot of time to realize this, with Definition 12.7

basically dating back to the late 1970s, with the beginning of the work of Jones, on it. Moral of the story, sometimes it takes a lot of skill, to come up with simple things.

Now given a subfactor $A_0 \subset A_1$, a first question is that of defining its index, measuring how big A_1 is, when compared to A_0. But this can be done as follows:

THEOREM 12.8. *Given a subfactor $A_0 \subset A_1$, the number*

$$N = \frac{\dim_{A_0} H}{\dim_{A_1} H}$$

is independent of the ambient Hilbert space H, and is called index.

PROOF. This is something quite standard, the idea being as follows:

(1) To start with, given a representation of a II_1 factor $A \subset B(H)$, we can talk about the corresponding coupling constant, as being a number as follows:

$$\dim_A H \in (0, \infty]$$

To be more precise, we can construct this coupling constant in the following way, with $u : H \to L^2(A) \otimes l^2(\mathbb{N})$ being an isometry satisfying $ux = (x \otimes 1)u$:

$$\dim_A H = tr(uu^*)$$

(2) Alternatively, we can use the following formula, after proving first that the number on the right is indeed independent of the choice on a nonzero vector $x \in H$:

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

This latter formula was in fact the original definition of the coupling constant, by Murray and von Neumann. However, technically speaking, it is better to use (1).

(3) Now with this in hand, given a subfactor $A_0 \subset A_1$, the fact that the index as defined above is indeed independent of the ambient Hilbert space H comes from the various basic properties of the coupling constant, established by Murray and von Neumann. $\square$

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. In general now, following Jones [**51**], let us start with:

PROPOSITION 12.9. *Given a subfactor $A_0 \subset A_1$, there is a unique linear map*

$$E : A_1 \to A_0$$

which is positive, unital, trace-preserving and which is such that, for any $a_1, a_2 \in A_0$:

$$E(a_1 b a_2) = a_1 E(b) a_2$$

This map is called conditional expectation from A_1 onto A_0.

Proof. We make use of the standard representation of the II_1 factor A_1, with respect to its unique trace $tr : A_1 \to \mathbb{C}$, namely:

$$A_1 \subset L^2(A_1)$$

If we denote by Ω the standard cyclic and separating vector of $L^2(A_1)$, we have an identification of vector spaces $A_0\Omega = L^2(A_0)$. Consider now the following projection:

$$e : L^2(A_1) \to L^2(A_0)$$

It follows from definitions that we have an inclusion $e(A_1\Omega) \subset A_0\Omega$. Thus the above projection e induces by restriction a certain linear map, as follows:

$$E : A_1 \to A_0$$

This linear map E and the orthogonal projection e are related by:

$$exe = E(x)e$$

But this shows that the linear map E satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map E as in the statement must come from a projection e. $\square$

We will be interested in what follows in the orthogonal projection $e : L^2(A_1) \to L^2(A_0)$ producing the expectation $E : A_1 \to A_0$, rather than in E itself:

Definition 12.10. *Associated to any subfactor $A_0 \subset A_1$ is the orthogonal projection*

$$e : L^2(A_1) \to L^2(A_0)$$

producing the conditional expectation $E : A_1 \to A_0$ via the following formula:

$$exe = E(x)e$$

This projection is called Jones projection for the subfactor $A_0 \subset A_1$.

Quite remarkably, the subfactor $A_0 \subset A_1$, as well as its commutant, can be recovered from the knowledge of this projection, in the following way:

Proposition 12.11. *Given a subfactor $A_0 \subset A_1$, with Jones projection e, we have*

$$A_0 = A_1 \cap \{e\}' \quad , \quad A_0' = (A_1' \cap \{e\})''$$

as equalities of von Neumann algebras, acting on the space $L^2(A_1)$.

Proof. The above two formulae both follow from $exe = E(x)e$, via some elementary computations, and for details here, we refer to Jones' paper [51]. $\square$

We are now ready to formulate a key definition, as follows:

Definition 12.12. *Associated to any subfactor $A_0\subset A_1$ is the basic construction*

$$A_0\subset_e A_1\subset A_2$$

with $A_2=<A_1,e>$ being the algebra generated by A_1 and by the Jones projection

$$e:L^2(A_1)\to L^2(A_0)$$

acting on the Hilbert space $L^2(A_1)$.

The idea now, following as before Jones [51], will be that the inclusion $A_1\subset A_2$ appears as a kind of "reflection" of the original inclusion $A_0\subset A_1$, and also that the basic construction can be iterated, with all this leading to non-trivial results. We first have:

Proposition 12.13. *Given a subfactor $A_0\subset A_1$ having finite index,*

$$[A_1:A_0]<\infty$$

the basic construction $A_0\subset_e A_1\subset A_2$ has the following properties:

(1) $A_2=JA_0'J$.
(2) $A_2=\overline{A_1+A_1eb}$.
(3) A_2 *is a* II_1 *factor.*
(4) $[A_2:A_1]=[A_1:A_0]$.
(5) $eA_2e=A_0e$.
(6) $tr(e)=[A_1:A_0]^{-1}$.
(7) $tr(xe)=tr(x)[A_1:A_0]^{-1}$, *for any $x\in A_1$.*

Proof. All this is standard, by using the same type of mathematics as in the proof of Proposition 12.9, and for details here, we refer to Jones' paper [51]. □

Let us perform now twice the basic construction, and see what we get. The result here, which is something more technical, at least at the first glance, is as follows:

Proposition 12.14. *Associated to $A_0\subset A_1$ is the double basic construction*

$$A_0\subset_e A_1\subset_f A_2\subset A_3$$

with $e:L^2(A_1)\to L^2(A_0)$ and $f:L^2(A_2)\to L^2(A_1)$ having the following properties:

$$fef=[A_1:A_0]^{-1}f\quad,\quad efe=[A_1:A_0]^{-1}e$$

Proof. We have two formulae to be proved, the idea being as follows:

(1) The first formula in the statement is clear, because we have:

$$fef=E(e)f=tr(e)f=[A_1:A_0]^{-1}f$$

(2) Regarding now the second formula, it is enough to check this on the dense subset $(A_1+A_1eA_1)\Omega$. Thus, we must show that for any $x,y,z\in A_1$, we have:

$$efe(x+yez)\Omega=[A_1:A_0]^{-1}e(x+yez)\Omega$$

But this is something which is routine as well. See Jones [51]. □

We can in fact perform the basic construction by recurrence, and we obtain:

THEOREM 12.15. *Associated to any subfactor $A_0 \subset A_1$ is the Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

with the Jones projections having the following properties:

(1) $e_i^2 = e_i = e_i^*$.
(2) $e_i e_j = e_j e_i$ *for* $|i-j| \geq 2$.
(3) $e_i e_{i\pm 1} e_i = [A_1 : A_0]^{-1} e_i$.
(4) $tr(we_{n+1}) = [A_1 : A_0]^{-1} tr(w)$, *for any word* $w \in < e_1, \ldots, e_n >$.

PROOF. This follows from Proposition 12.13 and Proposition 12.14, because the triple basic construction does not need in fact any further study. See [**51**]. $\square$

The relations found in Theorem 12.15 are in fact well-known, from the standard theory of the Temperley-Lieb algebra. This algebra, discovered by Temperley and Lieb in the context of statistical mechanics [**88**], has a very simple definition, as follows:

DEFINITION 12.16. *The Temperley-Lieb algebra of index $N \in [1, \infty)$ is defined as*

$$TL_N(k) = span(NC_2(k,k))$$

with product given by vertical concatenation, with the rule

$$\bigcirc = N$$

for the closed circles that might appear when concatenating.

In other words, the algebra $TL_N(k)$, depending on parameters $k \in \mathbb{N}$ and $N \in [1, \infty)$, is the linear span of the pairings $\pi \in NC_2(k,k)$. The product operation is obtained by linearity, for the pairings which span $TL_N(k)$ this being the usual vertical concatenation, with the conventions that things go "from top to bottom", and that each circle that might appear when concatenating is replaced by a scalar factor, equal to N.

In what concerns us, we will just need some elementary results. First, we have:

PROPOSITION 12.17. *The Temperley-Lieb algebra $TL_N(k)$ is generated by the diagrams*

$$\varepsilon_1 = {}^{\cup}_{\cap} \quad , \quad \varepsilon_2 = |{}^{\cup}_{\cap} \quad , \quad \varepsilon_3 = ||{}^{\cup}_{\cap} \quad , \quad \ldots$$

which are all multiples of projections, in the sense that their rescaled versions

$$e_i = N^{-1} \varepsilon_i$$

satisfy the abstract projection relations $e_i^2 = e_i = e_i^*$.

PROOF. We have two assertions here, the idea being as follows:

(1) The fact that the Temperley-Lieb algebra $TL_N(k)$ is indeed generated by the sequence $\varepsilon_1, \varepsilon_2, \ldots$ follows by drawing pictures, and more specifically by decomposing each basis element $\pi \in NC_2(k,k)$ as a product of such elements ε_i.

(2) Regarding now the projection assertion, when composing ε_i with itself we obtain ε_i itself, times a circle. Thus, according to our multiplication convention, we have:

$$\varepsilon_i^2 = N\varepsilon_i$$

Also, when turning upside-down ε_i, we obtain ε_i itself. Thus, according to our involution convention for the Temperley-Lieb algebra, we have the following formula:

$$\varepsilon_i^* = \varepsilon_i$$

We conclude that the rescalings $e_i = N^{-1}\varepsilon_i$ satisfy $e_i^2 = e_i = e_i^*$, as desired. $\square$

As a second result now, making the link with Theorem 12.15, we have:

PROPOSITION 12.18. *The standard generators $e_i = N^{-1}\varepsilon_i$ of the Temperley-Lieb algebra $TL_N(k)$ have the following properties, where tr is the trace obtained by closing:*

1. *$e_i e_j = e_j e_i$ for $|i-j| \geq 2$.*
2. *$e_i e_{i\pm 1} e_i = N^{-1} e_i$.*
3. *$tr(we_{n+1}) = N^{-1} tr(w)$, for any word $w \in < e_1, \ldots, e_n >$.*

PROOF. This follows indeed by doing some elementary computations with diagrams, in the spirit of those performed in the proof of Proposition 12.17. $\square$

With the above results in hand, and still following Jones' paper [51], we can now reformulate Theorem 12.15 into something more conceptual, as follows:

THEOREM 12.19. *Given a subfactor $A_0 \subset A_1$, construct its the Jones tower:*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

The rescaled sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ produces then a representation

$$TL_N \subset B(H)$$

of the Temperley-Lieb algebra of index $N = [A_1 : A_0]$.

PROOF. We know from Theorem 12.15 that the rescaled sequence of Jones projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves algebrically exactly as the following TL_N diagrams:

$$\varepsilon_1 = {}^{\cup}_{\cap} \quad , \quad \varepsilon_2 = |{}^{\cup}_{\cap} \quad , \quad \varepsilon_3 = ||{}^{\cup}_{\cap} \quad , \quad \ldots$$

But these diagrams generate TL_N, and so we have an embedding $TL_N \subset B(H)$, where H is the Hilbert space where our subfactor $A_0 \subset A_1$ lives, as claimed. $\square$

Let us make the following key observation, also from [51]:

THEOREM 12.20. *Given a finite index subfactor $A_0 \subset A_1$, the graded algebra $P = (P_k)$ formed by the sequence of higher relative commutants*

$$P_k = A_0' \cap A_k$$

contains the copy of the Temperley-Lieb algebra constructed above, $TL_N \subset P$. This graded algebra $P = (P_k)$ is called "planar algebra" of the subfactor.

PROOF. As a first observation, since the Jones projection $e_1 : A_1 \to A_0$ commutes with A_0, we have $e_1 \in P_2$. By translation we obtain, for any $k \in \mathbb{N}$:

$$e_1, \ldots, e_{k-1} \in P_k$$

Thus we have indeed an inclusion of graded algebras $TL_N \subset P$, as claimed. $\square$

As an interesting consequence of the above results, also from [51], we have:

THEOREM 12.21. *The index of subfactors $A \subset B$ is "quantized" in the $[1,4]$ range,*

$$N \in \left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n \geq 3\right\} \cup [4, \infty]$$

with the obstruction coming from the existence of the representation $TL_N \subset B(H)$.

PROOF. This comes from the basic construction, and more specifically from the combinatorics of the Jones projections $e_1, e_2, e_3, \ldots$, the idea being as folows:

(1) In order to best comment on what happens, when iterating the basic construction, let us record the first few values of the numbers in the statement:

$$4\cos^2\left(\frac{\pi}{3}\right) = 1 \quad , \quad 4\cos^2\left(\frac{\pi}{4}\right) = 2$$

$$4\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{2} \quad , \quad 4\cos^2\left(\frac{\pi}{6}\right) = 3$$

$$\ldots$$

(2) When performing a basic construction, we obtain, by trace manipulations on e_1:

$$N \notin (1, 2)$$

With a double basic construction, we obtain, by trace manipulations on $< e_1, e_2 >$:

$$N \notin \left(2, \frac{3+\sqrt{5}}{2}\right)$$

With a triple basic construction, we obtain, by trace manipulations on $< e_1, e_2, e_3 >$:

$$N \notin \left(\frac{3+\sqrt{5}}{2}, 3\right)$$

Thus, we are led to the conclusion in the statement, by a kind of recurrence, involving a certain family of orthogonal polynomials.

(3) In practice now, the most elegant way of proving the result is by using the fundamental fact, explained in Theorem 12.19, that that sequence of Jones projections $e_1, e_2, e_3, \ldots \subset B(H)$ generate a copy of the Temperley-Lieb algebra of index N:

$$TL_N \subset B(H)$$

With this result in hand, we must prove that such a representation cannot exist in index $N < 4$, unless we are in the following special situation:

$$N = 4\cos^2\left(\frac{\pi}{n}\right)$$

But this can be proved by using some suitable trace and positivity manipulations on TL_N, as in (2) above. For full details here, we refer to [51]. □

So long for basic subfactor theory. As a continuation of the story, the subfactors of index $N \leq 4$ are classified by the ADE graphs that we met in chapter 2. See [55].

12c. Planar algebras

Quite remarkably, the planar algebra structure of TL_N, taken in an intuitive sense, of composing diagrams, extends to a planar algebra structure on P. In order to discuss this, let us start with axioms for the planar algebras. Following Jones [56], we have:

Definition 12.22. *The planar algebras are defined as follows:*

(1) *We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.*
(2) *A labelled box is a box with $2n$ marked points on its boundary, n on its upper side, and n on its lower side, for some integer $n \in \mathbb{N}$.*
(3) *A tangle is labelled box, containing a number of labelled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.*
(4) *A planar algebra is a sequence of finite dimensional vector spaces $P = (P_n)$, together with linear maps $P_{n_1} \otimes \ldots \otimes P_{n_k} \to P_n$, one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [56]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

Let us also mention that Definition 12.22 is something quite simplified, based on [56]. As explained in [56], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added, and in view of this, it is perhaps better to start with something stronger than Definition 12.22, as basic axioms. However, as before with rectangles vs circles, our axiomatic choices here are mainly motivated by the concrete examples that we have in mind. More on this later.

As a basic example of a planar algebra, we have the Temperley-Lieb algebra:

Theorem 12.23. *The Temperley-Lieb algebra TL_N, viewed as graded algebra*

$$TL_N = (TL_N(n))_{n\in\mathbb{N}}$$

is a planar algebra, with the corresponding linear maps associated to the planar tangles

$$TL_N(n_1)\otimes\ldots\otimes TL_N(n_k)\to TL_N(n)$$

appearing by putting the various $TL_N(n_i)$ diagrams into the small boxes of the given tangle, which produces a $TL_N(n)$ diagram.

Proof. This is something trivial, which follows from definitions:

(1) Assume indeed that we are given a planar tangle π, as in Definition 12.22, consisting of a box having $2n$ marked points on its boundary, and containing k small boxes, having respectively $2n_1,\ldots,2n_k$ marked points on their boundaries, and then a total of $n+\Sigma n_i$ noncrossing strings, connecting the various $2n+\Sigma 2n_i$ marked points.

(2) We want to associate to this tangle π a linear map as follows:

$$T_\pi : TL_N(n_1)\otimes\ldots\otimes TL_N(n_k)\to TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams $\sigma_i\in TL_N(n_i)$, with $i=1,\ldots,k$:

$$T_\pi(\sigma_1\otimes\ldots\otimes\sigma_k)\in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams $\sigma_i\in TL_N(n_i)$ into the small boxes the given tangle π. Indeed, this procedure produces a certain diagram in $TL_N(n)$, that we can call $T_\pi(\sigma_1\otimes\ldots\otimes\sigma_k)$, as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of TL_N, and we are done. $\square$

As a conclusion to all this, $P = TL_N$ is indeed a planar algebra, but of somewhat "trivial" type, with the triviality coming from the fact that, in this case, the elements of P are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra TL_N is however an important planar algebra, because it is the "smallest" one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

Moving ahead now, here is our second basic example of a planar algebra, which is also "trivial" in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

Theorem 12.24. *The Fuss-Catalan algebra $FC_{N,M}$, which appears by coloring the Temperley-Lieb diagrams with black/white colors, clockwise, as follows*

$$\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots\ldots\circ\bullet\bullet\circ$$

and keeping those diagrams whose strings connect either $\circ-\circ$ or $\bullet-\bullet$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N,M}(n_1)\otimes\ldots\otimes FC_{N,M}(n_k)\to FC_{N,M}(n)$$

appearing by putting the various $FC_{N,M}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N,M}(n)$ diagram.

Proof. The proof here is nearly identical to the proof of Theorem 12.23, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is N. And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 12.23.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is N, and the value of the white circle is M. And it is this obvious composition rule that gives the planar algebra structure, as before for TL_N. $\square$

Getting back now to generalities, and to Definition 12.22, that of a general planar algebra, we have so far two illustrations for it, which, while both important, are both "trivial", with the planar structure simply coming from the fact that, in both these cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely "the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams".

In relation now with subfactors, the result, which extends Theorem 12.20, and which was found by Jones in [56], almost 20 years after [51], is as follows:

Theorem 12.25. *Given a subfactor $A_0\subset A_1$, the collection $P=(P_n)$ of linear spaces*

$$P_n=A_0'\cap A_n$$

has a planar algebra structure, extending the planar algebra structure of TL_N.

PROOF. We know from Theorem 12.20 that we have an inclusion as follows, coming from the basic construction, and with TL_N itself being a planar algebra:

$$TL_N \subset P$$

Thus, the whole point is that of proving that the trivial planar algebra structure of TL_N extends into a planar algebra structure of P. But this can be done via a long algebraic study, and for the full computation here, we refer to Jones' paper [55]. □

As a first illustration for the above result, we have:

THEOREM 12.26. *We have the following universality results:*

(1) *The Temperley-Lieb algebra TL_N appears inside the planar algebra of any subfactor $A_0 \subset A_1$ having index N.*
(2) *The Fuss-Catalan algebra $FC_{N,M}$ appears inside the planar algebra of any subfactor $A_0 \subset A_1$, in the presence of an intermediate subfactor $A_0 \subset B \subset A_1$.*

PROOF. Here the first assertion is something that we already know, from Theorem 12.20, and the second assertion is something quite standard as well, by carefully working out the basic construction for $A_0 \subset A_1$, in the presence of an intermediate subfactor $A_0 \subset B \subset A_1$. For details here, we refer to Bisch and Jones [19]. □

The above results raise the question on whether any planar algebra produces a subfactor. The answer here is yes, but with many subtleties, as follows:

THEOREM 12.27. *We have the following results:*

(1) *Any planar algebra with positivity produces a subfactor.*
(2) *In particular, we have TL and FC type subfactors.*
(3) *In the amenable case, and with $A_1 = R$, the correspondence is bijective.*
(4) *In general, we must take $A_1 = L(F_\infty)$, and we do not have bijectivity.*
(5) *The axiomatization of P, in the case $A_1 = R$, is not known.*

PROOF. All this is quite heavy, basically coming from the work of Popa in the 90s, using heavy functional analysis, the idea being as follows:

(1) As already mentioned in the comments after Definition 12.22, our planar algebra axioms here are something quite simplified, based on [56]. However, when getting back to Theorem 12.25, the conclusion is that the subfactor planar algebras there satisfy a number of supplementary "positivity" conditions, basically coming from the positivity of the II_1 factor trace. And the point is that, with these positivity conditions axiomatized, we reach to something which is equivalent to Popa's axiomatization of the lattice of higher relative commutants $A_i' \cap A_j$ of the finite index subfactors, obtained in the 90s via heavy functional analysis. For the full story here, and details, we refer to Jones' paper [56].

(2) The existence of the TL_N subfactors, also known as "A_∞ subfactors", is something which was known for some time, since some early work of Popa on the subject. As for the

existence of the $FC_{N,M}$ subfactors, this can be shown by using the intermediate subfactor picture, $A_0 \subset B \subset A_1$, by composing two A_∞ subfactors of suitable indices, $A_0 \subset B$ and $B \subset A_1$. For the full story here, we refer as before to Jones [56].

(3) This is something fairly heavy, as it is always the case with operator algebra results regarding hyperfiniteness and amenability, due to Popa. For the story here, see [56].

(4) This is something a bit more recent, obtained by further building on the above-mentioned constructions of Popa. Again, we refer here to [56] and related work.

(5) This is the big open question in subfactors. The story here goes back to Jones' original paper [51], which contains at the end the question, due to Connes, of finding the possible values of the index for the irreducible subfactors of R. This question, which certainly looks much easier than (5) in the statement, is in fact still open, now 40 years after its formulation, and with no one having any valuable idea in dealing with it. $\square$

12d. Graph symmetries

Getting back to quantum groups, all this machinery is interesting for us. We will need the construction of the tensor and spin planar algebras $\mathcal{T}_N, \mathcal{S}_N$. Let us start with:

DEFINITION 12.28. *The tensor planar algebra $\mathcal{T}_N$ is the sequence of vector spaces*

$$P_k = M_N(\mathbb{C})^{\otimes k}$$

with the multilinear maps $T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by the formula

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, exactly as in the easy quantum group case.

The fact that we have indeed a planar algebra, in the sense that the gluing of tangles corresponds to the composition of linear maps, as required by Definition 12.22, is something elementary, in the same spirit as the verification of the functoriality properties of the correspondence $\pi \to T_\pi$, from easiness, and we refer here to Jones [56].

Let us discuss now a second planar algebra of the same type, which is important as well for various reasons, namely the spin planar algebra $\mathcal{S}_N$. This planar algebra appears somehow as the "square root" of the tensor planar algebra $\mathcal{T}_N$. Let us start with:

Definition 12.29. *We write the standard basis of* $(\mathbb{C}^N)^{\otimes k}$ *in* $2\times k$ *matrix form,*

$$e_{i_1\ldots i_k}=\begin{pmatrix}i_1&i_1&i_2&i_2&i_3&\ldots&\ldots\\ i_k&i_k&i_{k-1}&\ldots&\ldots&\ldots&\ldots\end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra $\mathcal{S}_N$, also from [53], as follows:

Definition 12.30. *The spin planar algebra* $\mathcal{S}_N$ *is the sequence of vector spaces*

$$P_k=(\mathbb{C}^N)^{\otimes k}$$

written as above, with the multiplinear maps $T_\pi:P_{k_1}\otimes\ldots\otimes P_{k_r}\to P_k$ *being given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_r})=\sum_j\delta_\pi(i_1,\ldots,i_r:j)e_j$$

with the Kronecker symbols δ_π *being* 1 *if the indices fit, and being* 0 *otherwise.*

Here are some illustrating examples for the spin planar algebra calculus:

(1) The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y)=1$ being the pairs of the form (x,x), this tangle 1_k acts by the identity:

$$1_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(2) The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k\left(\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}\otimes\begin{pmatrix}l_1&\ldots&l_k\\ m_1&\ldots&m_k\end{pmatrix}\right)=\delta_{j_1m_1}\ldots\delta_{j_km_k}\begin{pmatrix}l_1&\ldots&l_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(3) The inclusion I_k is the $(k,k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y)=1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_l\begin{pmatrix}j_1&\ldots&j_k&l\\ i_1&\ldots&i_k&l\end{pmatrix}$$

(4) The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(5) The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting

from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

(6) The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap\,|\,|\,|\,\|\\ \|\qquad\ \ \|\\ \|\,|\,|\,|\,\Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\ldots i_k})=e_{i_2\ldots i_ki_1}$$

There are many other interesting examples of k-tangles, but in view of our present purposes, we can actually stop here, due to the following fact:

Theorem 12.31. *The multiplications, inclusions, expectations, Jones projections and rotations generate the set of all tangles, via the gluing operation.*

Proof. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list. See [56]. □

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. This is constructed as follows:

$$\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}^*=\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}$$

As before, we refer to Jones' paper [56] for more on all this. Getting back now to quantum groups, following [6], we have the following result:

Theorem 12.32. *Given $G\subset S_N^+$, consider the tensor powers of the associated coaction map on $C(X)$, where $X=\{1,\ldots,N\}$, which are the folowing linear maps:*

$$\Phi^k:C(X^k)\to C(X^k)\otimes C(G)$$

$$e_{i_1\ldots i_k}\to\sum_{j_1\ldots j_k}e_{j_1\ldots j_k}\otimes u_{j_1i_1}\ldots u_{j_ki_k}$$

The fixed point spaces of these coactions, which are by definition the spaces

$$P_k=\left\{x\in C(X^k)\Big|\Phi^k(x)=1\otimes x\right\}$$

are given by $P_k=Fix(u^{\otimes k})$, and form a subalgebra of the spin planar algebra $\mathcal{S}_N$.

Proof. Since the map Φ is a coaction, its tensor powers Φ^k are coactions too, and at the level of fixed point algebras we have the following formula:

$$P_k = Fix(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 12.31. Consider the rotation R_k. Rotating, then applying Φ^k, and rotating backwards by R_k^{-1} is the same as applying Φ^k, then rotating each k-fold product of coefficients of Φ. Thus the elements obtained by rotating, then applying Φ^k, or by applying Φ^k, then rotating, differ by a sum of Dirac masses tensored with commutators in $A = C(G)$:

$$\Phi^k R_k(x) - (R_k \otimes id)\Phi^k(x) \in C(X^k) \otimes [A, A]$$

Now let $\int_A$ be the Haar functional of A, and consider the conditional expectation onto the fixed point algebra P_k, which is given by the following formula:

$$\phi_k = \left(id \otimes \int_A \right) \Phi^k$$

Since $\int_A$ is a trace, it vanishes on commutators. Thus R_k commutes with ϕ_k:

$$\phi_k R_k = R_k \phi_k$$

The commutation relation $\phi_k T = T\phi_l$ holds in fact for any (l, k)-tangle T. These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle T:

$$\phi_k T \phi_l = T\phi_l$$

We conclude from this that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ of the planar operad consisting of tangles T satisfying the following condition, where $\phi = (\phi_k)$, and where $i(.)$ is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of Φ^k gives $M_k \in \mathcal{P}'$. Now since the planar operad $\mathcal{P}$ is generated by multiplications and annular tangles, it follows that we have $\mathcal{P}' = P$. Thus for any tangle T the corresponding multilinear map between spaces $P_k(X)$ restricts to a multilinear map between spaces P_k. In other words, the action of the planar operad $\mathcal{P}$ restricts to P, and makes it a subalgebra of $\mathcal{S}_N$, as claimed. $\square$

As a second result now, also from [6], completing our study, we have:

Theorem 12.33. *We have a bijection between quantum permutation groups and subalgebras of the spin planar algebra,*

$$(G \subset S_N^+) \quad \longleftrightarrow \quad (Q \subset \mathcal{S}_N)$$

given in one sense by the construction in Theorem 12.32, and in the other sense by a suitable modification of Tannakian duality.

PROOF. The idea is that this will follow by applying Tannakian duality to the annular category over Q. Let n,m be positive integers. To any element $T_{n+m}\in Q_{n+m}$ we associate a linear map $L_{nm}(T_{n+m}):P_n(X)\to P_m(X)$ in the following way:

$$L_{nm}\begin{pmatrix}|\,|\,|\\ T_{n+m}\\ |\,|\,|\end{pmatrix}:\begin{pmatrix}|\\ a_n\\ |\end{pmatrix}\to\begin{pmatrix}|\ \ |\ \ \cap\\ \ \ T_{n+m}|\\ |\,|\ \ |\ \ |\\ a_n|\ \ |\ \ |\\ \cup\ \ |\ \ |\end{pmatrix}$$

That is, we consider the planar $(n,n+m,m)$-tangle having an small input n-box, a big input $n+m$-box and an output m-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X)\otimes P_{n+m}(X)\to P_m(X)$$

If we put the element T_{n+m} in the big input box, we obtain in this way a certain linear map $P_n(X)\to P_m(X)$, that we call L_{nm}. With this convention, let us set:

$$Q_{nm}=\left\{L_{nm}(T_{n+m}):P_n(X)\to P_m(X)\Big|T_{n+m}\in Q_{n+m}\right\}$$

These spaces form a Tannakian category, so by [100] we obtain a Woronowicz algebra (A,u), such that the following equalities hold, for any m,n:

$$Hom(u^{\otimes m},u^{\otimes n})=Q_{mn}$$

We prove that u is a magic unitary. We have $Hom(1,u^{\otimes 2})=Q_{02}=Q_2$, so the unit of Q_2 must be a fixed vector of $u^{\otimes 2}$. But $u^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$\begin{aligned}u^{\otimes 2}(1)&=u^{\otimes 2}\left(\sum_i\begin{pmatrix}i&i\\ i&i\end{pmatrix}\right)\\ &=\sum_{ikl}\begin{pmatrix}k&k\\ l&l\end{pmatrix}\otimes u_{ki}u_{li}\\ &=\sum_{kl}\begin{pmatrix}k&k\\ l&l\end{pmatrix}\otimes(uu^t)_{kl}\end{aligned}$$

From $u^{\otimes 2}(1)=1\otimes 1$ ve get that uu^t is the identity matrix. Together with the unitarity of u, this gives the following formulae:

$$u^t=u^*=u^{-1}$$

Consider the Jones projection $E_1\in Q_3$. After isotoping, $L_{21}(E_1)$ looks as follows:

$$L_{21}\begin{pmatrix}|\cup\\ |\cap\end{pmatrix}:\begin{pmatrix}|\ \ |\\ i\ \ i\\ j\ \ j\\ |\ \ |\end{pmatrix}\to\begin{pmatrix}|\\ i\ \ i\ \supset\\ j\ \ j\\ |\end{pmatrix}=\delta_{ij}\begin{pmatrix}|\\ i\\ i\\ |\end{pmatrix}$$

In other words, the linear map $M = L_{21}(E_1)$ is the multiplication $\delta_i \otimes \delta_j \to \delta_{ij}\delta_i$:

$$M\begin{pmatrix} i & i \\ j & j \end{pmatrix} = \delta_{ij}\begin{pmatrix} i \\ i \end{pmatrix}$$

In order to finish, consider the following element of $C(X) \otimes A$:

$$(M \otimes id)u^{\otimes 2}\left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1\right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes u_{ki}u_{kj}$$

Since $M \in Q_{21} = Hom(u^{\otimes 2}, u)$, this equals the following element of $C(X) \otimes A$:

$$u(M \otimes id)\left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1\right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes \delta_{ij}u_{ki}$$

Thus we have $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ for any i, j, k, which shows that u is a magic unitary. Now if P is the planar algebra associated to u, we have $Hom(1, v^{\otimes n}) = P_n = Q_n$, as desired. As for the uniqueness, this is clear from the Peter-Weyl theory. □

All the above might seem a bit technical, but is worth learning, and for good reason, because it is extremely powerful. As an example of application, if you agree with the bijection $G \leftrightarrow Q$ in Theorem 12.33, then $G = S_N^+$ itself, which is the biggest object on the left, must correspond to the smallest object on the right, namely $Q = TL_N$.

Back now to our usual business, graphs, we have the following result:

THEOREM 12.34. *The planar algebra associated to $G^+(X)$ is equal to the planar algebra generated by d, viewed as a 2-box in the spin planar algebra $\mathcal{S}_N$, with $N = |X|$.*

PROOF. We recall from the above that any quantum permutation group $G \subset S_N^+$ produces a subalgebra $P \subset \mathcal{S}_N$ of the spin planar algebra, given by:

$$P_k = Fix(u^{\otimes k})$$

In our case, the idea is that $G = G^+(X)$ comes via the relation $d \in End(u)$, but we can view this relation, via Frobenius duality, as a relation of the following type:

$$\xi_d \in Fix(u^{\otimes 2})$$

Indeed, let us view the adjacency matrix $d \in M_N(0, 1)$ as a 2-box in $\mathcal{S}_N$, by using the canonical identification between $M_N(\mathbb{C})$ and the algebra of 2-boxes $\mathcal{S}_N(2)$:

$$(d_{ij}) \leftrightarrow \sum_{ij} d_{ij}\begin{pmatrix} i & i \\ j & j \end{pmatrix}$$

Let P be the planar algebra associated to $G^+(X)$ and let Q be the planar algebra generated by d. The action of $u^{\otimes 2}$ on d viewed as a 2-box is given by:

$$u^{\otimes 2}\left(\sum_{ij} d_{ij}\begin{pmatrix}i&i\\j&j\end{pmatrix}\right)=\sum_{ijkl} d_{ij}\begin{pmatrix}k&k\\l&l\end{pmatrix}\otimes u_{ki}u_{lj}=\sum_{kl}\begin{pmatrix}k&k\\l&l\end{pmatrix}\otimes (udu^t)_{kl}$$

Since v is a magic unitary commuting with d we have:

$$udu^t=duu^t=d$$

But this means that d, viewed as a 2-box, is in the algebra P_2 of fixed points of $u^{\otimes 2}$. Thus $Q\subset P$. As for $P\subset Q$, this follows from the duality found above. $\square$

Generally speaking, the above material, when coupled with what we did in this book about graphs, leads us into the classification of the subalgebras of the spin planar algebra generated by a 2-box. But this can be regarded as a particular case of the Bisch-Jones question of classifying the planar algebras generated by a 2-box [20]. We will see in Part IV how the notion of quantum graph can get us closer to the generality level of [20].

12e. Exercises

Tough chapter that we had here, which would normally require reading the complete works of von Neumann, Connes, Jones, Popa and others. As exercises, we have:

Exercise 12.35. *Learn some solid functional analysis, and operator theory.*

Exercise 12.36. *Learn about the reduction theory of von Neumann.*

Exercise 12.37. *Learn about the classification of factors, as much as you can.*

Exercise 12.38. *Learn about the main properties of the* II_1 *factors.*

Exercise 12.39. *Learn about the algebra of Temperley and Lieb.*

Exercise 12.40. *Learn more about knot invariants too, from Jones.*

Exercise 12.41. *Learn more about planar algebras, from Jones.*

Exercise 12.42. *Learn about the annular structure of subfactors, from Jones.*

As bonus exercise, learn more about the hyperfinite factor R, from Murray and von Neumann, then Connes, Jones, Popa. This is where the hard problems take place.

Part IV

Generalizations

Think about the way
That we live today
Think about the way
How some people play

CHAPTER 13

Twisted spaces

13a. Discussion

Welcome to quantum, take two. In this final Part IV we will go doubly quantum, by looking at certain quantum-flavored generalizations of the graphs X, and then at their quantum symmetry groups $G^+(X)$. With the hope that all this can be useful.

Sounds exciting, doesn't it? In practice, however, there are countless possible ways to be followed, especially if you don't know well quantum mechanics. And since I am precisely in this situation, it is probably safer to ask around, before getting to work.

And good news, cat is here, so let's ask him what he thinks about this:

CAT 13.1. *Sounds good, can we cats stop you humans from doing mathematics, but please attempt to meditate on how you choose X, and how you compute $G^+(X)$.*

Damn cat, not very useful advice, this was exactly the question I was asking about, what are the objects X that I should look at, and how exactly to define $G^+(X)$ for them. Guess I should ask him again, but go find cat, he dissapeared right after answering, guess by using one of these quantum mechanics tricks, that he's very familiar with.

Fortunately mouse is here too, not that I talk much to him in general, but I am really desperate at this point of writing, so let's ask him what he thinks too:

MOUSE 13.2. *As long as mathematics is formally correct, that can only help. By the way, you should work a bit less, and unionize, as us fellow mice do.*

Thanks mouse, I was sort of expecting this kind of advice from you, and I am a bit afraid that, in the lack of a good idea, I will just follow it. The first part I mean, after all any mathematical object X has a chance to be relevant to quantum mechanics, so picking one and working on it should be good work. We should all do this. Makes sense.

But wait, here comes dog, let's see what he thinks about this, too:

DOG 13.3. *No idea about quantum, and I'm not unionized either, but I'd rather agree here with cat. Flooding quantum mechanics with randomness will end up killing it.*

Thanks dog, interesting point, and actually reminds me of what happens in so many areas of mathematics. Formally correct stuff, done without any clear purpose in mind, ends up discouraging young people to join the subject, and eventually kills it.

So, what to do? In the lack of crocodile, who's nowhere to be found, but usually he's on the same wavelength with cat anyway, guess we will have to agree with cat.

And meditation, as cats suggests, gets us into the following good old principle:

Principle 13.4. *At very small scales, geometry should be free.*

To be more precise, this is something well-known since the 1970s, and whenever you hear someone, working on quarks, or tentatively at the Planck scale, or anywhere in between, talking about "freedom", "freeness", "high noncommutativity" and so on, it is about this general principle that they are talking about. Things being free at very small scales, no one really knowing how, and then our usual world, big particles and higher, appearing via some mysterious thermodynamic limits, from this freeness.

But you might probably ask, how come this very old principle hasn't evolved into something more concrete, in the meantime? Well, due to some form of greed, I guess. The end dream, going beyond what Principle 13.4 says, is to become fully deterministic, or Einstenian if you prefer, and most efforts in quantum mechanics since the 1970s, be them in string theory, or noncommutative geometry, or other related disciplines, have been way too deterministic. Punch above your weight, and you'll win nothing.

Anyway. Getting back now to our graph business, and to the present book, in view of Cat 13.1, of Dog 13.3 and of Principle 13.4, we will be very careful, as follows:

Plan 13.5. *We will talk in the remainder of this book about:*

1. *Quantum graphs X, chosen as basic as possible, and their symmetry groups $G^+(X)$, with the aim of enlarging our free geometry knowledge.*
2. *Carefully chosen block design type objects X, and their symmetry groups $G^+(X)$, again with the aim of enlarging our free geometry knowledge.*

Which sounds very good, enough for the philosophy I guess, but with the remark however that we forgot, in all this, to take into account the mysterious thermodynamic limits mentioned above, that are normally part of the theory that we want to develop, too. Nevermind, but for completeness, let us complement Plan 13.5 with:

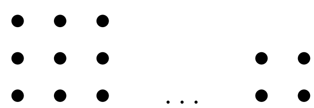
Plan B 13.6. *In case our plan leads into some statistical mechanics business, of quantum flavor, we should definitely go into that too, without hesitating.*

Getting started now, what is a quantum graph? Good question, and in this book we will go for the simplest axioms, with the idea in mind that "simple mathematics should correspond to true physics". So, let us axiomatize the quantum graphs. We need vertices and edges for them, and our story here will be as follows:

(1) First we need a vertex set, which must be a "finite quantum space". According to von Neumann, the quantum spaces are the duals of operator algebras, and when adopting this viewpoint, problem solved, because what we need is a finite dimensional C^*-algebra B, and then we can formally write $B = C(Z)$, and we have our vertex set Z.

(2) You might perhaps ask at this point, but what are the points of Z? Very natural question, with versions and duplicates of this regularly flooding the internet. So, in answer, let me ask you a question too, are you here for quantum, or not? In quantum we have no points, and not only this is not a problem, but we are proud of this.

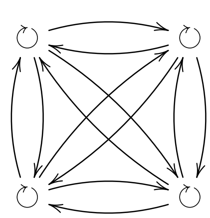
(3) This being said, something can be done here. Since our algebra B must be a direct sum of matrix algebras, $B = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, we can represent each matrix block as a square, and we end up with a picture like this, representing B:



(4) But looking at this picture, we can say that this represents Z itself. For instance the number of points is the correct one, $|Z| = \dim B$. Also, in the case $B = \mathbb{C}^N$, the picture that we get, $\bullet\ \bullet \ldots \bullet$, is the correct picture of Z, as a space of points. More generally, when $n_i = 1$, the associated point $\bullet$ is a true point of Z. And so on.

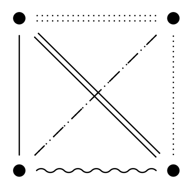
(5) With this done, let us turn now to edges. Here we need an adjacency matrix over the vertex set, $d \in \mathcal{L}(B) \simeq M_N(\mathbb{C})$, with $N = |Z| = \dim B$. Thus, good news, no need for complicated mathematics here, what we need is a usual matrix $d \in M_N(\mathbb{C})$.

(6) As further good news, we can represent $d \in M_N(\mathbb{C})$ on our picture of Z, by drawing an arrow $i \to j$ between any two points, and coloring it with the complex number $d_{ij} \in \mathbb{C}$. As an example here, in the case $B = M_2(\mathbb{C})$ we end up with a picture as follows, with all arrows being supposed to be colored, but with the colors missing, due to budget cuts:



(7) Of course, we can impose some natural conditions on d. For instance when assuming that d is symmetric, and has 0 on the diagonal, as the usual adjacency matrices of

graphs do, the generic picture looks as follows, this time without budget cuts:



(8) So done, we have now our definition of quantum graphs, along with pictures for them, which are very similar to the usual graphs, and can help us with our math.

We will talk about such quantum graphs $X = (Z, d)$ in what follows, in this chapter and in the next one, first by discussing their basic algebraic, geometric and analytic aspects, and then by focusing on the computation of the corresponding quantum symmetry groups $G^+(X)$, which will appear as follows, with a suitable definition for S_Z^+:

$$G^+(X) \subset S_Z^+$$

The theory of the quantum graphs $X = (Z, d)$ themselves will be quite elementary, to start with, usual linear algebra and graph theory, but regarded from a different perspective, as you can see from the above. However, when discussing the corresponding quantum symmetry groups $G^+(X)$, things will turn quite subtle, among others with several non-trivial connections with the planar algebra material from chapter 12.

This being said, one more thing. Remember thermodynamic limits and Plan B 13.6, and now that the plan for the present chapter and the next one is laid, with obviously not many connections with statistical mechanics, we will have to update Plan B 13.6 into:

PLAN C 13.7. *We will leave statistical mechanics for later, when discussing block designs. With a bit of luck, we will find some good objects X there, and how to define their $G^+(X)$ too, say in relation with the knot invariants discussed in chapter 4.*

All this might seem a bit too loose, and optimistic, but, well, the truth is that I've been regularly discussing with cat, about all this, since long, and not that I really understand what cat is talking about, quite complicated all that quantum and statistical mechanics, but I have this feeling that we can be optimistic here. More on this soon.

13b. Twisted spaces

So, quantum graphs, developed along the lines of the (1-8) program above. We first need to discuss, following [91], the quantum symmetry groups S_Z^+ of the finite quantum spaces Z, generalizing the quantum group S_N^+, coming from the space $Z = \{1, \ldots, N\}$. Coming a bit in advance, let us mention too that as a second basic example, we have the group SO_3, coming from the quantum space $Z = M_2$, given by $C(Z) = M_2(\mathbb{C})$.

In order to get started, we must talk about finite quantum spaces. In view of the general C^*-algebra theory explained in chapter 9, we have the following definition:

DEFINITION 13.8. *A finite quantum space Z is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(Z)=B$$

The formal number of elements of such a space is $|Z|=\dim B$. By decomposing the algebra B, we have a formula of the following type:

$$C(Z)=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

With $n_1=\ldots=n_k=1$ we obtain in this way the space $Z=\{1,\ldots,k\}$. Also, when $k=1$ the equation is $C(Z)=M_n(\mathbb{C})$, and the solution will be denoted $Z=M_n$.

In order to do some mathematics on such spaces, the very first observation is that we can talk about the formal number of points of such a space, as follows:

$$|Z|=\dim B$$

Alternatively, by decomposing the algebra B as a sum of matrix algebras, as in Definition 13.8, we have the following formula for the formal number of points:

$$|Z|=n_1^2+\ldots+n_k^2$$

Pictorially, this suggests representing Z as a set of $|Z|$ points in the plane, arranged in squares having sides $n_1,\ldots,n_k$, coming from the matrix blocks of B, as follows:

$$\begin{matrix}\circ&\circ&\circ\\ \circ&\circ&\circ\\ \circ&\circ&\circ\end{matrix}\qquad\ldots\qquad\begin{matrix}\circ&\circ\\ \circ&\circ\end{matrix}$$

Getting now to more advanced mathematics, what can we do with the above finite quantum spaces Z? The first thought goes to topology, and this due to the following well-known, elementary formula, valid for any two usual compact spaces X,Y:

$$C(X\sqcup Y)=C(X)\oplus C(Y)$$

Indeed, this suggests that direct sum of C^*-algebras corresponds to a disjoint union, at the level of corresponding quantum spaces, so that we can formulate:

DEFINITION 13.9. *Given a finite quantum space Z, appearing as above via*

$$C(Z)=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

with $n_1,\ldots,n_k\geq 1$, we write in this case the following disjoint union formula,

$$Z=M_{n_1}\sqcup\ldots\sqcup M_{n_k}$$

with M_n being, as before, the finite quantum space given by $C(M_n)=M_n(\mathbb{C})$.

Which leads us into the question on whether we should think of $Z = M_n$ as being connected, or not. Good question, indeed, requiring however a better knowledge of C^*-algebra theory, from a topological viewpoint, and we will leave such things for later.

As a continuation of this, let us look now more in detail at M_n, which is obviously the key to all finite quantum spaces, via disjoint unions. We are certainly very familiar with this space, or rather with the algebra of functions on it, which is $C(M_n) = M_n(\mathbb{C})$, and leaving aside the above connectedness question, here is a quite complete set of speculations that can be made about it, based on our usual linear algebra knowledge:

Speculation 13.10. *We can view the matrix algebra $M_n(\mathbb{C})$ as being the algebra of functions on a quantum space M_n, according to the following formula:*

$$M_n(\mathbb{C}) = C(M_n)$$

This quantum space M_n formally has $|M_n| = n^2$ points, and appears as a sort of twist of $\{1, \ldots, n^2\}$. Moreover, we can integrate over M_n, according to the formula

$$\int_{M_n} T = \frac{T_{11} + \ldots + T_{nn}}{n}$$

with the underlying measure being positive and of mass 1.

Obviously, many things going on here. In what regards the first assertion, this comes from Definition 13.8, and the same goes for the points count, which is as follows:

$$|M_n| = \dim_{\mathbb{C}} C(M_n) = \dim_{\mathbb{C}} M_n(\mathbb{C}) = n^2$$

Next, observe that we have an isomorphism of complex vector spaces, as follows:

$$C(M_n) = M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2} = C(1, \ldots, n^2)$$

But this isomorphism suggests that we should have a formula as follows, with $\sim$ standing for some sort of twisting operation, at the quantum space level:

$$M_n \sim \{1, \ldots, n^2\}$$

So, can we further justify this? Normally yes, the idea being that at the level of standard bases of $C(M_n) \simeq C(1, \ldots, n^2)$, the multiplication gets twisted as follows:

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \longleftrightarrow \quad e_je_k = \delta_{jk}e_j$$

We will come back later to this, with more details. Finally, in what regards the last assertion, this is something very intuitive, and rock-solid, expressing the standard fact that the normalized trace of $n \times n$ matrices $tr = Tr/n$ is unital and positive:

$$tr(1) = 1 \quad , \quad T \geq 0 \implies tr(T) \geq 0$$

Very nice all this, and as a conclusion to our study so far, many speculations can be made, and what seems most solid and useful are the integration aspects. So, let us go now for this. Getting back to the general framework of Definition 13.8, we have:

DEFINITION 13.11. *Given a finite quantum space Z, we construct the functional*

$$tr : C(Z) \to B(l^2(Z)) \to \mathbb{C}$$

obtained by applying the regular representation, and the normalized matrix trace, and we call it integration with respect to the normalized counting measure on Z.

To be more precise, consider the algebra $B = C(Z)$, which is by definition finite dimensional. We can make act B on itself, by left multiplication:

$$\pi : B \to \mathcal{L}(B) \quad , \quad a \to (b \to ab)$$

The target of π being a matrix algebra, $\mathcal{L}(B) \simeq M_N(\mathbb{C})$ with $N = \dim B$, we can further compose with the normalized matrix trace, and we obtain tr:

$$tr = \frac{1}{N} Tr \circ \pi$$

As basic examples, for both $Z = \{1, \ldots, N\}$ and $Z = M_n$ we obtain the usual trace. In general, with $C(Z) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, the weights of tr are:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Pictorially, this suggests fine-tuning our previous picture of Z, by adding to each point the unnormalized trace of the corresponding element of B, as follows:

$$\begin{matrix}
\bullet_{n_1} & \circ_0 & \circ_0 & & & \\
\circ_0 & \bullet_{n_1} & \circ_0 & \ldots & \bullet_{n_k} & \circ_0 \\
\circ_0 & \circ_0 & \bullet_{n_1} & & \circ_0 & \bullet_{n_k}
\end{matrix}$$

Here we have represented the points on the diagonals with solid circles, since they are of different nature from the off-diagonal ones, the attached numbers being nonzero. However, this picture is not complete either, and we can do better, as follows:

DEFINITION 13.12. *Given a finite quantum space Z, coming via a formula of type*

$$C(Z) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

we use the following equivalent conventions for drawing Z:

(1) *Triple indices. We represent Z as a set of $N = |Z|$ points, with each point being decorated with a triple index ija, coming from the standard basis $\{e_{ij}^a\} \subset B$.*
(2) *Double indices. As before, but by ignoring the index a, with the convention that i, j belong to various indexing sets, one for each of the matrix blocks of B.*
(3) *Single indices. As before, but with each point being now decorated with a single index, playing the role of the previous triple indices ija, or double indices ij.*

All the above conventions are useful, and in practice, we will be mostly using the single index convention from (3). As an illustration, consider the space $Z = \{1, \ldots, k\}$. According to our single index convention, we can represent this space as a set of k points, decorated by some indices, which must be chosen different. But the obvious choice for these k different indices is $1,\ldots,k$, and we are led to the following picture:

$$\bullet_1 \quad \bullet_2 \quad \ldots \quad \bullet_k$$

As another illustration, consider the space $Z = M_n$. Here the picture is as follows, using double indices, which can be regarded as well as being single indices:

$$\begin{matrix}\bullet_{11} & \circ_{12} & \circ_{13}\\ \circ_{21} & \bullet_{22} & \circ_{23}\\ \circ_{31} & \circ_{32} & \bullet_{33}\end{matrix}$$

As yet another illustration, for the space $Z = M_3 \sqcup M_2$, which appears by definition from the algebra $B = M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$, we are in need of triple indices, which can be of course regarded as single indices, in order to label all the points, and the picture is:

$$\begin{matrix}\bullet_{111} & \circ_{121} & \circ_{131} & & & \\ \circ_{211} & \bullet_{221} & \circ_{231} & & \bullet_{112} & \circ_{122}\\ \circ_{311} & \circ_{321} & \bullet_{331} & & \circ_{212} & \bullet_{222}\end{matrix}$$

Many more things can be said, along these lines, both elementary theorems and speculations. However, the best is to have a look at quantum group actions first, which can teach us many things, and come back to speculations afterwards.

13c. Symmetry groups

Let us study now the quantum group actions $G \curvearrowright Z$. If we denote by μ, η the multiplication and unit map of the algebra $C(Z)$, we have the following result:

Proposition 13.13. *Consider a linear map $\Phi : C(Z) \to C(Z) \otimes C(G)$, written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

with $\{e_i\}$ being a linear space basis of $C(Z)$, chosen orthonormal with respect to tr.

1. *Φ is a linear space coaction $\iff$ u is a corepresentation.*
2. *Φ is multiplicative $\iff$ $\mu \in Hom(u^{\otimes 2}, u)$.*
3. *Φ is unital $\iff$ $\eta \in Hom(1, u)$.*
4. *Φ leaves invariant tr $\iff$ $\eta \in Hom(1, u^*)$.*
5. *If these conditions hold, Φ is involutive $\iff$ u is unitary.*

Proof. This is similar to the proof for S_N^+ from chapter 9, as follows:

(1) There are two axioms to be processed here, and we have indeed:

$$(id \otimes \Delta)\Phi = (\Phi \otimes id)\Phi \iff \Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki}$$

$$(id \otimes \varepsilon)\Phi = id \iff \varepsilon(u_{ji}) = \delta_{ji}$$

(2) By using $\Phi(e_i)=u(e_i\otimes 1)$ we have the following identities, which give the result:

$$\Phi(e_ie_k)=u(\mu\otimes id)(e_i\otimes e_k\otimes 1)$$

$$\Phi(e_i)\Phi(e_k)=(\mu\otimes id)u^{\otimes 2}(e_i\otimes e_k\otimes 1)$$

(3) From $\Phi(e_i)=u(e_i\otimes 1)$ we obtain by linearity, as desired:

$$\Phi(1)=u(1\otimes 1)$$

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned}(tr\otimes id)\Phi(e_i)=tr(e_i)1 \quad&\iff\quad \sum_j tr(e_j)u_{ji}=tr(e_i)1\\ &\iff\quad \sum_j u_{ji}^*1_j=1_i\\ &\iff\quad (u^*1)_i=1_i\\ &\iff\quad u^*1=1\end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that Φ is involutive, we have:

$$\begin{aligned}(u^*u)_{ik}&=\sum_l u_{li}^*u_{lk}\\ &=\sum_{jl}tr(e_j^*e_l)u_{ji}^*u_{lk}\\ &=(tr\otimes id)\sum_{jl}e_j^*e_l\otimes u_{ji}^*u_{lk}\\ &=(tr\otimes id)(\Phi(e_i)^*\Phi(e_k))\\ &=(tr\otimes id)\Phi(e_i^*e_k)\\ &=tr(e_i^*e_k)1\\ &=\delta_{ik}\end{aligned}$$

Thus $u^*u=1$, and since we know from (1) that u is a corepresentation, it follows that u is unitary. The proof of the converse is standard too, by using a similar computation. □

Following now [6], [91], we have the following result, extending the basic theory of S_N^+ from chapter 9 to the present finite quantum space setting, with U_N^+ being the maximal compact matrix quantum group allowed by the Woronowicz axioms from chapter 9:

THEOREM 13.14. *Given a finite quantum space Z, there is a universal compact quantum group S_Z^+ acting on Z, and leaving the counting measure invariant. We have*

$$C(S_Z^+)=C(U_N^+)\Big/\Big\langle\mu\in Hom(u^{\otimes 2},u),\eta\in Fix(u)\Big\rangle$$

where $N=|Z|$, and where μ,η are the multiplication and unit maps of the algebra $C(Z)$. For the classical space $Z=\{1,\dots,N\}$ we have $S_Z^+=S_N^+$.

Proof. Here the first two assertions follow from Proposition 13.13, by using the standard fact that the complex conjugate of a unitary corepresentation is a unitary corepresentation too, and with the following definition for the quantum group U_N^+:

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^* = u^{-1},\, u^t = \bar{u}^{-1}\right)$$

To be more precise, observe first that the norms of generators being bounded by unitarity, $||u_{ij}|| \leq 1$, the above universal C^*-algebra is well-defined. But then, we can define maps Δ, ε, S as in chapter 9 by the following formulae, and universality:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

Thus, we have indeed a compact quantum group, and this gives the first two assertions. As for the last assertion, regarding S_N^+, this follows from the results in chapter 9. □

The above result is quite conceptual, and we will see some applications in a moment. However, for many concrete questions, nothing beats multimatrix bases and indices. So, following the original paper of Wang [91], let us discuss this. We first have:

Definition 13.15. *Given a finite quantum space Z, we let $\{e_i\}$ be the standard basis of $B = C(Z)$, so that the multiplication, involution and unit of B are given by*

$$e_i e_j = e_{ij} \quad , \quad e_i^* = e_{\bar{i}} \quad , \quad 1 = \sum_{i=\bar{i}} e_i$$

where $(i,j) \to ij$ is the standard partially defined multiplication on the indices, with the convention $e_\emptyset = 0$, and where $i \to \bar{i}$ is the standard involution on the indices.

To be more precise, let $\{e_{ab}^r\} \subset B$ be the multimatrix basis. We set $i = (abr)$, and with this convention, the multiplication, coming from $e_{ab}^r e_{cd}^p = \delta_{rp}\delta_{bc}e_{ad}^r$, is given by:

$$(abr)(cdp) = \begin{cases} (adr) & \text{if } b = c,\ r = p \\ \emptyset & \text{otherwise} \end{cases}$$

As for the involution, coming from $(e_{ab}^r)^* = e_{ba}^r$, this is given by:

$$\overline{(a,b,r)} = (b,a,r)$$

Finally, the unit formula comes from the following formula for the unit $1 \in B$:

$$1 = \sum_{ar} e_{aa}^r$$

Regarding now the generalized quantum permutation groups S_Z^+, the construction in Theorem 13.14 reformulates as follows, by using the above formalism:

PROPOSITION 13.16. *Given a finite quantum space Z, with basis $\{e_i\}\subset C(Z)$ as above, the algebra $C(S_Z^+)$ is generated by variables u_{ij} with the following relations,*

$$\sum_{ij=p}u_{ik}u_{jl}=u_{p,kl}\quad,\quad\sum_{kl=p}u_{ik}u_{jl}=u_{ij,p}$$

$$\sum_{i=\bar{i}}u_{ij}=\delta_{j\bar{j}}\quad,\quad\sum_{j=\bar{j}}u_{ij}=\delta_{i\bar{i}}$$

$$u_{ij}^*=u_{\bar{i}\bar{j}}$$

with the fundamental corepresentation being the matrix $u=(u_{ij})$. We call a matrix $u=(u_{ij})$ satisfying the above relations "generalized magic".

PROOF. We recall from Theorem 13.14 that the algebra $C(S_Z^+)$ appears as follows, where $N=|Z|$, and where μ,η are the multiplication and unit maps of $C(Z)$:

$$C(S_Z^+)=C(U_N^+)\Big/\Big\langle\mu\in Hom(u^{\otimes 2},u),\eta\in Fix(u)\Big\rangle$$

But the relations $\mu\in Hom(u^{\otimes 2},u)$ and $\eta\in Fix(u)$ produce the 1st and 4th relations in the statement, then the biunitarity of u gives the 5th relation, and finally the 2nd and 3rd relations follow from the 1st and 4th relations, by using the antipode. □

As an illustration, consider the case $Z=\{1,\ldots,N\}$. Here the index multiplication is $ii=i$ and $ij=\emptyset$ for $i\neq j$, and the involution is $\bar{i}=i$. Thus, our relations are as follows, corresponding to the standard magic conditions on a matrix $u=(u_{ij})$:

$$u_{ik}u_{il}=\delta_{kl}u_{ik}\quad,\quad u_{ik}u_{jk}=\delta_{ij}u_{ik}$$

$$\sum_i u_{ij}=1\quad,\quad\sum_j u_{ij}=1$$

$$u_{ij}^*=u_{ij}$$

As a second illustration now, which is something new, we have:

THEOREM 13.17. *For the space $Z=M_2$, coming via $C(Z)=M_2(\mathbb{C})$, we have*

$$S_Z^+=SO_3$$

with the action $SO_3\curvearrowright M_2(\mathbb{C})$ being the standard one, coming from $SU_2\to SO_3$.

PROOF. This is something quite tricky, the idea being as follows:

(1) First, we have an action by conjugation $SU_2\curvearrowright M_2(\mathbb{C})$, and this action produces, via the canonical quotient map $SU_2\to SO_3$, an action as follows:

$$SO_3\curvearrowright M_2(\mathbb{C})$$

(2) Then, it is routine to check, by using computations like those from the proof of $S_N^+=S_N$ at $N\leq 3$, from chapter 9, that any action $G\curvearrowright M_2(\mathbb{C})$ must come from a classical group. Thus the action $SO_3\curvearrowright M_2(\mathbb{C})$ is universal, as claimed.

(3) This was for the idea, and we will actually come back to this in a moment, in a more general setting, and with a new proof, complete this time. □

Let us develop now some basic theory for the quantum symmetry groups S_Z^+, and their closed subgroups $G\subset S_Z^+$. We have here the following key result, from [5]:

THEOREM 13.18. *The quantum groups S_Z^+ have the following properties:*

1. *The associated Tannakian categories are TL_N, with $N=|Z|$.*
2. *The main character follows the Marchenko-Pastur law π_1, when $|Z|\geq 4$.*
3. *The fusion rules for S_Z^+ with $|Z|\geq 4$ are the same as for SO_3.*

PROOF. This result is from [5], the idea being as follows:

(1) Let us pick our orthogonal basis $\{e_i\}$ as in Definition 13.15, so that we have, for a certain involution $i\to\bar{i}$ on the index set, the following formula:

$$e_i^*=e_{\bar{i}}$$

(2) With this convention, we have the following computation:

$$\begin{aligned}\Phi(e_i)=\sum_j e_j\otimes u_{ji}\quad&\implies\quad \Phi(e_i)^*=\sum_j e_j^*\otimes u_{ji}^*\\&\implies\quad \Phi(e_{\bar{i}})=\sum_j e_{\bar{j}}\otimes u_{ji}^*\\&\implies\quad \Phi(e_i)=\sum_j e_j\otimes u_{\bar{i}\bar{j}}^*\end{aligned}$$

(3) We conclude from this that we have $u_{ji}^*=u_{\bar{i}\bar{j}}$ for any two indices i,j, and so that we have, at the corepresentation level, an equivalence as follows:

$$u\sim\bar{u}$$

(4) Now with this result in hand, our claim is that the proof goes as for the proof for S_N^+, from chapter 9. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of the algebra $C(Z)$ that we are interested in here, can be modelled by the following two diagrams:

$$m=|\cup|\qquad,\qquad u=\cap$$

Indeed, this is certainly true algebrically, and well-known, with as an illustration here, the associativity formula $m(m\otimes id)=(id\otimes m)m$ being checked as follows:

$$\begin{matrix}|&\cup&|\,|&|\\|&&\cup&|\end{matrix}\quad=\quad\begin{matrix}|&|\,|&\cup&|\\|&\cup&&|\end{matrix}$$

As in what regards the $*$-structure, things here are fine too, because our choice for the trace from Definition 13.11 leads to the following formula regarding the adjoints, corresponding to $mm^* = N$, and so to the basic Temperley-Lieb calculus rule $\bigcirc = N$:

$$\mu\mu^* = N \cdot id$$

(5) We conclude that the Tannakian category associated to S_Z^+ is, as claimed:

$$\begin{aligned} C &= <\mu, \eta> \\ &= <m, u> \\ &= <|\cup|, \cap> \\ &= TL_N \end{aligned}$$

(6) Regarding now the second assertion in the statement, the probability one, the proof here is exactly as for S_N^+, by using moments. To be more precise, according to (5) these moments are the Catalan numbers, which are the moments of π_1.

(7) Finally, regarding the fusion rule assertion, also at $|Z| \geq 4$, once again the same proof as for S_N^+ works, by using the fact that the moments of χ are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules for the group SO_3. □

We can merge and reformulate our main results so far in the following way:

THEOREM 13.19. *The quantun groups S_Z^+ have the following properties:*

1. *For $Z = \{1, \ldots, N\}$ we have $S_Z^+ = S_N^+$.*
2. *For the space $Z = M_N$ we have $S_Z^+ = PO_N^+ = PU_N^+$.*
3. *In particular, for the space $Z = M_2$ we have $S_Z^+ = SO_3$.*
4. *The fusion rules for S_Z^+ with $|Z| \geq 4$ are independent of Z.*
5. *Thus, the fusion rules for S_Z^+ with $|Z| \geq 4$ are the same as for SO_3.*

PROOF. This is basically a compact form of what has been said above, with a new result added, and with some technicalities left aside, the idea being as follows:

(1) This is something that we know from Theorem 13.14.

(2) This is something related to the isomorphism $PO_N^+ = PU_N^+$, which is well-known, and that we will reprove here, with our study. To start with, we already know what the quantum group U_N^+ is, and the definition of O_N^+ is similar, as follows:

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u},\, u^t = u^{-1}\right)$$

Consider now the standard vector space action of the quantum group U_N^+, and its adjoint action, with a straightforward definition for the projective version on the right:

$$U_N^+ \curvearrowright \mathbb{C}^N \quad , \quad PU_N^+ \curvearrowright M_N(\mathbb{C})$$

By universality of $S_{M_N}^+$, we must have inclusions as follows:

$$PO_N^+ \subset PU_N^+ \subset S_{M_N}^+$$

On the other hand, the main character of O_N^+ with $N\geq 2$ being known to be semicircular, the main character of PO_N^+ must be Marchenko-Pastur. Thus the inclusion $PO_N^+\subset S_{M_N}^+$ has the property that it keeps fixed the law of main character, and by Peter-Weyl we conclude that this inclusion must be an isomorphism, as desired.

(3) This is something that we know from Theorem 13.18, and that can be deduced as well from (2), by using the formula $PO_2^+=SO_3$, which is something elementary. Alternatively, this follows without computations from (4) below, because the inclusion of quantum groups $SO_3\subset S_{M_2}^+$ has the property that it preserves the fusion rules.

(4) This is something that we know from Theorem 13.18.

(5) This follows from (3,4), as already pointed out in Theorem 13.18. □

As an application of our extended formalism, the Cayley theorem for the finite quantum groups holds in the S_Z^+ setting. We have indeed the following result:

THEOREM 13.20. *Any finite quantum group G has a Cayley embedding, as follows:*

$$G\subset S_G^+$$

However, there are finite quantum groups which are not quantum permutation groups.

PROOF. There are two statements here, the idea being as follows:

(1) We have an action $G\curvearrowright G$, which leaves invariant the Haar measure. Now since the counting measure is left and right invariant, so is the Haar measure. We conclude that $G\curvearrowright G$ leaves invariant the counting measure, and so $G\subset S_G^+$, as claimed.

(2) Regarding the second assertion, this is something non-trivial, the simplest counterexample being a certain quantum group G appearing as a split abelian extension associated to the factorization $S_4=\mathbb Z_4S_3$, having cardinality $|G|=24$. See [6]. □

Finally, some interesting phenomena appear in the "homogeneous" case, where our quantum space is of the form $Z=M_K\times\{1\ldots,L\}$. Here we first have:

PROPOSITION 13.21. *The classical symmetry group of $Z=M_K\times\{1\ldots,L\}$ is*

$$S_Z=PU_K\wr S_L$$

with on the right a wreath product, equal by definition to $PU_K^L\rtimes S_L$.

PROOF. This can be proved in two steps, as follows:

(1) The fact that we have an inclusion $PU_K\wr S_L\subset S_Z$ is standard, and follows as well by taking the classical version of the inclusion $PU_K^+\wr_* S_L^+\subset S_Z^+$, established below.

(2) As for the fact that this inclusion $PU_K\wr S_L\subset S_Z$ is an isomorphism, this can be proved by picking an arbitrary element $g\in S_Z$, and decomposing it. □

Quite surprisingly, the quantum analogue of the above result fails:

THEOREM 13.22. *The quantum symmetry group of $Z = M_K \times \{1\ldots,L\}$ satisfies:*

$$PU_K^+ \wr_* S_L^+ \subset S_Z^+$$

However, this inclusion is not an isomorphism at $K, L \geq 2$.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) The fact that we have $PU_K^+ \wr_* S_L^+ \subset S_Z^+$ is well-known and routine, by checking the fact that the matrix $w_{ija,klb} = u_{ij,kl}^{(a)} v_{ab}$ is a generalized magic unitary.

(2) The inclusion $PU_K^+ \wr_* S_L^+ \subset S_Z^+$ is not an isomorphism, for instance by using [86], along with the fact that $\pi_1 \boxtimes \pi_1 \neq \pi_1$ where π_1 is the Marchenko-Pastur law. □

We will be back to quantum spaces of type $Z = M_K \times \{1\ldots,L\}$ in the next chapter, when discussing the quantum graphs modelled on such spaces.

13d. Twisting results

Let us go back to the case $N = 4$. According to our various considerations above, the link between S_4^+ and SO_3 should come via some sort of twisting. To be more precise, since the classical space $\{1,2,3,4\}$ and the quantum space M_2 both have 4 elements, in the formal sense of Definition 13.8, we can expect to have a twisting result, as follows:

$$\{1,2,3,4\} \sim M_2$$

It is possible to be a bit more precise here, by developing some abstract algebra for this, but going ahead now towards what we are interested in, namely quantum permutation groups, this suggests that we should have a twisting relationship, as follows:

$$S_4^+ = S_{\{1,2,3,4\}}^+ \sim S_{M_2}^+ = SO_3$$

In order to discuss this, we will use the quantum group O_N^{-1} from chapter 10. Now going towards S_4^+, let us start with the following definition:

DEFINITION 13.23. *We let $SO_3^{-1} \subset O_3^{-1}$ be the subgroup coming from the relation*

$$\sum_{\sigma \in S_3} u_{1\sigma(1)} u_{2\sigma(2)} u_{3\sigma(3)} = 1$$

called twisted determinant one condition.

Normally, we should prove here that $C(SO_3^{-1})$ as constructed above is indeed a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because this follows from the following result:

THEOREM 13.24. *We have an isomorphism of compact quantum groups*

$$S_4^+ = SO_3^{-1}$$

given by the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Consider the following matrix, coming from the action of SO_3^{-1} on $\mathbb{C}^4$:

$$u^+ = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

We apply to this matrix the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$v = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & u_{12} & u_{13} \\ 0 & u_{21} & u_{22} & u_{23} \\ 0 & u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

This matrix is then magic, and vice versa, so the Fourier transform over K converts the relations in Definition 13.23 into the magic relations. But this gives the result. $\square$

There are many more things that can be said here, and we have:

Theorem 13.25. *The quantum group $S_4^+ = SO_3^{-1}$ has the following properties:*

1. *It appears as a cocycle twist of SO_3.*
2. *Its fusion rules are the same as for SO_3.*

Proof. These are more advanced results, the idea being as follows:

(1) This follows by suitably reformulating the definition of SO_3^{-1} given above in purely algebraic terms, using cocycles, and for details here, we refer to [9].

(2) This is something that we know well, via numerous proofs, and we can add to our trophy list one more proof, coming from (1), via standard cocycle twisting theory. $\square$

At a more advanced level now, we have the following result, making a connection with the various ADE considerations from this book, from chapters 2, 4 and 12:

Theorem 13.26. *The closed subgroups of $S_4^+ = SO_3^{-1}$ are as follows:*

1. *Infinite quantum groups: S_4^+, O_2^{-1}, $\widehat{D}_\infty$.*
2. *Finite groups: S_4, and its subgroups.*
3. *Finite group twists: S_4^{-1}, A_5^{-1}.*
4. *Series of twists: D_{2n}^{-1} $(n \geq 3)$, DC_{2n}^{-1} $(n \geq 2)$.*
5. *A group dual series: $\widehat{D}_n$, with $n \geq 3$.*

Moreover, these quantum groups are subject to an ADE classification result.

Proof. This is something quite technical, from [9]. Regarding the precise statement, the idea is that, with the convention that prime stands for twists, all unique in the cases below, and that double prime denotes pseudo-twists, the classification is as follows:

(A) $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, K, $\widehat{D}_n$ $(n = 2, 3, \ldots, \infty)$, S_4^+.

(D) $\mathbb{Z}_4$, D'_{2n}, D''_{2n} $(n = 2, 3, \ldots)$, H_2^+, D_1, S_3.

(E) A_4, S_4, S'_4, A'_5.

There are many comments to be made here, regarding our various conventions, and the construction of some of the above quantum groups, as follows:

– To start with, the 2-element group $\mathbb{Z}_2 = \{1, \tau\}$ can act in 2 ways on 4 points: either with the transposition τ acting without fixed point, and we use here the notation $\mathbb{Z}_2$, or with τ acting with 2 fixed points, and we use here the notation D_1.

– Similarly, the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ can act in 2 ways on 4 points: either with 2 non-trivial elements having 2 fixed points each, and we use here the notation K, or with all non-trivial elements having no fixed points, and we use here the notation $D_2 = \widehat{D}_2$.

– We have $D_4' = D_4$, and $D_4'' = G_0$, the Kac-Paljutkin quantum group. Besides being a pseudo-twist of D_{2n}, the quantum group D_{2n}'' with $n \geq 2$ is known to be as well a pseudo-twist of the dicyclic, or binary cyclic group DC_{2n}.

– Finally, the definition of D_{2n}', D_{2n}'' can be extended at $n = 1, \infty$, and we formally have $D_2' = D_2'' = K$, and $D_\infty' = D_\infty'' = H_2^+$, but these conventions are not very useful. Also, as explained in [9], the groups D_1, S_3 are a bit special at (D).

We refer to [9] for the construction and various properties of the various twists and pseudo-twists in the above ADE list, and for the proof of the result as well. □

Many other things can be said here, notably with a subfactor result, making a connection with the planar algebra results from chapter 12. In what concerns us, let us rather derive some simple consequences of Theorem 13.26, in relation with the notion of transitivity from chapter 11, as a concrete illustration for that material:

THEOREM 13.27. *The small order transitive quantum groups are as follows:*

(1) *At $N = 1, 2, 3$ we have $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, S_3.*
(2) *At $N = 4$ we have $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_4, A_4, S_4, O_2^{-1}, S_4^+$ and S_4^{-1}, A_5^{-1}.*

In particular, the inclusion $S_N \subset S_N^+$ is maximal at $N \leq 4$.

PROOF. This follows from Theorem 13.26, the idea being as follows:

(1) This follows from the fact that we have $S_N = S_N^+$ at $N \leq 3$, from [91].

(2) This follows from the ADE classification of the subgroups $G \subset S_4^+$, from Theorem 13.26, with all the twists appearing in the statement being standard twists.

(3) Finally, the last assertion follows from (1) and (2), with the remark that any intermediate subgroup $S_N \subset G \subset S_N^+$ must be transitive. □

The above result can be extended to $N = 5$, with a classification of the subgroups $G \subset S_5^+$, and a proof of the maximality of $S_5 \subset S_5^+$, by using recent progress in subfactor theory, via standard correspondences with quantum groups. We refer here to [6].

Getting back now to our regular twisting business for the quantum groups S_Z^+, an interesting extension of the formula $S_4^+ = SO_3^{-1}$ comes by looking at the general case $N = n^2$, with $n \in \mathbb{N}$. We will prove that we have a twisting result, as follows:

$$PO_n^+ = (S_N^+)^\sigma$$

In order to explain this material, which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

Proposition 13.28. *Given a finite group G, the algebra $C(S_{\widehat{G}}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h \in G$, with the following relations:*

$$x_{1g} = x_{g1} = \delta_{1g} \quad , \quad x_{s,gh} = \sum_{t\in G} x_{st^{-1},g} x_{th} \quad , \quad x_{gh,s} = \sum_{t\in G} x_{gt^{-1}} x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh}) = \sum_{s\in G} x_{gs} \otimes x_{sh} \quad , \quad \varepsilon(x_{gh}) = \delta_{gh} \quad , \quad S(x_{gh}) = x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. This follows indeed from a direct verification. □

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on G, which is by definition a map $\sigma : G \times G \to \mathbb{C}^*$ satisfying:

$$\sigma_{gh,s}\sigma_{gh} = \sigma_{g,hs}\sigma_{hs} \quad , \quad \sigma_{g1} = \sigma_{1g} = 1$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{G}_\sigma)$, as being the vector space $C(\widehat{G}) = C^*(G)$, with product $e_g e_h = \sigma_{gh} e_{gh}$. We have:

Proposition 13.29. *The algebra $C(S_{\widehat{G}_\sigma}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h \in G$, with the relations $x_{1g} = x_{g1} = \delta_{1g}$ and:*

$$\sigma_{gh} x_{s,gh} = \sum_{t\in G} \sigma_{st^{-1},t} x_{st^{-1},g} x_{th} \quad , \quad \sigma_{gh}^{-1} x_{gh,s} = \sum_{t\in G} \sigma_{t^{-1},ts}^{-1} x_{gt^{-1}} x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh}) = \sum_{s\in G} x_{gs} \otimes x_{sh} \quad , \quad \varepsilon(x_{gh}) = \delta_{gh} \quad , \quad S(x_{gh}) = \sigma_{h^{-1}h}\sigma_{g^{-1}g}^{-1} x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. Once again, this follows from a direct verification. □

In what follows, we will prove that the quantum groups $S_{\widehat{G}}^+$ and $S_{\widehat{G}_\sigma}^+$ are related by a cocycle twisting operation. Let A be a Hopf algebra. We recall that a left 2-cocycle is a convolution invertible linear map $\sigma : A \otimes A \to \mathbb{C}$ satisfying:

$$\sigma_{x_1y_1}\sigma_{x_2y_2,z} = \sigma_{y_1z_1}\sigma_{x,y_2z_2} \quad , \quad \sigma_{x1} = \sigma_{1x} = \varepsilon(x)$$

Note that σ is a left 2-cocycle if and only if σ^{-1}, the convolution inverse of σ, is a right 2-cocycle, in the sense that we have:

$$\sigma^{-1}_{x_1y_1,z}\sigma^{-1}_{x_1y_2}=\sigma^{-1}_{x,y_1z_1}\sigma^{-1}_{y_2z_2}\quad,\quad\sigma^{-1}_{x1}=\sigma^{-1}_{1x}=\varepsilon(x)$$

Given a left 2-cocycle σ on A, one can form the 2-cocycle twist A^σ as follows. As a coalgebra, $A^\sigma=A$, and an element $x\in A$, when considered in A^σ, is denoted $[x]$. The product in A^σ is then defined, in Sweedler notation, by:

$$[x][y]=\sum\sigma_{x_1y_1}\sigma^{-1}_{x_3y_3}[x_2y_2]$$

We can now state and prove a main theorem, as follows:

THEOREM 13.30. *If G is a finite group and σ is a 2-cocycle on G, the Hopf algebras*

$$C(S^+_{\widehat{G}})\quad,\quad C(S^+_{\widehat{G}_\sigma})$$

are 2-cocycle twists of each other, in the above sense.

PROOF. In order to prove this result, we use the following Hopf algebra map:

$$\pi:C(S^+_{\widehat{G}})\to C(\widehat{G})\quad,\quad x_{gh}\to\delta_{gh}e_g$$

Our 2-cocycle $\sigma:G\times G\to\mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma:C(\widehat{G})\otimes C(\widehat{G})\to\mathbb{C}$$

Consider now the following composition:

$$\alpha=\sigma(\pi\otimes\pi):C(S^+_{\widehat{G}})\otimes C(S^+_{\widehat{G}})\to C(\widehat{G})\otimes C(\widehat{G})\to\mathbb{C}$$

Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1}. Thus we can construct the twisted algebra $C(S^+_{\widehat{G}})^{\alpha^{-1}}$, and inside this algebra we have the following computation:

$$[x_{gh}][x_{rs}]=\alpha^{-1}(x_g,x_r)\alpha(x_h,x_s)[x_{gh}x_{rs}]=\sigma^{-1}_{gr}\sigma_{hs}[x_{gh}x_{rs}]$$

By using this, we obtain the following formula:

$$\sum_{t\in G}\sigma_{st^{-1},t}[x_{st^{-1},g}][x_{th}]=\sum_{t\in F}\sigma_{st^{-1},t}\sigma^{-1}_{st^{-1},t}\sigma_{gh}[x_{st^{-1},g}x_{th}]=\sigma_{gh}[x_{s,gh}]$$

Similarly, we have the following formula:

$$\sum_{t\in G}\sigma^{-1}_{t^{-1},ts}[x_{g,t^{-1}}][x_{h,ts}]=\sigma^{-1}_{gh}[x_{gh,s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\Phi:C(S^+_{\widehat{G}_\sigma})\to C(S^+_{\widehat{G}})^{\alpha^{-1}}\quad,\quad x_{gh}\to[x_{g,h}]$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. $\square$

Thus, we have proved our main twisting result. As a first consequence, we have:

PROPOSITION 13.31. *If G is a finite group and σ is a 2-cocycle on G, then*

$$\Phi(x_{g_1h_1}\ldots x_{g_mh_m})=\Omega(g_1,\ldots,g_m)^{-1}\Omega(h_1,\ldots,h_m)x_{g_1h_1}\ldots x_{g_mh_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1,\ldots,g_m)=\prod_{k=1}^{m-1}\sigma_{g_1\ldots g_k,g_{k+1}}$$

is a coalgebra isomorphism $C(S^+_{\widehat{G}_\sigma})\to C(S^+_{\widehat{G}})$, commuting with the Haar integrals.

PROOF. This is indeed just a technical reformulation of Theorem 13.30. □

Let us discuss now some concrete applications of the general results established above. Consider the group $G=\mathbb{Z}_n^2$, let $w=e^{2\pi i/n}$, and consider the following cocycle:

$$\sigma:G\times G\to\mathbb{C}^*\quad,\quad\sigma_{(ij)(kl)}=w^{jk}$$

In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i,j\in\mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. We first have:

PROPOSITION 13.32. *The linear map given by*

$$\psi(e_{(i,j)})=\sum_{k=0}^{n-1}w^{ki}E_{k,k+j}$$

defines an isomorphism of algebras $\psi:C(\widehat{G}_\sigma)\simeq M_n(\mathbb{C})$.

PROOF. Consider indeed the following linear map:

$$\psi'(E_{ij})=\frac{1}{n}\sum_{k=0}^{n-1}w^{-ik}e_{(k,j-i)}$$

It is routine to check that both ψ,ψ' are morphisms of algebras, and that these maps are inverse to each other. In particular, ψ is an isomorphism of algebras, as stated. □

Next in line, we have the following result:

PROPOSITION 13.33. *The algebra map given by*

$$\varphi(u_{ij}u_{kl})=\frac{1}{n}\sum_{a,b=0}^{n-1}w^{ai-bj}x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi:C(S^+_{M_n})\simeq C(S^+_{\widehat{G}_\sigma})$.

PROOF. Consider the universal coactions on the two algebras in the statement:

$$\begin{aligned}\alpha : M_n(\mathbb{C}) &\to M_n(\mathbb{C}) \otimes C(S^+_{M_n})\\ \beta : C(\widehat{G}_\sigma) &\to C(\widehat{G}_\sigma) \otimes C(S^+_{\widehat{G}_\sigma})\end{aligned}$$

In terms of the standard bases, these coactions are given by:

$$\begin{aligned}\alpha(E_{ij}) &= \sum_{kl} E_{kl} \otimes u_{ki}u_{lj}\\ \beta(e_{(i,j)}) &= \sum_{kl} e_{(k,l)} \otimes x_{(k,l),(i,j)}\end{aligned}$$

We use now the identification $C(\widehat{G}_\sigma) \simeq M_n(\mathbb{C})$ from Proposition 13.32. This identification produces a coaction map, as follows:

$$\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes C(S^+_{\widehat{G}_\sigma})$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij}) = \frac{1}{n} \sum_{ab} E_{ab} \otimes \sum_{kr} w^{ar-ik} x_{(r,b-a),(k,j-i)}$$

By comparing with the formula of α, we obtain the isomorphism in the statement. □

We will need one more result of this type, as follows:

PROPOSITION 13.34. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)}) = \frac{1}{n^2} \sum_{klrs} w^{ki+lj-ra-sb} p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho : C(S^+_{\widehat{G}}) \simeq C(S^+_G)$.

PROOF. We have a Fourier transform isomorphism, as follows:

$$C(\widehat{G}) \simeq C(G)$$

Thus the algebras in the statement are indeed isomorphic. □

As a conclusion to all this, we have the following result:

THEOREM 13.35. *Let* $n \geq 2$ *and* $w = e^{2\pi i/n}$. *Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n} \sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism $C(PO^+_n) \to C(S^+_{n^2})$ *commuting with the Haar integrals.*

PROOF. We know from before that we have identifications as follows, where the projective version of (A,u) is the pair (PA,v), with $PA=<v_{ij}>$ and $v=u\otimes\bar{u}$:

$$PO_n^+=PU_n^+=S_{M_n}^+$$

With this in hand, the result follows from Theorem 13.30 and Proposition 13.31, by combining them with the various isomorphisms established above. □

Summarizing, the main twisting formula $S_4^+=SO_3^{-1}$ ultimately comes from something of type $X_4\simeq M_2$, where $X_4=\{1,2,3,4\}$ and $M_2=Spec(M_2(\mathbb{C}))$, and at $N\geq 5$ there are some extensions of this result, and notably when $N=n^2$ with $n\geq 3$. For more on all this, details, examples, references and the whole story, we refer to [**6**], [**9**], [**13**].

13e. Exercises

We had a tough algebraic chapter here, and as exercises on this, we have:

EXERCISE 13.36. *Learn more about the finite dimensional C^*-algebras.*

EXERCISE 13.37. *Learn too about the finite dimensional von Neumann algebras.*

EXERCISE 13.38. *Work out all the details for the proof of $S_{M_2}^+=SO_3$.*

EXERCISE 13.39. *Learn about the ADE classification of the subgroups of SO_3.*

EXERCISE 13.40. *Learn too about the ADE classification of the subgroups of S_4^+.*

EXERCISE 13.41. *Find further twisting results for quantum automorphism groups.*

EXERCISE 13.42. *Learn more about the quantum Cayley theorem, and its failure.*

EXERCISE 13.43. *Work out details for the symmetry groups of homogeneous spaces.*

As bonus exercise, try understanding what happens when using other ground fields.

CHAPTER 14

Quantum graphs

14a. Quantum graphs

Time now to do some quantum graph work, by adding edges to the finite quantum spaces constructed in chapter 13. We have indeed the following straightforward extension of the usual notion of finite graph, from [41] and [86], obtained by using a finite quantum space as set of vertices, and something quite general as adjacency matrix:

Definition 14.1. *We call "finite quantum graph" a pair of type*

$$X=(Z,d)$$

with Z being a finite quantum space, and $d\in M_N(\mathbb{C})$ being a matrix, where $N=|Z|$.

This is of course something quite general. In the case $Z=\{1,\ldots,N\}$ for instance, what we have here is a directed graph, with the edges $i\to j$ colored by complex numbers $d_{ij}\in\mathbb{C}$, and with self-edges $i\to i$ allowed too, again colored by numbers $d_{ii}\in\mathbb{C}$. In the general case, however, where Z is arbitrary, the need for extra conditions of type $d=d^*$, or $d_{ii}=0$, or $d\in M_N(\mathbb{R})$, or $d\in M_N(0,1)$ and so on, is not very natural, as we will soon discover, and it is best to use Definition 14.1 as such, with no restrictions on d.

In general, a quantum graph can be represented as a colored oriented graph on $\{1,\ldots,N\}$, where $N=|Z|$, with the vertices being decorated by single indices i, and with the colors being complex numbers, namely the entries of d. This is similar to the formalism from before, but there is a discussion here in what regards the exact choice of the colors, which are usually irrelevant in connection with our symmetry problematics, and so can be true colors instead of complex numbers. More on this later.

With the above notion in hand, we have the following definition, also from [41]:

Definition 14.2. *The quantum automorphism group of $X=(Z,d)$ is the subgroup*

$$G^+(X)\subset S_Z^+$$

obtained via the relation $du=ud$, where $u=(u_{ij})$ is the fundamental corepresentation.

Again, this is something very natural, coming as a continuation of our previous constructions for usual graphs. We refer to [41], [86] for more on this notion, and for a number of advanced computations, in relation with free wreath products.

At an elementary level, to start with, a first problem is that of working out the basics of the correspondence $X \to G^+(X)$, in analogy with the material from chapter 10. There are several things to be done here, namely simplices, complementation, color independence, multi-simplices, and with a few twists, all this basically extends well.

Let us start with the simplices. As we will soon discover, things are quite tricky here, leading us in particular to the conclusion that the simplex based on an arbitrary finite quantum space Z is not a usual graph, with $d \in M_N(0,1)$ where $N = |Z|$, but rather a sort of "signed graph", with $d \in M_N(-1,0,1)$. Let us start our study with:

THEOREM 14.3. *Given a finite quantum space Z, we have*

$$G^+(Z_{empty}) = G^+(Z_{full}) = S_Z^+$$

where Z_{empty} is the empty graph on the vertex set Z, coming from the matrix $d = 0$, and where Z_{full} is the simplex on the vertex set Z, coming from the matrix

$$d = NP_1 - 1_N$$

where $N = |Z|$, and where P_1 is the orthogonal projection on the unit $1 \in C(Z)$.

PROOF. This is something quite tricky, the idea being as follows:

(1) First of all, the formula $G^+(Z_{empty}) = S_Z^+$ is clear from definitions, because the commutation of u with the matrix $d = 0$ is automatic.

(2) Regarding the formula $G^+(Z_{full}) = S_Z^+$, let us first discuss the classical case, $Z = \{1, \ldots, N\}$. Here the simplex Z_{full} is the graph having having edges between any two vertices, whose adjacency matrix is as follows, with $\mathbb{I}_N$ being the all-1 matrix:

$$d = \mathbb{I}_N - 1_N$$

The commutation of u with 1_N being automatic, and the commutation with $\mathbb{I}_N$ being automatic too, our magic matrix u being bistochastic, we have:

$$[u, d] = 0$$

Thus we have indeed $G^+(Z_{full}) = S_Z^+$ in this case, as claimed.

(3) In the general case, we know from chapter 13 that we have $\eta \in Fix(u)$, with $\eta : \mathbb{C} \to C(Z)$ being the unit map. We deduce from this that we have:

$$P_1 \in End(u)$$

Thus, $[u, P_1] = 0$ is automatic. Together with the fact that in the classical case we have $\mathbb{I}_N = NP_1$, this suggests to define the adjacency matrix of the simplex as being:

$$d = NP_1 - 1_N$$

And, with this definition, we have indeed $G^+(Z_{full}) = S_Z^+$, as claimed. $\square$

Let us study now the simplices Z_{full} found in Theorem 14.3. In the classical case, $Z = \{1, \ldots, N\}$, what we have is of course the usual simplex. However, in the general case things are more mysterious, the first result here being as follows:

PROPOSITION 14.4. *The adjacency matrix of the simplex Z_{full}, given by definition by $d = NP_1 - 1_N$, is a matrix $d \in M_N(-1, 0, 1)$, which can be computed as follows:*

1. *In single index notation, $d_{ij} = \delta_{i\bar{i}}\delta_{j\bar{j}} - \delta_{ij}$.*
2. *In double index notation, $d_{ab,cd} = \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}$.*
3. *In triple index notation, $d_{abp,cdq} = \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\delta_{pq}$.*

PROOF. According to our single index conventions, from chapter 13, the adjacency matrix of the simplex is the one in the statement, namely:

$$\begin{aligned} d_{ij} &= (NP_1 - 1_N)_{ij} \\ &= \bar{1}_i 1_j - \delta_{ij} \\ &= \delta_{i\bar{i}}\delta_{j\bar{j}} - \delta_{ij} \end{aligned}$$

In double index notation now, with $i = (ab)$ and $j = (cd)$, and a, b, c, d being usual matrix indices, each thought to be attached to the corresponding matrix block of $C(Z)$, the formula that we obtain in the second one in the statement, namely:

$$\begin{aligned} d_{ab,cd} &= \delta_{ab,ba}\delta_{cd,dc} - \delta_{ab,cd} \\ &= \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} \end{aligned}$$

Finally, in standard triple index notation, $i = (abp)$ and $j = (cdq)$, with a, b, c, d being now usual numeric matrix indices, ranging in $1, 2, 3, \ldots$, and with p, q standing for corresponding blocks of the algebra $C(Z)$, the formula that we obtain is:

$$\begin{aligned} d_{abp,cdq} &= \delta_{abp,bap}\delta_{cdq,dcq} - \delta_{abp,cdq} \\ &= \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\delta_{pq} \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

At the level of examples, for $Z = \{1, \ldots, N\}$ the best is to use the above formula (1). The involution on the index set is $\bar{i} = i$, and we obtain, as we should:

$$d_{ij} = 1 - \delta_{ij}$$

As a more interesting example now, for the quantum space $Z = M_n$, coming by definition via the formula $C(Z) = M_n(\mathbb{C})$, the situation is as follows:

PROPOSITION 14.5. *The simplex Z_{full} with $Z = M_n$ is as follows:*

1. *The vertices are n^2 points in the plane, arranged in square form.*
2. *Usual edges, worth 1, are drawn between distinct points on the diagonal.*
3. *In addition, each off-diagonal point comes with a self-edge, worth -1.*

Proof. Here the most convenient is to use the double index formula from Proposition 14.4 (2), which tells us that d is as follows, with indices $a, b, c, d \in \{1, \ldots, n\}$:

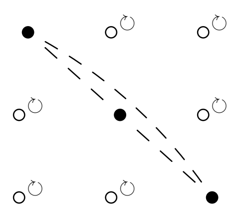
$$d_{ab,cd} = \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}$$

This quantity can be $-1, 0, 1$, and the study goes as follows:

– Case $d_{ab,cd} = 1$. This can only happen when $\delta_{ab}\delta_{cd} = 1$ and $\delta_{ac}\delta_{bd} = 0$, corresponding to a formula of type $d_{aa,cc} = 0$, with $a \neq c$, and so to the edges in (2).

– Case $d_{ab,cd} = -1$. This can only happen when $\delta_{ab}\delta_{cd} = 0$ and $\delta_{ac}\delta_{bd} = 1$, corresponding to a formula of type $d_{ab,ab} = 0$, with $a \neq b$, and so to the self-edges in (3). □

The above result is quite interesting, and as an illustration, here is the pictorial representation of the simplex Z_{full} on the vertex set $Z = M_3$, with the convention that the solid arrows are worth -1, and the dashed arrows are worth 1:



More generally, we can in fact compute Z_{full} for any finite quantum space Z, with the result here, which will be our final saying on the subject, being as follows:

Theorem 14.6. *Consider a finite quantum space Z, and write it as follows, according to the decomposition formula $C(Z) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ for its function algebra:*

$$Z = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

The simplex Z_{full} is then the classical simplex formed by the points lying on the diagonals of $M_{n_1}, \ldots, M_{n_k}$, with self-edges added, each worth -1, at the non-diagonal points.

Proof. The study here is quite similar to the one from the proof of Proposition 14.5, but by using this time the triple index formula from Proposition 14.4 (3), namely:

$$d_{abp,cdq} = \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\delta_{pq}$$

Indeed, this quantity can be $-1, 0, 1$, and the 1 case appears precisely as follows, leading to the classical simplex mentioned in the statement:

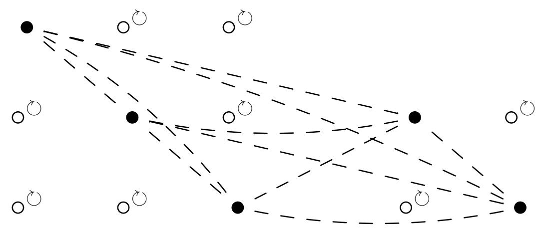
$$d_{aap,ccq} = 1 \quad , \quad \forall ap \neq cq$$

As for the remaining -1 case, this appears precisely as follows, leading this time to the self-edges worth -1, also mentioned in the statement:

$$d_{abp,abp} = 1 \quad , \quad \forall a \neq b$$

Thus, we are led to the conclusion in the statement. □

As an illustration, here is the simplex on the vertex set $Z = M_3 \sqcup M_2$, with again the convention that the solid arrows are worth -1, and the dashed arrows are worth 1:



Long story short, we know what the simplex Z_{full} is, and we have the formula $G^+(Z_{empty}) = G^+(Z_{full}) = S_Z^+$, exactly as in the $Z = \{1, \ldots, N\}$ case. Now with the above results in hand, we can talk as well about complementation, as follows:

THEOREM 14.7. *For any finite quantum graph X we have the formula*

$$G^+(X) = G^+(X^c)$$

where $X \to X^c$ is the complementation operation, given by $d_X + d_{X^c} = d_{Z_{full}}$.

PROOF. This follows from Theorem 14.3, and more specifically from the following commutation relation, which is automatic, as explained there:

$$[u, d_{Z_{full}}] = 0$$

Let us mention too that, in what concerns the pictorial representation of X^c, this can be deduced from what we have Theorem 14.6, in the obvious way. □

14b. Color independence

Following now [6], let us discuss an important theoretical point, namely the "independence on the colors" question. The idea indeed is that given a classical graph X with edges colored by complex numbers, or other types of colors, $G(X)$ does not change when changing the colors. This is obvious, and a quantum analogue of this fact, involving $G^+(X)$, can be shown to hold as well, as explained in [6], and in chapter 10.

In the quantum graph setting things are more complicated, as we will soon discover. Let us start with the following technical definition:

DEFINITION 14.8. *We call a quantum graph $X = (Z, d)$ washable if, with*

$$d = \sum_c c d_c$$

being the color decomposition of d, we have the equivalence

$$[u, d] = 0 \iff [u, d_c] = 0, \forall c$$

valid for any magic unitary matrix u, having size $|Z|$.

Obviously, this is something which is not very beautiful, but the point is that some quantum graphs are washable, and some other are not, and so we have to deal with the above definition, as stated. As a first observation, we have the following result:

PROPOSITION 14.9. *Assuming that $X = (Z, d)$ is washable, its quantum symmetry group $G^+(X)$ does not depend on the precise colors of X. That is, whenever we have another quantum graph $X' = (Z, d')$ with same color scheme, in the sense that*

$$d_{ij} = d_{kl} \iff d'_{ij} = d'_{kl}$$

we have $G^+(X) = G^+(X')$.

PROOF. This is something which is clear from the definition of $G^+(X)$, namely:

$$C(G^+(X)) = C(S_Z^+) \Big/ \Big\langle [u, d] = 0 \Big\rangle$$

Indeed, assuming that our graph is washable in the above sense, we have:

$$C(G^+(X)) = C(S_Z^+) \Big/ \Big\langle [u, d_c] = 0, \forall c \Big\rangle$$

Thus, we are led to the conclusion in the statement. □

As already mentioned, it was proved in [6] that in the classical case, $Z = \{1, \ldots, N\}$, all graphs are washable. This is a key result, and this for several reasons:

(1) First, this gives some intuition on what is going on with respect to colors, in analogy with what happens for $G(X)$. Also, it allows the use of true colors, like black, blue, red and so on, when drawing colored graphs, instead of complex numbers.

(2) Also, this can be combined with the fact that $G^+(X)$ is invariant as well via some similar changes in the spectral decomposition of d, at the level of eigenvalues, with all this leading to some powerful combinatorial methods for the computation of $G^+(X)$.

All these things do not necessarily hold in general, and to start with, we have:

THEOREM 14.10. *There are quantum graphs, such as the simplex in the homogeneous quantum space case, where*

$$Z = M_K \times \{1, \ldots, L\}$$

with $K, L \geq 2$, which are not washable.

PROOF. We know that the simplex, in the case $Z = M_K \times \{1, \ldots, L\}$, has as adjacency matrix a certain matrix $d \in M_N(-1, 0, 1)$, with $N = K^2 L$. Moreover, assuming $K, L \geq 2$ as above, entries of all types, $-1, 0, 1$, are possible. Thus, the color decomposition of the adjacency matrix is as follows, with all 3 components being nonzero:

$$d = -1 \cdot d_{-1} + 0 \cdot d_0 + 1 \cdot d_1$$

Now assume that our simplex $X = Z_{full}$ is washable, and let u be the fundamental corepresentation of $G^+(X)$. We have then the following commutation relations:

$$d_{-1}, d_0, d_1 \in End(u)$$

Now since the matrices d_{-1}, d_0, d_1 are all nonzero, we deduce from this that:

$$\dim(End(u)) \geq 3$$

On the other hand, we know that we have $G^+(X) = S_Z^+$. Also, we know from chapter 13 that the Tannakian category of S_Z^+ is the Temperley-Lieb category TL_N, with the index being $N = K^2L$ as above. By putting these two results together, we obtain:

$$\dim(End(u)) = \dim\left(span\left(\,|\;|\;,\;\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right)\right) \leq 2$$

Thus, we have a contradiction, and so our simplex is not washable, as claimed. $\square$

In order to come up with some positive results as well, the general idea will be that of using the method in [6]. We have the following statement, coming from there:

THEOREM 14.11. *The following matrix belongs to $End(u)$, for any $n \in \mathbb{N}$:*

$$d_{ij}^{\times n} = \sum_{i=k_1\ldots k_n} \sum_{j=l_1\ldots l_n} d_{k_1l_1} \ldots d_{k_nl_n}$$

In particular, in the classical case, $Z = \{1, \ldots, N\}$, all graphs are washable.

PROOF. We have two assertions here, the idea being as follows:

(1) Consider the multiplication and comultiplication maps of the algebra $C(Z)$, which in single index notation are given by:

$$\mu(e_i \otimes e_j) = e_{ij} \quad , \quad \gamma(e_i) = \sum_{i=jk} e_j \otimes e_k$$

Observe that we have $\mu^* = \gamma$, with the adjoint taken with respect to the scalar product coming from the canonical trace. We conclude that we have:

$$\mu \in Hom(u^{\otimes 2}, u) \quad , \quad \gamma \in Hom(u, u^{\otimes 2})$$

The point now is that we can consider the iterations $\mu^{(n)}, \gamma^{(n)}$ of μ, γ, constructed in the obvious way, and we have then, for any $n \in \mathbb{N}$:

$$\mu^{(n)} \in Hom(u^{\otimes n}, u) \quad , \quad \gamma^{(n)} \in Hom(u, u^{\otimes n})$$

Now if we assume that we have $d \in End(u)$, we have $d^{\otimes n} \in End(u^{\otimes n})$ for any $n \in \mathbb{N}$, and we conclude that we have the following formula:

$$\mu^{(n)} d^{\otimes n} \gamma^{(n)} \in End(u)$$

But, in single index notation, we have the following formula:

$$(\mu^{(n)}d^{\otimes n}\gamma^{(n)})_{ij}=\sum_{i=k_1\ldots k_n}\sum_{j=l_1\ldots l_n}d_{k_1l_1}\ldots d_{k_nl_n}$$

Thus, we are led to the conclusion in the statement.

(2) Assuming that we are in the case $Z=\{1,\ldots,N\}$, the matrix $d^{\times n}$ in the statement is simply the componentwise n-th power of d, given by:

$$d_{ij}^{\times n}=d_{ij}^n$$

As explained in [6] or in chapter 10, a simple analytic argument based on this, using $n\to\infty$ and then a recurrence on the number of colors, shows that we have washability indeed. Thus, we are led to the conclusions in the statement. □

In order now to further exploit the findings in Theorem 14.11, an idea would be that of assuming that we are in the homogeneous case, $Z=M_K\times\{1,\ldots,L\}$, and that the adjacency matrix is split, in the sense that one of the following happens:

$$d_{ab,cd}=e_{ab}f_{cd}\quad,\quad d_{ab,cd}=e_{ac}f_{bd}\quad,\quad d_{ab,cd}=e_{ad}f_{bc}$$

Normally the graph should be washable in this case, but the computations are quite complex, and there is no clear result known in this sense. To be more precise, we know from Theorem 14.11 that we have the following formula, in single index notation:

$$d_{ij}^{\times n}=\sum_{i=k_1\ldots k_n}\sum_{j=l_1\ldots l_n}d_{k_1l_1}\ldots d_{k_nl_n}$$

In double index notation, which is more convenient for our purposes here, we have:

$$\begin{aligned}d_{ab,cd}^{\times n}&=\sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}}d_{ax_1,cy_1}d_{x_1x_2,y_1y_2}d_{x_2x_3,y_2y_3}\ldots\ldots\\&\qquad\qquad\ldots\ldots d_{x_{n-2}x_{n-1},y_{n-2}y_{n-1}}d_{x_{n-1}b,y_{n-1}d}\end{aligned}$$

We have 3 cases to be investigated, and here are the computations:

(1) In the case $d_{ab,cd}=e_{ab}f_{cd}$ we have the following computation:

$$\begin{aligned}d_{ab,cd}^{\times n}&=\sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}}e_{ax_1}f_{cy_1}e_{x_1x_2}f_{y_1y_2}e_{x_2x_3}f_{y_2y_3}\ldots\ldots\\&\qquad\qquad\ldots\ldots e_{x_{n-2}x_{n-1}}f_{y_{n-2}y_{n-1}}e_{x_{n-1}b}f_{y_{n-1}d}\\&=\sum_{x_1\ldots x_{n-1}}e_{ax_1}e_{x_1x_2}e_{x_2x_3}\ldots\ldots e_{x_{n-2}x_{n-1}}e_{x_{n-1}b}\\&\quad\sum_{y_1\ldots y_{n-1}}f_{cy_1}f_{y_1y_2}f_{y_2y_3}\ldots\ldots f_{y_{n-2}y_{n-1}}f_{y_{n-1}d}\\&=(e^n)_{ab}(f^n)_{cd}\end{aligned}$$

(2) In the case $d_{ab,cd}=e_{ac}f_{bd}$ we have the following computation, where the $\times$ operation at the end is the usual componentwise product of the square matrices, and where E is the total sum of the entries of a given square matrix:

$$\begin{aligned}
d^{\times n}_{ab,cd} &= \sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}} e_{ac}f_{x_1y_1}e_{x_1y_1}f_{x_2y_2}e_{x_2y_2}f_{x_3y_3}\ldots\ldots\\
&\qquad\qquad\ldots\ldots e_{x_{n-2}y_{n-2}}f_{x_{n-1}y_{n-1}}e_{x_{n-1}y_{n-1}}f_{bd}\\
&= e_{ac}f_{bd}\sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}}(e\times f)_{x_1y_1}(e\times f)_{x_2y_2}\ldots\ldots(e\times f)_{x_{n-1}y_{n-1}}\\
&= e_{ac}e_{bd}E\left[(e\times f)^{n-1}\right]
\end{aligned}$$

(3) In the case $d_{ab,cd}=e_{ad}f_{bc}$ we have the following computation, which depends on the parity of n, and which comes, again, in a rather rough form:

$$\begin{aligned}
d^{\times n}_{ab,cd} &= \sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}} e_{ay_1}f_{x_1c}e_{x_1y_2}f_{x_2y_1}e_{x_2y_3}f_{x_3y_2}\ldots\ldots\\
&\qquad\qquad\ldots\ldots e_{x_{n-2}y_{n-1}}f_{x_{n-1}y_{n-2}}e_{x_{n-1}d}f_{by_{n-1}}\\
&= \sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}} e_{ay_1}(f^t)_{y_1x_2}e_{x_2y_3}\ldots\ldots(f^t)_{cx_1}e_{x_1y_2}(f^t)_{x_2y_3}\ldots\ldots\\
&= \left[(ef^t)^{n/2}\right]_{ad}\left[(f^te)^{n/2}\right]_{cb}
\end{aligned}$$

As a conclusion to all this, the basic theory of the quantum groups $G^+(X)$ from chapter 10 extends well to the present quantum graph setting, modulo some subtleties in connection with the colors. We will be back to this, later in this chapter.

14c. Twisted reflections

With the above technology in hand, we can talk about twisted quantum reflections. The idea will be that the twisted analogues of the quantum reflection groups $H_N^{s+}\subset S_{sN}^+$ will be the quantum automorphism groups $S^+_{Z\to Y}$ of the fibrations of finite quantum spaces $Z\to Y$, which correspond by definition to the Markov inclusions of finite dimensional C^*-algebras $C(Y)\subset C(Z)$. In order to discuss this, let us start with:

Definition 14.12. *A fibration of finite quantum spaces $Z\to Y$ corresponds to an inclusion of finite dimensional C^*-algebras*

$$C(Y)\subset C(Z)$$

which is Markov, in the sense that it commutes with the canonical traces.

Here the commutation condition with the canonical traces means that the composition $C(Y)\subset C(Z)\to\mathbb{C}$ should equal the canonical trace $C(Y)\to\mathbb{C}$. At the level of the corresponding quantum spaces, this means that the quotient map $Z\to Y$ must commute with the corresponding counting measures, and this is where our term "fibration" comes from. In order to talk now about the quantum symmetry groups $S^+_{Z\to Y}$, we will need:

PROPOSITION 14.13. *Given a fibration $Z \to Y$, a closed subgroup $G \subset S_Z^+$ leaves invariant Y precisely when its magic unitary $u = (u_{ij})$ satisfies the condition*

$$e \in End(u)$$

where $e : C(Z) \to C(Z)$ is the Jones projection, onto the subalgebra $C(Y) \subset C(Z)$.

PROOF. This is something that we know well, in the commutative case, where Z is a usual finite set, and the proof in general is similar. □

We can now talk about twisted quantum reflection groups, as follows:

THEOREM 14.14. *Any fibration of finite quantum spaces $Z \to Y$ has a quantum symmetry group, which is the biggest acting on Z by leaving Y invariant:*

$$S^+_{Z\to Y} \subset S^+_Z$$

At the level of algebras of functions, this quantum group $S^+_{Z\to Y}$ is obtained as follows, with $e : C(Z) \to C(Y)$ being the Jones projection:

$$C(S^+_{Z\to Y}) = C(S^+_Z)\Big/\Big\langle e \in End(u)\Big\rangle$$

We call these quantum groups $S^+_{Z\to Y}$ twisted quantum reflection groups.

PROOF. This follows indeed from Proposition 14.13, via some standard identifications for all the objects involved, as explained in the above. □

As a basic example, let us discuss the commutative case. Here we have:

PROPOSITION 14.15. *In the commutative case, the fibration $Z \to Y$ must be of the following special form, with N, s being certain integers,*

$$\{1,\ldots,N\} \times \{1,\ldots,s\} \to \{1,\ldots,N\} \quad , \quad (i,a) \to i$$

and we obtain the usual quantum reflection groups,

$$(S^+_{Z\to Y} \subset S^+_Z) \ = \ (H_N^{s+} \subset S^+_{sN})$$

via some standard identifications.

PROOF. In the commutative case our fibration must be a usual fibration of finite spaces, $\{1,\ldots,M\} \to \{1,\ldots,N\}$, commuting with the counting measures. But this shows that our fibration must be of the following special form, with $N, s \in \mathbb{N}$:

$$\{1,\ldots,N\} \times \{1,\ldots,s\} \to \{1,\ldots,N\} \quad , \quad (i,a) \to i$$

Regarding now the quantum symmetry group, we have the following formula for it, with $e : \mathbb{C}^N \otimes \mathbb{C}^s \to \mathbb{C}^N$ being the Jones projection for the inclusion $\mathbb{C}^N \subset \mathbb{C}^N \otimes \mathbb{C}^s$:

$$C(S^+_{Z\to Y}) = C(S^+_{sN})\Big/\Big\langle e \in End(u)\Big\rangle$$

On the other hand, recall that the quantum reflection group $H_N^{s+}\subset S_{sN}^+$ appears via the condition that the corresponding magic matrix must be sudoku:

$$u=\begin{pmatrix}a^0&a^1&\ldots&a^{s-1}\\ a^{s-1}&a^0&\ldots&a^{s-2}\\ \vdots&\vdots&&\vdots\\ a^1&a^2&\ldots&a^0\end{pmatrix}$$

But, as explained in [6], this is the same as saying that the quantum group $H_N^{s+}\subset S_{sN}^+$ appears as the symmetry group of the multi-simplex associated to the fibration $\{1,\ldots,N\}\times\{1,\ldots,s\}\to\{1,\ldots,N\}$, so we have an identification as follows:

$$(S_{Z\to Y}^+\subset S_Z^+)\ =\ (H_N^{s+}\subset S_{sN}^+)$$

Thus, we are led to the conclusions in the statement. □

Observe that in Proposition 14.15 the fibration $Z\to Y$ is "trivial", in the sense that it is of the following special form:

$$Y\times T\to Y\quad,\quad (i,a)\to i$$

However, in the general quantum case, there are many interesting fibrations $Z\to Y$ which are not trivial, and in what follows we will not make any assumption on our fibrations, and use Definition 14.13 and Theorem 14.14 as stated.

Following [6], we will prove now that the Tannakian category of $S_{Z\to Y}^+$, which is by definition a generalization of S_Z^+, is the Fuss-Catalan category, which is a generalization of the Temperley-Lieb category, introduced by Bisch and Jones.

Let us start with a suitable reformulation of what we have, as follows:

THEOREM 14.16. *Any Markov inclusion of finite dimensional algebras $D\subset B$ has a quantum symmetry group $S_{D\subset B}^+$. The corresponding Woronowicz algebra is generated by the coefficients of a biunitary matrix $v=(v_{ij})$ subject to the conditions*

$$m\in Hom(v^{\otimes 2},v)\quad,\quad u\in Hom(1,v)\quad,\quad e\in End(v)$$

where $m:B\otimes B\to B$ is the multiplication, $u:\mathbb{C}\to B$ is the unit and $e:B\to B$ is the projection onto D, with respect to the scalar product $<x,y>=tr(xy^*)$.

PROOF. This is a reformulation of Theorem 14.14, with several modifications made. Indeed, by using the algebras $D=C(Y)$, $B=C(Z)$ instead of the quantum spaces Y,Z used there, and also by calling the fundamental corepresentation $v=(v_{ij})$, in order to avoid confusion with the unit $u:\mathbb{C}\to B$, the formula in Theorem 14.14 reads:

$$C(S_{D\subset B}^+)=C(S_B^+)\Big/\Big\langle e\in End(v)\Big\rangle$$

Also, we know from chapter 13 that we have the following formula, again by using B instead of Z, and by calling the fundamental corepresentation $v = (v_{ij})$:

$$C(S_B^+) = C(U_N^+) \Big/ \Big\langle m \in Hom(v^{\otimes 2}, v), u \in Fix(v) \Big\rangle$$

Thus, we are led to the conclusion in the statement. $\square$

Let us first discuss in detail the Temperley-Lieb algebra, as a continuation of the material above. In the present context, we have the following definition:

DEFINITION 14.17. *The $\mathbb{N}$-algebra TL^2 of index $\delta > 0$ is defined as follows:*

(1) *The space $TL^2(m,n)$ consists of linear combinations of noncrossing pairings between $2m$ points and $2n$ points:*

$$TL^2(m,n) = \left\{ \sum \alpha \begin{array}{ccc} \cdots\cdots & \leftarrow & 2m \text{ points} \\ \mathfrak{W} & \leftarrow & m+n \text{ strings} \\ \cdot\cdot & \leftarrow & 2n \text{ points} \end{array} \right\}$$

(2) *The operations $\circ, \otimes, *$ are induced by the vertical and horizontal concatenation and the upside-down turning of diagrams:*

$$A \circ B = \binom{B}{A} \quad , \quad A \otimes B = AB \quad , \quad A^* = \forall$$

(3) *With the rule $\bigcirc = \delta$, erasing a circle is the same as multiplying by δ.*

Our first task will be that of finding a suitable presentation for this algebra. Consider the following two elements $u \in TL^2(0,1)$ and $m \in TL^2(2,1)$:

$$u = \delta^{-\frac{1}{2}} \cap \quad , \quad m = \delta^{\frac{1}{2}} \,|\cup|$$

With this convention, we have the following result, which is something well-known, and elementary, obtained by drawing diagrams:

THEOREM 14.18. *The following relations are a presentation of TL^2 by the above rescaled diagrams $u \in TL^2(0,1)$ and $m \in TL^2(2,1)$:*

(1) $mm^* = \delta^2$.
(2) $u^*u = 1$.
(3) $m(m \otimes 1) = m(1 \otimes m)$.
(4) $m(1 \otimes u) = m(u \otimes 1) = 1$.
(5) $(m \otimes 1)(1 \otimes m^*) = (1 \otimes m)(m^* \otimes 1) = m^*m$.

PROOF. This is something well-known, and elementary, obtained by drawing diagrams, and for details here, we refer to Bisch and Jones [19]. $\square$

In more concrete terms, the above result says that u,m satisfy the above relations, which is something clear, and that if C is a $\mathbb{N}$-algebra and $v\in C(0,1)$ and $n\in C(2,1)$ satisfy the same relations then there exists a $\mathbb{N}$-algebra morphism as follows:

$$TL^2\to C\quad,\quad u\to v\quad,\quad m\to n$$

Now let B be a finite dimensional C^*-algebra, with its canonical trace. We have a scalar product $<x,y>=tr(xy^*)$ on B, so B is an object in the category of finite dimensional Hilbert spaces. Consider the unit u and the multiplication m of B:

$$u\in\mathbb{N}B(0,1)\quad,\quad m\in\mathbb{N}B(2,1)$$

The relations in Theorem 14.18 are then satisfied, and one can deduce from this that in this case, the category of representations of S_B^+ is the completion of TL^2, as we already know. Getting now to Fuss-Catalan algebras, we have here:

Definition 14.19. *A Fuss-Catalan diagram is a planar diagram formed by an upper row of $4m$ points, a lower row of $4n$ points, both colored*

$$\circ\bullet\bullet\circ\circ\bullet\bullet\ldots$$

and by $2m+2n$ noncrossing strings joining these $4m+4n$ points, with the rule that the points which are joined must have the same color.

Fix $\beta>0$ and $\omega>0$. The $\mathbb{N}$-algebra FC is defined as follows. The spaces $FC(m,n)$ consist of linear combinations of Fuss-Catalan diagrams:

$$FC(m,n)=\left\{\sum\alpha\begin{matrix}\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots&\leftarrow&4m\ \text{ colored points}\\ &&m+n\text{ black strings}\\ \mathfrak{W}&\leftarrow&\text{and}\\ &&m+n\text{ white strings}\\ \circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots&\leftarrow&4n\text{ colored points}\end{matrix}\right\}$$

As before with the Temperley-Lieb algebra, the operations $\circ$, $\otimes$, $*$ are induced by vertical and horizontal concatenation and upside-down turning of diagrams, but this time with the rule that erasing a black/white circle is the same as multiplying by β/ω:

$$A\circ B=\begin{pmatrix}B\\ A\end{pmatrix}\quad,\quad A\otimes B=AB\quad,\quad A^*=\forall$$

$$\text{black}\to\bigcirc=\beta\qquad,\qquad\text{white}\to\bigcirc=\omega$$

Let $\delta=\beta\omega$. We have the following bicolored analogues of the elements u,m:

$$u=\delta^{-\frac{1}{2}}\ \bigcap\!\!\!\!\!\;|\ \ ,\quad m=\delta^{\frac{1}{2}}\ ||\bigcup\!\!\!\!\!\;|\,||$$

Consider also the black and white Jones projections, namely:

$$e=\omega^{-1}\ |\,{}^{\cup}_{\cap}\,|\quad,\quad f=\beta^{-1}\ |||\,{}^{\cup}_{\cap}\,|||$$

For simplifying writing we identify x and $x \otimes 1$. We have the following result:

THEOREM 14.20. *The following relations, with $f = \beta^{-2}(1 \otimes me)m^*$, are a presentation of FC by $m \in FC(2,1)$, $u \in FC(0,1)$ and $e \in FC(1)$:*

1. *The relations in Theorem 14.18, with $\delta = \beta\omega$.*
2. *$e = e^2 = e^*$, $f = f^*$ and $(1 \otimes f)f = f(1 \otimes f)$.*
3. *$eu = u$.*
4. *$mem^* = m(1 \otimes e)m^* = \beta^2$.*
5. *$mm(e \otimes e \otimes e) = emm(e \otimes 1 \otimes e)$.*

PROOF. As for any presentation result, we have to prove two assertions:

(I) The elements m, u, e satisfy the relations (1-5) and generate the $\mathbb{N}$-algebra FC.

(II) If M, U and E in a $\mathbb{N}$-algebra C satisfy the relations (1-5), then there exists a morphism of $\mathbb{N}$-algebras $FC \to C$ sending $m \to M$, $u \to U$, $e \to E$.

(I) First, the relations (1-5) are easily verified by drawing pictures. Let us show now that the $\mathbb{N}$-subalgebra $C =< m, u, e >$ of FC is equal to FC. First, C contains the infinite sequence of black and white Jones projections:

$$p_1 = e = \omega^{-1} \ | \overset{\cup}{\cap} |$$

$$p_2 = f = \beta^{-1} \ ||| \overset{\cup}{\cap} |||$$

$$p_3 = 1 \otimes e = \omega^{-1} \ ||||| \overset{\cup}{\cap} |$$

$$p_4 = 1 \otimes f = \beta^{-1} \ ||||||| \overset{\cup}{\cap} |||$$

$$\ldots$$

The algebra C contains as well the infinite sequence of bicolored Jones projections:

$$e_1 = uu^* = \delta^{-1} \overset{\bigcup}{\bigcap}$$

$$e_2 = \delta^{-2} m^* m = \delta^{-1} \ || \overset{\bigcup}{\bigcap} ||$$

$$e_3 = 1 \otimes uu^* = \delta^{-1} \ |||| \overset{\bigcup}{\bigcap}$$

$$e_4 = \delta^{-2}(1 \otimes m^* m) = \delta^{-1} \ |||||| \overset{\bigcup}{\bigcap} ||$$

$$\ldots$$

By standard results of Bisch and Jones, these latter projections generate the diagonal $\mathbb{N}$-algebra ΔFC. Thus we have inclusions as follows:

$$\Delta FC\subset C\subset FC$$

By definition of C, we have as well the following equality:

$$\Delta FC=\Delta C$$

Also, the existence of semicircles shows that the objects of C and FC are self-dual, and by Frobenius reciprocity we obtain that for $m+n$ even, we have:

$$\dim(C(m,n))=\dim(FC(m,n))$$

By tensoring with u and u^* we get embeddings as follows:

$$C(m,n)\subset C(m,n+1)\quad,\quad FC(m,n)\subset FC(m,n+1)$$

But this shows that the above dimension equalities hold for any m and n. Together with $\Delta FC\subset C\subset FC$, this shows that $C=FC$.

(II) Assume that M,U,E in a $\mathbb{N}$-algebra C satisfy the relations (1-5). We have to construct a morphism $FC\to C$ sending:

$$m\to M\quad,\quad u\to U\quad,\quad e\to E$$

As a first task, we would like to construct a morphism $\Delta FC\to\Delta C$ sending:

$$m^*m\to M^*M\quad,\quad uu^*\to UU^*\quad,\quad e\to E$$

By constructing the corresponding Jones projections E_i and P_i, we must send:

$$e_i\to E_i\quad,\quad p_i\to P_i$$

In order to construct these maps, we use now the standard fact, from the papers of Bisch and Jones, that the following relations are a presentation of ΔFC:

(a) $e_i^2=e_i$, $e_ie_j=e_je_i$ if $|i-j|\geq 2$ and $e_ie_{i\pm1}e_i=\delta^{-2}e_i$.

(b) $p_i^2=p_i$ and $p_ip_j=p_jp_i$.

(c) $e_ip_i=p_ie_i=e_i$ and $p_ie_j=e_jp_i$ if $|i-j|\geq 2$.

(d) $e_{2i\pm1}p_{2i}e_{2i\pm1}=\beta^{-2}e_{2i\pm1}$ and $e_{2i}p_{2i\pm1}e_{2i}=\omega^{-2}e_{2i}$.

(e) $p_{2i}e_{2i\pm1}p_{2i}=\beta^{-2}p_{2i\pm1}p_{2i}$ and $p_{2i\pm1}e_{2i}p_{2i\pm1}=\omega^{-2}p_{2i}p_{2i\pm1}$.

Thus, it remains to verify that we have the following implication, where m,u,e are now abstract objects, and we are no longer allowed to draw pictures:

$$(1-5)\implies(a-e)$$

But these relations are all easy to verify. The conclusion is that we constructed a certain $\mathbb{N}$-algebra morphism, as follows:

$$\Delta J : \Delta FC \to \Delta C$$

We have to extend now this morphism into a morphism $J : FC \to C$ sending $u \to U$ and $m \to M$. We will use a standard argument. For $w \geq k, l$ we define:

$$\phi : FC(l,k) \to FC(w)$$

$$x \to (u^{\otimes(w-k)} \otimes 1_k)\, x\, ((u^*)^{\otimes(w-l)} \otimes 1_l)$$

We can define as well a morphism as follows:

$$\theta : FC(w) \to FC(l,k)$$

$$x \to ((u^*)^{\otimes(w-k)} \otimes 1_k)\, x\, (u^{\otimes(w-l)} \otimes 1_l)$$

Here $1_k = 1^{\otimes k}$, and the convention $x = x \otimes 1$ is no longer used. We define Φ and Θ in C by similar formulae. We have $\theta\phi = \Theta\Phi = Id$. We define a map J by:

$$\begin{tikzcd} FC(l,k) \arrow[r, "J"] \arrow[d, "\phi"] & C(l,k) \\ FC(w) \arrow[r, "\Delta J"] & C(w) \arrow[u, "\Theta"'] \end{tikzcd}$$

Since $J(a)$ does now depend on the choice of w, these J maps are the components of a global map $J : FC \to C$, which sends $u \to U$ and $m \to M$, as desired. $\square$

Getting back now to the inclusions $D \subset B$, we have the following result:

THEOREM 14.21. *Given a Markov inclusion $D \subset B$, we have*

$$< m, u, e >= FC$$

as an equality of $\mathbb{N}$-algebras.

PROOF. It is routine to check that the linear maps m, u, e associated to an inclusion $D \subset B$ as in the statement satisfy the relations (1-5) in Theorem 14.20. Thus, we obtain a certain $\mathbb{N}$-algebra surjective morphism, as follows:

$$J : FC \to< m, u, e >$$

But it is routine to prove that this morphism J is faithful on ΔFC, and then by Frobenius reciprocity faithfulness has to hold on the whole FC. $\square$

Getting back now to quantum groups, we have:

THEOREM 14.22. *Given a Markov inclusion of finite dimensional algebras $D \subset B$, the category of representations of its quantum symmetry group*

$$S_{D\subset B}^+ \subset S_B^+$$

is the completion of the Fuss-Catalan category FC.

PROOF. Since $S_{D\subset B}^+$ comes by definition from the relations corresponding to m, u, e, its tensor category of corepresentations is the completion of the tensor category $< m, u, e >$. Thus Theorem 14.21 applies, and gives an isomorphism $< m, u, e >\simeq FC$. $\square$

In terms of finite quantum spaces and quantum graphs, the conclusion is that the quantum automorphism groups $S_{Z\to Y}^+$ of the Markov fibrations $Z \to Y$, which can be thought of as being the "twisted versions" of the quantum reflection groups H_N^{s+}, correspond to the Fuss-Catalan algebras. We refer here to [6] and related papers.

14d. Planar algebras

In order to further advance now, and make a more substantial link with the planar algebra material from chapter 12, the idea will be that of associating to the original inclusion $D \subset B$ a certain combinatorial planar algebra $P(D \subset B)$, appearing as a joint generalization of the spin and tensor planar algebras, which appear as follows:

$$\mathcal{S}_N = P(\mathbb{C} \subset \mathbb{C}^N) \quad , \quad \mathcal{T}_N = P(\mathbb{C} \subset M_N(\mathbb{C}))$$

In practice, all this will be something quite technical. We will need the notion of planar algebra of a bipartite graph, generalizing the spin and tensor planar algebras $\mathcal{S}_N, \mathcal{T}_N$, that we know well from chapter 12, constructed by Jones in [56].

Let us start with the following definition, which is standard in subfactor theory:

DEFINITION 14.23. *Associated to an inclusion $D \subset B$ of finite dimensional algebras are the following objects:*

1. *The column vector $(d_i) \in \mathbb{N}^s$ given by $D = \oplus_{i=1}^s M_{d_i}(\mathbb{C})$.*
2. *The column vector $(b_j) \in \mathbb{N}^t$ given by $B = \oplus_{j=1}^t M_{b_j}(\mathbb{C})$.*
3. *The inclusion matrix $(m_{ij}) \in M_{s\times t}(\mathbb{N})$, satisfying $m^t d = b$.*

To be more precise, in what regards the inclusion matrix, each minimal idempotent of the subalgebra $M_{d_i}(\mathbb{C}) \subset D$ splits, when regarded as element of B, as a sum of minimal idempotents of B, and $m_{ij} \in \mathbb{N}$ is the number of such idempotents from $M_{b_j}(\mathbb{C})$.

We have the following result, bringing traces into picture:

PROPOSITION 14.24. *For an inclusion $D \subset B$, the following are equivalent:*

1. *$D \subset B$ commutes with the canonical traces.*
2. *We have $mb = rd$, where $r = ||b||^2/||d||^2$.*

Proof. The weight vectors of the canonical traces of D,B are given by:

$$\tau_i=\frac{d_i^2}{||d||^2}\quad,\quad\tau_j=\frac{b_j^2}{||b||^2}$$

We can plug these values into the following standard compatibility formula:

$$\frac{\tau_i}{d_i}=\sum_j m_{ij}\cdot\frac{\tau_j}{b_j}$$

We obtain in this way the condition in the statement. □

We will need as well the following basic facts, which are also very standard:

Definition 14.25. *Associated to an inclusion of finite dimensional algebras $D\subset B$, with inclusion matrix $m\in M_{s\times t}(\mathbb N)$, are:*

(1) *The Bratteli diagram: this is the bipartite graph Γ having as vertices the sets $\{1,\ldots,s\}$ and $\{1,\ldots,t\}$, the number of edges between i,j being m_{ij}.*
(2) *The basic construction: this is the inclusion $B\subset D_1$ obtained from $D\subset B$ by reflecting the Bratteli diagram.*
(3) *The Jones tower: this is the tower of algebras $D\subset B\subset D_1\subset B_1\subset\ldots$ obtained by iterating the basic construction.*

We know that for a Markov inclusion $D\subset B$ we have $m^td=b$ and $mb=rd$, and so $mm^td=rd$, which gives an eigenvector for the square matrix $mm^t\in M_s(\mathbb N)$. When this latter matrix has positive entries, by Perron-Frobenius we obtain:

$$||mm^t||=r$$

This equality holds in fact without assumptions on m, and we have:

Theorem 14.26. *Let $D\subset B$ be Markov, with inclusion matrix $m\in M_{s\times t}(\mathbb N)$.*

(1) *$r=\dim(B)/\dim(D)$ is an integer.*
(2) *$||m||=||m^t||=\sqrt r$.*
(3) *$||\ldots mm^tmm^t\ldots||=r^{k/2}$, for any product of lenght k.*

Proof. Consider the vectors d,b, as in Definition 14.25. We have then:

$$b=m^td\quad,\quad mb=rd\quad,\quad r=||b||^2/||d||^2$$

(1) If we construct as above the Jones tower for $D\subset B$, we have, for any k:

$$\frac{\dim B_k}{\dim D_k}=\frac{\dim D_k}{\dim B_{k-1}}=r$$

On the other hand, we have as well the following well-known formula:

$$\lim_{k\to\infty}(\dim D_k)^{1/2k}=\lim_{k\to\infty}(\dim B_k)^{1/2k}=||mm^t||$$

By combining these two formulae we obtain the following formula:

$$||mm^t|| = r$$

But from $r \in \mathbb{Q}$ and $(mm^t)^k d = r^k d$ for any $k \in \mathbb{N}$, we get $r \in \mathbb{N}$, and we are done.

(2) This follows from the above equality $||mm^t|| = r$, and from the following standard equalities, for any real rectangular matrix r:

$$||m||^2 = ||m^t||^2 = ||mm^t||$$

(3) Let n be the length k word in the statement. First, by applying the norm and by using the formula $||m|| = ||m^t|| = \sqrt{r}$, we obtain the following inequality:

$$||n|| \leq r^{k/2}$$

As for the reverse inequality, $||n|| \geq r^{k/2}$, this is something very standard too. $\square$

The point now is that for a Markov inclusion, the basic construction and the Jones tower have a particularly simple form. Let us first work out the basic construction:

Proposition 14.27. *The basic construction for a Markov inclusion $i : D \subset B$ of index $r \in \mathbb{N}$ is the inclusion $j : B \subset D_1$ obtained as follows:*

1. *$D_1 = M_r(\mathbb{C}) \otimes D$, as an algebra.*
2. *$j : B \subset D_1$ is given by $mb = rd$.*
3. *$ji : D \subset D_1$ is given by $(mm^t)d = rd$.*

Proof. With notations from the above, the weight vector of the algebra D_1 appearing from the basic construction is $mb = rd$, and this gives the result. $\square$

We fix a Markov inclusion $i : D \subset B$. We have the following result:

Proposition 14.28. *The Jones tower associated to a Markov inclusion $i : D \subset B$, denoted as follows, with alternating letters,*

$$D \subset B \subset D_1 \subset B_1 \subset \ldots$$

is given by the following formulae:

1. *$D_k = M_r(\mathbb{C})^{\otimes k} \otimes D$.*
2. *$B_k = M_r(\mathbb{C})^{\otimes k} \otimes B$.*
3. *$D_k \subset B_k$ is $id_k \otimes i$.*
4. *$B_k \subset D_{k+1}$ is $id_k \otimes j$.*

Proof. This follows from Proposition 14.27, with the remark that if $i : D \subset B$ is Markov, then so is its basic construction $j : B \subset D_1$. $\square$

Regarding now the relative commutants for this tower, we have here:

THEOREM 14.29. *The relative commutants for the Jones tower*

$$D\subset B\subset D_1\subset B_1\subset\ldots$$

associated to a Markov inclusion $D\subset B$ are given by:

(1) $D_s'\cap A_{s+k}=M_r(\mathbb C)^{\otimes k}\otimes(D'\cap D)$.
(2) $D_s'\cap B_{s+k}=M_r(\mathbb C)^{\otimes k}\otimes(D'\cap B)$.
(3) $B_s'\cap A_{s+k}=M_r(\mathbb C)^{\otimes k}\otimes(B'\cap D)$.
(4) $B_s'\cap B_{s+k}=M_r(\mathbb C)^{\otimes k}\otimes(B'\cap B)$.

PROOF. The above assertions are all elementary, as follows:

(1,2) These assertions both follow from Proposition 14.28.

(3) In order to prove the formula in the statement, observe first that we have:

$$\begin{aligned}
B'\cap D_1 &= (B'\cap B_1)\cap D_1\\
&= (M_r(\mathbb C)\otimes Z(B))\cap(M_r(\mathbb C)\otimes D)\\
&= M_r(\mathbb C)\otimes(B'\cap D)
\end{aligned}$$

But this proves the assertion at $s=0,k=1$, and the general case follows from it.

(4) This is again clear, once again coming from Proposition 14.28. □

Good news, we can now construct our planar algebras. Inspired by the above, let Γ be a bipartite graph, with vertex set $\Gamma_d\cup\Gamma_b$, the basic example being the graph of $D\subset B$. Our first task is to define the graded vector space P. Since the elements of P will be subject to a planar calculus, it is convenient to introduce them "in boxes", as follows:

DEFINITION 14.30. *Associated to Γ is the abstract vector space P_k spanned by the $2k$-loops based at points of Γ_a. The basis elements of P_k will be denoted*

$$x=\begin{pmatrix}e_1&e_2&\ldots&e_k\\ e_{2k}&e_{2k-1}&\ldots&e_{k+1}\end{pmatrix}$$

where $e_1,e_2,\ldots,e_{2k}$ is the sequence of edges of the corresponding $2k$-loop.

Consider now the adjacency matrix of Γ, which is of the following type:

$$M=\begin{pmatrix}0&m\\ m^t&0\end{pmatrix}$$

We pick an M-eigenvalue $\gamma\neq0$, and then a γ-eigenvector, as follows:

$$\eta:\Gamma_a\cup\Gamma_b\to\mathbb C-\{0\}$$

With this data in hand, we have the following construction, due to Jones [56]:

DEFINITION 14.31. *Associated to any tangle is the multilinear map*

$$T(x_1 \otimes \ldots \otimes x_r) = \gamma^c \sum_x \delta(x_1, \ldots, x_r, x) \prod_m \mu(e_m)^{\pm 1} x$$

where the objects on the right are as follows:

(1) *The sum is over the basis of P_k, and c is the number of circles of T.*
(2) *$\delta = 1$ if all strings of T join pairs of identical edges, and $\delta = 0$ if not.*
(3) *The product is over all local maxima and minima of the strings of T.*
(4) *e_m is the edge of Γ labelling the string passing through m (when $\delta = 1$).*
(5) *$\mu(e) = \sqrt{\eta(e_f)/\eta(e_i)}$, where e_i, e_f are the initial and final vertex of e.*
(6) *The $\pm$ sign is $+$ for a local maximum, and $-$ for a local minimum.*

This looks quite similar to the calculus for the tensor and spin planar algebras. Let us work out the precise formula of the action, for 6 carefully chosen tangles:

(1) Let us look first at the identity 1_k. This tangle acts by the identity:

$$1_k \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix} = \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix}$$

(2) The multiplication tangle M_k acts as follows:

$$M_k \left(\begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix} \otimes \begin{pmatrix} h_1 & \ldots & h_k \\ g_1 & \ldots & g_k \end{pmatrix} \right) = \delta_{f_1 g_1} \ldots \delta_{f_k g_k} \begin{pmatrix} h_1 & \ldots & h_k \\ e_1 & \ldots & e_k \end{pmatrix}$$

(3) Regarding now the inclusion I_k, the formula here is:

$$I_k \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix} = \sum_g \begin{pmatrix} f_1 & \ldots & f_k & g \\ e_1 & \ldots & e_k & g \end{pmatrix}$$

(4) The expectation tangle U_k acts with a spin factor, as follows:

$$U_k \begin{pmatrix} f_1 & \ldots & f_k & h \\ e_1 & \ldots & e_k & g \end{pmatrix} = \delta_{gh} \mu(g)^2 \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix}$$

(5) For the Jones projection E_k, the formula is as follows:

$$E_k(1) = \sum_{egh} \mu(g)\mu(h) \begin{pmatrix} e_1 & \ldots & e_k & h & h \\ e_1 & \ldots & e_k & g & g \end{pmatrix}$$

(6) As for the shift J_k, its action is given by:

$$J_k \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix} = \sum_{gh} \begin{pmatrix} g & h & f_1 & \ldots & f_k \\ g & h & e_1 & \ldots & e_k \end{pmatrix}$$

Summarizing, we have here formulae which are quite similar to those for the tensor and spin planar algebras. We have the following result, from Jones' paper [56]:

THEOREM 14.32. *The graded linear space $P=(P_k)$, together with the action of the planar tangles given above, is a planar algebra.*

PROOF. This is routine, starting from the above study of the main planar algebra tangles. Also, let us mention that all this generalizes the previous constructions of the spin and tensor planar algebras $\mathcal{S}_N,\mathcal{T}_N$, which appear respectively from the Bratteli diagrams of the inclusions $\mathbb{C}\subset\mathbb{C}^N$ and $\mathbb{C}\subset M_N(\mathbb{C})$. For details on all this, we refer to [56]. $\square$

Let us go back now to the Markov inclusions $D\subset B$, as before, with D assumed to be commutative. We have here the following result, also from Jones' paper [56]:

THEOREM 14.33. *The planar algebra associated to the graph of $D\subset B$, with eigenvalue $\gamma=\sqrt{r}$ and eigenvector $\eta(i)=d_i/\sqrt{\dim D}$, $\eta(j)=b_j/\sqrt{\dim B}$, is as follows:*

1. *The graded algebra structure is given by $P_{2k}=D'\cap D_k$, $P_{2k+1}=D'\cap B_k$.*
2. *The elements e_k are the Jones projections for $D\subset B\subset D_1\subset B_1\subset\ldots$*
3. *The expectation and shift are given by the above formulae.*

PROOF. As a first observation, η is indeed a γ-eigenvector for the adjacency matrix of the graph. Indeed, we have the following formulae:

$$m^td=b\quad,\quad mb=rd\quad,\quad\sqrt{r}=||b||/||d||$$

By using these formulae, we have the following computation:

$$\begin{aligned}\begin{pmatrix}0&m\\ m^t&0\end{pmatrix}\begin{pmatrix}d/||d||\\ b/||b||\end{pmatrix}&=\begin{pmatrix}\gamma^2d/||b||\\ b/||d||\end{pmatrix}\\ &=\gamma\begin{pmatrix}\gamma d/||b||\\ b/\gamma||d||\end{pmatrix}\\ &=\gamma\begin{pmatrix}d/||d||\\ b/||b||\end{pmatrix}\end{aligned}$$

Since the algebra D was supposed commutative, the Jones tower algebras D_k,B_k are simply the span of the $4k$-paths, respectively $4k+2$-paths on Γ, starting at points of Γ_d. Thus, when taking commutants with D we have to just have to restrict attention from paths to loops, and we obtain the above spaces P_{2k},P_{2k+1}. See [56]. $\square$

In the particular case of the "abelian" inclusions, satisfying $[D,B]=0$, we have:

PROPOSITION 14.34. *The bipartite graph planar algebra $P(D\subset B)$ associated to an abelian inclusion $D\subset B$ can be described as follows:*

1. *As a graded algebra, this is the Jones tower $D\subset B\subset D_1\subset B_1\subset\ldots$*
2. *The Jones projections and expectations are the usual ones for this tower.*
3. *The shifts correspond to the canonical identifications $D_1'\cap P_{k+2}=P_k$.*

PROOF. The first two assertions are standard. Regarding the third assertion, we have indeed identifications $D_1' \cap D_{k+1} = D_k$ and $D_1' \cap B_{k+1} = B_k$. By using the path model for these algebras, as in the proof of Theorem 14.33, we obtain the result. □

In order to formulate now our results, regarding the subfactors associated to the compact quantum groups G, we will need a few abstract notions. Let us start with:

DEFINITION 14.35. *Let P_1, P_2 be two finite dimensional algebras, coming with coactions $\alpha_i : P_i \to P_i \otimes L^\infty(G)$, and let $T : P_1 \to P_2$ be a linear map.*
1. *We say that T is G-equivariant if $(T \otimes id)\alpha_1 = \alpha_2 T$.*
2. *We say that T is weakly G-equivariant if $T(P_1^G) \subset P_2^G$.*

Consider now a planar algebra $P = (P_k)$. The annular category over P is the collection of maps $T : P_k \to P_l$ coming from the annular tangles, having at most one input box. These maps form sets $Hom(k, l)$, and these sets form a category. We have:

DEFINITION 14.36. *A coaction of $L^\infty(G)$ on P is a graded algebra coaction*

$$\alpha : P \to P \otimes L^\infty(G)$$

such that the annular tangles are weakly G-equivariant.

This is something a bit technical, coming out of the known examples that we have. In fact, as we will show below, the examples are basically those coming from actions of quantum groups on Markov inclusions $D \subset B$, under the assumption $[D, B] = 0$. For the moment, at the generality level of Definition 14.36, we have:

PROPOSITION 14.37. *If G acts on a planar algebra P, then P^G is a planar algebra.*

PROOF. The weak equivariance condition tells us that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ consisting of tangles which leave invariant P^G. On the other hand the multiplicativity of α gives $M_k \in \mathcal{P}'$, for any k. Now since $\mathcal{P}$ is generated by multiplications and annular tangles, we get $\mathcal{P}' = \mathcal{P}$, and we are done. □

Let us go back now to the abelian inclusions. We have the following key result:

PROPOSITION 14.38. *If G acts on an abelian inclusion $D \subset B$, the canonical extension of this coaction to the Jones tower is a coaction of G on the planar algebra $P(D \subset B)$.*

PROOF. We know from the above that, as a graded algebra, $P = P(D \subset B)$ coincides with the Jones tower for our inclusion, denoted as follows:

$$D \subset B \subset D_1 \subset B_1 \subset \ldots$$

Thus the coaction in the statement can be regarded as a graded coaction, as follows:

$$\alpha : P \to P \otimes L^\infty(G)$$

In order to finish, we have to prove that the annular tangles are weakly equivariant. But, since the annular category is generated by I_k, E_k, U_k, J_k, we just have to prove that these 4 particular tangles are weakly equivariant, which is in turn clear. □

With the above result in hand, we can now prove:

PROPOSITION 14.39. *Assume that G acts on an abelian inclusion $D\subset B$. Then the graded vector space of fixed points $P(D\subset B)^G$ is a planar subalgebra of $P(D\subset B)$.*

PROOF. This follows indeed from Proposition 14.37 and Proposition 14.38. □

We are now in position of stating and proving a main result, as follows:

THEOREM 14.40. *In the abelian case, the planar algebra of the fixed point subfactor*

$$(D\otimes P)^G\subset(B\otimes P)^G$$

is the fixed point algebra $P(D\subset B)^G$ of the bipartite graph algebra $P(D\subset B)$.

PROOF. This basically follows from what we have, as follows:

(1) Let $P=P(D\subset B)$, and Q be the planar algebra of the fixed point subfactor. We have then $Q=P^G$, and it remains to prove that the planar algebra structure on Q coming from the fixed point subfactor agrees with the planar algebra structure of P.

(2) Since $\mathcal P$ is generated by the annular category $\mathcal A$ and by the multiplication tangles M_k, we just have to check that the annular tangles agree on P,Q. Moreover, since $\mathcal A$ is generated by I_k,E_k,U_k,J_k, we just have to check that these tangles agree on P,Q.

(3) But I_k,E_k,U_k agree on P,Q, and the only verification left is that for the shift J_k. And here, it is enough to show that the image of the subfactor shift J_k' coincides with that of the planar shift J_k. But this follows as in the proof of Proposition 14.38. □

14e. Exercises

We had a tough algebraic chapter here, quite often in relation with subfactor theory, planar algebras and related topics, and as exercises on all this, we have:

EXERCISE 14.41. *Find a criterion for $G^+(X)$ to be classical.*

EXERCISE 14.42. *Find a criterion for $G^+(X)$ to be a group dual.*

EXERCISE 14.43. *Futher build on our color independence computations above.*

EXERCISE 14.44. *Express in terms of quantum graphs the basic theory of orbitals.*

EXERCISE 14.45. *Check the details for the standard presentation of TL^2.*

EXERCISE 14.46. *Check the details for the standard presentation of ΔFC.*

EXERCISE 14.47. *Complete our proof of the formula $<m,u,e>=FC$.*

EXERCISE 14.48. *Fill in all the details for the Tannakian results at the end.*

As bonus exercise, learn some subfactors and knot theory from Jones [51], [52], [53], [54], [55], [56], [57]. When doing quantum graphs, this knowledge is very useful.

CHAPTER 15

Matrix models

15a. Matrix models

Time now to get into the second topic scheduled for this final Part IV, namely block design type generalizations of the graphs X, and their classical and quantum symmetry groups $G(X) \subset G^+(X)$, and with the hope that all this will improve our free geometry knowledge, and also, that there will be a connection with statistical mechanics.

All this might sound a bit messy, but in case you missed it, comes from the philosophical discussion at the beginning of chapter 13, which was quite straightforward. In practice now, however, messy all this remains, and it is not even clear how to get started. So, job for me to capture the cat, and ask him what he thinks, about all this.

And job done, with the help of a fish can, nothing beats those, and cat declares:

CAT 15.1. *You're too abstract with your mathematics, talking all the time about that u_{ij} coordinates, but do you have any idea of what these things are. Try modelling them with some random matrices, and with a bit of luck, statistical mechanics will be around the corner. As for block designs and stuff, that can wait a bit.*

Which sounds like a good advice, I don't know about you, but personally I've always been quite frustrated with my random matrix colleagues, and other probabilists, who are very at ease with statistical mechanics, while doing mathematics not very far from ours. So, good idea to go into that direction, by looking at matrix models for our algebras.

In practice now, no need to know anything probability in advance, because we can have as starting point something very natural and elementary, namely:

DEFINITION 15.2. *A random matrix model for a closed subgroup $G \subset U_N^+$ is a morphism of C^*-algebras of type*

$$\pi : C(G) \to M_K(C(T))$$

with $K \geq 1$ being an integer, and T being a compact probability space.

As a first comment, focusing on such models might look quite restrictive, and you might say, why not replacing the target algebras $M_K(C(T))$ by something more general, but we will soon discover that, with some know-how, we can do many things with such models. In addition, be said in passing, are you sure that, besides the random matrix algebras $M_K(C(T))$, there are other types of algebras that you know really well, to the point that you can trust them, as target algebras for your models? Not clear.

For the moment, let us develop some general theory for our matrix models, as axiomatized in Definition 15.2. The main question to be solved is as follows:

PROBLEM 15.3. *Under which faithfulness assumptions on a matrix model*

$$\pi : C(G) \to M_K(C(T))$$

does this model "remind" the quantum group G?

As we will see, in what follows, this problem is something quite tricky. The simplest situation is of course when π is faithful in the usual sense, and here π obviously reminds G. However, this is something quite restrictive, because in this case the algebra $C(G)$ must be quite small, admitting a random matrix embedding, as follows:

$$\pi : C(G) \subset M_K(C(T))$$

Technically, this means that $C(G)$ must be of type I, as an operator algebra, and we will discuss this in a moment, with the comment that this is indeed something quite restrictive. However, there are many interesting examples here, and all this is worth a detailed look. First, we have the following result, providing us with basic examples:

PROPOSITION 15.4. *The following closed subgroups $G \subset U_N^+$ have faithful models:*

(1) *The compact Lie groups $G \subset U_N$.*
(2) *The finite quantum groups $G \subset U_N^+$.*

In both cases, we can arrange for $\int_G$ to be restriction of the random matrix trace.

PROOF. These assertions are all elementary, the proofs being as follows:

(1) This is clear, because we can simply use here the identity map:

$$id : C(G) \to M_1(C(G))$$

(2) Here we can use the left regular representation $\lambda : C(G) \to M_{|G|}(\mathbb{C})$. Indeed, let us endow the linear space $H = C(G)$ with its usual scalar product, namely:

$$< a, b >= \int_G ab^*$$

We have then a representation of $*$-algebras, as follows:

$$\lambda : C(G) \to B(H) \quad , \quad a \to [b \to ab]$$

Now since we have $H \simeq \mathbb{C}^{|G|}$, we can view λ as a matrix model map, as above.

(3) Finally, our claim is that we can choose our model as for the following formula to hold, where $\int_T$ is the integration with respect to the probability measure on T:

$$\int_G=\left(tr\otimes\int_T\right)\pi$$

But this is clear for the model in (1), by definition, and is clear as well for the model in (2), by using the basic properties of the left regular representation. ☐

In the above result, the last assertion is quite interesting, and suggests formulating the following definition, somewhat independently on the notion of faithfulness:

Definition 15.5. *A matrix model $\pi:C(G)\to M_K(C(T))$ is called stationary when*

$$\int_G=\left(tr\otimes\int_T\right)\pi$$

where $\int_T$ is the integration with respect to the probability measure on T.

Here the term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and we will explain this later. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this.

We will see in a moment that stationarity implies faithfulness, so that stationarity can be regarded as being a useful, pragmatic version of faithfulness. But let us first discuss the examples. Besides those in Proposition 15.4, we can look at group duals. So, consider a discrete group Γ, and a model for the corresponding group algebra, as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

Since a representation of a group algebra must come from a unitary representation of the group, such a matrix model must come from a representation as follows:

$$\rho:\Gamma\to C(T,U_K)$$

With this identification made, we have the following result:

Proposition 15.6. *An matrix model $\rho:\Gamma\subset C(T,U_K)$ is stationary when:*

$$\int_T tr(g^x)dx=0,\forall g\neq 1$$

Moreover, the examples include all the abelian groups, and all finite groups.

Proof. Consider indeed a group embedding $\rho:\Gamma\subset C(T,U_K)$, which produces by linearity a matrix model, as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

It is enough to formulate the stationarity condition on the group elements $g \in C^*(\Gamma)$. Let us set $\rho(g) = (x \to g^x)$. With this notation, the stationarity condition reads:

$$\int_T tr(g^x)dx = \delta_{g,1}$$

Since this equality is trivially satisfied at $g = 1$, where by unitality of our representation we must have $g^x = 1$ for any $x \in T$, we are led to the condition in the statement. Regarding now the examples, these are both clear. More precisely:

(1) When Γ is abelian we can use the following trivial embedding:

$$\Gamma \subset C(\widehat{\Gamma}, U_1) \quad , \quad g \to [\chi \to \chi(g)]$$

(2) When Γ is finite we can use the left regular representation:

$$\Gamma \subset \mathcal{L}(\mathbb{C}\Gamma) \quad , \quad g \to [h \to gh]$$

Indeed, in both cases, the stationarity condition is trivially satisfied. □

Observe that Proposition 15.6 does in fact not add much to what we already knew from Proposition 15.4. However, when looking at the non-classical, non-dual classical, non-finite quantum groups, there are many interesting examples, having stationary models, as for instance the half-classical quantum groups. We will discuss them later.

In order to discuss now certain analytic aspects of the stationary models, let us go back to the von Neumann algebras, discussed in chapter 12. We recall from there that we have the following result, due to Murray-von Neumann and Connes:

THEOREM 15.7. *Given a von Neumann algebra $A \subset B(H)$, if we write its center as*

$$Z(A) = L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A = \int_X A_x \, dx$$

Moreover, the factors, $Z(A) = \mathbb{C}$, can be basically classified in terms of the II_1 factors, which are those satisfying $\dim A = \infty$, and having a faithful trace $tr : A \to \mathbb{C}$.

PROOF. This is something that we know to hold in finite dimensions, and in the commutative case too. In general, this is something heavy, the idea being as follows:

(1) The first assertion, regarding the decomposition into factors, is von Neumann's reduction theory main result, which is actually one of the heaviest results in fundamental mathematics, and whose proof uses advanced functional analysis techniques.

(2) The classification of factors, due to Murray-von Neumann and Connes, is again something heavy, the idea being that the II_1 factors are the "building blocks", with other factors basically appearing from them via crossed product type constructions. □

Back now to matrix models, as a first general result, which is something which is not exactly trivial, and whose proof requires some functional analysis, we have:

THEOREM 15.8. *Assuming that a closed subgroup $G \subset U_N^+$ has a stationary model*

$$\pi : C(G) \to M_K(C(T))$$

it follows that G must be coamenable, and that the model is faithful. Moreover, π extends into an embedding of von Neumann algebras, as follows,

$$L^\infty(G) \subset M_K(L^\infty(T))$$

which commutes with the canonical integration functionals.

PROOF. Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi : C(G) \to C(G)_{red} \subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, we have an embedding as follows:

$$C(G) \subset M_K(C(T))$$

Now observe that the GNS construction gives a better embedding, as follows:

$$L^\infty(G) \subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A = L^\infty(G)$. But this means that, when writing the center of this latter algebra as $Z(A) = L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G) = \int_X M_{K_x}(\mathbb{C})\,dx$$

In particular, we can see from this that $C(G) \subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of π consists of the identity, and of an inclusion. $\square$

In relation with the above, we have the following well-known result of Thoma:

THEOREM 15.9. *For a discrete group Γ, the following are equivalent:*

(1) *$C^*(\Gamma)$ is of type I, so that we have an embedding $\pi : C^*(\Gamma) \subset M_K(C(X))$, with X being a compact space.*
(2) *$C^*(\Gamma)$ has a stationary model of type $\pi : C^*(\Gamma) \to M_\Phi(C(L))$, with Φ being a finite group, and L being a compact abelian group.*
(3) *Γ is virtually abelian, in the sense that we have an abelian subgroup $\Lambda \triangleleft \Gamma$ such that the quotient group $\Phi = \Gamma/\Lambda$ is finite.*
(4) *Γ has an abelian subgroup $\Lambda \subset \Gamma$ whose index $K = [\Gamma : \Lambda]$ is finite.*

Proof. There are several proofs for this fact, the idea being as follows:

(1) $\implies$ (4) This is the non-trivial implication, that we will not prove here. We refer instead to the literature, either Thoma's orignal paper, or books like those of Dixmier, mixing advanced group theory and advanced operator algebra theory.

(4) $\implies$ (3) We choose coset representatives $g_i \in \Gamma$, and we set:

$$\Lambda' = \bigcap_i g_i \Gamma g_i^{-1}$$

Then $\Lambda' \subset \Lambda$ has finite index, and we have $\Lambda' \lhd \Gamma$, as desired.

(3) $\implies$ (2) This follows by using the theory of induced representations. We can define a model $\pi : C^*(\Gamma) \to M_\Phi(C(\widehat{\Lambda}))$ by setting:

$$\pi(g)(\chi) = Ind_\Lambda^\Gamma(\chi)(g)$$

Indeed, any character $\chi \in \widehat{\Lambda}$ is a 1-dimensional representation of Λ, and we can therefore consider the induced representation $Ind_\Lambda^\Gamma(\chi)$ of the group Γ. This representation is $|\Phi|$-dimensional, and so maps the group elements $g \in \Gamma$ into order $|\Phi|$ matrices $Ind_\Lambda^\Gamma(\chi)(g)$. Thus the above map π is well-defined, and the fact that it is a representation is clear as well. In order to check now the stationarity property of this representation, we can use the following well-known character formula, due to Frobenius:

$$Tr\left(Ind_\Lambda^\Gamma(\chi)(g)\right) = \sum_{x\in\Phi} \delta_{x^{-1}gx\in\Lambda}\chi(x^{-1}gx)$$

By integrating with respect to $\chi \in \widehat{\Lambda}$, we deduce from this that we have:

$$\begin{aligned}\left(Tr \otimes \int_{\widehat{\Lambda}}\right)\pi(g) &= \sum_{x\in\Phi} \delta_{x^{-1}gx\in\Lambda} \int_{\widehat{\Lambda}} \chi(x^{-1}gx)d\chi \\ &= \sum_{x\in\Phi} \delta_{x^{-1}gx\in\Lambda}\delta_{g,1} \\ &= |\Phi| \cdot \delta_{g,1}\end{aligned}$$

Now by dividing by $|\Phi|$ we conclude that the model is stationary, as claimed.

(2) $\implies$ (1) This is the trivial implication, with the faithfulness of π following from the abstract functional analysis arguments from the proof of Theorem 15.8. $\square$

Summarizing, the stationary models cover many interesting quantum groups G, but their range of applications is bounded by the fact that $\Gamma = \widehat{G}$ must be amenable, and more specifically must be virtually abelian, in a suitable quantum group sense.

15b. Inner faithfulness

Let us discuss now the general, non-coamenable case, with the aim of finding a weaker notion of faithfulness, which still does the job, namely that of "reminding" the quantum group. This might sound a bit crazy, but we will see that this is indeed possible.

The idea comes by looking at the group duals $G = \widehat{\Gamma}$. Consider indeed a general model for the associated group algebra, which can be written as follows:

$$\pi : C^*(\Gamma) \to M_K(C(T))$$

The point is that such a representation of the group algebra must come by linearization from a unitary group representation, as follows:

$$\rho : \Gamma \to C(T, U_K)$$

Now observe that when this group representation ρ is faithful, the representation π is in general not faithful, for instance because when $T = \{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" Γ, so can be used in order to fully understand Γ. Thus, we have an idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker.

This weaker notion, which will be of great interest for us, is called "inner faithfulness". The general theory here starts with the following definition:

Definition 15.10. *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

1. *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization $\pi : C(G) \to C(H) \to M_K(C(T))$.*
2. *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

The above notions are quite tricky, and having them well understood will take us some time. As a first example, motivated by the above discussion, in the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation, as follows:

$$\rho : \Gamma \to C(T, U_K)$$

Thus the minimal factorization in (1) is obtained by taking the image:

$$\rho : \Gamma \to \Lambda \subset C(T, U_K)$$

Thus, as a conclusion, in this case π is inner faithful precisely when we have:

$$\Gamma \subset C(T, U_K)$$

Dually now, given a compact Lie group G, and elements $g_1,\ldots,g_K\in G$, we have a diagonal representation $\pi:C(G)\to M_K(\mathbb C)$, appearing as follows:

$$f\to\begin{pmatrix}f(g_1)&&\\&\ddots&\\&&f(g_K)\end{pmatrix}$$

The minimal factorization of this representation π, as in Definition 15.10 (1), is then via the algebra $C(H)$, with H being the following closed subgroup of G:

$$H=\overline{<g_1,\ldots,g_K>}$$

Thus, as a conclusion, π is inner faithful precisely when we have:

$$G=H$$

Back to general theory now, in the framework of Definition 15.10, the existence and uniqueness of the Hopf image come by dividing $C(G)$ by a suitable ideal, with this being something standard. Alternatively, in Tannakian terms, we have:

THEOREM 15.11. *Assuming $G\subset U_N^+$, with fundamental corepresentation $u=(u_{ij})$, the Hopf image of a model $\pi:C(G)\to M_K(C(T))$ comes from the Tannakian category*

$$C_{kl}=Hom(U^{\otimes k},U^{\otimes l})$$

where $U_{ij}=\pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

PROOF. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(U^{\otimes k},U^{\otimes l})$$

More generally, we have such inclusions when replacing (G,u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions. On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H,v) given by:

$$Hom(v^{\otimes k},v^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. □

The above result is something very useful, in practice, and as already mentioned in the above, it can be taken as a definition for the Hopf image, or for the related notion of inner faithfulness. We will be back to it later, with some concrete applications.

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here is as follows:

THEOREM 15.12. *Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have*

$$\int_G = \lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k \int_G^r$$

with the truncations of the integration on the right being given by

$$\int_G^r = (\varphi\circ\pi)^{*r}$$

with $\phi*\psi = (\phi\otimes\psi)\Delta$, and with $\varphi = tr\otimes\int_T$ being the random matrix trace.

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in chapter 9. In fact, the above result holds more generally for any model $\pi : C(G) \to B$, with $\varphi \in B^*$ being a faithful trace.

(2) In order to prove now the result, we can proceed as in chapter 9. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that:

$$\int_G' = \int_G$$

It is enough to check this on the coefficients of the Peter-Weyl corepresentations, and if we let $v = u^{\otimes k}$ be one of these corepresentations, we must prove that we have:

$$\left(id\otimes\int_G'\right)v = \left(id\otimes\int_G\right)v$$

(3) In order to prove this, we already know, from the Haar measure theory from chapter 9, that the matrix on the right is the orthogonal projection onto $Fix(v)$:

$$\left(id\otimes\int_G\right)v = Proj\Big[Fix(v)\Big]$$

Regarding now the matrix on the left, the trick in [99] applied to the linear form $\varphi\pi$ tells us that this is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi\pi)v$:

$$\left(id\otimes\int_G'\right)v = Proj\Big[1\in(id\otimes\varphi\pi)v\Big]$$

(4) Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have the following formula:

$$(id\otimes\varphi\pi)v = (id\otimes\varphi)V$$

Thus, we can apply the trick in [99], and we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 15.11, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. $\square$

In practice, Theorem 15.12 is something quite powerful. As an illustration, regarding the law of the main character, we obtain here the following result:

PROPOSITION 15.13. *Assume that $\pi : C(G) \to M_K(C(T))$ is inner faithful, let*

$$\mu = law(\chi)$$

and let μ^r be the law of χ with respect to $\int_G^r = (\varphi \circ \pi)^{*r}$, where $\varphi = tr \otimes \int_T$.

(1) *We have the following convergence formula, in moments:*

$$\mu = \lim_{k \to \infty} \frac{1}{k} \sum_{r=0}^{k} \mu^r$$

(2) *The moments of μ^r are the numbers $c_\varepsilon^r = Tr(T_\varepsilon^r)$, where:*

$$(T_\varepsilon)_{i_1 \ldots i_p, j_1 \ldots j_p} = \left(tr \otimes \int_T \right) (U_{i_1 j_1}^{\varepsilon_1} \ldots U_{i_p j_p}^{\varepsilon_p})$$

PROOF. The above formulae are both elementary, by using the convergence result established in Theorem 15.12, the proof being as follows:

(1) This follows from the limiting formula in Theorem 15.12, by applying the linear forms there to the main character χ.

(2) This follows from the definitions of the measure μ^r and of the matrix T_e, by summing the entries of T_e over equal indices, $i_r = j_r$. □

Interestingly, the above results regarding the inner faithfulness have applications as well to the notion of stationarity introduced before, clarifying among others the use of the word "stationary". To be more precise, in order to detect the stationary models, we have the following useful criterion, mixing linear algebra and analysis:

THEOREM 15.14. *For a model $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and the Haar integration on it is:*

$$\psi = \left(tr \otimes \int_T \right) \pi$$

(2) *The linear form $\psi = (tr \otimes \int_T)\pi$ satisfies the idempotent state property:*

$$\psi * \psi = \psi$$

(3) *We have $T_e^2 = T_e$, $\forall p \in \mathbb{N}$, $\forall e \in \{1, *\}^p$, where:*

$$(T_e)_{i_1 \ldots i_p, j_1 \ldots j_p} = \left(tr \otimes \int_T \right) (U_{i_1 j_1}^{e_1} \ldots U_{i_p j_p}^{e_p})$$

If these conditions are satisfied, we say that π is stationary on its image.

PROOF. Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 15.10:

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that (1,2,3) above depend only on the factorized representation:

$$\nu : C(H) \to M_K(C(T))$$

Thus, we can assume in practice that we have $G = H$, which means that we can assume that π is inner faithful. With this assumption made, the formula in Theorem 15.12 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:

$$\psi * \psi = \psi$$

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$, and Theorem 15.12 gives $\int_G = \psi$. By using now Theorem 15.8, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}) = (T_e^r)_{i_1\ldots i_p,j_1\ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. $\square$

As a conclusion, with the help of the notion of inner faithfulness, our original matrix modelling theory has now evolved into something quite non-trivial, and powerful.

15c. Pauli and Weyl

Let us discuss now some further examples of stationary models, which are quite interesting, related to the Pauli matrices, and Weyl matrices. We first have:

DEFINITION 15.15. *Given a finite abelian group H, the associated Weyl matrices are*

$$W_{ia} : e_b \to < i, b > e_{a+b}$$

where $i \in H$, $a, b \in \widehat{H}$, and where $(i, b) \to < i, b >$ is the Fourier coupling $H \times \widehat{H} \to \mathbb{T}$.

As a basic example, consider the cyclic group $H=\mathbb{Z}_2=\{0,1\}$. Here the Fourier coupling is given by $<i,b>=(-1)^{ib}$, and so the Weyl matrices act via:

$$W_{00}:e_b\to e_b\quad,\quad W_{10}:e_b\to(-1)^be_b$$

$$W_{11}:e_b\to(-1)^be_{b+1}\quad,\quad W_{01}:e_b\to e_{b+1}$$

Thus, we have the following formulae for the Weyl matrices, in this case:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad,\quad W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad,\quad W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices. Now back to the general case, we have the following elementary result:

PROPOSITION 15.16. *The Weyl matrices are unitaries, and satisfy:*

(1) $W_{ia}^*=<i,a>W_{-i,-a}$.
(2) $W_{ia}W_{jb}=<i,b>W_{i+j,a+b}$.
(3) $W_{ia}W_{jb}^*=<j-i,b>W_{i-j,a-b}$.
(4) $W_{ia}^*W_{jb}=<i,a-b>W_{j-i,b-a}$.

PROOF. The unitarity follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned}W_{ia}^*&=\left(\sum_b<i,b>E_{a+b,b}\right)^*\\&=\sum_b<-i,b>E_{b,a+b}\\&=\sum_b<-i,b-a>E_{b-a,b}\\&=<i,a>W_{-i,-a}\end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned}W_{ia}W_{jb}&=\left(\sum_d<i,b+d>E_{a+b+d,b+d}\right)\left(\sum_d<j,d>E_{b+d,d}\right)\\&=\sum_d<i,b><i+j,d>E_{a+b+d,d}\\&=<i,b>W_{i+j,a+b}\end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned} W_{ia}W_{jb}^* &= <j,b> W_{ia}W_{-j,-b} \\ &= <j,b><i,-b> W_{i-j,a-b} \end{aligned}$$

We obtain as well the following formula:

$$\begin{aligned} W_{ia}^*W_{jb} &= <i,a> W_{-i,-a}W_{jb} \\ &= <i,a><-i,b> W_{j-i,b-a} \end{aligned}$$

But this gives the formulae in the statement, and we are done. $\square$

With $n = |H|$, we can use an isomorphism $l^2(\widehat{H}) \simeq \mathbb{C}^n$ as to view each W_{ia} as a usual matrix, $W_{ia} \in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia} \in U_n$. Also, given a vector ξ, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$. We have:

PROPOSITION 15.17. *Given a closed subgroup $E \subset U_n$, we have a representation*

$$\pi_H : C(S_N^+) \to M_N(C(E))$$

$$w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$

where $n = |H|, N = n^2$, and where W_{ia} are the Weyl matrices associated to H.

PROOF. The Weyl matrices being given by $W_{ia} : e_b \to <i,b> e_{a+b}$, we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i,a) = (0,0) \\ 0 & \text{if } (i,a) \neq (0,0) \end{cases}$$

Together with the formulae in Proposition 15.16, this shows that the Weyl matrices are pairwise orthogonal with respect to the following scalar product on $M_n(\mathbb{C})$:

$$<x,y> = tr(xy^*)$$

Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W = \left\{ W_{ia} \Big| i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb} = W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. $\square$

We will need the following well-known result:

PROPOSITION 15.18. *With $T = Proj(x_1)\ldots Proj(x_p)$ and $||x_i|| = 1$ we have*

$$<T\xi,\eta> = <\xi,x_p><x_p,x_{p-1}>\ldots<x_2,x_1><x_1,\eta>$$

for any ξ,η. In particular, we have:

$$Tr(T) = <x_1,x_p><x_p,x_{p-1}>\ldots<x_2,x_1>$$

PROOF. For $||x|| = 1$ we have $Proj(x)\xi = <\xi, x> x$. This gives:

$$\begin{aligned} T\xi &= Proj(x_1)\ldots Proj(x_p)\xi \\ &= Proj(x_1)\ldots Proj(x_{p-1}) <\xi, x_p> x_p \\ &= Proj(x_1)\ldots Proj(x_{p-2}) <\xi, x_p><x_p, x_{p-1}> x_{p-1} \\ &= \ldots \\ &= <\xi, x_p><x_p, x_{p-1}>\ldots <x_2, x_1> x_1 \end{aligned}$$

Now by taking the scalar product with η, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi = \eta = e_i$. $\square$

Now back to the Weyl matrix models, let us first compute T_p. We have:

PROPOSITION 15.19. *We have the formula*

$$\begin{aligned} (T_p)_{ia,jb} &= \frac{1}{N} <i_1, a_1 - a_p>\ldots <i_p, a_p - a_{p-1}><j_1, b_1 - b_2>\ldots <j_p, b_p - b_1> \\ & \int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU \end{aligned}$$

with all the indices varying in a cyclic way.

PROOF. By using the trace formula in Proposition 15.18, we obtain:

$$\begin{aligned} &(T_p)_{ia,jb} \\ = &\left(tr \otimes \int_E\right)\left(Proj(W_{i_1a_1}UW^*_{j_1b_1})\ldots Proj(W_{i_pa_p}UW^*_{j_pb_p})\right) \\ = &\frac{1}{N}\int_E <W_{i_1a_1}UW^*_{j_1b_1}, W_{i_pa_p}UW^*_{j_pb_p}>\ldots <W_{i_2a_2}UW^*_{j_2b_2}, W_{i_1a_1}UW^*_{j_1b_1}> dU \end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned} <W_{ia}UW^*_{jb}, W_{kc}UW^*_{ld}> &= tr(W_{jb}U^*W^*_{ia}W_{kc}UW^*_{ld}) \\ &= tr(W^*_{ia}W_{kc}UW^*_{ld}W_{jb}U^*) \\ &= <i, a-c><l, d-b> tr(W_{k-i,c-a}UW_{j-l,b-d}U^*) \end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. $\square$

Consider now the Weyl group $W = \{W_{ia}\} \subset U_n$, that we already met in the proof of Proposition 15.17. We have the following result, from [15]:

THEOREM 15.20. *For any compact group $W \subset E \subset U_n$, the model*

$$\pi_H : C(S_N^+) \to M_N(C(E))$$

$$w_{ia,jb} \to [U \to Proj(W_{ia}UW^*_{jb})]$$

constructed above is stationary on its image.

PROOF. We must prove that we have $T_p^2=T_p$. We have:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\sum_{kc}(T_p)_{ia,kc}(T_p)_{kc,jb}\\
=&\frac{1}{N^2}\sum_{kc}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><k_1,c_1-c_2>\ldots<k_p,c_p-c_1>\\
&<k_1,c_1-c_p>\ldots<k_p,c_p-c_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)dU\\
&\int_E tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\ldots tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\frac{1}{N^2}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E\int_E\sum_{kc}<k_1-k_p,c_1-c_p>\ldots<k_p-k_{p-1},c_p-c_{p-1}>\\
&tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\qquad\qquad\qquad\qquad\ldots\ldots\\
&tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

Let us denote by I the above double integral. By using $W_{kc}^*=<k,c>W_{-k,-c}$ for each of the couplings, and by moving as well all the U^* variables to the left, we obtain:

$$\begin{aligned}
I\ =&\int_E\int_E\sum_{kc}tr(U^*W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1})tr(W^*_{k_2-k_1,c_2-c_1}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\qquad\qquad\qquad\qquad\ldots\ldots\\
&tr(U^*W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p})tr(W^*_{k_1-k_p,c_1-c_p}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}
tr(AW_{kc})tr(W_{kc}^*B)&=\frac{1}{N}\sum_{qrst}A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st}B_{ts}\\
&=\frac{1}{N}\sum_{qrst}A_{qr}<k,q>\delta_{r-q,c}<k,-s>\delta_{t-s,c}B_{ts}\\
&=\frac{1}{N}\sum_{qs}<k,q-s>A_{q,q+c}B_{s+c,s}
\end{aligned}$$

If we denote by A_x, B_x the variables which appear in the formula of I, we have:

$$\begin{aligned} & I \\ = & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_2-k_1, q_1-s_1 > \ldots < k_1-k_p, q_p-s_p > \\ & (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p} \\ = & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1, q_p-s_p-q_1+s_1 > \ldots < k_p, q_{p-1}-s_{p-1}-q_p+s_p > \\ & (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p} \end{aligned}$$

Now observe that we can perform the sums over $k_1,\ldots,k_p$. We obtain in this way a multiplicative factor n^p, along with the condition:

$$q_1-s_1=\ldots=q_p-s_p$$

Thus we must have $q_x=s_x+a$ for a certain a, and the above formula becomes:

$$I=\frac{1}{n^p}\int_E\int_E\sum_{csa}(A_1)_{s_1+a,s_1+c_2-c_1+a}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{s_p+a,s_p+c_1-c_p+a}(B_p)_{s_p+c_1-c_p,s_p}$$

Consider now the variables $r_x=c_{x+1}-c_x$, which altogether range over the set Z of multi-indices having sum 0. By replacing the sum over c_x with the sum over r_x, which creates a multiplicative n factor, we obtain the following formula:

$$I=\frac{1}{n^{p-1}}\int_E\int_E\sum_{r\in Z}\sum_{sa}(A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\ldots(A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}$$

For an arbitrary multi-index r we have:

$$\delta_{\sum_i r_i,0}=\frac{1}{n}\sum_i < i,r_1 > \ldots < i,r_p >$$

Thus, we can replace the sum over $r\in Z$ by a full sum, as follows:

$$\begin{aligned} I \;=\; & \frac{1}{n^p}\int_E\int_E\sum_{rsia} < i,r_1 > (A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1} \\ & \ldots\ldots \\ & < i,r_p > (A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p} \end{aligned}$$

In order to "absorb" now the indices i,a, we can use the following formula:

$$\begin{aligned} & W_{ia}^*AW_{ia} \\ = & \left(\sum_b < i,-b > E_{b,a+b}\right)\left(\sum_{bc}E_{a+b,a+c}A_{a+b,a+c}\right)\left(\sum_c < i,c > E_{a+c,c}\right) \\ = & \sum_{bc} < i,c-b > E_{bc}A_{a+b,a+c} \end{aligned}$$

Thus we have the following formula:

$$(W_{ia}^*AW_{ia})_{bc} =< i, c-b > A_{a+b,a+c}$$

With this in hand, our formula becomes:

$$\begin{aligned}
& I \\
= \ & \frac{1}{n^p}\int_E\int_E\sum_{rsia}(W_{ia}^*A_1W_{ia})_{s_1,s_1+r_1}(B_1)_{s_1+r_1,s_1}\ldots(W_{ia}^*A_pW_{ia})_{s_p,s_p+r_p}(B_p)_{s_p+r_p,s_p} \\
= \ & \int_E\int_E\sum_{ia}tr(W_{ia}^*A_1W_{ia}B_1)\ldots\ldots tr(W_{ia}^*A_pW_{ia}B_p)
\end{aligned}$$

Now by replacing A_x, B_x with their respective values, we obtain:

$$\begin{aligned}
I \ = \ & \int_E\int_E\sum_{ia}tr(W_{ia}^*U^*W_{i_1-i_2,a_1-a_2}UW_{ia}VW_{j_2-j_1,b_2-b_1}V^*) \\
& \ldots\ldots \\
& tr(W_{ia}^*U^*W_{i_p-i_1,a_p-a_1}UW_{ia}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

By moving the $W_{ia}^*U^*$ variables at right, we obtain, with $S_{ia} = UW_{ia}V$:

$$\begin{aligned}
I \ = \ & \sum_{ia}\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}S_{ia}W_{j_2-j_1,b_2-b_1}S_{ia}^*) \\
& \ldots\ldots \\
& tr(W_{i_p-i_1,a_p-a_1}S_{ia}W_{j_1-j_p,b_1-b_p}S_{ia}^*)dUdV
\end{aligned}$$

Now since S_{ia} is Haar distributed when U, V are Haar distributed, we obtain:

$$I = N\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU$$

But this is exactly N times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 15.19. Since the N factor cancels with one of the two N factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. $\square$

As an illustration for the above result, regarding the quantum group S_4^+, we have:

THEOREM 15.21. *We have a stationary matrix model as follows,*

$$\pi : C(S_4^+) \subset M_4(C(SU_2))$$

given on the standard matrix coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_ixc_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

PROOF. As already explained in the comments following Definition 15.15, the Pauli matrices appear as particular cases of the Weyl matrices:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad,\quad W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad,\quad W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Thus, Theorem 15.20 produces in this case the model in the statement. $\square$

Observe that, since the projection $Proj(c_ixc_j)$ depends only on the image of x in the quotient $SU_2\to SO_3$, we can replace the model space SU_2 by the smaller space SO_3. This can be used in conjunction with the isomorphism $S_4^+\simeq SO_3^{-1}$ from chapter 13, and our model becomes in this way something more conceptual, as follows:

$$\pi:C(SO_3^{-1})\subset M_4(C(SO_3))$$

At a more advanced level now, we know from the above that we have a stationary matrix model for the algebra $C(S_4^+)$, and this suggests the following conjecture:

CONJECTURE 15.22. *Given a quantum permutation group of 4 points,*

$$G\subset S_4^+\simeq SO_3^{-1}$$

coming by twisting a usual ADE subgroup of the group SO_3,

$$H\subset SO_3$$

the restriction of the Pauli model for $C(S_4^+)$, with fibers coming from the elements of $H\subset SO_3$, has the algebra $C(G)$ as Hopf image.

As a first comment, we know from the above that the conjecture holds for the quantum group $G=S_4^+$ itself. Indeed, here we have by definition $H=SO_3$, so the corresponding restriction of the Pauli model for $C(S_4^+)$ is the Pauli model itself, and this model being stationary by Theorem 15.21, its Hopf image is the algebra $C(S_4^+)$ itself, as stated.

In general, the above conjecture does not look that scary, because the same methods used for S_4^+ can be used for any subgroup $G\subset S_4^+$. However, the problem is that, unless a global method in order to uniformly deal with the problem is found, this would need a case-by-case study depending on $G\subset S_4^+$, which looks quite time-consuming.

As a philosophical conclusion, to this and to some previous findings as well, no matter what we do, we always end up getting back to SU_2,SO_3. Thus, we are probably doing some physics here. This is indeed the case, the above computations being closely related to the standard computations for the Ising and Potts models. The general relation, however, between quantum permutations and lattice models, is not axiomatixed yet.

15d. Universal models

Following [15], let us discuss now the modelling problem for S_N^+, with $N \geq 5$. Here we cannot expect to have a stationary model, for the simple reason that this quantum group is not coamenable. However, we can still look for an inner faithful model.

Given a flat magic unitary, we can write it, in a non-unique way, as $u_{ij} = Proj(\xi_{ij})$. The array $\xi = (\xi_{ij})$ is then a "magic basis", in the sense that each of its rows and columns is an orthonormal basis of $\mathbb{C}^N$. We are therefore led to two spaces, as follows:

DEFINITION 15.23. *Associated to any $N \in \mathbb{N}$ are the following spaces:*

(1) *X_N, the space of all $N \times N$ flat magic unitaries $u = (u_{ij})$.*
(2) *K_N, the space of all $N \times N$ magic bases $\xi = (\xi_{ij})$.*

In order to understand this, geometrically, let us recall now that the rank 1 projections $p \in M_N(\mathbb{C})$ can be identified with the corresponding 1-dimensional subspaces $E \subset \mathbb{C}^N$, which are by definition the elements of the complex projective space $P^{N-1}_\mathbb{C}$.

In addition to this, if we consider the complex sphere, $S^{N-1}_\mathbb{C} = \{z \in \mathbb{C}^N | \sum_i |z_i|^2 = 1\}$, it follows from definitions that we have a quotient map as follows:

$$\pi : S^{N-1}_\mathbb{C} \to P^{N-1}_\mathbb{C} \quad , \quad z \to Proj(z)$$

Observe that we have $\pi(z) = \pi(z')$ precisely when $z' = wz$, for some $w \in \mathbb{T}$.

Finally, consider as well the embedding $U_N \subset (S^{N-1}_\mathbb{C})^N$ given by $x \to (x_1, \ldots, x_N)$, where $x_1, \ldots, x_N$ are the rows of x. Also, let us call an abstract square matrix stochastic/bistochastic when the entries on each row/each row and column sum up to 1.

With these notations and conventions, the abstract model spaces X_N, K_N that we are interested in, from Definition 15.23, and some related spaces, are as follows:

PROPOSITION 15.24. *We have inclusions and surjections as follows,*

$$\begin{matrix} K_N & \subset & U_N^N & \subset & M_N(S^{N-1}_\mathbb{C}) \\ \\ \downarrow & & \downarrow & & \downarrow \\ \\ X_N & \subset & Y_N & \subset & M_N(P^{N-1}_\mathbb{C}) \end{matrix}$$

where X_N, Y_N consist of bistochastic/stochastic matrices, and K_N is the lift of X_N.

PROOF. This follows from the above discussion. Indeed, the quotient map $S^{N-1}_\mathbb{C} \to P^{N-1}_\mathbb{C}$ induces the quotient map $M_N(S^{N-1}_\mathbb{C}) \to M_N(P^{N-1}_\mathbb{C})$ at right, and the lift of the space of stochastic matrices $Y_N \subset M_N(P^{N-1}_\mathbb{C})$ is then the rescaled group U_N^N, as claimed. $\square$

In order to get some insight into the structure of X_N, K_N, we can use some inspiration from the Sinkhorn algorithm, which starts with a $N\times N$ matrix having positive entries and produces, via successive averagings over rows/columns, a bistochastic matrix.

In our situation, we would like to have an "averaging" map $Y_N\to Y_N$, whose infinite iteration lands in the model space X_N. Equivalently, we would like to have an "averaging" map $U_N^N\to U_N^N$, whose infinite iteration lands in K_N. In order to construct such averaging maps, we use the orthogonalization procedure coming from the polar decomposition:

Proposition 15.25. *We have orthogonalization maps as follows,*

$$\begin{tikzcd}
(S^{N-1}_\mathbb C)^N \arrow[r, "\alpha"] \arrow[d] & (S^{N-1}_\mathbb C)^N \arrow[d] \\
(P^{N-1}_\mathbb C)^N \arrow[r, "\beta"] & (P^{N-1}_\mathbb C)^N
\end{tikzcd}$$

where $\alpha(x)_i=Pol([(x_i)_j]_{ij})$, *and* $\beta(p)=(P^{-1/2}p_iP^{-1/2})_i$, *with* $P=\sum_ip_i$.

Proof. This is something quite routine, as follows:

(1) Our first claim is that we have a factorization as in the statement. Indeed, pick $p_1,\ldots,p_N\in P^{N-1}_\mathbb C$, and write $p_i=Proj(x_i)$, with $||x_i||=1$. We can then apply α, as to obtain a vector $\alpha(x)=(x_i')_i$, and then set $\beta(p)=(p_i')$, where $p_i'=Proj(x_i')$.

(2) Our first task is to prove that β is well-defined. Consider indeed vectors $\tilde{x}_i$, satisfying $Proj(\tilde{x}_i)=Proj(x_i)$. We have then $\widetilde{x}_i=\lambda_ix_i$, for certain scalars $\lambda_i\in\mathbb T$, and so the matrix formed by these vectors is $\widetilde{M}=\Lambda M$, with $\Lambda=diag(\lambda_i)$. It follows that $Pol(\widetilde{M})=\Lambda Pol(M)$, and so $\tilde{x}_i'=\lambda_ix_i$, and finally $Proj(\tilde{x}_i')=Proj(x_i')$, as desired.

(3) It remains to prove that β is given by the formula in the statement. For this purpose, observe first that, given $x_1,\ldots,x_N\in S^{N-1}_\mathbb C$, with $p_i=Proj(x_i)$ we have:

$$\begin{aligned}
\sum_ip_i&=\sum_i[(\bar{x}_i)_k(x_i)_l]_{kl}\\
&=\sum_i(\bar{M}_{ik}M_{il})_{kl}\\
&=((M^*M)_{kl})_{kl}\\
&=M^*M
\end{aligned}$$

(4) We can now compute the projections $p_i'=Proj(x_i')$. Indeed, the coefficients of these projections are given by $(p_i')_{kl}=\bar{U}_{ik}U_{il}$ with $U=MP^{-1/2}$, and we get, as desired:

$$\begin{aligned}(p_i')_{kl}&=\sum_{ab}\bar{M}_{ia}P_{ak}^{-1/2}M_{ib}P_{bl}^{-1/2}\\&=\sum_{ab}P_{ka}^{-1/2}\bar{M}_{ia}M_{ib}P_{bl}^{-1/2}\\&=\sum_{ab}P_{ka}^{-1/2}(p_i)_{ab}P_{bl}^{-1/2}\\&=(P^{-1/2}p_iP^{-1/2})_{kl}\end{aligned}$$

(5) Alternatively, we can use the fact that the elements $p_i'=P^{-1/2}p_iP^{-1/2}$ are self-adjoint, and sum up to 1. The fact that these elements are indeed idempotents can be checked directly, via $p_iP^{-1}p_i=p_i$, because this equality holds on $\ker p_i$, and also on x_i. □

As an illustration, here is how the orthogonalization works at $N=2$:

Proposition 15.26. *At $N=2$ the orthogonalization procedure for $(Proj(x),Proj(y))$ amounts in considering the vectors $(x\pm y)/\sqrt{2}$, and then rotating by 45°.*

Proof. By performing a rotation, we can restrict attention to the case $x=(\cos t,\sin t)$ and $y=(\cos t,-\sin t)$, with $t\in(0,\pi/2)$. Here the computations are as follows:

$$\begin{aligned}M=\begin{pmatrix}\cos t&\sin t\\ \cos t&-\sin t\end{pmatrix}\quad&\implies\quad P=M^*M=\begin{pmatrix}2\cos^2 t&0\\0&2\sin^2 t\end{pmatrix}\\&\implies\quad P^{-1/2}=|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\cos t}&0\\0&\frac{1}{\sin t}\end{pmatrix}\\&\implies\quad U=M|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\end{aligned}$$

Thus the orthogonalization procedure replaces $(Proj(x),Proj(y))$ by the orthogonal projections on the vectors $(\frac{1}{\sqrt{2}}(1,1),\frac{1}{\sqrt{2}}(-1,1))$, and this gives the result. □

With these preliminaries in hand, let us discuss now the version that we need of the Sinkhorn algorithm. The orthogonalization procedure is as follows:

Theorem 15.27. *The orthogonalization maps α,β induce maps as follows,*

$$\begin{array}{ccc}U_N^N&\xrightarrow{\Phi}&U_N^N\\ \downarrow&&\downarrow\\ Y_N&\xrightarrow{\Psi}&Y_N\end{array}$$

which are the transposition maps on K_N,X_N, and which are projections at $N=2$.

PROOF. It follows from definitions that $\Phi(x)$ is obtained by putting the components of $x=(x_i)$ in a row, then picking the j-th column vectors of each x_i, calling M_j this matrix, then taking the polar part $x_j'=Pol(M_j)$, and finally setting $\Phi(x)=x'$. Thus:

$$\Phi(x)=Pol((x_{ij})_i)_j$$

$$\Psi(u)=(P_i^{-1/2}u_{ji}P_i^{-1/2})_{ij}$$

Thus, the first assertion is clear, and the second assertion is clear too. □

Our claim is that the algorithm converges, as follows:

CONJECTURE 15.28. *The above maps Φ,Ψ increase the volume,*

$$vol:U_N^N\to Y_N\to[0,1],\quad vol(u)=\prod_j|\det((u_{ij})_i)|$$

and respectively land, after an infinite number of steps, in K_N/X_N.

As a main application of the above conjecture, the infinite iteration $(\Phi^2)^\infty:U_N^N\to K_N$ would provide us with an integration on K_N, and hence on the quotient space $K_N\to X_N$ as well, by taking the push-forward measures, coming from the Haar measure on U_N^N.

In relation now with the matrix model problematics, we have:

CONJECTURE 15.29. *The universal $N\times N$ flat matrix representation*

$$\pi_N:C(S_N^+)\to M_N(C(X_N)),\quad \pi_N(w_{ij})=(u\to u_{ij})$$

is faithful at $N=4$, and is inner faithful at any $N\geq5$.

Here we already know the result at $N=4$, from Theorem 15.21, so the conjecture is that the same holds at $N\geq5$, with the stationarity replaced by inner faithfulness.

We first have the following definition, inspired by the above results:

DEFINITION 15.30. *Associated to $x\in M_N(S_{\mathbb{C}}^{N-1})$ is the $N^p\times N^p$ matrix*

$$(T_p^x)_{i_1\ldots i_p,j_1\ldots j_p}=\frac{1}{N}<x_{i_1j_1},x_{i_2j_2}><x_{i_2j_2},x_{i_3j_3}>\ldots\ldots<x_{i_pj_p},x_{i_1j_1}>$$

where the scalar products are the usual ones on $S_{\mathbb{C}}^{N-1}\subset\mathbb{C}^N$.

The first few values of these matrices, at $p=1,2,3$, are as follows:

$$\begin{aligned}(T_1^x)_{ia}&=\frac{1}{N}<x_{ia},x_{ia}>=\frac{1}{N}\\(T_2^x)_{ij,ab}&=\frac{1}{N}<x_{ia},x_{jb}><x_{jb},x_{ia}>=\frac{1}{N}|<x_{ia},x_{jb}>|^2\\(T_3^x)_{ijk,abc}&=\frac{1}{N}<x_{ia},x_{jb}><x_{jb},x_{kc}><x_{kc},x_{ia}>\end{aligned}$$

The interest in these matrices, in connection with Conjecture 15.28, comes from:

PROPOSITION 15.31. *For the universal model, the matrices T_p are*

$$T_p=\int_{K_N}T_p^x dx$$

where dx is the measure on the model space K_N coming from Conjecture 15.28.

PROOF. This is a trivial statement, because by definition of T_p, we have:

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p} &= tr(u_{i_1j_1}\ldots u_{i_pj_p})\\ &= \int_{K_N} tr(u^x_{i_1j_1}\ldots u^x_{i_pj_p})dx\\ &= \int_{K_N} tr(Proj(x_{i_1j_1})\ldots Proj(x_{i_px_p}))dx\\ &= \frac{1}{N}\int_{K_N} < x_{i_1j_1},x_{i_2j_2} > \ldots\ldots < x_{i_pj_p},x_{i_1j_1} > dx\\ &= \int_{K_N}(T_p^x)_{i_1\ldots i_p,j_1\ldots j_p}dx\end{aligned}$$

Thus the formula in the statement holds indeed. □

In fact, the matrices T_p^x are related to Conjecture 15.29 as well. We refer to the [**15**] and the literature for further details regarding the above conjectures.

Finally, you might wonder what happens to the graph problematics, in relation with all this. The main problem here to model $G^+(X)$, and there are many things that can be done here. However, at the level of valuable theorems, nothing much is known, so far:

(1) Generally speaking, the assumption $G\curvearrowright X$, which corresponds to $du=ud$, can be formulated directly in the model, thanks to Theorem 15.11.

(2) At the level of concrete examples, which actually concern quantum graphs rather than graphs, Conjecture 15.22 looks like a good statement.

(3) More generally, it is possible to formulate some interesting analogues of Conjecture 15.22, by using general Weyl matrices, instead of Pauli matrices.

(4) Finally, the above Sinkhorn considerations, for the quantum group S_N^+, are waiting too to be extended, to more general quantum groups of type $G^+(X)$.

In short, many interesting questions here, and good for you I guess, if looking for nice research problems. And finally, in case you were not very happy with the present chapter, due to the lack of graphs, sure I understand, but wait for it. We will be back to this, by using the matrix model technology that we learned here, in the next chapter.

15e. Exercises

We had an exciting analytic chapter here, and as exercises, we have:

EXERCISE 15.32. *Learn about half-liberations G_N^*, and their matrix models.*

EXERCISE 15.33. *Clarify the details in the proof of the Thoma theorem.*

EXERCISE 15.34. *Find some further examples of inner faithful models.*

EXERCISE 15.35. *Do some modeling work for the graph quantum symmetries.*

EXERCISE 15.36. *Do some modeling work too for the quantum graph symmetries.*

EXERCISE 15.37. *Learn more about the Weyl matrices, and their applications.*

EXERCISE 15.38. *Learn more about the Pauli model for S_4^+, and its applications.*

EXERCISE 15.39. *Work on the above-mentioned Sinkhorn related conjectures.*

As bonus exercise, learn some Random Matrix Theory (RMT).

CHAPTER 16

Block designs

16a. Hadamard matrices

We kept the best for the end, Hadamard matrices. These are fascinating matrices, and central objects in design theory, advanced linear algebra, coding theory, discrete Fourier analysis, operator algebras, subfactors, planar algebras, knot invariants, abstract statistical mechanics, and many more. And, remarkably, we will see that their symmetries, which are genuinely of "quantum" nature, are encoded by certain quantum groups, which can be of help in connection with the above considerations, and their applications.

All this, however, is not exactly beginner level, and in order to properly understand what an Hadamard matrix truly is, its precise mathematical and physical significance, I mean, and to correctly construct its quantum symmetry group, some considerable preliminary learning is necessary. So, here is the plan for the present chapter:

PLAN 16.1. *We will get introduced to the Hadamard matrices as follows:*

(1) *We will first talk about the real matrices, following Sylvester, Hadamard, Paley and others, making it clear that we are at the heart of design theory.*
(2) *Then, we will talk about the complex matrices, following Haagerup and others, making it clear that we are at the heart of discrete Fourier analysis.*
(3) *Then, we will discuss the operator algebra significance of these matrices, following a key observation of Popa, and then, some fundamental work of Jones.*
(4) *And finally, we will construct the symmetry groups of these matrices, compatible with all the above, and we will end with some computations.*

Which sounds good, but matter of doublechecking if this is good indeed, I should perhaps ask the audience. Unfortunately cat is sleeping on my lap, let's not disturb him, dog just says ok boss, without further comments, and crocodile gone as usual. Only mouse seems available, for a discussion, and he does not seem very happy with all this:

MOUSE 16.2. *What you say sounds like physics. We should launch instead a purely mathematical program, on this and related topics. I intend to complain about this.*

Well, cannot make everyone happy. This being said, frankly, dear mouse, I think that a bit of physical knowledge can help you with your mathematics. So, I know that you don't quite like cat, but once he will wake up, I prepared upstairs a cozy conference room for the two of you, so you can debate with him about mathematics and physics.

Getting started now, we first need to talk about the real Hadamard matrices, following Sylvester, Hadamard, Paley and others. As definition for them, we have:

DEFINITION 16.3. *An Hadamard matrix is a square matrix*

$$H\in M_N(\pm1)$$

whose rows are pairwise orthogonal.

Which looks like something very simple, and definitely belonging to design theory, because when comparing two rows, the number of matchings must equal the number of mismatchings. Here is a basic example, called first Walsh matrix:

$$W_2=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

This matrix is quite trivial, of size 2×2, but by taking tensor powers of it, we have as examples the higher Walsh matrices as well, having size $2^k\times2^k$, given by:

$$W_{2^k}=W_2^{\otimes k}$$

What happens then in arbitrary size $N\times N$? It is clear that we must have $2|N$, and along the same lines, it is easy to see, by playing around with the first rows, that once your matrix has $N\geq3$ rows, we must have $4|N$, the precise result being as follows:

PROPOSITION 16.4. *The size of an Hadamard matrix $H\in M_N(\pm1)$ must satisfy*

$$N\in\{2\}\cup4\mathbb N$$

with this coming from the orthogonality condition between the first 3 rows.

PROOF. By permuting the rows and columns or by multiplying them by -1, as to rearrange the first 3 rows, we can always assume that our matrix looks as follows:

$$H=\begin{pmatrix}1\ldots\ldots1&1\ldots\ldots1&1\ldots\ldots1&1\ldots\ldots1\\1\ldots\ldots1&1\ldots\ldots1&-1\ldots-1&-1\ldots-1\\1\ldots\ldots1&-1\ldots-1&1\ldots\ldots1&-1\ldots-1\\\underbrace{\ldots\ldots\ldots}_x&\underbrace{\ldots\ldots\ldots}_y&\underbrace{\ldots\ldots\ldots}_z&\underbrace{\ldots\ldots\ldots}_t\end{pmatrix}$$

Now if we denote by x,y,z,t the sizes of the block columns, as indicated, the orthogonality conditions between the first 3 rows give the following system of equations:

$$(1\perp2)\quad:\quad x+y=z+t$$

$$(1\perp3)\quad:\quad x+z=y+t$$

$$(2\perp3)\quad:\quad x+t=y+z$$

The numbers x,y,z,t being such that the average of any two equals the average of the other two, and so equals the global average, the solution of our system is $x=y=z=t$. Thus the matrix size $N=x+y+z+t$ must be a multiple of 4, as claimed. $\square$

The above result is something quite interesting, and the point is that a similar analysis with 4 rows or more does not give any further restriction on the possible values of the size $N \in \mathbb{N}$. In fact, we are led in this way to the following famous conjecture:

CONJECTURE 16.5 (Hadamard). *There is an Hadamard matrix of order N,*

$$H \in M_N(\pm 1)$$

for any $N \in 4\mathbb{N}$.

Normally this is an analytic question, because in practice the number of Hadamard matrices grows exponentially with N, and so in order to prove the conjecture, you just need a modest lower estimate on this number. But, no one knows how to do this, and this despite the Hadamard conjecture being open for more than 100 years.

This being said, let us verify this at small values of $N \in 4\mathbb{N}$. And here, with $N = 4, 8$ being solved by the Walsh matrices, we are faced with constructing a matrix at $N = 12$. In order to solve this question, let $q = p^k$ be an odd prime power, and set:

$$\chi(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a = b^2, b \neq 0 \\ -1 & \text{otherwise} \end{cases}$$

Then set $Q_{ab} = \chi(b - a)$, with indices in $\mathbb{F}_q$. With these conventions, the Paley construction of Hadamard matrices, which works well at $N = 12$, is as follows:

THEOREM 16.6. *Given an odd prime power $q = p^k$, construct $Q_{ab} = \chi(b-a)$ as above. We have then constructions of Hadamard matrices, as follows:*

(1) *Paley 1: if $q = 3(4)$ we have a matrix of size $N = q + 1$, as follows:*

$$P_N^1 = 1 + \begin{pmatrix} 0 & 1 & \ldots & 1 \\ -1 & & & \\ \vdots & & Q & \\ -1 & & & \end{pmatrix}$$

(2) *Paley 2: if $q = 1(4)$ we have a matrix of size $N = 2q + 2$, as follows:*

$$P_N^2 = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & & & \\ \vdots & & Q & \\ 1 & & & \end{pmatrix} \quad : \quad 0 \to \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad , \quad \pm 1 \to \pm \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

These matrices are skew-symmetric ($H + H^t = 2$), respectively symmetric ($H = H^t$).

PROOF. In order to simplify the presentation, we denote by 1 all the identity matrices, of any size, and by $\mathbb{I}$ all the rectangular all-one matrices, of any size as well. It is elementary to check that the matrix $Q_{ab} = \chi(a-b)$ has the following properties:

$$QQ^t = q1 - \mathbb{I} \quad , \quad Q\mathbb{I} = \mathbb{I}Q = 0$$

In addition, we have the following formulae, which are elementary as well, coming from the fact that -1 is a square in $\mathbb{F}_q$ precisely when $q = 1(4)$:

$$q = 1(4) \implies Q = Q^t \quad , \quad q = 3(4) \implies Q = -Q^t$$

With these observations in hand, the proof goes as follows:

(1) With our above conventions for 1 and $\mathbb{I}$, the matrix in the statement is:

$$P_N^1 = \begin{pmatrix} 1 & \mathbb{I} \\ -\mathbb{I} & 1+Q \end{pmatrix}$$

With this formula in hand, the Hadamard matrix condition follows from:

$$\begin{aligned} P_N^1(P_N^1)^t &= \begin{pmatrix} 1 & \mathbb{I} \\ -\mathbb{I} & 1+Q \end{pmatrix} \begin{pmatrix} 1 & -\mathbb{I} \\ \mathbb{I} & 1-Q \end{pmatrix} \\ &= \begin{pmatrix} N & 0 \\ 0 & \mathbb{I}+1-Q^2 \end{pmatrix} \\ &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

(2) If we denote by G, F the 2×2 matrices in the statement, which replace respectively the $0, 1$ entries, then we have the following formula for our matrix:

$$P_N^2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes F + 1 \otimes G$$

With this formula in hand, the Hadamard matrix condition follows from:

$$\begin{aligned} (P_N^2)^2 &= \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix}^2 \otimes F^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes G^2 + \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes (FG+GF) \\ &= \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \otimes 2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 2 + \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes 0 \\ &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

Finally, the last assertion is clear, from the above formulae relating Q, Q^t. $\square$

In practice, with Walsh and Paley, the next problem is at $N = 92$. But here, we have:

THEOREM 16.7. *Assuming that $A,B,C,D\in M_K(\pm 1)$ are circulant, symmetric, pairwise commute and satisfy the condition*

$$A^2+B^2+C^2+D^2=4K$$

the following $4K\times 4K$ matrix is Hadamard, called of Williamson type:

$$H=\begin{pmatrix}A&B&C&D\\-B&A&-D&C\\-C&D&A&-B\\-D&-C&B&A\end{pmatrix}$$

Moreover, matrices A,B,C,D as above exist at $K=23$, where $4K=92$.

PROOF. Consider the quaternion units $1,i,j,k\in M_4(0,1)$. These describe the positions of the A,B,C,D entries in our matrix H, so this matrix can be written as:

$$H=A\otimes 1+B\otimes i+C\otimes j+D\otimes k$$

Assuming now that A,B,C,D are symmetric, we have:

$$\begin{aligned}HH^t&=(A\otimes 1+B\otimes i+C\otimes j+D\otimes k)\\&\quad\,(A\otimes 1-B\otimes i-C\otimes j-D\otimes k)\\&=(A^2+B^2+C^2+D^2)\otimes 1-([A,B]-[C,D])\otimes i\\&\quad\,-([A,C]-[B,D])\otimes j-([A,D]-[B,C])\otimes k\end{aligned}$$

Now assume that our matrices A,B,C,D pairwise commute, and satisfy the condition in the statement. In this case, it follows from the above formula that we have:

$$HH^t=4K$$

Thus, we obtain indeed an Hadamard matrix, as claimed. However, finding such matrices is in general a difficult task, and this is where Williamson's extra assumption in the statement, that A,B,C,D should be taken circulant, comes from. Finally, regarding the $K=23$ and $N=92$ example, this comes via a computer search. □

At higher N things become more technical, and more complicated constructions, along the lines of those of Paley and Williamson, are needed. Quite curiously, as of now, early 21th century, the human knowledge stops at the number of the beast, namely:

$$\mathfrak{N}=666$$

But hey, the story is not over here. We will not let the Devil win, and as a further twist to the plot, bringing some sort of solution to this, we have:

THEOREM 16.8. *When enlarging the attention to the complex Hadamard matrices, $H \in M_N(\mathbb{T})$ having the rows pairwise orthogonal, the Fourier matrix,*

$$F_N = \begin{pmatrix}1&1&1&\ldots&1\\ 1&w&w^2&\ldots&w^{N-1}\\ 1&w^2&w^4&\ldots&w^{2(N-1)}\\ \vdots&\vdots&\vdots&&\vdots\\ 1&w^{N-1}&w^{2(N-1)}&\ldots&w^{(N-1)^2}\end{pmatrix}$$

with $w = e^{2\pi i/N}$, provides an example of such a matrix, at any $N \in \mathbb{N}$. Thus, the Hadamard Conjecture problematics dissapears, in the complex setting.

PROOF. We have seen in chapter 2 that the rescaling $U = F_N/\sqrt{N}$ is unitary. Thus the rows of U are pairwise orthogonal, and so follow to be the rows of F_N. □

In view of the above result, let us study more in detail the complex Hadamard matrices. Many examples can be constructed, quite often by using the combinatorics of roots of unity, and as a basic example here, we have the tensor products of Fourier matrices:

$$F_{N_1,\ldots,N_k} = F_{N_1} \otimes \ldots \otimes F_{N_k}$$

Of course, not all examples of complex Hadamard matrices come from roots of unity, as shown by the following quite exotic looking result, due to Björck and Fröberg:

PROPOSITION 16.9. *The following is a complex Hadamard matrix,*

$$BF_6 = \begin{pmatrix}1&ia&-a&-i&-\bar{a}&i\bar{a}\\ i\bar{a}&1&ia&-a&-i&-\bar{a}\\ -\bar{a}&i\bar{a}&1&ia&-a&-i\\ -i&-\bar{a}&i\bar{a}&1&ia&-a\\ -a&-i&-\bar{a}&i\bar{a}&1&ia\\ ia&-a&-i&-\bar{a}&i\bar{a}&1\end{pmatrix}$$

where $a \in \mathbb{T}$ is one of the roots of $a^2 + (\sqrt{3}-1)a + 1 = 0$.

PROOF. The matrix in the statement is circulant, in the sense that the rows appear by cyclically permuting the first row. Thus, we only have to check that the first row is orthogonal to the other 5 rows. But this follows from $a^2 + (\sqrt{3}-1)a + 1 = 0$. □

Leaving aside such monsters, we have as well deformations, as shown by:

THEOREM 16.10. *The only Hadamard matrices at $N = 4$ are, up to equivalence*

$$F_4^q = \begin{pmatrix}1&1&1&1\\ 1&-1&1&-1\\ 1&q&-1&-q\\ 1&-q&-1&q\end{pmatrix}$$

with $q \in \mathbb{T}$, which appear as suitable deformations of $W_4 = F_2 \otimes F_2$.

Proof. This is something quite self-explanatory, and we will leave working out all this, namely finding the correct meaning of the equivalence relation, and of the deformation notion used, along of course with the proof, as an instructive exercise. □

Getting now into further classification matters, at $N = 5$ things are still manageable, as shown by the following remarkable result, due to Haagerup:

Theorem 16.11. *The only complex Hadamard matrix at $N = 5$ is the Fourier matrix,*

$$F_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 \\ 1 & w^2 & w^4 & w & w^3 \\ 1 & w^3 & w & w^4 & w^2 \\ 1 & w^4 & w^3 & w^2 & w \end{pmatrix}$$

with $w = e^{2\pi i/5}$, up to the standard equivalence relation for such matrices.

Proof. This is something quite tricky, with the key lemma used for the proof concerning an Hadamard matrix $H \in M_5(\mathbb{T})$, chosen dephased, as follows:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & a & x & * & * \\ 1 & y & b & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \end{pmatrix}$$

The key lemma states then that a, b, x, y must satisfy the following equation:

$$(x - y)(x - ab)(y - ab) = 0$$

And, by applying this lemma all across the matrix, we get the result. □

At $N = 6$ the complex Hadamard matrices are not classifiable, due to the presence of the Björck-Fröberg matrix from Proposition 16.9, and of many other bizarre matrices, of the same type, or even more complicated. However, with a suitable notion of "regularity", basically stating that the scalar products between rows must appear as rotated versions of vanishing sums of roots of unity, a classification is possible at $N = 6$.

At $N = 7$ we have the Fourier matrix F_7, and a matrix P_7^q found by Petrescu, and the classification, under the above-mentioned notion of regularity, is only available in conjectural form. At $N = 8$ and higher things become severly complicated, and there are no conjectures available, so far. For more on all this, you can check my book [7].

16b. Algebras, subfactors

Let us discuss now the relation with von Neumann algebras, subfactor theory, and planar algebras. As a starting point, we have the following key observation of Popa:

THEOREM 16.12. *Up to a conjugation by a unitary, the pairs of orthogonal MASA in the simplest factor, namely the matrix algebra $M_N(\mathbb{C})$, are as follows,*

$$A=\Delta\quad,\quad B=H\Delta H^*$$

with $\Delta\subset M_N(\mathbb{C})$ being the diagonal matrices, and with $H\in M_N(\mathbb{C})$ being Hadamard.

PROOF. Any maximal abelian subalgebra (MASA) in $M_N(\mathbb{C})$ being conjugated to Δ, we can assume, up to conjugation by a unitary, that we have, with $U\in U_N$:

$$A=\Delta\quad,\quad B=U\Delta U^*$$

Now observe that given two diagonal matrices $D,E\in\Delta$, we have:

$$\begin{aligned}tr(D\cdot UEU^*)&=\frac{1}{N}\sum_i(DUEU^*)_{ii}\\&=\frac{1}{N}\sum_{ij}D_{ii}U_{ij}E_{jj}\bar{U}_{ij}\\&=\frac{1}{N}\sum_{ij}D_{ii}E_{jj}|U_{ij}|^2\end{aligned}$$

Thus, the orthogonality condition $A\perp B$ reformulates as follows:

$$\frac{1}{N}\sum_{ij}D_{ii}E_{jj}|U_{ij}|^2=\frac{1}{N^2}\sum_{ij}D_{ii}E_{jj}$$

Thus, we must have $|U_{ij}|=\frac{1}{\sqrt{N}}$, for any i,j. But this condition tells us precisely that the rescaled matrix $H=\sqrt{N}U$ must be Hadamard, as claimed. □

Along the same lines, but at a more advanced level, we have:

THEOREM 16.13. *Given a complex Hadamard matrix $H\in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal MASA, namely*

$$\begin{tikzcd}\Delta \arrow[r] & M_N(\mathbb{C})\\ \mathbb{C} \arrow[u]\arrow[r] & H\Delta H^* \arrow[u]\end{tikzcd}$$

is a commuting square in the sense of subfactor theory, in the sense that the expectations onto $\Delta,H\Delta H^*$ commute, and their product is the expectation onto $\mathbb{C}$.

PROOF. The expectation $E_\Delta : M_N(\mathbb{C}) \to \Delta$ is the operation $M \to M_\Delta$ which consists in keeping the diagonal, and erasing the rest. Consider now the other expectation:

$$E_{H\Delta H^*} : M_N(\mathbb{C}) \to H\Delta H^*$$

It is better to identify this with the following expectation, with $U = H/\sqrt{N}$:

$$E_{U\Delta U^*} : M_N(\mathbb{C}) \to U\Delta U^*$$

This latter expectation must be of the form $M \to UX_\Delta U^*$, with X satisfying:

$$< M, UDU^* >=< UX_\Delta U^*, UDU^* > \quad , \quad \forall D \in \Delta$$

The scalar products being given by $< a, b >= tr(ab^*)$, this condition reads:

$$tr(MUD^*U^*) = tr(X_\Delta D^*) \quad , \quad \forall D \in \Delta$$

Thus $X = U^*MU$, and the formulae of our two expectations are as follows:

$$\begin{aligned} E_\Delta(M) &= M_\Delta \\ E_{U\Delta U^*}(M) &= U(U^*MU)_\Delta U^* \end{aligned}$$

With these formulae in hand, an elementary computation gives the result. $\square$

The point now is that any commuting square C produces a subfactor of the Murray-von Neumann hyperfinite II_1 factor R. Consider indeed such a square:

$$\begin{tikzcd} C_{01} \arrow[r] & C_{11} \\ C_{00} \arrow[r] \arrow[u] & C_{10} \arrow[u] \end{tikzcd}$$

Under suitable assumptions on $C_{00} \subset C_{10}, C_{01} \subset C_{11}$, we can perform the basic construction, in finite dimensions, and we obtain a whole array of squares, as follows:

$$\begin{tikzcd} A_0 & A_1 & A_2 & \\ C_{02} \arrow[r] \arrow[u, dotted] & C_{12} \arrow[r] \arrow[u, dotted] & C_{22} \arrow[r, dotted] \arrow[u, dotted] & B_2 \\ C_{01} \arrow[r] \arrow[u] & C_{11} \arrow[r] \arrow[u] & C_{21} \arrow[r, dotted] \arrow[u] & B_1 \\ C_{00} \arrow[r] \arrow[u] & C_{10} \arrow[r] \arrow[u] & C_{20} \arrow[r, dotted] \arrow[u] & B_0 \end{tikzcd}$$

Here the various A, B letters stand for the von Neumann algebras obtained in the limit, which are all isomorphic to the hyperfinite II_1 factor R, and we have:

THEOREM 16.14. *In the context of the above diagram, the following happen:*
(1) *$A_0 \subset A_1$ is a subfactor, and $\{A_i\}$ is the Jones tower for it.*
(2) *The corresponding planar algebra is given by $A_0' \cap A_k = C_{01}' \cap C_{k0}$.*
(3) *A similar result holds for the "horizontal" subfactor $B_0 \subset B_1$.*

PROOF. This is something very standard, the idea being as follows:

(1) This is something routine, coming from the subfactor basics.

(2) This is a subtle result, called Ocneanu compactness theorem.

(3) This follows indeed from (1,2), simply by flipping the diagram. □

Getting back now to the Hadamard matrices, we can extend our lineup of results, namely Theorem 16.12 and Theorem 16.13, with an advanced result, as follows:

THEOREM 16.15. *Given a complex Hadamard matrix $H \in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal MASA is a commuting square in the sense of subfactor theory, and the associated planar algebra $P = (P_k)$ is given by the following formula,*

$$T \in P_k \iff T^\circ G^2 = G^{k+2} T^\circ$$

where the objects on the right are constructed as follows:
(1) *$T^\circ = id \otimes T \otimes id$.*
(2) *$G_{ia}^{jb} = \sum_k H_{ik} \bar{H}_{jk} \bar{H}_{ak} H_{bk}$.*
(3) *$G^k_{i_1 \ldots i_k, j_1 \ldots j_k} = G^{j_k j_{k-1}}_{i_k i_{k-1}} \ldots G^{j_2 j_1}_{i_2 i_1}$.*

PROOF. We have two assertions here, the idea being as follows:

(1) The fact that we have indeed a commuting square is something that we already know, and generalizing the original MASA observation of Popa, from Theorem 16.13.

(2) The computation of the associated planar algebra is possible thanks to the Ocneanu compactness formula from Theorem 16.14 (2), and we refer here to Jones [56]. □

All the above was of course quite short, especially in what regards Theorem 16.15. But, we will see that all this can be recovered by using quantum groups.

16c. Symmetry groups

In relation with quantum groups, the starting observation is as follows:

PROPOSITION 16.16. *If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections*

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

PROOF. This is clear, the verification for the rows being as follows:

$$\begin{aligned}\left\langle\frac{H_i}{H_j},\frac{H_i}{H_k}\right\rangle &= \sum_l\frac{H_{il}}{H_{jl}}\cdot\frac{H_{kl}}{H_{il}}\\ &= \sum_l\frac{H_{kl}}{H_{jl}}\\ &= N\delta_{jk}\end{aligned}$$

As for the verification for the columns, this is similar, as follows:

$$\begin{aligned}\left\langle\frac{H_i}{H_j},\frac{H_k}{H_j}\right\rangle &= \sum_l\frac{H_{il}}{H_{jl}}\cdot\frac{H_{jl}}{H_{kl}}\\ &= \sum_l\frac{H_{il}}{H_{kl}}\\ &= N\delta_{ik}\end{aligned}$$

Thus, we have indeed a magic unitary, as claimed. □

We can now proceed in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction:

DEFINITION 16.17. *To any Hadamard matrix $H\in M_N(\mathbb C)$ we associate the quantum permutation group $G\subset S_N^+$ given by the fact that $C(G)$ is the Hopf image of*

$$\pi:C(S_N^+)\to M_N(\mathbb C)\quad,\quad u_{ij}\to Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1,\ldots,H_N\in\mathbb T^N$ are the rows of H.

Summarizing, we have a construction $H\to G$, and our claim is that this construction is something really useful, with G encoding the combinatorics of H. To be more precise, our claim is that "H can be thought of as being a kind of Fourier matrix for G".

This is of course quite interesting, philosophically speaking. There are several results supporting this, with the main evidence coming from the following result, which collects the basic known results regarding the construction $H\to G$:

THEOREM 16.18. *The construction $H\to G$ has the following properties:*

(1) *For a Fourier matrix $H=F_G$ we obtain the group G itself, acting on itself.*
(2) *For $H\notin\{F_G\}$, the quantum group G is not classical, nor a group dual.*
(3) *For a tensor product $H=H'\otimes H''$ we obtain a product, $G=G'\times G''$.*

PROOF. All this material is standard, and elementary, as follows:

(1) Let us first discuss the cyclic group case, where our Hadamard matrix is a usual Fourier matrix, $H = F_N$. Here the rows of H are given by $H_i = \rho^i$, where:

$$\rho = (1, w, w^2, \ldots, w^{N-1})$$

Thus, we have the following formula, for the associated magic basis:

$$\frac{H_i}{H_j} = \rho^{i-j}$$

It follows that the corresponding rank 1 projections $P_{ij} = Proj(H_i/H_j)$ form a circulant matrix, all whose entries commute. Since the entries commute, the corresponding quantum group must satisfy $G \subset S_N$. Now by taking into account the circulant property of $P = (P_{ij})$ as well, we are led to the conclusion that we have:

$$G = \mathbb{Z}_N$$

In the general case now, where $H = F_G$, with G being an arbitrary finite abelian group, the result can be proved either by extending the above proof, of by decomposing $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ and using (3) below, whose proof is independent from the rest.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative.

(3) Assume that we have a tensor product $H = H' \otimes H''$, and let G, G', G'' be the associated quantum permutation groups. We have then a diagram as follows:

$$\begin{tikzcd}
C(S_{N'}^+) \otimes C(S_{N''}^+) \arrow[r] & C(G') \otimes C(G'') \arrow[r] & M_{N'}(\mathbb{C}) \otimes M_{N''}(\mathbb{C}) \arrow[d] \\
C(S_N^+) \arrow[u] \arrow[r] & C(G) \arrow[r] & M_N(\mathbb{C})
\end{tikzcd}$$

Here all the maps are the canonical ones, with those on the left and on the right coming from $N = N'N''$. At the level of standard generators, the diagram is as follows:

$$\begin{tikzcd}
u'_{ij} \otimes u''_{ab} \arrow[r] & w'_{ij} \otimes w''_{ab} \arrow[r] & P'_{ij} \otimes P''_{ab} \arrow[d] \\
u_{ia,jb} \arrow[u] \arrow[r] & w_{ia,jb} \arrow[r] & P_{ia,jb}
\end{tikzcd}$$

Now observe that this diagram commutes. We conclude that the representation associated to H factorizes indeed through $C(G') \otimes C(G'')$, and this gives the result. $\square$

At a more abstract level, one interesting question is that of abstractly characterizing the magic matrices coming from the complex Hadamard matrices. We have here:

PROPOSITION 16.19. *Given an Hadamard matrix $H \in M_N(\mathbb{C})$, the vectors*

$$\xi_{ij} = \frac{H_i}{H_j}$$

on which the magic unitary entries P_{ij} project, have the following properties:

(1) *$\xi_{ii} = \xi$ is the all-one vector.*
(2) *$\xi_{ij}\xi_{jk} = \xi_{ik}$, for any i, j, k.*
(3) *$\xi_{ij}\xi_{kl} = \xi_{il}\xi_{kj}$, for any i, j, k, l.*

PROOF. All these assertions are trivial, by using the formula $\xi_{ij} = H_i/H_j$. $\square$

Let us call now magic basis of a given Hilbert space H any square array of vectors $\xi \in M_N(H)$, all whose rows and columns are orthogonal bases of H. With this convention, the above observations lead to the following result, at the magic basis level:

THEOREM 16.20. *The magic bases $\xi \in M_N(S_{\mathbb{C}}^{N-1})$ coming from the complex Hadamard matrices are those having the following properties:*

(1) *We have $\xi_{ij} \in \mathbb{T}^N$, after a suitable rescaling.*
(2) *The conditions in Proposition 16.19 are satisfied.*

PROOF. By using the multiplicativity conditions (1,2,3) in Proposition 16.19, we conclude that, up to a rescaling, we must have $\xi_{ij} = \xi_i/\xi_j$, where $\xi_1, \ldots, \xi_N$ is the first row of the magic basis. Together with our assumption $\xi_{ij} \in \mathbb{T}^N$, this gives the result. $\square$

At the general level now, regarding the representation theory of the quantum groups associated to the Hadamard matrices, we have the following result:

THEOREM 16.21. *The Tannakian category of the quantum group $G \subset S_N^+$ associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ is given by*

$$T \in Hom(u^{\otimes k}, u^{\otimes l}) \iff T^\circ G^{k+2} = G^{l+2} T^\circ$$

where the objects on the right are constructed as follows:

(1) *$T^\circ = id \otimes T \otimes id$.*
(2) *$G_{ia}^{jb} = \sum_k H_{ik}\bar{H}_{jk}\bar{H}_{ak}H_{bk}$.*
(3) *$G^k_{i_1\ldots i_k, j_1\ldots j_k} = G^{j_k j_{k-1}}_{i_k i_{k-1}} \ldots G^{j_2 j_1}_{i_2 i_1}$.*

PROOF. We use the Tannakian result for the Hopf image of a representation, discussed in chapter 15. With the notations here, we have the following formula:

$$Hom(u^{\otimes k}, u^{\otimes l}) = Hom(U^{\otimes k}, U^{\otimes l})$$

Here the vector space on the right consists by definition of the complex $N^l \times N^k$ matrices T satisfying the following relation:

$$TU^{\otimes k} = U^{\otimes l} T$$

If we denote this equality by $L = R$, the left term L is given by:

$$\begin{aligned} L_{ij} &= (TU^{\otimes k})_{ij} \\ &= \sum_a T_{ia} U^{\otimes k}_{aj} \\ &= \sum_a T_{ia} U_{a_1 j_1} \ldots U_{a_k j_k} \end{aligned}$$

As for the right term R, this is given by the following formula:

$$\begin{aligned} R_{ij} &= (U^{\otimes l} T)_{ij} \\ &= \sum_b U^{\otimes l}_{ib} T_{bj} \\ &= \sum_b U_{i_1 b_1} \ldots U_{i_l b_l} T_{bj} \end{aligned}$$

Consider now the vectors $\xi_{ij} = H_i / H_j$. Since these vectors span the ambient Hilbert space, the equality $L = R$ is equivalent to the following equality:

$$< L_{ij} \xi_{pq}, \xi_{rs} > = < R_{ij} \xi_{pq}, \xi_{rs} >$$

We use now the following well-known formula, expressing a product of rank one projections $P_1, \ldots, P_k$ in terms of the corresponding image vectors $\xi_1, \ldots, \xi_k$:

$$< P_1 \ldots P_k x, y > = < x, \xi_k >< \xi_k, \xi_{k-1} > \ldots\ldots < \xi_2, \xi_1 >< \xi_1, y >$$

This gives the following formula for L:

$$\begin{aligned} < L_{ij} \xi_{pq}, \xi_{rs} > &= \sum_a T_{ia} < P_{a_1 j_1} \ldots P_{a_k j_k} \xi_{pq}, \xi_{rs} > \\ &= \sum_a T_{ia} < \xi_{pq}, \xi_{a_k j_k} > \ldots < \xi_{a_1 j_1}, \xi_{rs} > \\ &= \sum_a T_{ia} G^{q j_k}_{p a_k} G^{j_k j_{k-1}}_{a_k a_{k-1}} \ldots G^{j_2 j_1}_{a_2 a_1} G^{j_1 s}_{a_1 r} \\ &= \sum_a T_{ia} G^{k+2}_{rap, sjq} \\ &= (T^\circ G^{k+2})_{rip, sjq} \end{aligned}$$

As for the right term R, this is given by:

$$\begin{aligned}
< R_{ij}\xi_{pq}, \xi_{rs} > &= \sum_b < P_{i_1b_1} \ldots P_{i_lb_l}\xi_{pq}, \xi_{rs} > T_{bj}\\
&= \sum_b < \xi_{pq}, \xi_{i_lb_l} > \ldots < \xi_{i_1b_1}, \xi_{rs} > T_{bj}\\
&= \sum_b G_{pi_l}^{qb_l} G_{i_li_{l-1}}^{b_lb_{l-1}} \ldots G_{i_2i_1}^{b_2b_1} G_{i_1r}^{b_1s} T_{bj}\\
&= \sum_b G_{rip,sbq}^{l+2} T_{bj}\\
&= (G^{l+2}T^\circ)_{rip,sjq}
\end{aligned}$$

Thus, we obtain the formula in the statement. $\square$

The point now is that all the above is very similar to Theorem 16.15, and we have:

THEOREM 16.22. *Let $H \in M_N(\mathbb{C})$ be a complex Hadamard matrix.*

(1) *The planar algebra associated to H is given by $P_k = Fix(u^{\otimes k})$, where $G \subset S_N^+$ is the associated quantum permutation group.*
(2) *The corresponding Poincaré series $f(z) = \sum_k \dim(P_k) z^k$ equals the Stieltjes transform $\int_G \frac{1}{1-z\chi}$ of the law of the main character $\chi = \sum_i u_{ii}$.*

PROOF. This follows by comparing the quantum group and subfactor results:

(1) As mentioned above, this simply follows by comparing Theorem 16.21 with the subfactor computation in Theorem 16.15. For full details here, we refer to [7].

(2) This is a consequence of (1), and of the Peter-Weyl type results from [99], which tell us that fixed points can be counted by integrating characters. $\square$

Regarding now the subfactor itself, the result here is as follows:

THEOREM 16.23. *The subfactor associated to $H \in M_N(\mathbb{C})$ is of the form*

$$A^G \subset (\mathbb{C}^N \otimes A)^G$$

with $A = R \rtimes \widehat{G}$, where $G \subset S_N^+$ is the associated quantum permutation group.

PROOF. This is something more technical, the idea being that the basic construction procedure for the commuting squares, explained before Theorem 16.14, can be performed in an "equivariant setting", for commuting squares having components as follows:

$$D \otimes_G E = (D \otimes (E \rtimes \widehat{G}))^G$$

To be more precise, starting with a commuting square formed by such algebras, we obtain by basic construction a whole array of commuting squares as follows, with $\{D_i\}, \{E_i\}$

being by definition Jones towers, and with D_∞, E_∞ being their inductive limits:

$$\begin{tikzcd}
D_0\otimes_G E_\infty & D_1\otimes_G E_\infty & D_2\otimes_G E_\infty & \\
D_0\otimes_G E_2 \arrow[r] \arrow[u, dotted] & D_1\otimes_G E_2 \arrow[r] \arrow[u, dotted] & D_2\otimes_G E_2 \arrow[r, dotted] \arrow[u, dotted] & D_\infty\otimes_G E_2 \\
D_0\otimes_G E_1 \arrow[r] \arrow[u] & D_1\otimes_G E_1 \arrow[r] \arrow[u] & D_2\otimes_G E_1 \arrow[r, dotted] \arrow[u] & D_\infty\otimes_G E_1 \\
D_0\otimes_G E_0 \arrow[r] \arrow[u] & D_1\otimes_G E_0 \arrow[r] \arrow[u] & D_2\otimes_G E_0 \arrow[r, dotted] \arrow[u] & D_\infty\otimes_G E_0
\end{tikzcd}$$

The point now is that this quantum group picture works in fact for any commuting square having $\mathbb{C}$ in the lower left corner. In the Hadamard matrix case, that we are interested in here, the corresponding commuting square is as follows:

$$\begin{tikzcd}
\mathbb{C}\otimes_G \mathbb{C}^N \arrow[r] & \mathbb{C}^N\otimes_G \mathbb{C}^N \\
\mathbb{C}\otimes_G \mathbb{C} \arrow[r] \arrow[u] & \mathbb{C}^N\otimes_G \mathbb{C} \arrow[u]
\end{tikzcd}$$

Thus, the subfactor obtained by vertical basic construction appears as follows:

$$\mathbb{C}\otimes_G E_\infty \subset \mathbb{C}^N\otimes_G E_\infty$$

But this gives the conclusion in the statement, with the II_1 factor appearing there being by definition a crossed product, as follows:

$$A = E_\infty \rtimes \widehat{G}$$

Observe also that we have $E_\infty \simeq R$, the hyperfinite factor. See [7]. □

There are many other things that can be said about the Hadamard matrices and their relation with operator algebras, and about the subfactors associated to the Hadamard matrices, and their generalizations, and we refer to [7], [56] and related papers.

16d. Fourier matrices

Getting back now to Theorem 16.18, going beyond it is a quite delicate task. The next simplest models appear by deforming the Fourier matrices, or rather the tensor products of such matrices, $F_{G\times H} = F_G \otimes F_H$, via the following construction, due to Diţă:

Definition 16.24. *Given two finite abelian groups G, H, we consider the corresponding deformed Fourier matrix, given by the formula*

$$(F_G \otimes_Q F_H)_{ia,jb} = Q_{ib}(F_G)_{ij}(F_H)_{ab}$$

and we factorize the associated representation π_Q of the algebra $C(S_{G\times H}^+)$,

$$\begin{tikzcd} C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[dr] & & M_{G\times H}(\mathbb{C}) \\ & C(G_Q) \arrow[ur, "\pi"'] & \end{tikzcd}$$

with $C(G_Q)$ being the Hopf image of this representation π_Q.

Explicitely computing the above quantum permutation group $G_Q \subset S_{G\times H}^+$, as function of the parameter matrix $Q \in M_{G\times H}(\mathbb{T})$, will be our main purpose, in what follows. In order to do so, following [10], we first have the following elementary result:

Proposition 16.25. *We have a factorization as follows,*

$$\begin{tikzcd} C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[dr] & & M_{G\times H}(\mathbb{C}) \\ & C(H \wr_* G) \arrow[ur, "\pi"'] & \end{tikzcd}$$

given on the standard generators by the formulae

$$U_{ab}^{(i)} = \sum_j W_{ia,jb} \quad , \quad V_{ij} = \sum_a W_{ia,jb}$$

independently of b, where W is the magic matrix producing π_Q.

Proof. With $K = F_G, L = F_H$ and $M = |G|, N = |H|$, the formula of the magic matrix $W \in M_{G\times H}(M_{G\times H}(\mathbb{C}))$ associated to $H = K \otimes_Q L$ is as follows:

$$\begin{aligned}(W_{ia,jb})_{kc,ld} &= \frac{1}{MN}\cdot\frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}}\cdot\frac{K_{ik}K_{jl}}{K_{il}K_{jk}}\cdot\frac{L_{ac}L_{bd}}{L_{ad}L_{bc}} \\ &= \frac{1}{MN}\cdot\frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}}\cdot K_{i-j,k-l}L_{a-b,c-d}\end{aligned}$$

Our claim now is that the representation π_Q constructed in Definition 16.24 can be factorized in three steps, up to the factorization in the statement, as follows:

$$\begin{tikzcd}
C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[d] & & M_{G\times H}(\mathbb{C}) \\
C(S_H^+ \wr_* S_G^+) \arrow[r] \arrow[urr, dotted] & C(S_H^+ \wr_* G) \arrow[r] \arrow[ur, dotted] & C(H \wr_* G) \arrow[u, dotted]
\end{tikzcd}$$

Indeed, the construction of the map on the left is standard. Regarding the second factorization, this comes from the fact that since the elements V_{ij} depend on $i-j$, they satisfy the defining relations for the quotient algebra $C(S_G^+) \to C(G)$. Finally, regarding the third factorization, observe that $W_{ia,jb}$ depends only on i,j and on $a-b$. By summing over j we obtain that the elements $U_{ab}^{(i)}$ depend only on $a-b$, and we are done. $\square$

Still following [**10**], we can further refine Proposition 16.25, as follows:

Proposition 16.26. *We have a factorization as follows,*

$$\begin{tikzcd}
C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[dr] & & M_{G\times H}(\mathbb{C}) \\
& C^*(\Gamma_{G,H}) \rtimes C(G) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where the group on the bottom is given by:

$$\Gamma_{G,H} = H^{*G} \Big/ \left\langle [c_1^{(i_1)} \ldots c_s^{(i_s)}, d_1^{(j_1)} \ldots d_s^{(j_s)}] = 1 \Big| \sum_r c_r = \sum_r d_r = 0 \right\rangle$$

Proof. Assume that we have a representation, as follows:

$$\pi : C^*(\Gamma) \rtimes C(G) \to M_L(\mathbb{C})$$

Let Λ be a G-stable normal subgroup of Γ, so that G acts on Γ/Λ, and we can form the product $C^*(\Gamma/\Lambda) \rtimes C(G)$, and assume that π is trivial on Λ. Then π factorizes as:

$$\begin{tikzcd}
C^*(\Gamma) \rtimes C(G) \arrow[rr, "\pi"] \arrow[dr] & & M_L(\mathbb{C}) \\
& C^*(\Gamma/\Lambda) \rtimes C(G) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

With $\Gamma = H^{*G}$, this gives the result. $\square$

We have now all the needed ingredients for proving a main result, as follows:

THEOREM 16.27. *When Q is generic, the minimal factorization for π_Q is*

$$\begin{tikzcd}
C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[dr] & & M_{G\times H}(\mathbb{C}) \\
& C^*(\Gamma_{G,H}) \rtimes C(G) \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where on the bottom

$$\Gamma_{G,H} \simeq \mathbb{Z}^{(|G|-1)(|H|-1)} \rtimes H$$

is the discrete group constructed above.

PROOF. Consider the factorization in Proposition 16.26, which is as follows, where L denotes the Hopf image of π_Q:

$$\theta : C^*(\Gamma_{G,H}) \rtimes C(G) \to L$$

To be more precise, this morphism produces the following commutative diagram:

$$\begin{tikzcd}
C(S_{G\times H}^+) \arrow[rr, "\pi_Q"] \arrow[dr] \arrow[ddr, bend right] & & M_{G\times H}(\mathbb{C}) \\
& L \arrow[ur] & \\
& C^*(\Gamma_{G,H}) \rtimes C(G) \arrow[u, dashed, "\theta"'] \arrow[uur, bend right, "\pi"'] &
\end{tikzcd}$$

The first observation is that the injectivity assumption on $C(G)$ holds by construction, and that for $f \in C(G)$, the matrix $\pi(f)$ is "block scalar". Now for $r \in \Gamma_{G,H}$ with $\theta(r \otimes 1) = \theta(1 \otimes f)$ for some $f \in C(G)$, we see, using the commutative diagram, that $\pi(r \otimes 1)$ is block scalar. Thus, modulo some standard algebra, we are done. $\square$

Summarizing, we have computed the quantum permutation groups associated to the Diţă deformations of the tensor products of Fourier matrices, in the case where the deformation matrix Q is generic. For some further computations, in the case where the deformation matrix Q is no longer generic, we refer to [**10**] and follow-up papers.

Let us compute now the Kesten measure $\mu = law(\chi)$, in the case where the deformation matrix is generic, as before. Our results here will be a combinatorial moment formula, a geometric interpretation of it, and an asymptotic result. We first have:

THEOREM 16.28. *We have the moment formula*

$$\int\chi^p=\frac{1}{|G|\cdot|H|}\#\left\{\begin{matrix}i_1,\ldots,i_p\in G\\ d_1,\ldots,d_p\in H\end{matrix}\Bigg|\begin{matrix}[(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]\\ =[(i_1,d_p),(i_2,d_1),\ldots,(i_p,d_{p-1})]\end{matrix}\right\}$$

where the sets between square brackets are by definition sets with repetition.

PROOF. According to the various formulae above, the factorization found in Theorem 16.27 is, at the level of standard generators, as follows:

$$\begin{matrix}C(S_{G\times H}^+)&\to&C^*(\Gamma_{G,H})\otimes C(G)&\to&M_{G\times H}(\mathbb C)\\ u_{ia,jb}&\to&\frac{1}{|H|}\sum_cF_{b-a,c}c^{(i)}\otimes v_{ij}&\to&W_{ia,jb}\end{matrix}$$

Thus, the main character of the quantum permutation group that we found in Theorem 16.27 is given by the following formula:

$$\begin{aligned}\chi&=\frac{1}{|H|}\sum_{iac}c^{(i)}\otimes v_{ii}\\ &=\sum_{ic}c^{(i)}\otimes v_{ii}\\ &=\left(\sum_{ic}c^{(i)}\right)\otimes\delta_1\end{aligned}$$

Now since the Haar functional of $C^*(\Gamma)\rtimes C(H)$ is the tensor product of the Haar functionals of $C^*(\Gamma),C(H)$, this gives the following formula, valid for any $p\geq 1$:

$$\int\chi^p=\frac{1}{|G|}\int_{\widehat{\Gamma}_{G,H}}\left(\sum_{ic}c^{(i)}\right)^p$$

Consider the elements $S_i=\sum_cc^{(i)}$. With standard notations, we have:

$$S_i=\sum_c(b_{i0}-b_{ic},c)$$

Now observe that these elements multiply as follows:

$$S_{i_1}\ldots S_{i_p}=\sum_{c_1\ldots c_p}\begin{pmatrix}b_{i_10}-b_{i_1c_1}+b_{i_2c_1}-b_{i_2,c_1+c_2}&&\\ +b_{i_3,c_1+c_2}-b_{i_3,c_1+c_2+c_3}+\ldots\ldots&,&c_1+\ldots+c_p\\ \ldots\ldots+b_{i_p,c_1+\ldots+c_{p-1}}-b_{i_p,c_1+\ldots+c_p}&&\end{pmatrix}$$

In terms of the new indices $d_r=c_1+\ldots+c_r$, this formula becomes:

$$S_{i_1}\ldots S_{i_p}=\sum_{d_1\ldots d_p}\begin{pmatrix}b_{i_10}-b_{i_1d_1}+b_{i_2d_1}-b_{i_2d_2}&&\\ +b_{i_3d_2}-b_{i_3d_3}+\ldots\ldots&,&d_p\\ \ldots\ldots+b_{i_pd_{p-1}}-b_{i_pd_p}&&\end{pmatrix}$$

Now by integrating, we must have $d_p=0$ on one hand, and on the other hand:

$$[(i_1,0),(i_2,d_1),\ldots,(i_p,d_{p-1})]=[(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]$$

Equivalently, we must have $d_p=0$ on one hand, and on the other hand:

$$[(i_1,d_p),(i_2,d_1),\ldots,(i_p,d_{p-1})]=[(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]$$

Thus, by translation invariance with respect to d_p, we obtain:

$$\int_{\widehat{\Gamma}_{G,H}}S_{i_1}\ldots S_{i_p}=\frac{1}{|H|}\#\left\{d_1,\ldots,d_p\in H\Big|\begin{matrix}[(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]\\=[(i_1,d_p),(i_2,d_1),\ldots,(i_p,d_{p-1})]\end{matrix}\right\}$$

It follows that we have the following moment formula:

$$\int_{\widehat{\Gamma}_{G,H}}\left(\sum_iS_i\right)^p=\frac{1}{|H|}\#\left\{\begin{matrix}i_1,\ldots,i_p\in G\\d_1,\ldots,d_p\in H\end{matrix}\Big|\begin{matrix}[(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]\\=[(i_1,d_p),(i_2,d_1),\ldots,(i_p,d_{p-1})]\end{matrix}\right\}$$

Now by dividing by $|G|$, we obtain the formula in the statement. □

The formula in Theorem 16.28 can be further interpreted as follows:

THEOREM 16.29. *With $M=|G|,N=|H|$ we have the formula*

$$law(\chi)=\left(1-\frac{1}{N}\right)\delta_0+\frac{1}{N}law(A)$$

where the matrix

$$A\in C(\mathbb{T}^{MN},M_M(\mathbb{C}))$$

is given by $A(q)=$ Gram matrix of the rows of q.

PROOF. According to Theorem 16.28, we have the following formula:

$$\begin{aligned}\int\chi^p&=\frac{1}{MN}\sum_{i_1\ldots i_p}\sum_{d_1\ldots d_p}\delta_{[i_1d_1,\ldots,i_pd_p],[i_1d_p,\ldots,i_pd_{p-1}]}\\&=\frac{1}{MN}\int_{\mathbb{T}^{MN}}\sum_{i_1\ldots i_p}\sum_{d_1\ldots d_p}\frac{q_{i_1d_1}\ldots q_{i_pd_p}}{q_{i_1d_p}\ldots q_{i_pd_{p-1}}}\,dq\\&=\frac{1}{MN}\int_{\mathbb{T}^{MN}}\sum_{i_1\ldots i_p}\left(\sum_{d_1}\frac{q_{i_1d_1}}{q_{i_2d_1}}\right)\left(\sum_{d_2}\frac{q_{i_2d_2}}{q_{i_3d_2}}\right)\ldots\left(\sum_{d_p}\frac{q_{i_pd_p}}{q_{i_1d_p}}\right)dq\end{aligned}$$

Consider now the Gram matrix in the statement, namely:

$$A(q)_{ij}=<R_i,R_j>$$

Here $R_1,\ldots,R_M$ are the rows of the following matrix:

$$q\in\mathbb{T}^{MN}\simeq M_{M\times N}(\mathbb{T})$$

We have then the following computation:

$$\begin{aligned}\int\chi^p &= \frac{1}{MN}\int_{\mathbb{T}^{MN}}<R_{i_1},R_{i_2}><R_{i_2},R_{i_3}>\ldots<R_{i_p},R_{i_1}>\\ &= \frac{1}{MN}\int_{\mathbb{T}^{MN}}A(q)_{i_1i_2}A(q)_{i_2i_3}\ldots A(q)_{i_pi_1}\\ &= \frac{1}{MN}\int_{\mathbb{T}^{MN}}Tr(A(q)^p)dq\\ &= \frac{1}{N}\int_{\mathbb{T}^{MN}}tr(A(q)^p)dq\end{aligned}$$

But this gives the formula in the statement, and we are done. □

In general, the moments of the Gram matrix A are given by a quite complicated formula, and we cannot expect to have a refinement of Theorem 16.29, with A replaced by a plain, non-matricial random variable, say over a compact abelian group.

Asymptotically, however, things are quite simple, and we will explain this now. Let us go back indeed to the general case, where $M,N\in\mathbb{N}$ are both arbitrary. The problem that we would like to solve now is that of finding the good regime, of the following type, where the measure in Theorem 16.28 converges, after some suitable manipulations:

$$M=f(K)\quad,\quad N=g(K)\quad,\quad K\to\infty$$

In order to do so, we have to do some combinatorics. Let $NC(p)$ be the set of noncrossing partitions of $\{1,\ldots,p\}$, and for $\pi\in P(p)$ we denote by $|\pi|\in\{1,\ldots,p\}$ the number of blocks. With these conventions, we have the following result:

Proposition 16.30. *With $M=\alpha K,N=\beta K$, $K\to\infty$ we have:*

$$\frac{c_p}{K^{p-1}}\simeq\sum_{r=1}^p\#\left\{\pi\in NC(p)\Big||\pi|=r\right\}\alpha^{r-1}\beta^{p-r}$$

In particular, with $\alpha=\beta$ we have:

$$c_p\simeq\frac{1}{p+1}\binom{2p}{p}(\alpha K)^{p-1}$$

Proof. We use the combinatorial formula in Theorem 16.28. Our claim is that, with $\pi=\ker(i_1,\ldots,i_p)$, the corresponding contribution to c_p is:

$$C_\pi\simeq\begin{cases}\alpha^{|\pi|-1}\beta^{p-|\pi|}K^{p-1}&\text{if }\pi\in NC(p)\\ O(K^{p-2})&\text{if }\pi\notin NC(p)\end{cases}$$

As a first observation, the number of choices for a multi-index $(i_1,\ldots,i_p)\in X^p$ satisfying $\ker i=\pi$ is given by the following formula:

$$M(M-1)\ldots(M-|\pi|+1)\simeq M^{|\pi|}$$

Thus, we have the following estimate:

$$C_\pi\simeq M^{|\pi|-1}N^{-1}\#\left\{d_1,\ldots,d_p\in Y\Big|[d_\alpha|\alpha\in b]=[d_{\alpha-1}|\alpha\in b],\forall b\in\pi\right\}$$

The contribution of $\sigma=\ker d$ to the above quantity C_π is given by:

$$\Delta(\pi,\sigma)N(N-1)\ldots(N-|\sigma|+1)\simeq\Delta(\pi,\sigma)N^{|\sigma|}$$

Here the quantities on the right are by definition as follows:

$$\Delta(\pi,\sigma)=\begin{cases}1&\text{if }|b\cap c|=|(b-1)\cap c|,\forall b\in\pi,\forall c\in\sigma\\0&\text{otherwise}\end{cases}$$

We use now the standard fact that for $\pi,\sigma\in P(p)$ satisfying $\Delta(\pi,\sigma)=1$ we have:

$$|\pi|+|\sigma|\leq p+1$$

In addition, the equality case is well-known to happen when $\pi,\sigma\in NC(p)$ are inverse to each other, via Kreweras complementation. This shows that for $\pi\notin NC(p)$ we have:

$$C_\pi=O(K^{p-2})$$

Also, this shows that for $\pi\in NC(p)$ we have:

$$\begin{aligned}C_\pi&\simeq M^{|\pi|-1}N^{-1}N^{p-|\pi|-1}\\&=\alpha^{|\pi|-1}\beta^{p-|\pi|}K^{p-1}\end{aligned}$$

Thus, we have obtained the result. □

Still following [**10**], we can now formulate a main result, as follows:

THEOREM 16.31. *With $M=\alpha K,N=\beta K$, $K\to\infty$ we have*

$$\mu=\left(1-\frac{1}{\alpha\beta K^2}\right)\delta_0+\frac{1}{\alpha\beta K^2}D_{\frac{1}{\beta K}}(\pi_{\alpha/\beta})$$

and in particular with $\alpha=\beta$ we have

$$\mu=\left(1-\frac{1}{\alpha^2K^2}\right)\delta_0+\frac{1}{\alpha^2K^2}D_{\frac{1}{\alpha K}}(\pi_1)$$

where D is the dilation operation, given by $D_r(law(X))=law(rX)$.

PROOF. At $\alpha=\beta$, this follows from Proposition 16.30. In general now, we have:

$$\begin{aligned}\frac{c_p}{K^{p-1}} &\simeq \sum_{\pi\in NC(p)}\alpha^{|\pi|-1}\beta^{p-|\pi|}\\ &= \frac{\beta^p}{\alpha}\sum_{\pi\in NC(p)}\left(\frac{\alpha}{\beta}\right)^{|\pi|}\\ &= \frac{\beta^p}{\alpha}\int x^p d\pi_{\alpha/\beta}(x)\end{aligned}$$

When $\alpha\geq\beta$, where $d\pi_{\alpha/\beta}(x)=\varphi_{\alpha/\beta}(x)dx$ is continuous, we obtain:

$$\begin{aligned}c_p &= \frac{1}{\alpha K}\int(\beta Kx)^p\varphi_{\alpha/\beta}(x)dx\\ &= \frac{1}{\alpha\beta K^2}\int x^p\varphi_{\alpha/\beta}\left(\frac{x}{\beta K}\right)dx\end{aligned}$$

But this gives the formula in the statement. When $\alpha\leq\beta$ the computation is similar, with a Dirac mass as 0 dissapearing and reappearing, and gives the same result. $\square$

Summarizing, the simplest deformed Fourier matrices lead to free Poisson laws. For more on all this, we refer to [**10**] and related papers.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter. However, for further reading, you have many possible books, on graphs and related topics. We have referenced some below, and in the hope that you will like some of them.

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