Improper integrals of the second kind

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ABSTRACT: This document briefly discusses improper integrals of the second kind.

I. Introduction

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of the second kind.

II. Definition

1. If $f(x)$ is continuous on $(a,b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c\longrightarrow a^+} \int_c^b f(x)\,dx \tag{1}$$

2. If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b then

$$\int_a^b f(x)\,dx = \lim_{c\longrightarrow b^-} \int_a^c f(x)\,dx \tag{2}$$

3. If $f(x)$ is discontinuous at c , where $a < c < b$, and continuous on $[a, c) \bigcup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{3}$$

In each case , if the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist , the integral diverges.

III. Special improper integrals of the second kind

1.

$$\int_a^b \frac{1}{(x-a)^p}\,dx \text{ converges if } p<1 \text{ and diverges if } p \geq 1 \tag{4}$$

2.

$$\int_a^b \frac{1}{(b-x)^p}\,dx \text{ converges if } p<1 \text{ and diverges if } p \geq 1 \tag{5}$$

These can be called p integrals of the second kind. Note that when $p \leq 0$ the integrals are proper.

IV. Convergence tests

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leq x \leq b$. Similar tests are available if $f(x)$ is unbounded at $x = b$ or $x = c$ where $a < c < b$.

1. Comparison test for integrals with non-negative integrands.

1.1. Convergence. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)\, dx$ converges. Then if $0 \leq f(x) \leq g(x)$ for $a < x \leq b$, $\int_a^b f(x)\, dx$ also converges.

1.2. Divergence. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)\, dx$ diverges. Then if $f(x) \geq g(x)$ for $a < x \leq b$, $\int_a^b f(x)\, dx$ also diverges.

2. Quotient test for integrals with non-negative integrands.

2.1. If $f(x) \geq 0$ and $g(x) \geq 0$ for $a < x \leq b$, and if $\lim_{x\longrightarrow a}\left(\frac{f(x)}{g(x)}\right) = A \neq 0$ or ∞, then $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ either both converge or both diverge.

2.2. If $A = 0$ in 2.1, and $\int_a^b g(x)\, dx$ converges, then $\int_a^b f(x)\, dx$ converges.

2.3. If $A = \infty$ in 2.1, and $\int_a^b g(x)\, dx$ diverges, then $\int_a^b f(x)\, dx$ diverges.

V. Examples

1. Example 1.

$$I = \int_0^1 \frac{2}{\sqrt{1-x^2}}\, dx \tag{6}$$

$$a = 0,\ b = 1,\ f(x) = \frac{2}{\sqrt{1-x^2}},\ f(1) \longrightarrow \infty,\ g(x) = \frac{1}{\sqrt{1-x}},\ \lim_{x\longrightarrow 1} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow 1} \frac{2}{\sqrt{1+x}} = \sqrt{2} \tag{7}$$

$$\int_0^1 g(x)\, dx = 2 \tag{8}$$

$$(7) \wedge (8) \Longrightarrow \int_0^1 \frac{2}{\sqrt{1-x^2}}\, dx \text{ converges} \tag{9}$$

2. Example 2.

$$I = \int_0^{\sqrt{2}} f(x)\, dx \tag{10}$$

where

$$f(x) = \sqrt{-1 + \frac{1}{2\sqrt{3}}\sqrt{\frac{4}{x^2} + \frac{16\cdot 2^{1/3}}{h(x)} + \frac{h(x)}{2^{1/3}\,x^4}} + \frac{1}{2}\sqrt{t(x) - \frac{16\cdot 2^{1/3}}{3\,h(x)} - \frac{h(x)}{3\cdot 2^{1/3}\,x^4} + \frac{\sqrt{3}\ s(x)}{4\sqrt{\frac{4}{x^2} + \frac{16\cdot 2^{1/3}}{h(x)} + \frac{h(x)}{2^{1/3}\,x^4}}}}} \tag{11}$$

$$h(x) = \left(27\,x^4 + 128\,x^6 + \sqrt{729\,x^8 + 6912\,x^{10}}\right)^{1/3} \tag{12}$$

$$t(x) = 8 - \frac{2\left(-1 + 3\,x^2\right)}{x^2} - \frac{2\left(-x^2 + 3\,x^4\right)}{3\,x^4} \tag{13}$$

$$s(x) = -64 + \frac{32\left(-1 + 3\,x^2\right)}{x^2} - \frac{8\left(-1 - 4\,x^2 + 4\,x^4\right)}{x^4} \tag{14}$$

$$a = 0,\ b = \sqrt{2},\ f(0) \longrightarrow \infty,\ g(x) = \frac{1}{x^{3/4}},\ \lim_{x\longrightarrow 0} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow 0} x^{3/4}\, f(x) = 0 \tag{15}$$

$$\int_0^{\sqrt{2}} g(x)\, dx = 4 \cdot 2^{1/8} \tag{16}$$

$$(15) \wedge (16) \Longrightarrow \int_0^{\sqrt{2}} f(x)\, dx \text{ converges} \tag{17}$$

Recall that

$$\pi = \int_0^1 \frac{2}{\sqrt{1-x^2}}\, dx \tag{18}$$

$$\pi = \int_0^{\sqrt{2}} \sqrt{-1 + \frac{1}{2\sqrt{3}} \sqrt{\frac{4}{x^2} + \frac{16 \cdot 2^{1/3}}{h(x)} + \frac{h(x)}{2^{1/3} x^4}} + \frac{1}{2} \sqrt{t(x) - \frac{16 \cdot 2^{1/3}}{3\, h(x)} - \frac{h(x)}{3 \cdot 2^{1/3} x^4} + \frac{\sqrt{3}\, s(x)}{4 \sqrt{\frac{4}{x^2} + \frac{16 \cdot 2^{1/3}}{h(x)} + \frac{h(x)}{2^{1/3} x^4}}}}}\, dx \tag{19}$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \tag{20}$$

VI. References

1. Apelblat, A.: Tables of Integrals and Series. Verlag Harri Deutsch, 1996.
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3. Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.
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