Chirality (Electroweak interaction) using Geometric (real Clifford) Algebra $Cl_{3,0}$

Jesús Sánchez

Independent Researcher, Bilbao, Spain
Email: jesus.sanchez.bilbao@gmail.com
https://www.researchgate.net/profile/Jesus-Sanchez-21
https://vixra.org/author/jesus_sanchez
ORCID 0000-0002-5631-8195

Abstract

In this paper, we will obtain the left and the right-handed representation (chirality) of the wavefunction using Geometric (real Clifford) Algebra $Cl_{3,0}$. Having the wavefunction ψ:

$$\psi = \psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} + \psi^{123} e_{123}$$

In Chiral basis, the separation between left and right-handed elements is explicit:

$$\begin{aligned} \psi_L &= \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{123} e_{123} \\ \psi_R &= \psi^0 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} \end{aligned}$$

In Pauli/Dirac basis, this explicit separation is not possible, and the result is as follows:

$$\begin{aligned} \psi_L = \frac{1}{2}\big((\psi^3 + \psi^0) &+ (+\psi^1 + \psi^{31}) e_1 + (\psi^2 - \psi^{23}) e_2 + (\psi^3 + \psi^0) e_3 \\ &+ (-\psi^2 + \psi^{23}) e_{23} + (\psi^1 + \psi^{31}) e_{31} + (\psi^{123} + \psi^{12}) e_{12} \\ &+ (\psi^{123} + \psi^{12}) e_{123}\big) \end{aligned}$$

$$\begin{aligned} \psi_R = \frac{1}{2}\big((-\psi^3 + \psi^0) &+ (+\psi^1 - \psi^{31}) e_1 + (\psi^2 + \psi^{23}) e_2 + (\psi^3 - \psi^0) e_3 \\ &+ (\psi^2 + \psi^{23}) e_{23} + (-\psi^1 + \psi^{31}) e_{31} + (-\psi^{123} + \psi^{12}) e_{12} \\ &+ (\psi^{123} - \psi^{12}) e_{123}\big) \end{aligned}$$

Also, a summary of how all the interactions can be calculated and represented using Geometric (real Clifford) Algebra is shown.

Keywords

Geometric Algebra, Real Clifford Algebras, Electroweak interaction, Chirality

1. Introduction

In this paper, we will deal with the chirality in Geometric (real Clifford) Algebra representation. We will obtain the left and the right-handed representation of the wavefunction using Geometric (real Clifford) Algebra $Cl_{3,0}$.

2. Geometric (real Clifford) Algebra $Cl_{3,0}$. Basis vectors

There is a discipline in mathematics that is called real Geometric Algebra also known as real Clifford Algebras [1][3].

In the specific Geometric Algebra $Cl_{3,0}$, it is considered a three-dimensional space, so we need three independent vectors to define a basis. The classical definition of a basis is as follows:

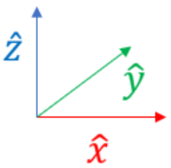
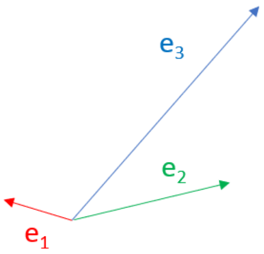


Fig. 1 Basis vectors in three-dimensional space.

In this paper we will use the nomenclature e_i (without any hat or vector sign) to name these three vectors instead the classical $\hat{x}\ \hat{y}\ \hat{z}$. Above, I have considered an orthonormal basis as an example.

But in the general case, this is not even necessary. The only necessary constraint to form a basis is that the three vectors are linearly independent (this is, they do not lie on the same plane). An example below:



In Geometric algebra, it is defined an operation called the geometric product. The geometric product is not represented by any symbol. It is the implicit operation when two vectors are represented one after the other.

Its definition is:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

Being:

$$e_i \cdot e_j = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

The classical definition of the scalar product. The product of the two norms (the length) of the vectors by the cosine of the angle formed by them (we have called it α_{ij} in this case).

The result of the scalar product is a number, a scalar. An important property of the scalar product is that it is commutative:

$$e_i \cdot e_j = e_j \cdot e_i = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

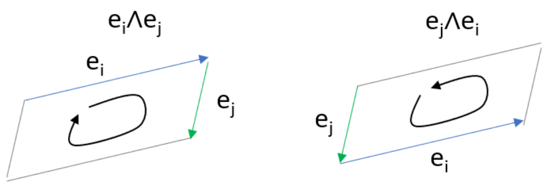
As the cosine of the angle is included in the product, you can check that when e_i and e_j are perpendicular (right angle), the scalar product is zero. And if the vectors are colinear (the angle is zero), the scalar product is just the product of the modules of the vectors.

The other element of the geometric product above is:

$$e_i \wedge e_j$$

What it is called the outer, exterior or wedge product of the two vectors.

The result of this operation is not a number. It is another entity that is not a number and not a vector. It is called a bivector. The bivector is an entity that represents an oriented surface area (in a same way that a vector “represents” an oriented line segment).



It can be checked above that the module (area of the surface) when reversing the order of the exterior product is the same. But the orientation (its sign) changes. So, the exterior product is anticommutative:

$$e_i \wedge e_j = -e_j \wedge e_i$$

The module (area of the surface) of the exterior product is:

$$\|e_i \wedge e_j\| = \|e_j \wedge e_i\| = \|e_i\| \|e_j\| \sin(\alpha_{ij})$$

You can see that when the vectors are colinear (the angle is zero), the exterior product result is zero. And when the vectors are perpendicular, the module of the exterior product is the product of the modules of the vectors.

Coming back to the definition of the geometric product:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

We can see that when we perform the square of a vector, this is, the product of a vector by itself (the vector is colinear with itself, its angle is zero) the result is:

$$(e_i)^2 = e_i e_i = e_i \cdot e_i + e_i \wedge e_i = \|e_i\| \|e_i\| \cdot 1 + 0 = \|e_i\| \|e_i\| = \|e_i\|^2$$

So, the square of a vector is its norm squared. The important thing here, is that the result is just a number. It is not a vector, it is not a bivector, it is just a number. We have converted a vector to a number just multiplying it by itself.

If now, we multiply (geometric product) two perpendicular vectors (the angle between them is a right angle):

$$e_ie_j = e_i \cdot e_j + e_i \wedge e_j = 0 + e_i \wedge e_j = e_i \wedge e_j$$

So, you can see that the result is a pure bivector. It does not include vectors or scalars, just a bivector.

If we reverse the product, we have:

$$e_je_i = e_j \cdot e_i + e_j \wedge e_i = 0 + e_j \wedge e_i = e_j \wedge e_i = -e_i \wedge e_j = -e_ie_j$$

So, when two vectors are perpendicular, not only the exterior product, but also the geometric product is anticommutative.

From the equations above we can obtain the following equations.

$$e_i \cdot e_j = \frac{1}{2}\left(e_ie_j + e_je_i\right)$$

$$e_i \wedge e_j = \frac{1}{2}\left(e_ie_j - e_je_i\right)$$

The demonstration comes directly from the definition of the geometric product. If we sum a geometric product by its reverse, we put the definition of geometric product, we take into account that the scalar product is commutative and the exterior product anticommutative:

$$\begin{aligned} e_ie_j + e_je_i &= e_i \cdot e_j + e_i \wedge e_j + e_j \cdot e_i + e_j \wedge e_i = e_i \cdot e_j + e_i \wedge e_j + e_i \cdot e_j - e_i \wedge e_j \\ &= 2\left(e_i \cdot e_j\right) \end{aligned}$$

$$e_i \cdot e_j = \frac{1}{2}\left(e_ie_j + e_je_i\right)$$

If instead of summing, we subtract:

$$\begin{aligned} e_ie_j - e_je_i &= e_i \cdot e_j + e_i \wedge e_j - e_j \cdot e_i - e_j \wedge e_i = e_i \cdot e_j + e_i \wedge e_j - e_i \cdot e_j + e_i \wedge e_j \\ &= 2\left(e_i \wedge e_j\right) \end{aligned}$$

$$e_i \wedge e_j = \frac{1}{2}\left(e_ie_j - e_je_i\right)$$

We will see in next chapters that when we apply the exterior product instead of the geometric product of two vectors, this means that we want only the result that appears in the plane they form (in the bivector they form). And we “remove” from the result the scalars (that will appear with the scalar product of the vectors) and also, we remove the possible result in vectors (in more complicated products that we will see in next chapters).

Another point to comment is that the exterior product of bivectors (instead of vectors) is defined in the opposite way (summing instead of subtracting). I am not going to enter into details, you can check it in [3].

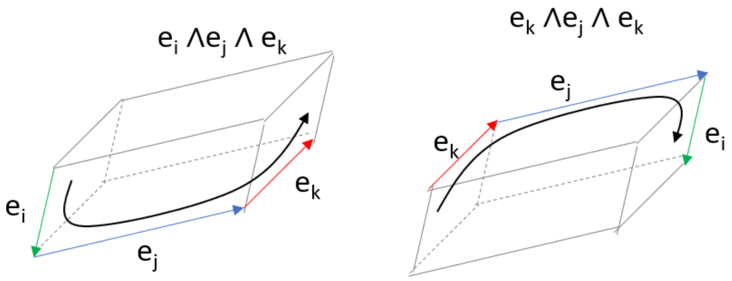
$$\left(e_ie_j\right) \wedge \left(e_re_s\right) = \frac{1}{2}\left(e_ie_je_re_s + e_re_se_je_i\right)$$

The same way, the scalar product of bivectors is also defined as the opposite of vectors. See [3].

$$(e_ie_j)\cdot(e_re_s)=\frac{1}{2}(e_ie_je_re_s-e_re_se_je_i)$$

Also, to remark that the geometric product is always associative and distributive as you can see in [3]. But in general, is not commutative or anticommutative as commented (it depends on the specific product) We will see more examples in the following chapters.

To conclude this chapter about geometric algebra, we will define the trivector. When two vectors are exterior multiplied, they form a bivector as seen above. The same way, when three vectors are exterior multiplied, they create an oriented volume, called the trivector:



You can see again, that when we reverse the vectors, we get the same volume (module of the trivector) but with different orientation (sign):

$$e_i\wedge e_j\wedge e_k=-e_k\wedge e_j\wedge e_i$$

We will check more thing regarding reversion and change of signs in the next chapter.

3. Geometric Algebra $Cl_{3,0}$. Different types of bases

3.1 Orthonormal basis

In an orthonormal basis, the norm of the basis vectors is equal to one. And the basis vectors are perpendicular to each other.

So, from the properties commented in chapter 2, we can get obtain the following equations (for orthonormal basis):

$$(e_i)^2=e_ie_i=e_i\cdot e_i=1$$
$$e_ie_j=e_i\wedge e_j=-e_j\wedge e_i=-e_je_i\quad (when\ i\neq j)$$
$$e_i\cdot e_j=e_j\cdot e_i=0\quad (when\ i\neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2=e_1e_1=1$$
$$(e_2)^2=e_2e_2=1$$
$$(e_3)^2=e_3e_3=1$$
$$e_1e_2=-e_2e_1$$
$$e_2e_3=-e_3e_2$$
$$e_3e_1=-e_1e_1$$

We can define the inverse of a vector and name it e^i , as the vector that fulfills (Einstein summation is not implied here):

$$(e_i)^{-1}e_i\equiv e^ie_i=1=e_i(e_i)^{-1}\equiv e_ie^i$$

To calculate e^i we can post multiply by e_i:

$$(e_i)^{-1}e_ie_i \equiv e^ie_ie_i = 1 \cdot e_i$$
$$e^i(e_i)^2 = e_i$$
$$e^i \cdot 1 = e_i$$
$$e^i = e_i = (e_i)^{-1}$$

So, in orthonormal metric the inverse of a basis vector is itself. It is important to remark here that in Geometric Algebra there are no covectors (or 1-forms). There are only scalars, bivectors, trivectors… We will see that the concept of covector in Geometric Algebra is just a vector that is the inverse of another vector.

In traditional algebra you cannot define the inverse of a vector, so it is used a different type of element. In Geometric Algebra, the covectors are also vectors. And in fact, the product of inverse vectors by vectors outputs scalars as it would be expected by the product of a covector by a vector.

3.2. Geometric Algebra $Cl_{3,0}$. Orthogonal but not orthonormal basis

In an orthogonal basis, the vectors are perpendicular to each other. But in general, the norm of the vectors is not one. In Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from zero.

The 3 in the name $Cl_{3,0}$, makes reference to that there are 3 basis vectors with positive norm. The 0 in the name $Cl_{3,0}$, makes reference to that there are no basis vectors with negative norm. And the absence of a third number makes reference to that there are no basis vectors with zero norm.

From the properties commented in chapter 2, we can obtain the following equations (for orthogonal, not orthonormal basis):

$$(e_i)^2 = e_ie_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$
$$e_ie_j = e_i \wedge e_j = -e_j \wedge e_i = -e_je_i \qquad (when\ i \neq j)$$
$$e_i \cdot e_j = e_j \cdot e_i = 0 \qquad (when\ i \neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1e_1 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = e_2e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3e_3 = \|e_3\|^2 = g_{33}$$
$$e_1e_2 = -e_2e_1$$
$$e_2e_3 = -e_3e_2$$
$$e_3e_1 = -e_1e_1$$

Where the g_{ii} makes reference to the metric tensor components. See paper [2]. Take into account that when you multiply two colinear vectors (and a vector is colinear with itself), its geometric product is equal to the scalar product. And this is exactly the definition of g_{ii} (the scalar product of e_i with itself).

The definition of the inverse of a vector, and naming it e^i , is the vector that fulfills (not Einstein summation is implied here):

$$(e_i)^{-1}e_i \equiv e^ie_i = 1 = e_i(e_i)^{-1} \equiv e_ie^i$$

To calculate e^i we can post multiply by e_i:

$$(e_i)^{-1}e_ie_i \equiv e^ie_ie_i = 1 \cdot e_i$$
$$e^i(e_i)^2 = e_i$$
$$e^i\|e_i\|^2 = e_i$$
$$e^ig_{ii} = e_i$$
$$e^i = \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2} = (e_i)^{-1}$$

So, in orthogonal metric the inverse of a basis vector is itself divided by its norm squared (by g_{ii}). Everything commented regarding covectors in 3.1 applies also here.

One important consequence of this, is that if the basis vectors are orthogonal (as in this chapter), all the basis vectors and all the inverse of the basis vectors are also orthogonal among them (when i≠j). this is:

$$e^i \cdot e_j = \frac{e_i}{g_{ii}} \cdot e_j = \frac{1}{g_{ii}}(e_i \cdot e_j) = \frac{1}{2g_{ii}}(e_i e_j + e_j e_i) = 0$$
$$e^i \cdot e^j = \frac{e_i}{g_{ii}} \cdot \frac{e_j}{g_{jj}} = \frac{1}{2g_{ii}g_{jj}}(e_i \cdot e_j) = \frac{1}{2g_{ii}g_{jj}}(e_i e_j + e_j e_i) = 0$$

In the last equation (but when i=j) we get:

$$e^i \cdot e^i = (e^i)^2 = \frac{e_i}{g_{ii}} \cdot \frac{e_i}{g_{ii}} = \frac{1}{g_{ii}g_{ii}}(e_i \cdot e_i) = \frac{1}{g_{ii}g_{ii}}(e_i e_i) = \frac{1}{(g_{ii})^2} \cdot 1 = \frac{1}{(g_{ii})^2}$$

These last properties apply also to chapter 3.1 (orthonormal basis) but in that case the elements g_{ii} or g_{jj} are always 1.

3.3. Geometric Algebra $Cl_{3,0}$. Non-Orthogonal (and therefore not orthonormal) basis

In a non-orthogonal basis, the vectors are not perpendicular from each other. And in general, the norm of the vectors is not one. As commented in 3.2, in Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from zero.

From the properties commented in chapter 2 and also in [2], we can get obtain the following equations (for orthogonal, not orthonormal basis):

$$(e_i)^2 = e_i e_i = \|e_i\|^2 = g_{ii}$$
$$e_i e_j = 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i$$
$$e_i \cdot e_j = e_j \cdot e_i = g_{ij} = g_{ji}$$
$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = g_{ij} + e_i \wedge e_j$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$
$$e_1 e_2 = 2g_{12} - e_2 e_1 = 2g_{21} - e_2 e_1$$
$$e_2 e_3 = 2g_{23} - e_3 e_2 = 2g_{32} - e_3 e_2$$
$$e_3 e_1 = 2g_{31} - e_1 e_3 = 2g_{13} - e_1 e_3$$

Where the g_{ij} makes reference again to the metric tensor components (the scalar products of the basis vectors). See paper [2] for more information. You can obtain the above equations from the definition of scalar product in geometric algebra as commented in chapter 2.

$$e_i \cdot e_j = g_{ij} = \frac{1}{2}(e_i e_j + e_j e_i)$$

Multiplying by 2:

$$2g_{ij} = e_i e_j + e_j e_i$$

Rearranging terms (and knowing that the metric tensor is symmetric):

$$e_i e_j = 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i$$

Now, we will define again the inverse of the basis vectors and name them e^i. To obtain the inverse of the basis vectors is this case, you have to get the inverse of the metric tensor, so you are able to define a vector e^i that fulfills for every i and every j the following (Einstein summation does not apply):

$$(e_i)^{-1}e_i \equiv e^i e_i = 1 = e_i(e_i)^{-1} \equiv e_i e^i$$

$$e^i \cdot e_j = e_i \cdot e^j = \frac{1}{2}(e_i e^j + e^j e_i) = 0 \quad for\ i \neq j$$

In general, this is written as:

$$e^i \cdot e_j = \delta^i_j$$

Where δ^i_j is the Kronecker Delta, that is equal to 1 when i=j and 0 when i≠j.

If we multiply two inverse vectors between them, in non-orthogonal metric, we do not obtain zero as a general case. See below:

$$e^i \cdot e^j = \frac{1}{2}(e^i e^j + e^j e^i) = g^{ij} = g^{ji}$$

So:

$$e^i e^j = 2g^{ij} - e^j e^i$$

And:

$$e^i e^i = (e^i)^2 = e^i \cdot e^i = g^{ii}$$

In this paper, we will work mainly with orthogonal (or orthonormal basis), so do not worry about these above points. For more info regarding how to invert the metric you have a lot of literature [58][59][60][61][62][64].

What we will do in general, is to make all the calculations with orthogonal metrics and then try to generalize to the case of non-orthogonal metric applying the above relations.

3.4. Geometric (real Clifford) Algebra $Cl_{3,0}$. Expanding the basis

One of the properties of the Geometric Algebra is that the number of elements that conform the algebra of a certain realm are more than the number of dimensions of that realm. In three dimensions we have three basis vectors as commented, but we have 8 different elements that conform that algebra, that are:

- The scalars
- The three vectors
- The three bivectors
- One trivector

We will call these elements with these names:

$$\begin{gathered} e_0 \to 1\ (scalars) \\ e_1 \\ e_2 \\ e_3 \\ e_4 = e_{23} \equiv e_2 e_3 \\ e_5 = e_{31} \equiv e_3 e_1 \\ e_6 = e_{12} \equiv e_1 e_2 \\ e_7 = e_{123} \equiv e_1 e_2 e_3 \end{gathered}$$

In this paper we will consider e_0 directly as the scalar 1. In other contexts (influenced by gravitation or non-Euclidean metrics), the value could be a scalar but with norm $\|e_0\|^2$ different than 1. We will not consider this here (check [75] for more information).

The elements e_4, e_5, e_6 are bivectors whose square is negative, as we will see now. And e_7 is the trivector whose square is also negative, as we will see.

In general, we will work with orthogonal (not necessarily orthonormal) basis. About the non-orthogonal case, we will talk explicitly in certain points of the paper. If nothing is said, along the paper we will work with orthogonal metric that fulfills the following, already commented, relations:

$$(e_i)^2 = e_ie_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$
$$e_{ij} = e_ie_j = e_i \wedge e_j = -e_j \wedge e_i = -e_je_i = -e_{ji}$$
$$e_i \cdot e_j = e_j \cdot e_i = 0 \qquad (when\ i \neq j)$$

This is, in 3 dimensions:

$$e_0 \to 1$$
$$(e_1)^2 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = \|e_3\|^2 = g_{33}$$
$$e_{12} = -e_{21}$$
$$e_{23} = -e_{32}$$
$$e_{31} = -e_{13}$$
$$e_{123} = -e_{321}$$

The last three equations are key in orthogonal metric and are the ones that will make working with bivectors or the trivector much easier. Because they permit us to swap the order of the vectors in any geometric product, just adding a minus sign for each swap. These means that the result will be the same if we make an even number of swaps. And will be the negative of the original if we make an odd number of swaps.

As commented, all these swapping's with changing of sign can only be applied in orthogonal bases. In non-orthogonal bases you should apply the equations in the beginning of chapter. 3.3.

Knowing this rule, I would just show the squares of the bivectors and the trivector to check that they are in fact negative:

$$(e_4)^2 = (e_2e_3)^2 = e_2e_3e_2e_3 = -e_2e_3e_3e_2 = -e_2g_{33}e_2 = -g_{33}e_2e_2 = -g_{33}g_{22}$$
$$(e_5)^2 = (e_3e_1)^2 = e_3e_1e_3e_1 = -e_3e_1e_1e_3 = -e_3g_{11}e_3 = -g_{11}e_3e_3 = -g_{11}g_{33}$$
$$(e_6)^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_2e_2e_1 = -e_1g_{22}e_1 = -g_{22}e_1e_1 = -g_{22}g_{11}$$
$$(e_7)^2 = (e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = +e_1e_2e_3e_3e_1e_2 = g_{33}e_1e_2e_1e_2 = -g_{33}e_1e_1e_2e_2 = -g_{33}g_{11}g_{22}$$

Remind that the g_{ij} are just numbers, so you can move them as you want along the product. I keep the order obtained in the operations to facilitate the understanding, but you can swap them as you want not changing the sign or the result.

Just to close the chapter, I will comment that an entity that is composed by the sum of scalars, vectors, bivectors etc… is called a multivector. As an example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3e_1$$

This entity A is called a multivector. We will see that in Geometric Algebra any object can be defined by a multivector expression.

The most important comment of this section is the following. In Geometric Algebra, once you have defined the number of dimensions (in this case 3) and the consequent degrees of freedom (or different basis vectors and their combinations, in this case 8, from e_0 to e_7), it does not matter how many operations (sums, geometric products, even exponentials etc…) you do, the number of basis vectors and their combinations are always the same (8 in this case). You can multiply the times you want any multivector by another one, you will only finish with 8 coefficients that multiply 8 basis vectors from e_0 to e_7 (considering also basis vectors their product combinations). Nothing else. This is key in Geometric Algebra and its power.

If you are familiarized with matrices, tensors or tensors products, you know that in those cases the number of elements could grow to infinite (the number of dimensions also). In Geometric Algebra, there is a limit. And this KEY as we will see.

3.5. Geometric Algebra $Cl_{3,0}$. Comments about e_0 and e_7

Regarding e_7 the important property as commented is this:

$$(e_7)^2 = (e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = -g_{33}g_{11}g_{22}$$

This means, its square is negative, and it is a "neutral" vector. Meaning "neutral" that it does not have any "preferred" direction or orientation. The bivectors e_4, e_5, e_6 have also negative square but with "preferred" directions.

$$(e_4)^2 = (e_2e_3)^2 = e_2e_3e_2e_3 = -g_{33}g_{22}$$
$$(e_5)^2 = (e_3e_1)^2 = e_3e_1e_3e_1 = -g_{11}g_{33}$$
$$(e_6)^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -g_{22}g_{11}$$

But e_7 has a negative square and does not point anywhere specific. It applies to the volume in general (not a surface or a line). If you have read the papers [4][5][6] probably you have already seen the possibility that the time vector can be associated with e_7 (the trivector). The reason is that the square of e_7 is negative and that taking this consideration is completely coherent with Dirac Equation, Maxwell equations and Gell-Mann matrices [5][6][26][63].

In previous papers [4][5][6][26][63], we saw that depending on the context, the scalars e_0 (as considered in APS[43][74]) or the trivector e_7 could represent time depending on the context. We will see later, but first we need to understand the spinor in Geometric Algebra to understand the different possible contexts.

What we will keep from previous papers [4][5][6][26][63]is that as the square of e_7 is negative and does not have any preferred direction. So, when the imaginary unit *i* is used in traditional algebra, we will substitute it in Geometric Algebra by the trivector e_7. The reason is that in Geometric Algebra there are already elements as e_7 (appearing in a natural way) whose square is negative.

And the imaginary unit *i* is used in traditional algebra as an "unknown or generic" element whose square is negative. In Geometric Algebra, what you have to do is, depending on the context, to use the corresponding already existing element in the Algebra (of all the ones whose square is negative) instead of using *i*. As commented, we will used e_7 for the reasons commented above.

4. Special operations in Geometric Algebra

4.1. The reverse of a multivector and the reverse product

If we have multivector, the reverse of it can be defined as a multivector with the same coefficients but where all the products of basis vectors are reversed. An example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3e_1 + 2e_2e_3 - 5e_1e_2e_3$$

Its reverse will be:

$$A^{\dagger} = 3 + 2e_1 - 3e_1 + 7e_1e_3 + 2e_2e_3 - 5e_3e_2e_1$$

This, in orthogonal metric (not in general) can be converted using chapter 3.2 equations into:

$$A^{\dagger} = 3 + 2e_1 - 3e_1 - 7e_3e_1 - 2e_2e_3 + 5e_1e_2e_3 = A^{*}$$

Being A^* the conjugate multivector. This means, in orthogonal metric the reverse of a multivector is the same as a conjugate of the multivector. The conjugate means changing the sign of the elements whose square is negative (this means: bivectors and trivector) and keeping the same sign for scalars and vectors (whose square is positive)

In a non-orthogonal metric, you should use equations in chapter 3.3 instead of those in chapter 3.2, so in a general case, reverse and conjugate will not be the same.

Anyhow, as commented, in this paper we will focus on orthogonal basis, so here reverse and conjugate will be the same in most cases (but this is not true for a general case).

Calculating the reverse for the different basis vectors, we have (orthogonal basis):

$$e_0^{\dagger} = e_0$$
$$e_1^{\dagger} = e_1$$
$$e_2^{\dagger} = e_2$$
$$e_3^{\dagger} = e_3$$
$$e_4^{\dagger} = (e_2e_3)^{\dagger} = e_3e_2 = -e_2e_3$$
$$e_5^{\dagger} = (e_3e_1)^{\dagger} = e_1e_3 = -e_3e_1$$
$$e_6^{\dagger} = (e_1e_2)^{\dagger} = e_2e_1 = -e_1e_2$$
$$e_7^{\dagger} = (e_1e_2e_3)^{\dagger} = e_3e_2e_1 = -e_1e_2e_3$$

One important property is that a product of basis vectors multiplied by its reverse is always positive definite (also in non-orthogonal metrics):

$$e_0e_0^{\dagger} = e_0e_0 = \|e_0\|^2 = g_{00}$$
$$e_1e_1^{\dagger} = e_1e_1 = \|e_1\|^2 = g_{11}$$
$$e_2e_2^{\dagger} = e_2e_2 = \|e_2\|^2 = g_{22}$$
$$e_3e_3^{\dagger} = e_3e_3 = \|e_3\|^2 = g_{33}$$
$$e_4e_4^{\dagger} = e_2e_3(e_2e_3)^{\dagger} = e_2e_3e_3e_2 = e_2g_{33}e_2 = g_{33}e_2e_2 = g_{33}g_{22} \equiv g_{44}$$
$$e_5e_5^{\dagger} = e_3e_1(e_3e_1)^{\dagger} = e_3e_1e_1e_3 = e_3g_{11}e_3 = g_{11}e_3e_3 = g_{11}g_{33} \equiv g_{55}$$
$$e_6e_6^{\dagger} = e_1e_2(e_1e_2)^{\dagger} = e_1e_2e_2e_1 = e_1g_{22}e_1 = g_{22}e_1e_1 = g_{22}g_{11} \equiv g_{66}$$
$$e_7e_7^{\dagger} = e_1e_2e_3(e_1e_2e_3)^{\dagger} = e_1e_2e_3e_3e_2e_1 = g_{33}e_1e_2e_2e_1 = g_{33}g_{22}e_1e_1 = g_{33}g_{22}g_{11} \equiv g_{77}$$

Where I have defined the g_{ii} as the result of these products also for basis vectors with i>3. And also, as commented it is defined a g_{00} as the square for e_0 to have one degree of freedom more (even that very probably defining it as 1, should be ok, meaning just a that pre-normalization has been de-facto done).

As you can guess, the reverse product is just defined as multivector by the reverse of other (or the same) multivector following the rules commented above.

An important thing to comment, is that the reverse should not be mixed up with the inverse.

The inverse of a product of basis vectors is defined as the inverse of each basis vector in reverse order. This is, for example:

$$(e_7)^{-1} = (e_1e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1}(e_1)^{-1} = e^3e^2e^1 = e^7$$

Where in the last steps above, I have used the definition of the superscripts as defined in chapters 3.1, 3.2 and 3.3, as the inverse of the basis vectors. We can check that this hold:

$$e_7e^7 = e_1e_2e_3e^3e^2e^1 = e_1e_2 \cdot 1 \cdot e^2e^1 = e_1 \cdot 1 \cdot e^1 = 1$$

So, in fact, it corresponds to the inverse of e_7. The same applies, to the rest of vectors:

$$(e_1)^{-1} = e^1$$

$$(e_2)^{-1} = e^2$$

$$(e_3)^{-1} = e^3$$

$$(e_4)^{-1} = (e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1} = e^3e^2 = e^4$$

$$(e_5)^{-1} = (e_3e_1)^{-1} = (e_1)^{-1}(e_3)^{-1} = e^1e^3 = e^5$$

$$(e_6)^{-1} = (e_1e_2)^{-1} = (e_2)^{-1}(e_1)^{-1} = e^2e^1 = e^6$$

$$(e_7)^{-1} = (e_1e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1}(e_1)^{-1} = e^3e^2e^1 = e^7$$

So, you can see that the inverse, also reverses the order, but besides that, it inverses the basis vectors (converts the subscripts in superscripts and vice-versa).

4.2. Clifford conjugation

Another special operation is the Clifford conjugation that it has not be confused with the reverse or with the standard conjugation (see 4.1).

The Clifford conjugation [73] is represented by a bar above the vectors. It changes the sign of the vectors and reverses the product. This is:

$$\begin{aligned}
\bar{e}_0 &= e_0 = 1 \\
\bar{e}_1 &= -e_1 \\
\bar{e}_2 &= -e_2 \\
\bar{e}_3 &= -e_3 \\
\bar{e}_4 &= \overline{(e_2e_3)} = (-e_3)(-e_2) = e_3e_2 = -e_2e_3 \\
\bar{e}_5 &= \overline{(e_3e_1)} = (-e_1)(-e_3) = e_1e_3 = -e_3e_1 \\
\bar{e}_6 &= \overline{(e_1e_2)} = (-e_2)(-e_1) = e_2e_1 = -e_1e_2 \\
\bar{e}_6 &= \overline{(e_1e_2e_3)} = (-e_3)(-e_2)(-e_1) = -e_3e_2e_1 = e_1e_2e_3
\end{aligned}$$

It changes the signs of the vectors and bivectors and keep the sign of the scalars and the trivector.

4.3. Grade automorphism

It is the combination of the 4.1 and 4.2 [73]:

$$\begin{aligned}
\bar{e}_0{}^\dagger &= e_0 = 1 \\
\bar{e}_1{}^\dagger &= -e_1 \\
\bar{e}_2{}^\dagger &= -e_2 \\
\bar{e}_3{}^\dagger &= -e_3 \\
\bar{e}_4{}^\dagger &= \overline{(e_2e_3)}^\dagger = e_2e_3 \\
\bar{e}_5{}^\dagger &= \overline{(e_3e_1)}^\dagger = e_3e_1 \\
\bar{e}_6{}^\dagger &= \overline{(e_1e_2)}^\dagger = e_1e_2 \\
\bar{e}_6{}^\dagger &= \overline{(e_1e_2e_3)}^\dagger = -e_1e_2e_3
\end{aligned}$$

It changes the sign of the vectors and the trivector (odd grade elements). It keeps the sign of the scalar and the bivector (even grade elements).

5. Spinor in Geometric Algebra $Cl_{3,0}$

A spinor in matrix notation has this form:

$$\psi = \begin{pmatrix} \psi_{1r} + \psi_{1i} i \\ \psi_{2r} + \psi_{2i} i \\ \psi_{3r} + \psi_{3i} i \\ \psi_{4r} + \psi_{4i} i \end{pmatrix}$$

As you can see, it has eight parameters:

$$\psi_{1r}\ \ \psi_{1i}\ \ \psi_{2r}\ \ \psi_{2i}\ \ \psi_{3r}\ \ \psi_{3i}\ \ \psi_{4r}\ and\ \psi_{4i}$$

In Geometric Algebra, the spinor has this form:

$$\psi = \psi^{\mu} e_{\mu} = \psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7$$

Where the e_i are the elements (scalars, vectors, bivectors and trivector) as defined in chapter 3.5.

The ψ^i are the coefficients of the spinor or wavefunction. You can see that they are also eight as in the matrix notation. You can find a relation between both in [5] [31]and [63]. There you can find that that relation is coherent with Dirac Equation and Strong Force Interaction (Gell-Mann matrices).

For this paper we will just stick to that these 8 coefficients are sufficient to define a spinor or wavefunction. And calculating them is what we need to define the state of a particle or a related filed.

6. Probability density and probability current

As we saw in [63] we can calculate probability density and probability current multiplying the reverse of the wavefunction by itself, this way:

$$\psi^{\dagger}\psi = \left(\psi^0 e_0^{\dagger} + \psi^1 e_1^{\dagger} + \psi^2 e_2^{\dagger} + \psi^3 e_3^{\dagger} + \psi^4 e_4^{\dagger} + \psi^5 e_5^{\dagger} + \psi^6 e_6^{\dagger} + \psi^7 e_7^{\dagger}\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7)$$

Where all the vectors, bivectors and the trivector and their reverses, are as defined in chapter 4 and previous ones.

Only in the case of orthogonal metric (not in the general case), this can be simplified as (the reverse is the same as the conjugate):

$$\psi^{\dagger}\psi = \psi^{*}\psi = (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7)$$

As you can see in Annex A2, the result of this multiplication is for the orthogonal case is:

$$\psi^{\dagger}\psi = \rho + \vec{j}$$

Being:

$$\rho = (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33}$$

$$\vec{j} = 2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33})e_1 \\ + 2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11})e_2 \\ + 2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11}g_{22})e_3$$

Being ρ the probability and $\vec{j}$ the fermionic current.

But we can say that even in the general case where the basis is not orthogonal or even if the product above is defined another way, the result will have for sure have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

In Annexes A1, A2, A3 and A4, you can find that in whatever metric you are or however this product is defined (in A4 it is shown an example using the inverse product instead of the reverse product), the result will always have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where μ and ν go from 0 to 7 in the most general case. This means, independently of the metric, independently if the product is correctly defined or are some elements pending (see Annexes A1, A2, A3 and A4 for details), what it is true is that the result, will have the form above.

Even if we calculate wrongly the coefficients of j^μ, we can continue with our study as these coefficients will represent a general case. In case they change the value, we will change the operations done, but the study following will be perfectly correct as the meaning of the coefficients j^μ is general. This is the power of geometric algebra. We know the form of the results even if we have calculated them wrong. We know that the result will have 8 components j^μ (very important, scalar coefficients or functions that output a scalar) multiplying 8 basis vectors (considering their product combinations also, this means, considering them from e_o to e_7).

Last comment to make are the measuring units of this $j^\mu e_\nu$. For the j^0 component the units are density of probability in 3D space, this means probability/cubic length. Probability does not have units, so it is L^{-3}.

The components j^1 to j^3 are called the probability current and its units are density of probability multiplied by velocity. As probability does not have units, the density has L^{-3} and the speed has LT^{-1}, the total units are $L^{-2}T^{-1}$. To make these units coherent with j^0, we have to multiply j^0 by c (the speed of light) or the opposite, to divide the components of j^1 to j^3 by it.

As commented, for orthonormal or orthogonal bases, j^μ only has components from 0 to 3. For the general case, it would have components from 0 to 7 and the measuring units should be harmonized with the units that have the components from 0 to 3. But we will not care about that now, we will just consider that we can find a coherent following expression with coherent units:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Just to finalize, I will comment that to be consequent with certain papers in the literature [57], sometimes I will use the following nomenclature, but you can check that the concept is the same, just changing the name of j to V, and the dummy index form μ to ρ:

$$\psi^\dagger\psi = j^\mu e_\mu = V^\rho e_\rho$$

7. Chirality (Electroweak interaction) in Geometric (real Clifford) Algebra $Cl_{3.0}$

In this chapter we will explain how to deal with the chirality in Geometric (real Clifford) Algebra $Cl_{3,0}$. In the Feynmann diagrams where electroweak interaction is involved, the following chiral projection operators appear [48][77][78]:

$$\frac{1}{2}(1-\gamma^5)$$

$$\frac{1}{2}(1+\gamma^5)$$

Where γ^5 is defined as:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

And being γ^i the gamma matrices as defined in [48]:

In Dirac-Pauli representation[48][77][78] this is:

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In Chiral representation [77][78]:

$$\gamma^5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. 1. Chirality (Electroweak interaction) in Geometric (real Clifford) Algebra $Cl_{3.0}$ in chiral basis

The chiral representation is the one which has a clearer map-to-map relation with Geometric (real Clifford) Algebra representation as we will see now. Anyhow, in previous papers [63][75] we have always worked with Dirac-Pauli representation [48], so we will calculate with the two options.

We start with chiral representation. Left operator:

$$\frac{1}{2}(I-\gamma^5) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Right operator.

$$\frac{1}{2}(I+\gamma^5)=\frac{1}{2}\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$$
$$=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

Using the following definitions:

$$\psi=\begin{pmatrix}\psi^{1r}+i\psi^{1i}\\\psi^{2r}+i\psi^{2i}\\\psi^{3r}+i\psi^{3i}\\\psi^{4r}+i\psi^{4i}\end{pmatrix}$$

$$\begin{gathered}e_0\to 1\to scalars\\e_1\\e_2\\e_3\\e_4\equiv e_{23}\equiv e_2e_3\\e_5\equiv e_{31}\equiv e_3e_1\\e_6\equiv e_{12}\equiv e_1e_2\\e_7\equiv e_{123}\equiv e_1e_2e_3\end{gathered}$$

And defininng the wave function as in previous papers [63][75]:

$$\begin{aligned}\psi=\psi^\mu e_\mu=\psi^0e_0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^4e_4+\psi^5e_5+\psi^6e_6+\psi^7e_7\\=\psi^0e_0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^{23}e_{23}+\psi^{31}e_{31}+\psi^{12}e_{12}\\+\psi^{123}e_{123}\end{aligned}$$

The one-to-one map we obtained between matrix representation and Clifford Algebras representation (in the calculation of the probability and fermionic current and Dirac equation (Annex A1-A4, [63][75]) was:

$$\begin{gathered}\psi^{1r}=-\psi^2\\\psi^{1i}=-\psi^1\\\psi^{2r}=-\psi^{123}\\\psi^{2i}=\psi^3\\\psi^{3r}=-\psi^{23}\\\psi^{3i}=\psi^{31}\\\psi^{4r}=\psi^{12}\\\psi^{4i}=-\psi^0\end{gathered}$$

$$\begin{gathered}\psi^0=-\psi^{4i}\\\psi^1=-\psi^{1i}\\\psi^2=-\psi^{1r}\\\psi^3=\psi^{2i}\\\psi^{23}=-\psi^{3r}\\\psi^{31}=\psi^{3i}\\\psi^{12}=\psi^{4r}\\\psi^{123}=-\psi^{2r}\end{gathered}$$

Putting this mapping in the matrix representation:

$$\begin{pmatrix} \psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i} \end{pmatrix} = \begin{pmatrix} -\psi^{2} - i\psi^{1} \\ -\psi^{123} + i\psi^{3} \\ -\psi^{23} + i\psi^{31} \\ \psi^{12} - i\psi^{0} \end{pmatrix}$$

This is:

$$\psi = \psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} + \psi^{123} e_{123}$$

$$\psi = -\psi^{4i} - \psi^{1i} e_1 - \psi^{1r} e_2 + \psi^{2i} e_3 - \psi^{3r} e_{23} + \psi^{3i} e_{31} + \psi^{4r} e_{12} - \psi^{2r} e_{123}$$

So, defining the left hand projected wavefunction in Clifford Algebras as:

$$\psi_L = {\psi^0}_L e_0 + {\psi^1}_L e_1 + {\psi^2}_L e_2 + {\psi^3}_L e_3 + {\psi^{23}}_L e_{23} + {\psi^{31}}_L e_{31} + {\psi^{12}}_L e_{12} + {\psi^{123}}_L e_{123}$$

$${\psi^0}_L = -{\psi^{4i}}_L$$
$${\psi^1}_L = -{\psi^{1i}}_L$$
$${\psi^2}_L = -{\psi^{1r}}_L$$
$${\psi^3}_L = {\psi^{2i}}_L$$
$${\psi^{23}}_L = -{\psi^{3r}}_L$$
$${\psi^{31}}_L = {\psi^{3i}}_L$$
$${\psi^{12}}_L = {\psi^{4r}}_L$$
$${\psi^{123}}_L = -{\psi^{2r}}_L$$

$$\psi_L = -{\psi^{4i}}_L - {\psi^{1i}}_L e_1 - {\psi^{1r}}_L e_2 + {\psi^{2i}}_L e_3 - {\psi^{3r}}_L e_{23} + {\psi^{3i}}_L e_{31} + {\psi^{4r}}_L e_{12} - {\psi^{2r}}_L e_{123}$$

So, the naming of the elements in the matrix representation would correspond to:

$$\psi_L = \begin{pmatrix} {\psi^{1r}}_L + i{\psi^{1i}}_L \\ {\psi^{2r}}_L + i{\psi^{2i}}_L \\ {\psi^{3r}}_L + i{\psi^{3i}}_L \\ {\psi^{4r}}_L + i{\psi^{4i}}_L \end{pmatrix} \rightarrow \begin{pmatrix} -{\psi^{2}}_L - i{\psi^{1}}_L \\ -{\psi^{123}}_L + i{\psi^{3}}_L \\ -{\psi^{23}}_L + i{\psi^{31}}_L \\ {\psi^{12}}_L - i{\psi^{0}}_L \end{pmatrix}$$

And now calculating ψ_L, from ψ using γ^5, we get:

$$\psi_L = \frac{1}{2}(I - \gamma^5)\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i} \end{pmatrix} = \begin{pmatrix} \psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ 0 \\ 0 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} -\psi^{2} - i\psi^{1} \\ -\psi^{123} + i\psi^{3} \\ 0 \\ 0 \end{pmatrix}$$

So, the transformation is:

$${\psi^{1r}}_L = \psi^{1r} \rightarrow -\psi^2$$
$${\psi^{1i}}_L = \psi^{1i} \rightarrow \psi^1$$
$${\psi^{2r}}_L = \psi^{2r} \rightarrow -\psi^{123}$$
$${\psi^{2i}}_L = \psi^{2i} \rightarrow \psi^3$$
$${\psi^{3r}}_L = 0$$
$${\psi^{3i}}_L = 0$$

$$\psi^{4r}{}_L = 0$$
$$\psi^{4i}{}_L = 0$$

$$\psi_L = -\psi^{4i}{}_L - \psi^{1i}{}_L e_1 - \psi^{1r}{}_L e_2 + \psi^{2i}{}_L e_3 - \psi^{3r}{}_L e_{23} + \psi^{3i}{}_L e_{31} + \psi^{4r}{}_L e_{12} - \psi^{2r}{}_L e_{123}$$

$$\psi_L = -\psi^{1i} e_1 - \psi^{1r} e_2 + \psi^{2i} e_3 - \psi^{2r} e_{123}$$

$$\psi_L = \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{123} e_{123}$$

First point is to remark that the definition of γ^5 used for this result is the chirality one. This is incoherent with the previous papers and [48], but just to check that depending on the definitions chosen, the results can be highly simplified. For example, in this case, we see that only the odd grade elements of the algebra (vectors and trivector) are left with this definition. Again, the signs can be changed depending on the convention and definitions chosen.

Later, we will use the Dirac-Pauli definition that is more coherent with the paper [48], but the result will include more elements.

First let's calculate the right-handed wavefunction:

$$\psi_R = \begin{pmatrix} \psi^{1r}{}_R + i\psi^{1i}{}_R \\ \psi^{2r}{}_R + i\psi^{2i}{}_R \\ \psi^{3r}{}_R + i\psi^{3i}{}_R \\ \psi^{4r}{}_R + i\psi^{4i}{}_R \end{pmatrix} \to \begin{pmatrix} -\psi^{2}{}_R - i\psi^{1}{}_R \\ -\psi^{123}{}_R + i\psi^{3}{}_R \\ -\psi^{23}{}_R + i\psi^{31}{}_R \\ \psi^{12}{}_R - i\psi^{0}{}_R \end{pmatrix}$$

$$\psi_R = \frac{1}{2}(I + \gamma^5)\psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i} \end{pmatrix}$$

$$\psi = \psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} + \psi^{123} e_{123}$$

$$\psi = -\psi^{4i} - \psi^{1i} e_1 - \psi^{1r} e_2 + \psi^{2i} e_3 - \psi^{3r} e_{23} + \psi^{3i} e_{31} + \psi^{4r} e_{12} - \psi^{2r} e_{123}$$

$$\psi^0{}_R = -\psi^{4i}{}_R$$
$$\psi^1{}_R = -\psi^{1i}{}_R$$
$$\psi^2{}_R = -\psi^{1r}{}_R$$
$$\psi^3{}_R = \psi^{2i}{}_R$$
$$\psi^{23}{}_R = -\psi^{3r}{}_R$$
$$\psi^{31}{}_R = \psi^{3i}{}_R$$
$$\psi^{12}{}_R = \psi^{4r}{}_R$$
$$\psi^{123}{}_R = -\psi^{2r}{}_R$$

$$\psi_R = \psi^0{}_R e_0 + \psi^1{}_R e_1 + \psi^2{}_R e_2 + \psi^3{}_R e_3 + \psi^{23}{}_R e_{23} + \psi^{31}{}_R e_{31} + \psi^{12}{}_R e_{12} + \psi^{123}{}_R e_{123}$$
$$\psi_R = -\psi^{4i}{}_R - \psi^{1i}{}_R e_1 - \psi^{1r}{}_R e_2 + \psi^{2i}{}_R e_3 - \psi^{3r}{}_R e_{23} + \psi^{3i}{}_R e_{31} + \psi^{4r}{}_R e_{12} - \psi^{2r}{}_R e_{123}$$

$$\psi^{1r}{}_R = 0$$
$$\psi^{1i}{}_R = 0$$
$$\psi^{2r}{}_R = 0$$
$$\psi^{2i}{}_R = 0$$

$$\psi^{3r}{}_R = \psi^{3r} \to -\psi^{23}$$
$$\psi^{3i}{}_R = \psi^{3i} \to \psi^{31}$$
$$\psi^{4r}{}_R = \psi^{4r} \to \psi^{12}$$
$$\psi^{4i}{}_R = \psi^{4i} \to -\psi^{0}$$

$$\psi_R = -\psi^{4i}{}_R - \psi^{1i}{}_R e_1 - \psi^{1r}{}_R e_2 + \psi^{2i}{}_R e_3 - \psi^{3r}{}_R e_{23} + \psi^{3i}{}_R e_{31} + \psi^{4r}{}_R e_{12} - \psi^{2r}{}_R e_{123}$$

$$\psi_R = -\psi^{4i} - \psi^{3r} e_{23} + \psi^{3i} e_{31} + \psi^{4r} e_{12}$$

$$\psi_R = \psi^0 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12}$$

We can see that only the even grade elements are left (scalars and bivectors).
We see that with the chiral representation the separation between left-handed and right-handed elements In Geometric Algebra is pretty straight forward. The odd grade elements are the left-handed part and the even grade ones the right-handed part.

This means that in the chiral basis we can use a simple operation in geometric algebra to separate the left and the right-handed elements.
In fact, applying the grade automorphism (see chapter 4.3 and [73]) to ψ:

$$\psi = \psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} + \psi^{123} e_{123}$$
$$\bar{\psi}^{\dagger} = \psi^0 - \psi^1 e_1 - \psi^2 e_2 - \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} - \psi^{123} e_{123}$$

$$\begin{aligned}\psi_L = \frac{1}{2}(1-\gamma^5)\psi = \frac{1}{2}\left(1 - \left({}^{-\,\dagger}\right)\right)\psi = \frac{1}{2}(\psi - \bar{\psi}^{\dagger}) \\ = \big(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} \\ + \psi^{123} e_{123} \\ - (\psi^0 - \psi^1 e_1 - \psi^2 e_2 - \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} \\ - \psi^{123} e_{123})\big) = \frac{1}{2}\big(2(\psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{123} e_{123})\big) \\ = \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{123} e_{123}\end{aligned}$$

$$\begin{aligned}\psi_R = \frac{1}{2}(1+\gamma^5)\psi = \frac{1}{2}\left(1 + \left({}^{-\,\dagger}\right)\right)\psi = \frac{1}{2}(\psi + \bar{\psi}^{\dagger}) \\ = \big(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} \\ + \psi^{123} e_{123} \\ + (\psi^0 - \psi^1 e_1 - \psi^2 e_2 - \psi^3 e_3 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12} \\ - \psi^{123} e_{123})\big) = \frac{1}{2}\big(2(\psi^0 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12})\big) \\ = \psi^0 + \psi^{23} e_{23} + \psi^{31} e_{31} + \psi^{12} e_{12}\end{aligned}$$

So, we can conclude that in the chiral basis [48][77][[78] the γ^5 is equivalent in Clifford

Algebras to the grade automorphism operation. Regretfully, this will not be true in the case of the Pauli/Dirac basis, as we will see now.

7. 2. Chirality (Electroweak interaction) in Geometric (real Clifford) Algebra $Cl_{3.0}$ in Pauli/Dirac basis

Now, let's complicate the things using the Pauli Dirac definition of gammas as used during the paper [48]. Let's calculate the following factors:

$$\frac{1}{2}(I-\gamma^5)=\frac{1}{2}\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}$$

$$\frac{1}{2}(I+\gamma^5)=\frac{1}{2}\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}$$

Now, let's start calculating the left-handed elements of the wavefunction representation in Geometric Algebra $Cl_{3,0}$. For that, let's start with the matrix representation and then make the mapping to the Geometric Algebra representation:

$$\psi_L=\begin{pmatrix}\psi^{1r}{}_L+i\psi^{1i}{}_L\\\psi^{2r}{}_L+i\psi^{2i}{}_L\\\psi^{3r}{}_L+i\psi^{3i}{}_L\\\psi^{4r}{}_L+i\psi^{4i}{}_L\end{pmatrix}=\frac{1}{2}(I-\gamma^5)\psi=\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}\psi^{1r}+i\psi^{1i}\\\psi^{2r}+i\psi^{2i}\\\psi^{3r}+i\psi^{3i}\\\psi^{4r}+i\psi^{4i}\end{pmatrix}$$

$$\begin{pmatrix}\psi^{1r}{}_L+i\psi^{1i}{}_L\\\psi^{2r}{}_L+i\psi^{2i}{}_L\\\psi^{3r}{}_L+i\psi^{3i}{}_L\\\psi^{4r}{}_L+i\psi^{4i}{}_L\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\psi^{1r}+i\psi^{1i}-\psi^{3r}-i\psi^{3i}\\\psi^{2r}+i\psi^{2i}-\psi^{4r}-i\psi^{4i}\\-\psi^{1r}-i\psi^{1i}+\psi^{3r}+i\psi^{3i}\\-\psi^{2r}-i\psi^{2i}+\psi^{4r}+i\psi^{4i}\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\psi^{1r}-\psi^{3r}+i(\psi^{1i}-\psi^{3i})\\\psi^{2r}-\psi^{4r}+i(\psi^{2i}-\psi^{4i})\\-\psi^{1r}+\psi^{3r}+i(-\psi^{1i}+\psi^{3i})\\-\psi^{2r}+\psi^{4r}+i(-\psi^{2i}+\psi^{4i})\end{pmatrix}$$

$$\psi^{1r}{}_L=\frac{1}{2}(\psi^{1r}-\psi^{3r})$$

$$\psi^{1i}{}_L=\frac{1}{2}(\psi^{1i}-\psi^{3i})$$

$$\psi^{2r}{}_L=\frac{1}{2}(\psi^{2r}-\psi^{4r})$$

$$\psi^{2i}{}_L=\frac{1}{2}(\psi^{2i}-\psi^{4i})$$

$$\psi^{3r}{}_L = \frac{1}{2}(-\psi^{1r} + \psi^{3r})$$

$$\psi^{3i}{}_L = \frac{1}{2}(-\psi^{1i} + \psi^{3i})$$

$$\psi^{4r}{}_L = \frac{1}{2}(-\psi^{2r} + \psi^{4r})$$

$$\psi^{4i}{}_L = \frac{1}{2}(-\psi^{2i} + \psi^{4i})$$

$$\psi_L = -\psi^{4i}{}_L - \psi^{1i}{}_L e_1 - \psi^{1r}{}_L e_2 + \psi^{2i}{}_L e_3 - \psi^{3r}{}_L e_{23} + \psi^{3i}{}_L e_{31} + \psi^{4r}{}_L e_{12} - \psi^{2r}{}_L e_{123}$$

$$\begin{aligned}\psi_L = \frac{1}{2}(&-(-\psi^{2i} + \psi^{4i}) - (\psi^{1i} - \psi^{3i})e_1 - (\psi^{1r} - \psi^{3r})e_2 + (\psi^{2i} - \psi^{4i})e_3 \\ &- (-\psi^{1r} + \psi^{3r})e_{23} + (-\psi^{1i} + \psi^{3i})e_{31} + (-\psi^{2r} + \psi^{4r})e_{12} \\ &- (\psi^{2r} - \psi^{4r})e_{123})\end{aligned}$$

$$\begin{aligned}\psi^{1r} &= -\psi^2 \\ \psi^{1i} &= -\psi^1 \\ \psi^{2r} &= -\psi^{123} \\ \psi^{2i} &= \psi^3 \\ \psi^{3r} &= -\psi^{23} \\ \psi^{3i} &= \psi^{31} \\ \psi^{4r} &= \psi^{12} \\ \psi^{4i} &= -\psi^0\end{aligned}$$

$$\begin{aligned}\psi_L = \frac{1}{2}(&-(-\psi^3 - \psi^0) - (-\psi^1 - \psi^{31})e_1 - (-\psi^2 + \psi^{23})e_2 + (\psi^3 + \psi^0)e_3 \\ &- (\psi^2 - \psi^{23})e_{23} + (\psi^1 + \psi^{31})e_{31} + (\psi^{123} + \psi^{12})e_{12} \\ &- (-\psi^{123} - \psi^{12})e_{123})\end{aligned}$$

$$\begin{aligned}\psi_L = \frac{1}{2}\big(&(\psi^3 + \psi^0) + (+\psi^1 + \psi^{31})e_1 + (\psi^2 - \psi^{23})e_2 + (\psi^3 + \psi^0)e_3 \\ &+ (-\psi^2 + \psi^{23})e_{23} + (\psi^1 + \psi^{31})e_{31} + (\psi^{123} + \psi^{12})e_{12} \\ &+ (\psi^{123} + \psi^{12})e_{123}\big)\end{aligned}$$

We can see that the result is not as straight forward as in 7.1. Here the left-handed elements are a combination of the elements in the original wavefunction. There is not a simple operation to calculate ψ_L apart from a mapping as above. The γ^5 in this case is not easily converted to a factor or an operation in Geometric algebra.

Now, let's go with the right-handed:

$$\psi_R = \begin{pmatrix}\psi^{1r}{}_R + i\psi^{1i}{}_R \\ \psi^{2r}{}_R + i\psi^{2i}{}_R \\ \psi^{3r}{}_R + i\psi^{3i}{}_R \\ \psi^{4r}{}_R + i\psi^{4i}{}_R\end{pmatrix} = \frac{1}{2}(I + \gamma^5)\psi = \frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}\psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i}\end{pmatrix}$$

$$\begin{pmatrix} \psi^{1r}{}_R + i\psi^{1i}{}_R \\ \psi^{2r}{}_R + i\psi^{2i}{}_R \\ \psi^{3r}{}_R + i\psi^{3i}{}_R \\ \psi^{4r}{}_R + i\psi^{4i}{}_R \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \psi^{1r} + i\psi^{1i} + \psi^{3r} + i\psi^{3i} \\ \psi^{2r} + i\psi^{2i} + \psi^{4r} + i\psi^{4i} \\ +\psi^{1r} + i\psi^{1i} + \psi^{3r} + i\psi^{3i} \\ +\psi^{2r} + i\psi^{2i} + \psi^{4r} + i\psi^{4i} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \psi^{1r} + \psi^{3r} + i(\psi^{1i} + \psi^{3i}) \\ \psi^{2r} + \psi^{4r} + i(\psi^{2i} + \psi^{4i}) \\ \psi^{1r} + \psi^{3r} + i(\psi^{1i} + \psi^{3i}) \\ \psi^{2r} + \psi^{4r} + i(\psi^{2i} + \psi^{4i}) \end{pmatrix}$$

$$\psi^{1r}{}_R = \frac{1}{2}(\psi^{1r} + \psi^{3r})$$

$$\psi^{1i}{}_R = \frac{1}{2}(\psi^{1i} + \psi^{3i})$$

$$\psi^{2r}{}_R = \frac{1}{2}(\psi^{2r} + \psi^{4r})$$

$$\psi^{2i}{}_R = \frac{1}{2}(\psi^{2i} + \psi^{4i})$$

$$\psi^{3r}{}_R = \frac{1}{2}(\psi^{1r} + \psi^{3r})$$

$$\psi^{3i}{}_R = \frac{1}{2}(\psi^{1i} + \psi^{3i})$$

$$\psi^{4r}{}_R = \frac{1}{2}(\psi^{2r} + \psi^{4r})$$

$$\psi^{4i}{}_R = \frac{1}{2}(\psi^{2i} + \psi^{4i})$$

$$\psi_R = -\psi^{4i}{}_R - \psi^{1i}{}_R e_1 - \psi^{1r}{}_R e_2 + \psi^{2i}{}_R e_3 - \psi^{3r}{}_R e_{23} + \psi^{3i}{}_R e_{31} + \psi^{4r}{}_R e_{12} - \psi^{2r}{}_R e_{123}$$

$$\psi_R = \frac{1}{2}(-(\psi^{2i} + \psi^{4i}) - (\psi^{1i} + \psi^{3i})e_1 - (\psi^{1r} + \psi^{3r})e_2 + (\psi^{2i} + \psi^{4i})e_3 - (\psi^{1r} + \psi^{3r})e_{23} + (\psi^{1i} + \psi^{3i})e_{31} + (\psi^{2r} + \psi^{4r})e_{12} - (\psi^{2r} + \psi^{4r})e_{123})$$

$$\psi^{1r} = -\psi^2$$
$$\psi^{1i} = -\psi^1$$
$$\psi^{2r} = -\psi^{123}$$
$$\psi^{2i} = \psi^3$$
$$\psi^{3r} = -\psi^{23}$$
$$\psi^{3i} = \psi^{31}$$
$$\psi^{4r} = \psi^{12}$$
$$\psi^{4i} = -\psi^0$$

$$\psi_R = \frac{1}{2}(-(\psi^3 - \psi^0) - (-\psi^1 + \psi^{31})e_1 - (-\psi^2 - \psi^{23})e_2 + (\psi^3 - \psi^0)e_3 - (-\psi^2 - \psi^{23})e_{23} + (-\psi^1 + \psi^{31})e_{31} + (-\psi^{123} + \psi^{12})e_{12} - (-\psi^{123} + \psi^{12})e_{123})$$

$$\psi_R = \frac{1}{2}\big((-\psi^3 + \psi^0) + (+\psi^1 - \psi^{31})e_1 + (\psi^2 + \psi^{23})e_2 + (\psi^3 - \psi^0)e_3 + (\psi^2 + \psi^{23})e_{23} + (-\psi^1 + \psi^{31})e_{31} + (-\psi^{123} + \psi^{12})e_{12} + (\psi^{123} - \psi^{12})e_{123}\big)$$

We have a similar solution to the right-handed elements. Just a mapping between matrix and geometric algebra. No easy way to represent γ^5 in Geometric Algebra for the Pauli/Dirac basis. In chiral basis γ^5 represented just the grade automorphism operation (see 7.1).

8. APS, STA and Chirality

In my previous papers [5][6][63][75] I have always considered the time as the trivector in $Cl_{3,0}$. In the last papers I was considering that the time could be the trivector or the scalar depending on context.

In APS [43][73][74] (Algebra of Physical Space) $Cl_{3,0}$ the time is considered to be the scalars. In STA $Cl_{1,3}$ [1][3] time is considered to be a separate vector independent of the space vectors.

In fact, there is a mapping between STA and APS.

In STA we have the following vectors (that are related to the gamma matrices). The square of γ^0 is +1 while the square of γ^i is -1 when i=1,2 or 3.

$$\gamma^0 \to t$$
$$\gamma^1 \to x$$
$$\gamma^2 \to y$$
$$\gamma^3 \to z$$

If we post multiply above vectors by γ^0 we get the correspondence with APS:

$$\gamma^0\gamma^0 \to (\gamma^0)^2 \to 1\ (scalars) \to t$$
$$\gamma^1\gamma^0 \to e_1 \to x$$
$$\gamma^2\gamma^0 \to e_2 \to y$$
$$\gamma^3\gamma^0 \to e_3 \to z$$

So, let's say in APS the following elements define an entity or event (they include time via the scalars and space direction via the three vectors):

$$1 \to t$$
$$e_1 \to x$$
$$e_2 \to y$$
$$e_3 \to z$$

In my previous papers I used a similar approach, but instead of using the scalars as time I used the trivector:

$$e_{123} \to t$$
$$e_1 \to x$$
$$e_2 \to y$$
$$e_3 \to z$$

After what I have checked, what I see is that somehow both are correct because we can consider that any wavefunction has two parts that we can separate either the APS way like this:

One part: $1 \ e_1 \ e_3 \ e_3$
Second part: $e_{123}(1 \ e_1 \ e_2 \ e_3) = e_{123} \ e_{23} \ e_{31} \ e_{12}$

Or you can separate this way:

One part: $e_1 \ e_3 \ e_3 \ \ e_{123}$ (odd grade elements)
Second part: $1 \ e_{23} \ e_{31} \ e_{12}$ (even grade elements)

The important thing is that you use the eight elements and you do not use only one of the parts (you do not use only four of them),

In this paper (chapter 7), we have seen that in chiral basis[48][77][78], the elements that behave left-handed are the odd grade elements:
One part: $e_1 \ e_3 \ e_3 \ \ e_{123}$

And the ones that behave like right-handed elements are the even grade elements:
Second part: $1 \, e_{23} \ e_{31} \ e_{12}$

But what I have checked is that for example in the Dirac equation, the APS separation (scalars and vectors in one side and bivectors and trivector in the other side) has more sense.
So, depending in the context both could be used. In fact, if we consider that one part corresponds to a particle and the other part to an entangled antiparticle, this "mixing" in the separation of elements depending on the interaction, could explain that somehow, they are not never really separated. And the effects in one are affecting the other, as they have their elements mixed depending on context. In fact, one of the most common interpretations is that both particles are the same particle but one going into the future and the antiparticle going reverse in time, in a Tenet-like way. Or could be that both particles go in the same direction of time as the result of the density probability always being positive because of the way that is calculated (original wavefunction by its reverse) independently whatever values of the wavefunction.

As last comment, just to say that STA $Cl_{1,3}$ has been the option that have pushed the Geometric Algebra universally, with David Hestenes as main head [1][3]. And for that, all who we love Geometric Algebra will have an eternal doubt with him and all that have contributed to it.

Anyhow, STA leads to 16 free parameters that are not necessary in all the interactions or disciplines I have checked (electromagnetism, Strong Force, Weak force, Gravity, Dirac equation, quantum probability and fermionic current). In fact, only the even subalgebra coming from $Cl_{1,3}$ is used. This means, only 8 parameters of the 16 are used.

So, the 8 free parameters coming directly from $Cl_{3,0}$ are sufficient in all the areas I have checked. $Cl_{3,0}$ includes the time as an emergent phenomenon emerging for the 3 spatial dimensions (whether it is the scalar or the trivector) and it is not necessary to add it as an ad-hoc new dimension as made in STA.

Following the Occam razor, $Cl_{3,0}$ fits better for the purpose.

Another final point was regarding the possibility of using $Cl_{0,3}$ instead of $Cl_{3,0}$. This is disregarded, as in $Cl_{0,3}$ rotations are available, but Lorentz boosts are not possible. As there are no elements whose square is +1, hyperbolic functions cannot be formed, so we will lack tools to make all the possible Lorentz Transformations. See [63] and [78 Eigenchris spinor series].

9. Summary of interactions in Geometric (real Clifford) Algebra $Cl_{3,0}$

Maxwell Equation [26].

$$\nabla F = \bar{J}$$

Being:

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} + \frac{\partial}{\partial t}$$

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y}$$

$$J = J_x\hat{x} + J_y\hat{y} + J_z\hat{z} + J_0$$

Lorentz Force equation [Annex A7 and [6]]

$$\frac{d\bar{p}}{d\tau} = IqFU$$

With:

$$\frac{dp}{d\tau} = \frac{dp_0}{d\tau} + \frac{dp_{yz}}{d\tau}\hat{x} + \frac{dp_{zx}}{d\tau}\hat{y} + \frac{dp_{xy}}{d\tau}\hat{z} + \frac{dp_x}{d\tau}\hat{y}\hat{z} + \frac{dp_y}{d\tau}\hat{z}\hat{x} + \frac{dp_z}{d\tau}\hat{x}\hat{y} + \frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z}$$

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y}$$

$$U = U_0 + U_x\hat{x} + U_y\hat{y} + U_z\hat{z}$$

$$I = \hat{x}\hat{y}\hat{z}$$

Dirac Equation [Annex 5, [3][5][43][73][74]]

$$\bar{\partial}\psi I e_3 = m\bar{\psi}^{\dagger}$$

Being:

$$\partial = \frac{\partial}{\partial e^0} + e_1\frac{\partial}{\partial e^1} + e_2\frac{\partial}{\partial e^2} + e_3\frac{\partial}{\partial e^3}$$

$$\psi = \psi^0 + e_1\psi^1 + e_2\psi^2 + e_3\psi^3 + e_{12}\psi^{12} + e_{23}\psi^{23} + e_{31}\psi^{31} + e_{123}\psi^{123}$$
$$I = e_{123}$$

Probability and fermionic Current [Annex A1, A2, A3, A4 [63][75]:

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j}$$

Being:

$$\psi = \psi^0 + e_1\psi^1 + e_2\psi^2 + e_3\psi^3 + e_{12}\psi^{12} + e_{23}\psi^{23} + e_{31}\psi^{31} + e_{123}\psi^{123}$$

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 + 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

Strong Force (Gell-Mann matrices) [63]:

Being:

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

The new ψ' obtained when applying each of the Gell-Mann matrices λ_i is:

$$\psi' = (\lambda_1 \to \psi) = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_2 \to \psi) = \psi_0 + \psi_{zx}\hat{x} - \psi_{yz}\hat{y} - \psi_y\hat{y}\hat{z} + \psi_x\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_3 \to \psi) = \psi_0 + \psi_x\hat{x} - \psi_y\hat{y} + \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_4 \to \psi) = \psi_0 + \psi_z\hat{x} + \psi_x\hat{z} + \psi_{xy}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_5 \to \psi) = \psi_0 + \psi_{xy}\hat{x} - \psi_{yz}\hat{z} - \psi_z\hat{y}\hat{z} + \psi_x\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_6 \to \psi) = \psi_0 + \psi_z\hat{y} + \psi_y\hat{z} + \psi_{xy}\hat{z}\hat{x} + \psi_{zx}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_7 \to \psi) = \psi_0 + \psi_{xy}\hat{y} - \psi_{zx}\hat{z} - \psi_z\hat{z}\hat{x} + \psi_y\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_8 \to \psi) = \psi_0 + \frac{1}{\sqrt{3}}\psi_x\hat{x} + \frac{1}{\sqrt{3}}\psi_y\hat{y} - \frac{2}{\sqrt{3}}\psi_z\hat{z} + \frac{1}{\sqrt{3}}\psi_{yz}\hat{y}\hat{z} + \frac{1}{\sqrt{3}}\psi_{zx}\hat{z}\hat{x} - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (29)$$

Chirality (Weak interaction) [Chapter 7]:

$$\psi = \psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{23}e_{23} + \psi^{31}e_{31} + \psi^{12}e_{12} + \psi^{123}e_{123}$$

In Chiral basis:

$$\psi_L = \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^{123}e_{123}$$
$$\psi_R = \psi^0 + \psi^{23}e_{23} + \psi^{31}e_{31} + \psi^{12}e_{12}$$

In Pauli/Dirac basis:

$$\psi_L = \frac{1}{2}\big((\psi^3 + \psi^0) + (+\psi^1 + \psi^{31})e_1 + (\psi^2 - \psi^{23})e_2 + (\psi^3 + \psi^0)e_3 + (-\psi^2 + \psi^{23})e_{23} + (\psi^1 + \psi^{31})e_{31} + (\psi^{123} + \psi^{12})e_{12} + (\psi^{123} + \psi^{12})e_{123}\big)$$

$$\psi_R = \frac{1}{2}\big((-\psi^3+\psi^0)+(+\psi^1-\psi^{31})e_1+(\psi^2+\psi^{23})e_2+(\psi^3-\psi^0)e_3+(\psi^2+\psi^{23})e_{23} + (-\psi^1+\psi^{31})e_{31}+(-\psi^{123}+\psi^{12})e_{12}+(\psi^{123}-\psi^{12})e_{123}\big)$$

Einstein equations and non-Euclidean metric [75][76][Chapters 3.2 and 3.3]:

In the paper [75] different candidates for the Einstein equations are defined being the more coherent ones the following:

$$\frac{8\pi G}{c^4}T_{\mu\nu}\left(1-\frac{\hbar^2}{m^2c^2}R\right)=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}$$

$$\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^\beta\nabla_\beta\big(\nabla_\alpha(\psi^\dagger\psi)e^\alpha\big)\right)+\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R-mc^2\right)\psi^\dagger\psi-\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)=0$$

The important thing here is that the sub and super indices do not go from 0 to 3. They go from 0 to 7. This means, they include equations in the vectors and time (as usually) but they also include four equations more in the 3 bivectors and the trivector.

This is one of the biggest walls to join quantum mechanics and gravitation. In the Einstein equations we normally work with spacetime (4 dimensions) while the quantum world has 8 dimensions (see Dirac spinor for example). In the above equations the eight dimensions (scalars, 3 vectors, 3 bivectors and the trivector) have to be used. See [75] for more details.

Another finding of using Einstein equations in Geometric Algebra $Cl_{3,0}$ ids that the Energy-momentum relation is modified adding a new element [75] [76]:

$$E^2=m^2c^4+p^2c^2-R\hbar^2c^2$$

The other point is ones you have obtained the gravitational effects in one area of space using above equations, you have to use the following product relation among vectors and bivectors.

$$\begin{aligned}(e_1)^2&=e_1e_1=\|e_1\|^2=g_{11}\\(e_2)^2&=e_2e_2=\|e_2\|^2=g_{22}\\(e_3)^2&=e_3e_3=\|e_3\|^2=g_{33}\\e_1e_2&=2g_{12}-e_2e_1\\e_2e_3&=2g_{23}-e_3e_2\\e_3e_1&=2g_{31}-e_1e_3\end{aligned}$$

10. Conclusions

In this paper, we have obtained the left and the right-handed representation (chirality) of the wavefunction using Geometric (real Clifford) algebra $Cl_{3,0}$. Having the wavefunction ψ:

$$\psi=\psi^0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^{23}e_{23}+\psi^{31}e_{31}+\psi^{12}e_{12}+\psi^{123}e_{123}$$

In Chiral basis, the separation between left and right-handed elements is explicit:

$$\begin{aligned}\psi_L&=\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^{123}e_{123}\\\psi_R&=\psi^0+\psi^{23}e_{23}+\psi^{31}e_{31}+\psi^{12}e_{12}\end{aligned}$$

In Pauli/Dirac basis, this explicit separation is not possible, and the result is as follows:

$$\psi_L = \frac{1}{2}\big((\psi^3+\psi^0)+(+\psi^1+\psi^{31})e_1+(\psi^2-\psi^{23})e_2+(\psi^3+\psi^0)e_3+(-\psi^2+\psi^{23})e_{23}+(\psi^1+\psi^{31})e_{31} + (\psi^{123}+\psi^{12})e_{12}+(\psi^{123}+\psi^{12})e_{123}\big)$$

$$\psi_R = \frac{1}{2}\big((-\psi^3+\psi^0)+(+\psi^1-\psi^{31})e_1+(\psi^2+\psi^{23})e_2+(\psi^3-\psi^0)e_3+(\psi^2+\psi^{23})e_{23} + (-\psi^1+\psi^{31})e_{31}+(-\psi^{123}+\psi^{12})e_{12}+(\psi^{123}-\psi^{12})e_{123}\big)$$

Also, a summary of how all the interactions can be calculated and represented using Geometric (real Clifford) Algebra is shown.

Bilbao, 31st May 2024 (viXra-v1).

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AAAAÁBCCCDEEIIILLLLLMMMOOOPSTU

If you consider this helpful, do not hesitate to drop your BTC here:

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A1. Annex A1. Bra-Ket product in Euclidean metric

The bra-ket product of a reversed spinor (in orthogonal metrics is the same as reverse) can be calculated as:

$$
\begin{gathered}
\psi^\dagger\psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 \\
+ \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \psi^*\psi = \\
= (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 \\
+ \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \\
= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_2 e_3 - \psi^5 e_3 e_1 - \psi^6 e_1 e_2 - \psi^7 e_1 e_2 e_3)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 \\
+ \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) = \\
(\psi^0)^2 + \psi^0\psi^1 e_1 + \psi^0\psi^2 e_2 + \psi^0\psi^3 e_3 + \psi^0\psi^4 e_2 e_3 + \psi^0\psi^5 e_3 e_1 + \psi^0\psi^6 e_1 e_2 + \psi^0\psi^7 e_1 e_2 e_3 + \\
\psi^1\psi^0 e_1 + (\psi^1)^2 + \psi^1\psi^2 e_1 e_2 - \psi^1\psi^3 e_3 e_1 + \psi^1\psi^4 e_1 e_2 e_3 - \psi^1\psi^5 e_3 + \psi^1\psi^6 e_2 + \psi^1\psi^7 e_2 e_3 + \\
\psi^2\psi^0 e_2 - \psi^2\psi^1 e_1 e_2 + (\psi^2)^2 + \psi^2\psi^3 e_2 e_3 + \psi^2\psi^4 e_3 + \psi^2\psi^5 e_1 e_2 e_3 - \psi^2\psi^6 e_1 + \psi^2\psi^7 e_3 e_1 + \\
\psi^3\psi^0 e_3 + \psi^3\psi^1 e_3 e_1 - \psi^3\psi^2 e_2 e_3 + (\psi^3)^2 - \psi^3\psi^4 e_2 + \psi^3\psi^5 e_1 + \psi^3\psi^6 e_1 e_2 e_3 + \psi^3\psi^7 e_1 e_2 \\
-\psi^4\psi^0 e_2 e_3 - \psi^4\psi^1 e_1 e_2 e_3 + \psi^4\psi^2 e_3 - \psi^4\psi^3 e_2 + (\psi^4)^2 + \psi^4\psi^5 e_1 e_2 - \psi^4\psi^6 e_3 e_1 + \psi^4\psi^7 e_1 - \\
-\psi^5\psi^0 e_3 e_1 - \psi^5\psi^1 e_3 - \psi^5\psi^2 e_1 e_2 e_3 + \psi^5\psi^3 e_1 - \psi^5\psi^4 e_1 e_2 + (\psi^5)^2 + \psi^5\psi^6 e_2 e_3 + \psi^5\psi^7 e_2 - \\
-\psi^6\psi^0 e_1 e_2 + \psi^6\psi^1 e_2 - \psi^6\psi^2 e_1 - \psi^6\psi^3 e_1 e_2 e_3 + \psi^6\psi^4 e_3 e_1 - \psi^6\psi^5 e_2 e_3 + (\psi^6)^2 + \psi^6\psi^7 e_3 - \\
-\psi^7\psi^0 e_1 e_2 e_3 - \psi^7\psi^1 e_2 e_3 - \psi^7\psi^2 e_3 e_1 - \psi^7\psi^3 e_1 e_2 + \psi^7\psi^4 e_1 + \psi^7\psi^5 e_2 + \psi^7\psi^6 e_3 + (\psi^7)^2
\end{gathered}
$$

Please, take into account that for simplification I have considered directly $e_0 = 1$. If in the end, it has another value, it has just to be considered in the operations.

Continuing with the operation. If we separate from the result above only the scalars, we have:

$$(\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

We will call this sum ρ (probability density):

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

If we separate the components that multiply by e_1 we get:

$$
\begin{aligned}
&\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7 + \psi^5\psi^3 - \psi^6\psi^2 + \psi^7\psi^4 \\
&= 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)
\end{aligned}
$$

In e_2 we get:

$$
\begin{aligned}
&\psi^0\psi^2 + \psi^1\psi^6 + \psi^2\psi^0 - \psi^3\psi^4 - \psi^4\psi^3 + \psi^5\psi^7 + \psi^6\psi^1 + \psi^7\psi^5 \\
&= 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)
\end{aligned}
$$

In e_3 we get:

$$\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^3\psi^0 + \psi^4\psi^2 - \psi^5\psi^1 + \psi^6\psi^7 + \psi^7\psi^6$$
$$= 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)$$

In e_2e_3 :

$$\psi^0\psi^4 + \psi^1\psi^7 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6 - \psi^6\psi^5 - \psi^7\psi^1 = 0$$

In e_3e_1:

$$\psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi_{xyz} + \psi^3\psi^1 - \psi^4\psi^6 - \psi^5\psi^0 + \psi^6\psi^4 - \psi^7\psi^2 = 0$$

In e_1e_2:

$$\psi^0\psi^6 + \psi^1\psi^2 - \psi^2\psi^1 + \psi^3\psi^7 + \psi^4\psi^5 - \psi^5\psi^4 - \psi^6\psi^0 - \psi^7\psi^3 = 0$$

In $e_1e_2e_3$:

$$\psi^0\psi^7 + \psi^1\psi^4 + \psi^2\psi^5 + \psi^3\psi^6 - \psi^4\psi^1 - \psi^5\psi^2 - \psi^6\psi^3 - \psi^7\psi^0 = 0$$

If we call vector $\vec{j}$ (fermionic current) the sum in e_1, e_2 and e_3 , we get:

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2$$
$$+ 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

So, in total we have:

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j} \qquad (29.1)$$

With:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

And:

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2$$
$$+ 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A2. Annex A2. Bra-Ket product in non-Euclidean metric (Orthogonal but not orthonormal)

We apply the following relations, when performing the multiplication:

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

$$(e_1)^2 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = \|e_3\|^2 = g_{33}$$

$$e_0e_i = e_ie_0$$

$$e_2e_3 = -e_3e_2$$
$$e_3e_1 = -e_1e_3$$
$$e_1e_2 = -e_2e_1$$

For simplification we will consider directly $e_0 = 1$. If in the end, it has another value, it just will have to be considered in the operations.

$$\psi^\dagger\psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = \left(\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) =$$

$$(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) =$$
$${\psi^0}^2 + \psi^0\psi^1 e_1 + \psi^0\psi^2 e_2 + \psi^0\psi^3 e_3 + \psi^0\psi^4 e_2 e_3 + \psi^0\psi^5 e_3 e_1 + \psi^0\psi^6 e_1 e_2 + \psi^0\psi^7 e_1 e_2 e_3 +$$
$$\psi^1\psi^0 e_1 + {\psi^1}^2\|e_1\|^2 + \psi^1\psi^2 e_1 e_2 - \psi^1\psi^3 e_3 e_1 + \psi^1\psi^4 e_1 e_2 e_3 - \psi^1\psi^5\|e_1\|^2 e_3 + \psi^1\psi^6\|e_1\|^2 e_2 + \psi^1\psi^7\|e_1\|^2 e_2 e_3 +$$
$$\psi^2\psi^0 e_2 - \psi^2\psi^1 e_1 e_2 + {\psi^2}^2\|e_2\|^2 + \psi^2\psi^3 e_2 e_3 + \psi^2\psi^4\|e_2\|^2 e_3 + \psi^2\psi^5 e_1 e_2 e_3 - \psi^2\psi^6\psi_{xy} e_1 e_2 e_3 + \psi^3\psi^7\|e_3\|^2 e_1 e_2$$
$$-\psi^4\psi^0 e_2 e_3 - \psi^4\psi^1 e_1 e_2 e_3 + \psi^4\psi^2\|e_2\|^2 e_3 - \psi^4\psi^3\|e_3\|^2 e_2 + {\psi^4}^2\|e_2\|^2\|e_3\|^2 + \psi^4\psi^5\|e_3\|^2 e_1 e_2 - \psi^4\psi^6\|e_2\|^2 e_3 e_1 + \psi^4\psi^7\|e_2\|^2\|e_3\|^2 e_1 -$$
$$-\psi^5\psi^0 e_3 e_1 - \psi^5\psi^1\|e_1\|^2 e_3 - \psi^5\psi^2 e_1 e_2 e_3 + \psi^5\psi^3\|e_3\|^2 e_1 - \psi^5\psi^4\|e_3\|^2 e_1 e_2 + {\psi^5}^2\|e_3\|^2\|e_1\|^2 + \psi^5\psi^6\|e_1\|^2 e_2 e_3 + \psi^5\psi^7\|e_3\|^2\|e_1\|^2 e_2 -$$
$$-\psi^6\psi^0 e_1 e_2 + \psi^6\psi^1\|e_1\|^2 e_2 - \psi^6\psi^2\|e_2\|^2 e_1 - \psi^6\psi^3 e_1 e_2 e_3 + \psi^6\psi^4\|e_2\|^2 e_3 e_1 - \psi^6\psi^5\|e_1\|^2 e_2 e_3 + {\psi^6}^2\|e_1\|^2\|e_2\|^2 + \psi^6\psi^7\|e_1\|^2\|e_2\|^2 e_3 -$$
$$-\psi^7\psi^0 e_1 e_2 e_3 - \psi^7\psi^1\|e_1\|^2 e_2 e_3 - \psi^7\psi^2\|e_2\|^2 e_3 e_1 - \psi^7\psi^3\|e_3\|^2 e_1 e_2 + \psi^7\psi^4\|e_2\|^2\|e_3\|^2 e_1 + \psi^7\psi^5\|e_1\|^2\|e_3\|^2 e_2 + \psi^7\psi^6\|e_1\|^2\|e_2\|^2 e_3$$
$$+ {\psi^7}^2\|e_1\|^2\|e_2\|^2\|e_3\|^2$$

If we separate from the result above only the scalars, we have:

$$\rho = (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33}$$

We will call above sum ρ (probability density).

Now, if we separate by e_1 :

$$\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6\|e_2\|^2 + \psi^3\psi^5\|e_3\|^2 + \psi^4\psi^7\|e_2\|^2\|e_3\|^2 + \psi^5\psi^3\|e_3\|^2 - \psi^6\psi^2\|e_2\|^2 + \psi^7\psi^4\|e_2\|^2\|e_3\|^2$$

$$2(\psi^0\psi^1 - \psi^2\psi^6\|e_2\|^2 + \psi^3\psi^5\|e_3\|^2 + \psi^4\psi^7\|e_2\|^2\|e_3\|^2)$$

$$\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22} g_{33} + \psi^5\psi^3 g_{33} - \psi^6\psi^2 g_{22} + \psi^7\psi^4 g_{22} g_{33}$$

$$2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22} g_{33})$$

By e_2 :

$$+\psi^0\psi^2 + \psi^1\psi^6\|e_1\|^2 + \psi^2\psi^0 - \psi^3\psi^4\|e_3\|^2 - \psi^4\psi^3\|e_3\|^2 + \psi^5\psi^7\|e_3\|^2\|e_1\|^2 + \psi^6\psi^1\|e_1\|^2 + \psi^7\psi^5\|e_1\|^2\|e_3\|^2$$

$$2(+\psi^0\psi^2 + \psi^1\psi^6\|e_1\|^2 - \psi^3\psi^4\|e_3\|^2 + \psi^5\psi^7\|e_3\|^2\|e_1\|^2)$$

$$+\psi^0\psi^2 + \psi^1\psi^6 g_{11} + \psi^2\psi^0 - \psi^3\psi^4 g_{33} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33} g_{11} + \psi^6\psi^1 g_{11} + \psi^7\psi^5 g_{11} g_{33}$$
$$2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33} g_{11})$$

By e_3 :

$$+\psi^0\psi^3 - \psi^1\psi^5\|e_1\|^2 + \psi^2\psi^4\|e_2\|^2 + \psi^3\psi^0 + \psi^4\psi^2\|e_2\|^2 - \psi^5\psi^1\|e_1\|^2 + \psi^6\psi^7\|e_1\|^2\|e_2\|^2 + \psi^7\psi^6\|e_1\|^2\|e_2\|^2$$

$$2(+\psi^0\psi^3 - \psi^1\psi^5\|e_1\|^2 + \psi^2\psi^4\|e_2\|^2 + \psi^6\psi^7\|e_1\|^2\|e_2\|^2)$$

$$+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^3\psi^0 + \psi^4\psi^2 g_{22} - \psi^5\psi^1 g_{11} + \psi^6\psi^7 g_{11} g_{22} + \psi^7\psi^6 g_{11} g_{22}$$

$$2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11} g_{22})$$

In $e_2 e_3$ plane:

$$+\psi^0\psi^4 + \psi^1\psi^7\|e_1\|^2 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6\|e_1\|^2 - \psi^6\psi^5\|e_1\|^2 - \psi^7\psi^1\|e_1\|^2 = 0$$

In $e_3 e_1$ plane:

$$+\psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi^7\|e_2\|^2 + \psi^3\psi^1 - \psi^4\psi^6\|e_2\|^2 - \psi^5\psi^0 + \psi^6\psi^4\|e_2\|^2 - \psi^7\psi^2\|e_2\|^2 = 0$$

In $e_1 e_2$ plane:

$$+\psi^0\psi^6 + \psi^1\psi^2 - \psi^2\psi^1 + \psi^3\psi^7\|e_3\|^2 + \psi^4\psi^5\|e_3\|^2 - \psi^5\psi^4\|e_3\|^2 - \psi^6\psi^0 - \psi^7\psi^3\|e_3\|^2 = 0$$

In $e_1 e_2 e_3$ plane:

$$+\psi^0\psi^7 + \psi^1\psi^4 + \psi^2\psi^5 + \psi^3\psi^6 - \psi^4\psi^1 - \psi^5\psi^2 - \psi^6\psi^3 - \psi^7\psi^0 = 0$$

So, in this case, we can sum up the result as:

$$\psi^\dagger\psi = \rho + \vec{j}$$

Being:

$$\rho = (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33}$$

$$\vec{j} = 2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33})e_1 \\ + 2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11})e_2 \\ + 2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11}g_{22})e_3$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A3. Annex A3. Bra-Ket product between the reverse of a spinor and a spinor in non-Euclidean metric (Non orthogonal and non orthonormal).

We should do the following operation again:

$$\psi^\dagger\psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 \\ + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \\ (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) =$$

But using the following rules commented in chapter 3.3.

$$(e_i)^2 = e_i e_i = \|e_i\|^2 = g_{ii}$$
$$e_i e_j = 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i$$
$$e_i \cdot e_j = e_j \cdot e_i = g_{ij} = g_{ji}$$
$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = g_{ij} + e_i \wedge e_j$$

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$
$$e_1 e_2 = 2g_{12} - e_2 e_1 = 2g_{21} - e_2 e_1$$
$$e_2 e_3 = 2g_{23} - e_3 e_2 = 2g_{32} - e_3 e_2$$
$$e_3 e_1 = 2g_{31} - e_1 e_3 = 2g_{13} - e_1 e_3$$

I am not going to do it (you have a start of these calculations in[63]), but anyhow, you can understand that the result, whatever it is, will have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A4. Annex A4. Bra-Ket product between the inverse of a spinor and a spinor in non-Euclidean metric (Orthogonal but not orthonormal).

If instead of multiplying by the reverse, we multiply by the inverse (in orthogonal but not orthonormal metric), we should use the following rules from previous chapters:

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

$$(e_1)^2 = \|e_1\|^2 = g_{11}$$

$$(e_2)^2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = \|e_3\|^2 = g_{33}$$

$$e_0 e_i = e_i e_0$$

$$e_2 e_3 = -e_3 e_2$$
$$e_3 e_1 = -e_1 e_3$$
$$e_1 e_2 = -e_2 e_1$$

$$(e_i)^{-1} = e^i = \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2}$$

$$\left(e_i e_j\right)^{-1} = \frac{e_j e_i}{\left\|e_j\right\|^2 \|e_i\|^2} = \frac{e_j e_i}{g_{jj} g_{ii}}$$

Where all the above relation we have seen in previous chapters.
Operating:

$$\psi^{-1}\psi = \left(\psi^0 + \psi^1 \frac{e_1}{\|e_1\|^2} + \psi^2 \frac{e_2}{\|e_2\|^2} + \psi^3 \frac{e_3}{\|e_3\|^2} + \psi^4 \frac{e_3 e_2}{\|e_2\|^2\|e_3\|^2} + \psi^5 \frac{e_1 e_3}{\|e_3\|^2\|e_1\|^2} + \psi^6 \frac{e_2 e_1}{\|e_1\|^2\|e_2\|^2} + \psi^7 \frac{e_3 e_2 e_1}{\|e_1\|^2\|e_2\|^2\|e_3\|^2}\right)$$
$$(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3)$$
$$=$$

$$(\psi^0)^2 + \psi^1\psi^0 \frac{e_1}{\|e_1\|^2} + \psi^2\psi^0 \frac{e_2}{\|e_2\|^2} + \psi^3\psi^0 \frac{e_3}{\|e_3\|^2} - \psi^4\psi^0 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^5\psi^0 \frac{e_3 e_1}{\|e_3\|^2\|e_1\|^2} - \psi^6\psi^0 \frac{e_1 e_2}{\|e_1\|^2\|e_2\|^2} - \psi^7\psi^0 \frac{e_1 e_2 e_3}{\|e_1\|^2\|e_2\|^2\|e_3\|^2} +$$

$$\psi^0\psi^1 e_1 + (\psi^1)^2 - \psi^2\psi^1 e_1 \frac{e_2}{\|e_2\|^2} + \psi^3\psi^1 \frac{e_3}{\|e_3\|^2} e_1 - \psi^4\psi^1 e_1 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^5\psi^1 \frac{e_3}{\|e_3\|^2} + \psi^6\psi^1 \frac{e_2}{\|e_2\|^2} - \psi^7\psi^1 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} +$$

$$\psi^0\psi^2 e_2 + \psi^1\psi^2 \frac{e_1}{\|e_1\|^2} e_2 + (\psi^2)^2 - \psi^3\psi^2 e_2 \frac{e_3}{\|e_3\|^2} + \psi^4\psi^2 \frac{e_3}{\|e_3\|^2} - \psi^5\psi^2 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^6\psi^2 \frac{e_1}{\|e_1\|^2} - \psi^7\psi^2 \frac{e_3}{\|e_3\|^2} \frac{e_1}{\|e_1\|^2} +$$

$$\psi^0\psi^3 e_3 - \psi^1\psi^3 e_3 \frac{e_1}{\|e_1\|^2} + \psi^2\psi^3 \frac{e_2}{\|e_2\|^2} e_3 + (\psi^3)^2 - \psi^4\psi^3 \frac{e_2}{\|e_2\|^2} + \psi^5\psi^3 \frac{e_1}{\|e_1\|^2} - \psi^6\psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2} e_3 - \psi^7\psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2}$$

$$+\psi^0\psi^4 e_2 e_3 + \psi^1\psi^4 \frac{e_1}{\|e_1\|^2} e_2 e_3 + \psi^2\psi^4 e_3 - \psi^3\psi^4 e_2 + (\psi^4)^2 - \psi^5\psi^4 \frac{e_1}{\|e_1\|^2} e_2 + \psi^6\psi^4 e_3 \frac{e_1}{\|e_1\|^2} + \psi^7\psi^4 \frac{e_1}{\|e_1\|^2} +$$

$$+\psi^0\psi^5 e_3 e_1 - \psi^1\psi^5 e_3 + \psi^2\psi^5 e_1 \frac{e_2}{\|e_2\|^2} e_3 + \psi^3\psi^5 e_1 + \psi^4\psi^5 e_1 \frac{e_2}{\|e_2\|^2} + (\psi^5)^2 - \psi^6\psi^5 \frac{e_2}{\|e_2\|^2} e_3 + \psi^7\psi^6 \frac{e_2}{\|e_2\|^2} +$$

$$+\psi^0\psi^6 e_1 e_2 + \psi^1\psi^6 e_2 - \psi^2\psi^6 e_1 + \psi^3\psi^6 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^4\psi^6 \frac{e_3}{\|e_3\|^2} e_1 + \psi^5\psi^6 e_2 \frac{e_3}{\|e_3\|^2} + (\psi^6)^2 + \psi^7\psi^6 \frac{e_3}{\|e_3\|^2} +$$

$$+\psi^0\psi^7 e_1 e_2 e_3 + \psi^1\psi^7 e_2 e_3 + \psi^2\psi^7 e_3 e_1 + \psi^3\psi^7 e_1 e_2 + \psi^4\psi^7 e_1 + \psi^5\psi^7 e_2 + \psi^6\psi^7 e_3 + \left(\psi^7\right)^2$$

The scalar part is the same as the one multiplying by the reverse in a Euclidean orthonormal metric:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

This could be a hint, that probably this is the real operation that has to be done in general, instead of the reverse. The issue is that in orthonormal metric, the inverse and the reverse are the same operation. But this is not true in general, in non-orthonormal metrics.

If continuing with the operation, for example, we separate by e_1 we can see that the result is not as compact and in orthonormal or orthogonal solutions.

$$\psi^1\psi^0 \frac{e_1}{\|e_1\|^2} + \psi^0\psi^1 e_1 - \psi^6\psi^2 \frac{e_1}{\|e_1\|^2} + \psi^5\psi^3 \frac{e_1}{\|e_1\|^2} + \psi^7\psi^4 \frac{e_1}{\|e_1\|^2} + \psi^3\psi^5 e_1 - \psi^2\psi^6 e_1 + \psi^4\psi^7 e_1$$

Even we can see that the result in the planes is not zero. Example $e_2 e_3$:

$$-\psi^4\psi^0 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^7\psi^1 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^3\psi^2 e_2 \frac{e_3}{\|e_3\|^2} + \psi^2\psi^3 \frac{e_2}{\|e_2\|^2} e_3 + \psi^0\psi^4 e_2 e_3 - \psi^6\psi^5 \frac{e_2}{\|e_2\|^2} e_3 + \psi^5\psi^6 e_2 \frac{e_3}{\|e_3\|^2} + \psi^1\psi^7 e_2 e_3$$

Or $e_1 e_2 e_3$, also different from zero:

$$-\psi^7\psi^0\frac{e_1e_2e_3}{\|e_1\|^2\|e_2\|^2\|e_3\|^2}-\psi^4\psi^1e_1\frac{e_2e_3}{\|e_2\|^2\|e_3\|^2}-\psi^5\psi^2e_1e_2\frac{e_3}{\|e_3\|^2}-\psi^6\psi^3\frac{e_1}{\|e_1\|^2}\frac{e_2}{\|e_2\|^2}e_3+\psi^1\psi^4\frac{e_1}{\|e_1\|^2}e_2e_3+\psi^2\psi^5e_1\frac{e_2}{\|e_2\|^2}e_3$$
$$+\psi^3\psi^6e_1e_2\frac{e_3}{\|e_3\|^2}+\psi^0\psi^7e_1e_2e_3$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^{-1}\psi = j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

In case that we perform this operation (multiplying by the inverse) in an orthonormal metric, we will get the same result as in Annex A1 (as the inverse is the same as the reverse in this case).

In case, that we perform this operation in a non-orthogonal (and therefore non-orthogonal case), we will have to follow the rules in chapter 3.3.

Anyhow, the result will always have this form:

$$\psi^{-1}\psi = j^\mu e_\mu$$

A5. Annex A5. Dirac equation

In [5] I already made a mapping between Matrix Algebra and Geometric Algebra $Cl_{3,0}$. I will put it here again with another nomenclature (1,2,3 instead of x,y,z) and using the operations we commented in chapter 4 to make it even more clear. Similar approaches have been done by Baylis, APS [43][73][74].

A5.1. Dirac equation in Geometric Algebra

We will start with the Dirac Equation as defined by APS [43][73][74], similar one defined by [3][5]:

$$\bar{\partial}\psi I e_3 = m\bar{\psi}^\dagger$$

Being I the trivector. Now just operating and changing to the nomenclatures of this paper:

$$\bar{\partial}\psi I e_3 - m\bar{\psi}^\dagger = 0$$
$$\bar{\partial}\psi I - m\bar{\psi}^\dagger e_3 = 0$$
$$I\bar{\partial}\psi - m\bar{\psi}^\dagger e_3 = 0$$
$$e_{123}\bar{\partial}\psi - m\bar{\psi}^\dagger e_3 = 0$$

Now, if deploy the equation element by element:

$$\left(e_{123}\frac{\partial}{\partial e^0}-e_{23}\frac{\partial}{\partial e^1}-e_{31}\frac{\partial}{\partial e^2}-e_{12}\frac{\partial}{\partial e^3}\right)(\psi^0+e_1\psi^1+e_2\psi^2+e_3\psi^3+e_{12}\psi^{12}+e_{23}\psi^{23}+e_{31}\psi^{31}$$
$$+e_{123}\psi^{123})$$
$$+m(-\psi^0+e_1\psi^1+e_2\psi^2+e_3\psi^3-e_{12}\psi^{12}-e_{23}\psi^{23}-e_{31}\psi^{31}+e_{123}\psi^{123})e_3=0$$

Making the multiplication element by element, we get:

$$e_{123}\frac{\partial\psi^0}{\partial e^0}+e_{23}\frac{\partial\psi^1}{\partial e^0}+e_{31}\frac{\partial\psi^2}{\partial e^0}+e_{12}\frac{\partial\psi^3}{\partial e^0}-e_3\frac{\partial\psi^{12}}{\partial e^0}-e_1\frac{\partial\psi^{23}}{\partial e^0}-e_2\frac{\partial\psi^{31}}{\partial e^0}-\frac{\partial\psi^{123}}{\partial e^0}-$$
$$-e_{23}\frac{\partial\psi^0}{\partial e^1}-e_{123}\frac{\partial\psi^1}{\partial e^1}+e_3\frac{\partial\psi^2}{\partial e^1}-e_2\frac{\partial\psi^3}{\partial e^1}-e_{31}\frac{\partial\psi^{12}}{\partial e^1}+\frac{\partial\psi^{23}}{\partial e^1}+e_{12}\frac{\partial\psi^{31}}{\partial e^1}+e_1\frac{\partial\psi^{123}}{\partial e^1}-$$

$$-e_{31}\frac{\partial\psi^0}{\partial e^2}-e_3\frac{\partial\psi^1}{\partial e^2}-e_{123}\frac{\partial\psi^2}{\partial e^2}+e_1\frac{\partial\psi^3}{\partial e^2}+e_{23}\frac{\partial\psi^{12}}{\partial e^2}-e_{12}\frac{\partial\psi^{23}}{\partial e^2}+\frac{\partial\psi^{31}}{\partial e^2}+e_2\frac{\partial\psi^{123}}{\partial e^2}-$$
$$-e_{12}\frac{\partial\psi^0}{\partial e^3}+e_2\frac{\partial\psi^1}{\partial e^3}-e_1\frac{\partial\psi^2}{\partial e^3}-e_{123}\frac{\partial\psi^3}{\partial e^3}+\frac{\partial\psi^{12}}{\partial e^3}+e_{31}\frac{\partial\psi^{23}}{\partial e^3}-e_{23}\frac{\partial\psi^{31}}{\partial e^3}+e_3\frac{\partial\psi^{123}}{\partial e^3}-$$
$$-m\psi^0e_3-e_{31}m\psi^1+e_{23}m\psi^2+m\psi^3-e_{123}m\psi^{12}-e_2m\psi^{23}+e_1m\psi^{31}+e_{12}m\psi^{123}$$
$$=0$$

If we separate the equations depending on the element they are multiplying (the vector, bivector, trivector or scalars) we get these 8 equations:

$$\frac{\partial\psi^0}{\partial e^0}-\frac{\partial\psi^1}{\partial e^1}-\frac{\partial\psi^2}{\partial e^2}-\frac{\partial\psi^3}{\partial e^3}-m\psi^{12}=0$$
$$\frac{\partial\psi^1}{\partial e^0}-\frac{\partial\psi^0}{\partial e^1}+\frac{\partial\psi^{12}}{\partial e^2}-\frac{\partial\psi^{31}}{\partial e^3}+m\psi^2=0$$
$$\frac{\partial\psi^2}{\partial e^0}-\frac{\partial\psi^{12}}{\partial e^1}-\frac{\partial\psi^0}{\partial e^2}+\frac{\partial\psi^{23}}{\partial e^3}-m\psi^1=0$$
$$\frac{\partial\psi^3}{\partial e^0}+\frac{\partial\psi^{31}}{\partial e^1}-\frac{\partial\psi^{23}}{\partial e^2}-\frac{\partial\psi^0}{\partial e^3}+m\psi^{123}=0$$
$$-\frac{\partial\psi^{12}}{\partial e^0}+\frac{\partial\psi^2}{\partial e^1}-\frac{\partial\psi^1}{\partial e^2}+\frac{\partial\psi^{123}}{\partial e^3}-m\psi^0=0$$
$$-\frac{\partial\psi^{23}}{\partial e^0}+\frac{\partial\psi^{123}}{\partial e^1}+\frac{\partial\psi^3}{\partial e^2}-\frac{\partial\psi_y}{\partial e^3}+m\psi^{31}=0$$
$$-\frac{\partial\psi^{31}}{\partial e^0}-\frac{\partial\psi_z}{\partial e^1}+\frac{\partial\psi^{123}}{\partial e^2}+\frac{\partial\psi^1}{\partial e^3}-m\psi^{23}=0$$
$$-\frac{\partial\psi^{123}}{\partial e^0}+\frac{\partial\psi^{23}}{\partial e^1}+\frac{\partial\psi^{31}}{\partial e^2}+\frac{\partial\psi^{12}}{\partial e^3}+m\psi^3=0$$

A5.2. Dirac equation in Matrix Algebra

For this chapter 5.2 I will use the old nomenclature in which I used t,x,y,z instead of 0,1,2,3 and used subscripts instead of superscripts.

In matrix algebra the solution to Dirac equation has this form:

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}$$

Where the ψ_k are complex functions. If we consider that they can be divided in the real and the imaginary part of the function, the wavefunction would have the form:

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}=\begin{pmatrix}\psi_{1r}+i\psi_{1i}\\\psi_{2r}+i\psi_{2i}\\\psi_{3r}+i\psi_{3i}\\\psi_{4r}+i\psi_{4i}\end{pmatrix}$$

Now, we apply the Dirac equation in matrix algebra according [48]:

$$\begin{pmatrix} i\frac{\partial}{\partial t}-m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\\ 0 & i\frac{\partial}{\partial t}-m & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z}\\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t}-m & 0\\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t}-m \end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Applying the division in real and imaginary parts commented, we have:

$$\begin{pmatrix} i\frac{\partial}{\partial t}-m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t}-m & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t}-m & 0 \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t}-m \end{pmatrix} \begin{pmatrix} \psi_{1r}+i\psi_{1i} \\ \psi_{2r}+i\psi_{2i} \\ \psi_{3r}+i\psi_{3i} \\ \psi_{4r}+i\psi_{4i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And now, performing the matrix multiplication, we have for the first line:

$$\left(i\frac{\partial}{\partial t}-m\right)(\psi_{1r}+i\psi_{1i})+\left(i\frac{\partial}{\partial z}\right)(\psi_{3r}+i\psi_{3i})+\left(i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(\psi_{4r}+i\psi_{4i})=0$$

$$i\frac{\partial\psi_{1r}}{\partial t}-\frac{\partial\psi_{1i}}{\partial t}-m\psi_{1r}-im\psi_{1i}+i\frac{\partial\psi_{3r}}{\partial z}-\frac{\partial\psi_{3i}}{\partial z}+i\frac{\partial\psi_{4r}}{\partial x}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}+i\frac{\partial\psi_{4i}}{\partial y} = 0$$

Dividing in two equations, one for the real part and another one for the imaginary part, we get:

$$-\frac{\partial\psi_{1i}}{\partial t}-m\psi_{1r}-\frac{\partial\psi_{3i}}{\partial z}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}=0$$

$$\frac{\partial\psi_{1r}}{\partial t}-m\psi_{1i}+\frac{\partial\psi_{3r}}{\partial z}+\frac{\partial\psi_{4r}}{\partial x}+\frac{\partial\psi_{4i}}{\partial y}=0$$

For the second line of the matrix, we get:

$$\left(i\frac{\partial}{\partial t}-m\right)(\psi_{2r}+i\psi_{2i})+\left(i\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(\psi_{3r}+i\psi_{3i})+\left(-i\frac{\partial}{\partial z}\right)(\psi_{4r}+i\psi_{4i})=0$$

$$i\frac{\partial\psi_{2r}}{\partial t}-\frac{\partial\psi_{2i}}{\partial t}-m\psi_{2r}-im\psi_{2i}+i\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}-i\frac{\partial\psi_{3i}}{\partial y}-i\frac{\partial\psi_{4r}}{\partial z}+\frac{\partial\psi_{4i}}{\partial z} = 0$$

Again, dividing in two equations (real and imaginary part):

$$-\frac{\partial\psi_{2i}}{\partial t}-m\psi_{2r}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}+\frac{\partial\psi_{4i}}{\partial z}=0$$

$$\frac{\partial\psi_{2r}}{\partial t}-m\psi_{2i}+\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial y}-\frac{\partial\psi_{4r}}{\partial z}=0$$

For the third line of the equation, you have

$$\left(-i\frac{\partial}{\partial z}\right)(\psi_{1r}+i\psi_{1i})+\left(-i\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(\psi_{2r}+i\psi_{2i})+\left(-i\frac{\partial}{\partial t}-m\right)(\psi_{3r}+i\psi_{3i})=0$$

$$-i\frac{\partial\psi_{1r}}{\partial z}+\frac{\partial\psi_{1i}}{\partial z}-i\frac{\partial\psi_{2r}}{\partial x}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}-i\frac{\partial\psi_{2i}}{\partial y}-i\frac{\partial\psi_{3r}}{\partial t}+\frac{\partial\psi_{3i}}{\partial t}-m\psi_{3r}-im\psi_{3i} = 0$$

Dividing in real and imaginary part, we get:

$$\frac{\partial\psi_{1i}}{\partial z}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}+\frac{\partial\psi_{3i}}{\partial t}-m\psi_{3r}=0$$

$$-\frac{\partial \psi_{1r}}{\partial z}-\frac{\partial \psi_{2r}}{\partial x}-\frac{\partial \psi_{2i}}{\partial y}-\frac{\partial \psi_{3r}}{\partial t}-m\psi_{3i}=0$$

And for the fourth line:

$$\left(-i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(\psi_{1r}+i\psi_{1i})+\left(i\frac{\partial}{\partial z}\right)(\psi_{2r}+i\psi_{2i})+\left(-i\frac{\partial}{\partial t}-m\right)(\psi_{4r}+i\psi_{4i})=0$$

$$-i\frac{\partial \psi_{1r}}{\partial x}+\frac{\partial \psi_{1i}}{\partial x}+\frac{\partial \psi_{1r}}{\partial y}+i\frac{\partial \psi_{1i}}{\partial y}+i\frac{\partial \psi_{2r}}{\partial z}-\frac{\partial \psi_{2i}}{\partial z}-i\frac{\partial \psi_{4r}}{\partial t}+\frac{\partial \psi_{4i}}{\partial t}-m\psi_{4r}-im\psi_{4i}$$
$$=0$$

Getting these two equations:

$$\frac{\partial \psi_{1i}}{\partial x}+\frac{\partial \psi_{1r}}{\partial y}-\frac{\partial \psi_{2i}}{\partial z}+\frac{\partial \psi_{4i}}{\partial t}-m\psi_{4r}=0$$

$$-\frac{\partial \psi_{1r}}{\partial x}+\frac{\partial \psi_{1i}}{\partial y}+\frac{\partial \psi_{2r}}{\partial z}-\frac{\partial \psi_{4r}}{\partial t}-m\psi_{4i}=0$$

Putting all the equations together:

$$-\frac{\partial \psi_{1i}}{\partial t}-m\psi_{1r}-\frac{\partial \psi_{3i}}{\partial z}-\frac{\partial \psi_{4i}}{\partial x}+\frac{\partial \psi_{4r}}{\partial y}=0$$

$$\frac{\partial \psi_{1r}}{\partial t}-m\psi_{1i}+\frac{\partial \psi_{3r}}{\partial z}+\frac{\partial \psi_{4r}}{\partial x}+\frac{\partial \psi_{4i}}{\partial y}=0$$

$$-\frac{\partial \psi_{2i}}{\partial t}-m\psi_{2r}-\frac{\partial \psi_{3i}}{\partial x}-\frac{\partial \psi_{3r}}{\partial y}+\frac{\partial \psi_{4i}}{\partial z}=0$$

$$\frac{\partial \psi_{2r}}{\partial t}-m\psi_{2i}+\frac{\partial \psi_{3r}}{\partial x}-\frac{\partial \psi_{3i}}{\partial y}-\frac{\partial \psi_{4r}}{\partial z}=0$$

$$\frac{\partial \psi_{1i}}{\partial z}+\frac{\partial \psi_{2i}}{\partial x}-\frac{\partial \psi_{2r}}{\partial y}+\frac{\partial \psi_{3i}}{\partial t}-m\psi_{3r}=0$$

$$-\frac{\partial \psi_{1r}}{\partial z}-\frac{\partial \psi_{2r}}{\partial x}-\frac{\partial \psi_{2i}}{\partial y}-\frac{\partial \psi_{3r}}{\partial t}-m\psi_{3i}=0$$

$$\frac{\partial \psi_{1i}}{\partial x}+\frac{\partial \psi_{1r}}{\partial y}-\frac{\partial \psi_{2i}}{\partial z}+\frac{\partial \psi_{4i}}{\partial t}-m\psi_{4r}=0$$

$$-\frac{\partial \psi_{1r}}{\partial x}+\frac{\partial \psi_{1i}}{\partial y}+\frac{\partial \psi_{2r}}{\partial z}-\frac{\partial \psi_{4r}}{\partial t}-m\psi_{4i}=0$$

If we transform the nomenclature txyz to 0123 and we put superscripts to be coherent with the rest of the paper and order the elements we have:

$$-\frac{\partial \psi^{1i}}{\partial e^0}-\frac{\partial \psi^{4i}}{\partial e^1}+\frac{\partial \psi^{4r}}{\partial e^2}-\frac{\partial \psi^{3i}}{\partial e^3}-m\psi^{1r}=0$$

$$\frac{\partial \psi^{1r}}{\partial e^0}+\frac{\partial \psi^{4r}}{\partial e^1}+\frac{\partial \psi^{4i}}{\partial e^2}+\frac{\partial \psi^{3r}}{\partial e^3}-m\psi^{1i}=0$$

$$-\frac{\partial \psi^{2i}}{\partial e^0}-\frac{\partial \psi^{3i}}{\partial e^1}-\frac{\partial \psi^{3r}}{\partial e^2}+\frac{\partial \psi^{4i}}{\partial e^3}-m\psi^{2r}=0$$

$$\frac{\partial \psi^{2r}}{\partial e^0}+\frac{\partial \psi^{3r}}{\partial e^1}-\frac{\partial \psi^{3i}}{\partial e^2}-\frac{\partial \psi^{4r}}{\partial e^3}-m\psi^{2i}=0$$

$$\frac{\partial \psi^{3i}}{\partial e^0}+\frac{\partial \psi^{2i}}{\partial e^1}-\frac{\partial \psi^{2r}}{\partial e^2}+\frac{\partial \psi^{1i}}{\partial e^3}-m\psi^{3r}=0$$

$$-\frac{\partial\psi^{3r}}{\partial e^0}-\frac{\partial\psi^{2r}}{\partial e^1}-\frac{\partial\psi^{2i}}{\partial e^2}-\frac{\partial\psi^{1r}}{\partial e^3}-m\psi^{3i}=0$$
$$\frac{\partial\psi^{4i}}{\partial e^0}+\frac{\partial\psi^{1i}}{\partial e^1}+\frac{\partial\psi^{1r}}{\partial e^2}-\frac{\partial\psi^{2i}}{\partial e^3}-m\psi^{4r}=0$$
$$-\frac{\partial\psi^{4r}}{\partial e^0}-\frac{\partial\psi^{1r}}{\partial e^1}+\frac{\partial\psi^{1i}}{\partial e^2}+\frac{\partial\psi^{2r}}{\partial e^3}-m\psi^{4i}=0$$

A5.3. Matching Dirac equation in Geometric Algebra with Matrix Algebra

If we put the equations obtained in 5.1 and 5.2 together (changing sometimes the order for better understanding), from 5.1:

$$\frac{\partial\psi^{0}}{\partial e^0}-\frac{\partial\psi^{1}}{\partial e^1}-\frac{\partial\psi^{2}}{\partial e^2}-\frac{\partial\psi^{3}}{\partial e^3}-m\psi^{12}=0$$
$$\frac{\partial\psi^{1}}{\partial e^0}-\frac{\partial\psi^{0}}{\partial e^1}+\frac{\partial\psi^{12}}{\partial e^2}-\frac{\partial\psi^{31}}{\partial e^3}+m\psi^{2}=0$$
$$\frac{\partial\psi^{2}}{\partial e^0}-\frac{\partial\psi^{12}}{\partial e^1}-\frac{\partial\psi^{0}}{\partial e^2}+\frac{\partial\psi^{23}}{\partial e^3}-m\psi^{1}=0$$
$$\frac{\partial\psi^{3}}{\partial e^0}+\frac{\partial\psi^{31}}{\partial e^1}-\frac{\partial\psi^{23}}{\partial e^2}-\frac{\partial\psi^{0}}{\partial e^3}+m\psi^{123}=0$$
$$-\frac{\partial\psi^{12}}{\partial e^0}+\frac{\partial\psi^{2}}{\partial e^1}-\frac{\partial\psi^{1}}{\partial e^2}+\frac{\partial\psi^{123}}{\partial e^3}-m\psi^{0}=0$$
$$-\frac{\partial\psi^{23}}{\partial e^0}+\frac{\partial\psi^{123}}{\partial e^1}+\frac{\partial\psi^{3}}{\partial e^2}-\frac{\partial\psi_{y}}{\partial e^3}+m\psi^{31}=0$$
$$-\frac{\partial\psi^{31}}{\partial e^0}-\frac{\partial\psi_{z}}{\partial e^1}+\frac{\partial\psi^{123}}{\partial e^2}+\frac{\partial\psi^{1}}{\partial e^3}-m\psi^{23}=0$$
$$-\frac{\partial\psi^{123}}{\partial e^0}+\frac{\partial\psi^{23}}{\partial e^1}+\frac{\partial\psi^{31}}{\partial e^2}+\frac{\partial\psi^{12}}{\partial e^3}+m\psi^{3}=0$$

From 5.2:

$$\frac{\partial\psi^{4i}}{\partial e^0}+\frac{\partial\psi^{1i}}{\partial e^1}+\frac{\partial\psi^{1r}}{\partial e^2}-\frac{\partial\psi^{2i}}{\partial e^3}-m\psi^{4r}=0$$
$$-\frac{\partial\psi^{1i}}{\partial e^0}-\frac{\partial\psi^{4i}}{\partial e^1}+\frac{\partial\psi^{4r}}{\partial e^2}-\frac{\partial\psi^{3i}}{\partial e^3}-m\psi^{1r}=0$$
$$\frac{\partial\psi^{1r}}{\partial e^0}+\frac{\partial\psi^{4r}}{\partial e^1}+\frac{\partial\psi^{4i}}{\partial e^2}+\frac{\partial\psi^{3r}}{\partial e^3}-m\psi^{1i}=0$$
$$-\frac{\partial\psi^{2i}}{\partial e^0}-\frac{\partial\psi^{3i}}{\partial e^1}-\frac{\partial\psi^{3r}}{\partial e^2}+\frac{\partial\psi^{4i}}{\partial e^3}-m\psi^{2r}=0$$
$$-\frac{\partial\psi^{4r}}{\partial e^0}-\frac{\partial\psi^{1r}}{\partial e^1}+\frac{\partial\psi^{1i}}{\partial e^2}+\frac{\partial\psi^{2r}}{\partial e^3}-m\psi^{4i}=0$$
$$-\frac{\partial\psi^{3r}}{\partial e^0}-\frac{\partial\psi^{2r}}{\partial e^1}-\frac{\partial\psi^{2i}}{\partial e^2}-\frac{\partial\psi^{1r}}{\partial e^3}-m\psi^{3i}=0$$
$$\frac{\partial\psi^{3i}}{\partial e^0}+\frac{\partial\psi^{2i}}{\partial e^1}-\frac{\partial\psi^{2r}}{\partial e^2}+\frac{\partial\psi^{1i}}{\partial e^3}-m\psi^{3r}=0$$
$$\frac{\partial\psi^{2r}}{\partial e^0}+\frac{\partial\psi^{3r}}{\partial e^1}-\frac{\partial\psi^{3i}}{\partial e^2}-\frac{\partial\psi^{4r}}{\partial e^3}-m\psi^{2i}=0$$

You can see that there is a one-to-one map that corresponds to:

$$\psi^{1r} = -\psi^2$$
$$\psi^{1i} = -\psi^1$$
$$\psi^{2r} = \psi^{123}$$
$$\psi^{2i} = \psi^3$$
$$\psi^{3r} = -\psi^{23}$$
$$\psi^{3i} = \psi^{31}$$
$$\psi^{4r} = \psi^{12}$$
$$\psi^{4i} = \psi^0$$

This means considering the solution in geometric algebra as:

$$\psi = \psi^0 + e_1\psi^1 + e_2\psi^2 + e_3\psi^3 + e_{12}\psi^{12} + e_{23}\psi^{23} + e_{31}\psi^{31} + e_{123}\psi^{123}$$

And the solution in matrix algebra as:

$$\psi = \begin{pmatrix} \psi^{1r} + i\psi^{1i} \\ \psi^{2r} + i\psi^{2i} \\ \psi^{3r} + i\psi^{3i} \\ \psi^{4r} + i\psi^{4i} \end{pmatrix}$$
$$\psi = \begin{pmatrix} -\psi^2 - i\psi^1 \\ \psi^{123} + i\psi^3 \\ -\psi^{23} + i\psi^{31} \\ \psi^{12} + i\psi^0 \end{pmatrix}$$

To be noted that when we try to make other mappings like getting the fermionic current as the result of a wavefunction by its reverse Annexes A1-A4 [63] we obtain the mapping of ψ^0 and ψ^{123} with different sign as follows. The mapping to use will depend on the context:

$$\psi^{2r} = -\psi^{123}$$
$$\psi^{4i} = -\psi^0$$

A6. Relation between standard nomenclature and Geometric Algebra

During my studies of Geometric Algebra $Cl_{3,0}$. I have got to the following relations. When a $\pm$ means that could be a + or – sign depending on context. This mainly happens with γ^0 and γ^5 depending on if they pre or post multiply in the original equations before making the mapping. The σ_i are the sigma matrices and the γ^i the Dirac matrices.

The imaginary unit is in general converted to the trivector. But when it is implying a direction (like can be in linear momentum in quantum mechanics) could be substituted by a bivector (but as commented this is a very special case, trivector should work in general).

$$\sigma_1 = \gamma^1\gamma^0 \to e_1$$
$$\sigma_2 = \gamma^2\gamma^0 \to e_2$$
$$\sigma_3 = \gamma^3\gamma^0 \to e_3$$
$$i \to \pm\sigma_1\sigma_2\sigma_3 = \pm\gamma^0\gamma^1\gamma^2\gamma^3 \to \pm e_{123} \equiv \pm e_1e_2e_3$$
$$i \to \pm e_{jk} \equiv \pm e_je_k \quad j \neq k \quad (for\ very\ specific\ cases)$$
$$\gamma^0 \to \pm 1$$

$$\gamma^1 \rightarrow \pm e_1 \;\; or \;\; \pm e_2 e_3 \;\; (depending\ on\ context)$$
$$\gamma^2 \rightarrow \pm e_2 \;\; or \;\; \pm e_3 e_1 \;\; (depending\ on\ context)$$
$$\gamma^3 \rightarrow \pm e_3 \;\; or \;\; \pm e_1 e_2 \;\; (depending\ on\ context)$$
$$\gamma^5 \rightarrow \pm(^{-\,\dagger}) \;\; (or\ another\ special\ transformation\ depending\ on\ context\ see\ chapter\ 7)$$

A7. Lorentz Force

In [6] I used a different equation that was:

$$\frac{dp}{d\tau} = qFU$$

Where U was defined as:

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} \qquad (18)$$

In fact, an exact equation that does not oblige us to change the naming of the U elements is the following:

$$\frac{d\bar{p}}{d\tau} = IqFU$$

With:

$$\frac{dp}{d\tau} = \frac{dp_0}{d\tau} + \frac{dp_{yz}}{d\tau}\hat{x} + \frac{dp_{zx}}{d\tau}\hat{y} + \frac{dp_{xy}}{d\tau}\hat{z} + \frac{dp_x}{d\tau}\hat{y}\hat{z} + \frac{dp_y}{d\tau}\hat{z}\hat{x} + \frac{dp_z}{d\tau}\hat{x}\hat{y} + \frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z}$$

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y}$$

$$U = U_0 + U_x\hat{x} + U_y\hat{y} + U_z\hat{z}$$

$$I = \hat{x}\hat{y}\hat{z}$$

As we can do (the trivector is commutative):

$$\frac{d\bar{p}}{d\tau} = qFIU$$

Getting the magnitude:

$$IU = U_0\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y}$$

So, obtaining the same result in the end.

The other difference is the Clifford reversion in the momentum $\bar{p}$. This is to be able to accommodate the final equations, without having to change any sign, see [6]:

$$\frac{dp_0}{d\tau} = \frac{dp_4}{d\tau} \qquad \frac{d\overline{p_x}}{d\tau} = -\frac{dp_x}{d\tau} = -\frac{dp_1}{d\tau} \qquad \frac{d\overline{p_y}}{d\tau} = -\frac{dp_y}{d\tau} = -\frac{dp_2}{d\tau} \qquad \frac{d\overline{p_z}}{d\tau} = -\frac{dp_z}{d\tau} = -\frac{dp_3}{d\tau}$$

So:

$$\frac{dp_0}{d\tau} = \frac{dp_4}{d\tau} \qquad \frac{dp_x}{d\tau} = \frac{dp_1}{d\tau} \qquad \frac{dp_y}{d\tau} = \frac{dp_2}{d\tau} \qquad \frac{dp_z}{d\tau} = \frac{dp_3}{d\tau}$$