BEYOND GÖDEL

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ABSTRACT

In 1930 Gödel wrote a landmark paper showing that in any formal system there will always be statements that cannot be proven. But the deficiency of formal systems goes much deeper. The same logically valid statement can be used in conjunction with two different sets simultaneously proving a true statement and a false statement. This result is profound. It explains why people can use the same sound argument to prove two contradictory statements. It is no wonder the most lucid arguments still sometimes result in hung juries and earnest people can disagree on the most fundamental issues. Truth is a much deeper concept than logical validity.

Keywords inconsistency · logical validity

1 Introduction.

Inconsistency in mathematical systems has a long heritage. From the earliest times division by zero led to many contradictions [1], which could only be rectified by banning division by zero.

1) $x = 1$

2) $x^2 = x$; multiply by x

3) $x^2 - 1 = x - 1$; subtract 1

4) $x + 1 = 1$; divide by $x - 1$ (0)

5) $x = 0$; subtract 1

Gottlob Frege developed the earliest formal system of arithmetic at the beginning of the twentieth century. But, it contained a contradictory self-referential set [2]: the set of all sets that do not contain themselves. Does it contain itself? If it does, then it does not. If it does not, then it does. Gödel proved that any formal system containing arithmetic though not necessarily inconsistent is incomplete [3].

We make a logically valid argument concluding that sets of rationals $0 \leq a < b$ with $b < 100$ have largest elements. We can do this because the exact same argument applied to the natural numbers is valid. We explain our reasoning in the Conclusion section of this paper.

2 Inconsistency.

We establish a collection of nested sets of rational numbers in a descending hierarchy. The sets higher in the descending hierarchy contain element(s) that are not in the sets below them in the hierarchy. Given such a descending set hierarchy, it is easy to develop two arguments that contradict each other.

For rational numbers a in $(0, 100)$ let the collection of Ra sets be $\{$ y is a rational number $\mid 0 \leq y < a$ $\}$

Argument #1: No Ra set contains a largest element.

1) Suppose there is a largest element a' in some individual Ra.

2) $a' < (a' + a)/2 < a$.

3) Let $b = (a' + a)/2$.

4) Then b is in Ra and $a' < b$.

5) Therefore, no R*a* set contains a largest element.

When a largest element is assumed in Argument#1, it leads to a contradiction; so there is no largest element. Every R*a* set element is in one of the proper subsets below R*a* in the set hierarchy. It is a valid proof by contradiction.

Argument #2: Each R*a* set contains a largest element.

1) Below each R*a* for all rationals $x < a$ is a collection of R*x* subsets$\{$ *y* is a rational number | $0 \leq y < x$ $\}$.

2) Each R*a* and its collection of R*x* subsets comprise a descending nested set hierarchy with R*a* at the top.

3) Each R*x* is missing its index *"x"*. R*a* contains all the *"x"* indices.

4) Since the union of the R*x* set collection does not contain any element greater than the elements in all the individual R*x* sets and R*a* (at the top of a nested set hierarchy with the collection of R*x* sets below it) includes all the missing *"x"* indices in the R*x* sets, the union of the R*x* set collection does not equal R*a*.

5) There exists at least one R*a* set element *s* $\geq$ (all values of) ***x*.**

6) Let *c* and *d* be two elements of a single R*a* set with $c > d$.

7) *d* is an element of R*c*, which is a proper subset of R*a*.

8) For any two elements in R*a* the smaller element is contained in a R*x* subset of R*a.*

9) By steps 6) 7) and 8), **there is at most one R*a* set element** missing from all the R*x* subsets.

10) By steps 5) 9), **each R*a* set contains a largest element *a'*** not in a R***x*** set below in the hierarchy.

11) There is no $b = (a' + a)/2$. It would be a second element not in a R***x*** set below R*a* in the hierarchy. We know by step 8) that isn't possible.

3 Conclusion.

Argument #1 is generally considered correct and its conclusion is true. The first three statements of Argument #2 are generally accepted as true. It's the latter part of statement *#4* stating "the union of the collection of R*x* sets does not equal R*a*" that is a false statement. This causes most people to dismiss Argument #2. This is because they are unable to view Argument #1 and Argument #2 independently. The logical validity of each argument stands alone. We cannot consider Argument #1, when evaluating Argument #2. The latter part of statement *#4* is a valid logical deduction drawn from the true statements that directly proceed it. Moreover, if the rationals in *(0, 100)* are replaced by the natural numbers in *(0, 100),* Augument #2 remains exactly the same and no one questions its validity.

Think about it. We use the first three true statements and the first part of the fourth statement in Argument #2, and we conclude the Ra sets of natural numbers have a largest element, which is true they do. However, the first three true statements and the first part of the fourth statement in Argument #2 are equally true for the Ra sets of rational numbers. If it is a valid deduction to conclude that the Ra sets of natural numbers have a largest element, then like it or not; it is an equally valid deduction to conclude that the Ra sets of rational numbers have a largest element.

Containing a false statement does not keep the latter part of statement #4 from being a valid logical deduction from previous true statements. It simply means that in all formal systems containing sets, arithmetic, and rational numbers this false statement can be deduced and such formal systems are therefore inconsistent. Likewise, statements #5, #9, #10, and #11 are valid logical deductions from previous statements.

Many people will find argument #2 repugnant because a false statement has been validly deduced from previous true statements. However, perhaps people will become more tolerant, when they realize that in discussing anything there may be a logically valid argument using the same premises arriving at an opposite conclusion. In the future we will have to find more than just a logically valid argument to conclusively determine whether a statement is true or false. There may well be another logically valid argument that comes to an opposite conclusion. This also shows that truth is a deeper concept than simple logical validity.

References

[1] Bunch B. Mathematical fallacies and paradoxes. page 13, 1982.

[2] Clegg B. Are numbers real? pages 175–176, 2016.

[3] Kline M. Mathematics: The loss of certainty. pages 260–264, 1980.