THE *abc* CONJECTURE IS TRUE

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract. In this paper, we consider the *abc* conjecture. Assuming the conjecture $c < rad^2(abc)$ is true, we give the proof of the *abc* conjecture for $\epsilon \geq 1$. For the case $\epsilon \in]0,1[$, we consider that the *abc* conjecture is false, from the proof, we arrive in a contradiction.

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1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1.1. ***(abc Conjecture)****: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant depending only of ϵ.

The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [4]. It is the key to resolve the abc conjecture. In the following, assuming the conjecture $c < rad^2(abc)$ holds, I propose an elementary proof of the abc conjecture.

2. The Proof of the abc conjecture

Proof. We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$.

2.1. **Case :** $\epsilon \geq 1$**.** As $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e, \;\epsilon \geq 1 \tag{2.1}$$

Then the abc conjecture is true.

2.2. **Case:** $0 < \epsilon < 1$**.** For the cases $c < R$, it is trivial that the abc conjecture is true. In the following we consider that $c > R$. From the statement of the abc conjecture 1.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \Longrightarrow LogK(\epsilon) + (1+\epsilon)LogR - Logc > 0$.

For our proof, we proceed by contradiction of the abc conjecture. We suppose that the abc conjecture is false:

$$\begin{array}{c} \exists\, \epsilon_0 \in]0,1[, \forall\, K(\epsilon) > 0, \quad \exists\, c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that} \\ c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \end{array} \tag{2.2}$$

We choose the constant $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR_0 - Logc_0, \epsilon \in]0,1[\tag{2.3}$$

From the above explications, if we will obtain $\forall \epsilon \;\in]0,1[, Y_{c_0}(\epsilon) > 0 \Longrightarrow c_0 < K(\epsilon)R_0^{1+\epsilon} \Longrightarrow c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (2.2).

About the function Y_{c_0}, we have:

$$lim_{\epsilon \longrightarrow 1} Y_{c_0}(\epsilon) = 1 + Log(R_0^2/c_0) = \lambda > 0$$
$$lim_{\epsilon \longrightarrow 0} Y_{c_0}(\epsilon) = +\infty$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in]0,1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \tag{2.4}$$

$Y'_{c_0}(\epsilon) = 0 \Longrightarrow \epsilon = \epsilon' = \sqrt[3]{\dfrac{2}{LogR_0}} \in]0,1[$ for $R_0 \geq 8$.

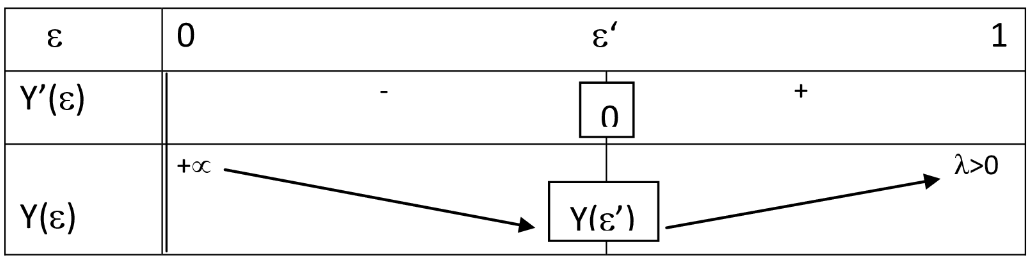

FIGURE 1. Table of variations

Discussion from the table (Fig.: 1):

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the *abc* conjecture is false can not hold. Hence the *abc* conjecture is true for $\epsilon \in]0,1[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists\ 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain:

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2} \tag{2.5}$$

We recall the following definition:

Definition 2.1. *The number ξ is called algebraic number if there is at least one polynomial:*

$$l(x) = l_0 + l_1 x + \cdots + l_m x^m, \quad l_m \neq 0 \tag{2.6}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$c_0 = K(\epsilon_1) R_0^{1+\epsilon_1} \Longrightarrow \frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0 b_0)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} \tag{2.7}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$\frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0 b_0)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{2.8}$$

From the theorem (see theorem 3, page 196 in [2]):

Theorem 2.2. $e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_n^{\beta_n}$ *is transcendental for any nonzero algebraic numbers* $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$.

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (2.8) is transcendental, but the term $\dfrac{\mu_{c_0}}{rad(a_0 b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon') < 0$ is impossible. It follows $Y_{c_0}(\epsilon') \geq 0$ then the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, then $1/(\epsilon_1^2)$ is transcendental. If not, $1/(\epsilon_1^2)$ is an algebraic number and from the definition (2.1) above, we find a contradiction. As $R_0 > 0$ is an algebraic number, then $LogR_0$ is transcendental. We rewrite the equation (2.5) as:

$$\frac{c_0}{R_0} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\frac{1}{\epsilon_2^2}} R_0^{\epsilon_2} \Longrightarrow \frac{c_0}{R_0} = e^{\frac{1}{\epsilon_1^2}+\epsilon_1 LogR_0} = e^{\frac{1}{\epsilon_2^2}+\epsilon_2 LogR_0} \tag{2.9}$$

As e is transcendental and let $z = \frac{1}{\epsilon_1^2} + \epsilon_1 LogR_0 > 0$, then e^z is transcendental [5], it follows the contradiction with c_0/R_0 an algebraic number. It follows that $Y_{c_0}(\epsilon') \geq 0$ and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. Assuming $c < R^2$ is true, we obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{2.10}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

□

3. Conclusion

Assuming $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 3.1. *Assuming $c < R^2$ is true, the abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1}$$

where K is a constant depending of ϵ.

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