A COMPLETE PROOF OF THE abc CONJECTURE: IT IS EASY AS ABC!

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To the memory of my Parents,
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

ABSTRACT. In this paper, we consider the abc conjecture. Assuming that the conjecture $c < rad^{1.63}(abc)$ is true, we give the proof that the abc conjecture is true.

1. INTRODUCTION AND NOTATIONS

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 1.1. *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 1.2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{5}$$
$$abc < rad^{4.42}(abc) \tag{6}$$

In the following, we assume that the conjecture giving by the equation (5) is true that constitutes the key to obtain the proof of the abc conjecture.

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2. THE PROOF OF THE *ABC* CONJECTURE

Proof. :

2.1. **Case** $\epsilon \geq (0.63 = \epsilon_0)$**.** In this case, we choose $K(\epsilon) = 1$ and let a, b, c be positive integers, relatively prime, with $c = a + b$, $1 \leq b < a, R = rad(abc)$, then $c < R^{1+\epsilon_0} \leq K(\epsilon).R^{1+\epsilon} \Longrightarrow c < K(\epsilon).R^{1+\epsilon}$ and the *abc* conjecture is true.

2.2. **Case:** $\epsilon < (0.63 = \epsilon_0)$**.** We suppose that the abc conjecture is false, then it exists $\epsilon' \in]0, \epsilon_0[$ and for all parameter $K' = K'(\epsilon) > 0$, it exists at least one triplet (a', b', c') so a', b', c' be positive integers relatively prime with $c' = a' + b'$ and c' verifies :

$$c' > K'(\epsilon').R^{1+\epsilon'} \tag{7}$$

In the above equation, c' depends of the value of $K'(\epsilon')$ but not of the value of $K'(\tau)$ with $\tau \neq \epsilon'$. We can choose $K'(\epsilon)$ as a smooth increasing function for $\epsilon \in]0, \epsilon_0[$. Let $\bar{\epsilon} = \epsilon' - \Delta\epsilon$ with $0 < \Delta\epsilon \ll \epsilon'$ so that the *abc* conjecture is verified : it exists $K(\bar{\epsilon})$ and:

$$c' < K(\bar{\epsilon}).R^{1+\bar{\epsilon}} \tag{8}$$

We remark here that c' is independent of $K(\bar{\epsilon})$. The equation (7) can be written as:

$$c' > K'(\epsilon')R^{1+\epsilon'} > K'(\epsilon' - \Delta\epsilon).R^{1+\epsilon'-\Delta\epsilon} \Longrightarrow \\ c' > K'(\bar{\epsilon}).R^{1+\bar{\epsilon}} \tag{9}$$

Now, as the parameter $K'(\epsilon)$ is arbitrary, we choose in the last equation above (9), $K'(\bar{\epsilon}) = K(\bar{\epsilon})$, it follows using the equation (8):

$$K'(\bar{\epsilon}).R^{1+\bar{\epsilon}} < c' < K(\bar{\epsilon}).R^{1+\bar{\epsilon}} \Longrightarrow \\ K'(\bar{\epsilon}).R^{1+\bar{\epsilon}} < c' < K'(\bar{\epsilon}).R^{1+\bar{\epsilon}} \Longrightarrow 1 < 1 \tag{10}$$

Then the contradiction. It follows that the assumption that the *abc* conjecture is false on $]0, 0.63[$ is not verified and the *abc* conjecture is true for all $\epsilon \in]0, 0.63[$.

Finally, the *abc* conjecture is true for all $\epsilon > 0$.

Q.F.D

□

We can announce the theorem:

Theorem 2.1. ***(The** abc **Theorem)*** *We assume that the conjecture $c < R^{1.63}$ is true. For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).R^{1+\epsilon} \tag{11}$$

where K is a constant depending only of ϵ.

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