An Elementary Proof of the Explicit Formula of Bernoulli Numbers

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Abstract : The aim of this paper is to give an elementary proof to a well-known explicit formula of Bernoulli numbers.

Keywords : Stirling numbers of the second kind, Bernoulli numbers, Bernoulli polynomials.

1 Introduction

The numbers :

$$b_0 = 1, \qquad b_2 = \frac{1}{6}, \qquad b_4 = -\frac{1}{30}, \qquad b_6 = \frac{1}{42},$$

$$b_8 = -\frac{1}{30} \qquad \ldots, \qquad b_1 = -\frac{1}{2}, \qquad b_3 = b_5 = b_7 = b_9 = \cdots = 0$$

are called Bernoulli numbers, they can be defined by the following exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$$

where $|t| < 2\pi$.

It was shown in the 19th century that an explicit formula for b_n is[1]:

$$b_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{i=0}^{k} \binom{k}{i} (-1)^i i^n \qquad (1)$$

Many proofs have been given to formula (1), but we will present here the most simplest of them [2, 3].

2 Stirling numbers of the second kind

Let Y be a function of x, and set :

$$\vartheta^n Y = \underbrace{x(\ldots x(x(x}_{n} Y \overbrace{')')' \ldots)'}^{n}$$

If we expand $D^n Y$ for $n = 1, 2, 3, 4$, we find :

$\vartheta Y = xY'$

$\vartheta^2 Y = xY' + x^2 Y''$

$\vartheta^3 Y = xY' + 3x^2 Y'' + x^3 Y^{(3)}$

$\vartheta^4 Y = xY' + 7x^2 Y'' + 6x^3 Y^{(3)} + x^4 Y^{(4)}$
…

We see that :

$$\vartheta^n Y = S_n^0 Y + S_n^1 xY' + S_n^2 x^2 Y'' + \cdots + S_n^n x^n Y^{(n)} \quad (2)$$

In fact, the numbers S_n^k are called Stirling numbers of the second kind. Formula (2) is called Grunert's formula.

3 The explicit formula of Stirling numbers of the second kind

If we put $Y = e^x$ in the formula (2) we obtain :

$$\vartheta^n e^x = e^x \sum_{k=0}^{n} S_n^k x^k \quad \Longrightarrow \quad e^{-x}.\,\vartheta^n e^x = \sum_{k=0}^{n} S_n^k x^k$$

$$\Longrightarrow \quad \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{\vartheta^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k x^k$$

One can easily prove that $\vartheta^n x^i = i^n x^i$, so :

$$\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{i^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k\, x^k$$

If we expand the left-hand side we obtain :

$$\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\frac{(-1)^{k-i}\binom{k}{i}i^n}{k!}\right)x^k = \sum_{k=0}^{n} S_n^k\, x^k$$

Comparing coefficients in both summations we conclude that :

$$S_n^k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\, i^n \quad (3)$$

4 Relation between Stirling numbers of the second kind and Bernoulli numbers

Putting $Y = x^y$ in the formula (2), we get :

$$\vartheta^n x^y = \sum_{k=0}^{n} S_n^k\, x^k (x^y)^{(k)}$$

We know that $(x^y)^{(k)} = y(y-1)\ldots(y-k+1)x^{y-k}$ and $\vartheta^n x^y = y^n x^y$ so we get :

$$y^n = \sum_{k=0}^{n} S_n^k y(y-1)\ldots(y-k+1) \quad (4)$$

The polynomial $y(y-1)\ldots(y-k+1)$ is called the falling factorial of order k of y. Pochhammer used the symbol $(y)_k$ to denote it, so the formula (4) becomes using Pochhammer symbol :

$$y^n = \sum_{k=0}^{n} S_n^k (y)_k \quad (4')$$

One interesting property of the falling factorial function is the following :

Proposition 1

Let n and y be non-negative integers, then :

$$(y+1)_{n+1} - (y)_{n+1} = (n+1)(y)_n$$

Proof

$$
\begin{aligned}
(y+1)_{n+1}-(y)_{n+1} &= (y+1)y(y-1)\ldots(y-n+1)-y(y-1)\ldots(y-n+1)(y-n) \\
&= [(y+1)-(y-n)]y(y-1)\ldots(y-n+1) \\
&= (n+1)(y)_n
\end{aligned}
$$

We are going to use this property in the proof of the following proposition.

Proposition 2

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$. We have :

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1} \quad (5)$$

Proof

If we sum for y in the formula (4') we find :

$$
\begin{aligned}
\sum_{y=0}^{m-1} y^n = \sum_{y=0}^{m-1}\left(\sum_{k=0}^{n} S_n^k\,(y)_k\right) \quad &\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\sum_{y=0}^{m-1}(y)_k\right) \\
&\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\sum_{y=0}^{m-1}\frac{(y+1)_{k+1}-(y)_{k+1}}{k+1}\right) \\
&\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\frac{(m)_{k+1}-(0)_{k+1}}{k+1}\right)
\end{aligned}
$$

Therefore :

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1}$$

Definition

Let $n \in \mathbb{N}$
Bernoulli's polynomials $B_n(x)$ are defined by the following exponential generating function :

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

One interesting observation to make about Bernoulli's polynomials is that if we put $x = 0$ we get :

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(0) \frac{t^n}{n!}$$

This generating function corresponds to the generating function of Bernoulli numbers b_n. Hence for all $n \in \mathbb{N}$, we have :

$$B_n(0) = b_n$$

Another interesting property of the Bernoulli polynomials is the following :

<u>Proposition 3</u>

Let $n \in \mathbb{N}$

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

<u>Proof</u>

On the one hand :

$$\begin{aligned} \sum_{n=0}^{\infty} \{B_n(x+1) - B_n(x)\} \frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} B_n(x+1) \frac{t^n}{n!}\right) - \left(\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}\right) \\ &= \frac{te^{t(x+1)}}{e^t - 1} - \frac{te^{tx}}{e^t - 1} \\ &= \frac{te^{tx}.e^t - te^{tx}}{e^t - 1} \\ &= \frac{te^{tx}(e^t - 1)}{e^t - 1} \\ &= te^{tx} \end{aligned}$$

On the other hand :

$$\begin{aligned} \sum_{n=0}^{\infty} nx^{n-1} \frac{t^n}{n!} &= \sum_{n=1}^{\infty} t \frac{(xt)^{n-1}}{(n-1)!} \\ &= t \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \\ &= te^{xt} \end{aligned}$$

Comparing coefficients of both summations we conclude that for all $n \in \mathbb{N}$:

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Proposition 4

Let $n \in \mathbb{N}$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$$

Proof

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} &= \frac{te^{tx}}{e^t - 1} \\
&= \frac{t}{e^t - 1} . e^{tx} \\
&= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n-k} \frac{t^{n-k}}{(n-k)!} . \frac{(xt)^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n-k} \binom{n}{k} x^k \right) \frac{t^n}{n!}
\end{aligned}$$

Therefore :

$$B_n(x) = \sum_{k=0}^{n} b_{n-k} \binom{n}{k} x^k$$

Summing for y in the relation $B_{n+1}(y+1) - B_{n+1}(y) = (n+1)y^n$ we obtain :

$$\begin{aligned}
(n+1) \sum_{y=0}^{m-1} y^n &= \sum_{y=0}^{m-1} \left\{ B_{n+1}(y+1) - B_{n+1}(y) \right\} \\
&= B_{n+1}(m) - B_{n+1}(0) \\
&= B_{n+1}(m) - b_{n+1}
\end{aligned}$$

Thus :

$$(n+1) \sum_{y=0}^{m-1} y^n = B_{n+1}(m) - b_{n+1} \quad (6)$$

Comparing formula (5) with formula (6) we conclude that :

$$B_{n+1}(m) - b_{n+1} = (n+1)\sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1} \quad (7)$$

If we develop the expression of $(X)_{k+1}$ in terms of the powers of X we find :

$$\begin{aligned}
(X)_{k+1} &= X(X-1)\dots(X-k) \\
&= X\left(X^k - \frac{k(k+1)}{2}X^{k-1} + \dots + (-1)^k k!\right) \\
&= X\sum_{j=0}^{k} c_j\, X^j \\
&= \sum_{j=0}^{k} c_j\, X^{j+1}
\end{aligned}$$

Therefore :

$$(X)_{k+1} = \sum_{j=0}^{k} c_j\, X^{j+1}$$

If we apply the above formula for $(m)_{k+1}$ in the formula (7) we find:

$$B_{n+1}(m) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1}$$

Substituting also $B_{n+1}(m)$ by its explicit expression, we finally get :

$$\begin{aligned}
\left(\sum_{k=0}^{n+1}\binom{n+1}{k} b_{n+1-k}m^k\right) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1} \quad &\Rightarrow \quad \sum_{k=1}^{n+1}\binom{n+1}{k} b_{n+1-k}m^k = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1} \\
&\Rightarrow \quad \sum_{j=0}^{n}\binom{n+1}{j+1} b_{n-j}m^{j+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1} \\
&\Rightarrow \quad \sum_{j=0}^{n}\left(\binom{n+1}{j+1} b_{n-j}\right)m^j = \sum_{j=0}^{n}\left(\sum_{k=j}^{n} S_n^k \frac{n+1}{k+1} c_j\right)m^j
\end{aligned}$$

We have equality between two polynomials in m, both of degree n, so the coefficients of the terms of the same degree are equal. In particular for $j = 0$ we have :

$$\binom{n+1}{1} b_n = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} c_0 \quad \Rightarrow \quad b_n = \sum_{k=0}^{n} S_n^k \frac{(-1)^k k!}{k+1} \quad (8)$$

To get the explicit expression of b_n, we substitute S_n^k in the above identity by its explicit expression, and after simplification we obtain the remarkable formula (1) for the Bernoullian numbers.

5 Comments

From formula (6) we can deduce Faulhaber's formula, we have :

$$\begin{aligned}
\sum_{y=0}^{m-1} y^n &= \frac{1}{n+1}\{B_{n+1}(m) - b_{n+1}\} \\
&= \frac{1}{n+1}\left\{\left(\sum_{k=0}^{n+1}\binom{n+1}{k} b_{n+1-k}m^k\right) - b_{n+1}\right\} \\
&= \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k} b_{n+1-k}m^k \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j+1} b_{n-j}m^{j+1} \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j} b_j m^{n-j+1}
\end{aligned}$$

We can deduce identity (8) directly from the explicit formula of Stirling numbers of the second kind. We know from formula (3) that for all $0 \leq k \leq n$:

$$k!\,S_n^k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n$$

If we invert the above formula using the binomial inversion theorem we find that :

$$k^n = \sum_{i=0}^{k} S_n^i (k)_i$$

This formula is similar to formula (4') with the exception that the sum is taken here from 0 to k, this is valid only for $k \in \{0, 1, \dots, n\}$, while in formula (4') the sum was taken from 0 to n, and that was valid for every real number y.

Now summing for k in the last formula we obtain :

$$\begin{aligned}
\sum_{k=0}^{n} k^n &= \sum_{k=0}^{n}\left(\sum_{i=0}^{k} S_n^i (k)_i\right) \\
&= \sum_{i=0}^{n} S_n^i \sum_{k=i}^{n} (k)_i \\
&= \sum_{i=0}^{n} S_n^i \left(\frac{(n+1)_{i+1} - (i)_{i+1}}{i+1}\right) \\
&= \sum_{i=0}^{n} S_n^i \frac{(n+1)_{i+1}}{i+1} \\
&= \sum_{i=0}^{n} S_n^i \frac{1}{i+1} \sum_{j=0}^{i} c_j \,(n+1)^{i+1} \\
&= \sum_{j=0}^{n}\left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1}\right)(n+1)^{j+1}
\end{aligned}$$

Thus we have :

$$\sum_{k=0}^{n} k^n = \sum_{j=0}^{n}\left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1}\right)(n+1)^{j+1}$$

Using Faulhaber's formula we conclude that :

$$\sum_{j=0}^{n}\left(\frac{\binom{n+1}{j+1}}{n+1} b_{n-j}\right)(n+1)^{j+1} = \sum_{j=0}^{n}\left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1}\right)(n+1)^{j+1}$$

The coefficients of $(n+1)$ in both representations are equal so :

$$\frac{\binom{n+1}{1}}{n+1} b_n = \sum_{i=0}^{n} S_n^i \frac{c_0}{i+1} \quad \Rightarrow \quad b_n = \sum_{i=0}^{n} S_n^i \frac{(-1)^i i!}{i+1}$$

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